The optimum aeroplane and beyond

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ABSTRACT

A summary of the ways in which aviation impacts the environment is presented and the ratio of the energy liberated during a flight to the revenue work done (*ETRW*) is identified as a key indicator in the assessment of environmental impact. Using the 'Breguet' range equation, a number of theorems relating to *ETRW* are derived and discussed. This is followed by an approximate analysis to produce estimates for the *ETRW* of aircraft currently in service. It is found that the global fleet average value for *ETRW* is much higher than those estimated for existing individual aircraft. An explanation of the difference is presented, with the contributions from airline operations and air traffic management identified and quantified. Consideration is then given to the potential for future reduction in *ETRW* through advances in materials, alternative fuels, structures, aerodynamics and propulsion technologies and the likely benefits are quantified. The improvement in *ETRW* that could be achieved if this parameter was minimised in the design process with the current level of technology is also considered. Finally, the likelihood of performance improvements being introduced in the short, medium and long term is briefly discussed.

NOMENCLATURE

EI	Emission Index (mass of emission/mass of fuel burned)
ETRW	ratio of energy liberated to revenue work done
g	acceleration due to gravity (9·81m/s^{-2})
LCV	lower calorific value of fuel (≈ 43MJ/kg for kerosene)
L/*D*	aircraft lift to drag ratio
LF	load factor (actual payload mass/maximum payload mass)
LFC	cargo load factor (actual cargo mass/maximum possible cargo mass)
LFP	passenger load factor (actual passenger number/maximum possible)
MFNC	mass of fuel carried on journey, but not consumed (reserve + tankered)
ML	mass of aircraft on landing
MMF	mass of the mission fuel (fuel actually burned on trip)
MMP	maximum payload mass (passengers + cargo)
MMTO	maximum permitted take-off mass
MMZF	maximum zero fuel mass (maximum mass of aircraft + payload)
MOE	aircraft operational empty mass (no payload and no fuel)
MP	payload mass (passengers + cargo)
MZF	zero fuel mass (mass of aircraft + payload only)
R	great circle distance between departure point and destination (km)
X	non-dimensional range ($R.g/(LCV.\eta_o L/D)$)
α	mission fuel mass/take-off mass
β	mass of fuel carried, but not consumed/take-off mass
ε	fuel burned minus fuel needed for optimum cruise over same distance
η	efficiency

1.0 INTRODUCTION

This year is being celebrated as the 100th anniversary of British aviation. It is British aviation in the sense of piloted, heavier-than-air machines propelled by internal combustion engines flying through British skies. Alliott Roe was, unofficially, but widely acknowledged to be, the first British citizen to fly a British built aircraft in Britain, having made several short hops at Brooklands in June 1908. However, it was Samuel Cody, an American, who, on 16 October, made the first substantial, officially recognised flight in a British built machine. At the same time, in addition to these practical developments, British aviation science took an important step forward with the publication of Frederick Lanchester's book *Aerodonetics*[1]. This was a companion to his book *Aerodynamics*, which had been published in 1907. These two volumes constituted the first formal attempt to summarise the state of knowledge in the theoretical and practical subject of aeronautics. All these events served to establish Great Britain's strong presence in the field of aeronautics and provided the solid foundations upon which a great many more British developments would be based.

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However, although this year is the 100th anniversary of British flying machines, it is not the 100th anniversary of British aviation vision. Two hundred years ago, Sir George Cayley[2] was developing his concept of aviation. He recognised that, in order to realise its full potential, the Industrial Revolution needed transportation — transportation to bring raw materials to the factories and transportation to take the products to the market places and to customers. He saw that, no matter how sophisticated, or how well developed, the traditional methods of road, rail and sea might become, they were all ultimately limited in their capability. However, Cayley realised that the one medium that joined every place on Earth was the air and that the development of a practical air transportation system would bring huge economic benefits.

It was Cayley who set in motion the process that would eventually lead to a world linked by fleets of aircraft, travelling great distances at high speed and enabling and supporting wealth creation on a global scale. While it has taken 200 years to materialise, Cayley's vision of global air transport has become a reality. Today, civil aviation is a major international business and it is an essential part of the world's commercial infrastructure.

2.0 AVIATION TODAY AND IN THE FUTURE

During a 25-year working life, a typical, commercial aircraft will fly for 100,000 hours and cover some 40m miles. It will support wealth creation by providing rapid access to traditional markets and open up new markets that are completely inaccessible by other forms of transport. Aviation brings significant benefits to some of the world's poorest countries through tourism (the world's largest industry) and by giving local producers access to markets in richer parts of the world. Travel fosters better international understanding and facilitates cultural exchange, both of which help to reduce the risk of dispute or conflict. It also allows important social links to be maintained and last, but by no means least, it contributes to improved quality of life by providing a wider, affordable choice of holiday destinations.

Aviation's benefits have resulted in an ever increasing demand for air transport. In 1995, the global airline industry flew approximately 3·5 trillion available seat kilometres for passengers and approximately 200bn available tonne kilometres for freight, with 50% of the freight capacity being provided by dedicated freighter aircraft. By 2025, the forecasts[3,4] suggest that these figures will have risen to 12 trillion available seat kilometres and 1 trillion available tonne kilometres — a very substantial increase by any standards. This growth is also expected to vary from region to region e.g. China is expected to grow at almost 9% pa and Southeast Asia by 8% pa, while North America is expected to grow at 3% pa and Europe by 3·5% pa.

Aircraft operators and manufacturers expect that, in the process of delivering this increase in capacity, 50% of the current fleet of 26,000 aircraft will be replaced and 17,000 additional aircraft will be needed. According to Refs 3 and 4, these new aircraft are expected to be distributed along the following lines — regional jets (10%), single aisle (65%), twin aisle (20%) and very large aircraft (5%).

3.0 AVIATION AND THE ENVIRONMENT

Over the past 20 years, there has been a growing awareness of, and a growing concern about, the impact of anthropological activity on the environment. In 1988, the World Meteorological Organization and the United Nations Environment Programme established the Intergovernmental Panel on Climate Change (IPCC). This body provides independent information through a series of reports based upon scientific evidence and it reflects the consensus (or as near consensus as possible) views of the broad scientific community. IPCC links scientists, and other experts, from all over the world and their work is subject to strenuous, and wide ranging, peer review.

Since 1988, the IPCC has issued four assessment reports with the most recent being published in 2007. These have charted a progressive convergence of evidence based, scientific opinion that anthropological emissions of greenhouse and other gases are having a major impact on the characteristics of the Earth's atmosphere. It has been concluded that the greenhouse effect is responsible for the observed steady increase of the global average surface temperature; the parameter that is a measureable physical characteristic of climate change. The inescapable conclusion is that the continued release of greenhouse gases through human activity is producing a situation in which major and irreversible changes are likely to occur e.g. the melting of the Arctic ice and the Greenland ice sheet. These events will be accompanied by a rising sea level and will probably produce major climate change.

According to the IPCC[5], carbon dioxide is the most important anthropogenic greenhouse gas, with a current annual emission rate of about 38 gigatonnes ($3{\cdot}8 \times 1{,}010$ tonnes). Approximately 73% of this comes from the burning of fossil fuels and about 20% of all fossil fuel is used to power transportation systems.

Carbon dioxide emissions are important because a significant percentage of the gas from each emission event (of the order 50%) stays in the atmosphere for a very long time, possibly for centuries. Hence, for all practical purposes, carbon dioxide accumulates in the atmosphere and its climate effects persist long after the emission itself took place.

The total mass of air in the global atmosphere is about 5,200 teratonnes ($5{\cdot}2 \times 10^{15}$ tonnes) and chemical analysis of present day air indicates that it contains about 3 teratonnes of carbon dioxide. The IPCC research suggests that, if global mean surface temperature rise is to be stabilised at about 3°C above pre-industrial levels, with a corresponding rise of about 1m in the mean sea level, then the amount of carbon dioxide in the atmosphere should not be allowed to exceed approximately 3·8 teratonnes. This is an absolute level for carbon dioxide from all sources. The clear implication is that a simple reduction in the rate of emission of anthropological carbon dioxide may not be sufficient to contain the problem and that emissions may need to go to zero at some point in the not too distant future.

At present, civil aviation uses the kerosene burning, gas turbine as its prime mover. The kerosene that is used is derived, in the main, from oil, but a little is synthesised from coal. Combustion of kerosene produces carbon dioxide, water vapour, a mixture of nitric oxide and nitrogen dioxide (NO_X) and some aerosols in the form of carbon and sulphur particulates. The relationship between the mass of fuel consumed and the mass of the individual species produced is determined by the chemistry of the combustion process. The ratio of masses is usually expressed as an Emission Index, or *EI*, and in the case of carbon dioxide —

$$EI_{CO_2} = \frac{massCO_2}{MassFuel} = \frac{44{\cdot}01}{\left(12{\cdot}01 + \left(Y/X\right)\right)} \quad \ldots (1)$$

where Y is the average number of hydrogen atoms and X the average number of carbon atoms in a 'molecule' of hydrocarbon fuel. In the case of kerosene, $(Y/X) \approx 1{\cdot}91$ and

$$EI_{CO_2} \approx 3{\cdot}16 \quad \ldots (2)$$

Therefore, for every kilogramme of kerosene burned, 3·16kg of carbon dioxide are produced and emitted into the atmosphere.

This year the civil aviation industry will consume about 220m tonnes of kerosene and this will produce 700m tonnes (0·0007 teratonnes) of carbon dioxide. This is equivalent to a 0·023% increase in the level of carbon dioxide in the atmosphere.

If we now consider the anticipated demand for air transport, the industry forecasts suggest a global average traffic growth rate of 5% pa. However, as new, higher technology aircraft are introduced and older aircraft are retired, the global fleet becomes progressively more efficient and so the growth rate for fuel burn is expected to be somewhat lower. Historical data suggest the growth rate for fuel

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burn is about 1·5% less than that for the traffic, i.e. a fuel burn growth rate of 3·5% is expected. Therefore, by 2030, we could see the annual aviation fuel burn increase to 470m tonnes and, if this growth rate continued until 2050, the fuel burn rate could be almost 1bn tonnes per annum. In terms of carbon dioxide emissions, the annual rate would be 1·5bn tonnes in 2030, potentially rising to over 3bn tonnes by 2050. Furthermore, since the carbon dioxide would be accumulating, by 2050 aviation could have increased the total amount on carbon dioxide in the atmosphere by some 70bn tonnes (0·07 teratonnes), or a 2·3% increase over the current level. Therefore, if there was an international agreement to limit the total amount of carbon dioxide in the atmosphere to 3·8 teratonnes, then between now and 2050, if kerosene continues as the fuel of choice, global aviation alone could account for up to 8% of the remaining 0·8 teratonnes quota.

The other emissions from the kerosene burning, gas turbine do not remain in the atmosphere for long and, consequently, only have an environmental impact for a short time. Nevertheless, while they last, the magnitude of the impact can be large. NO_X has an indirect effect on global warming because it affects the concentrations of two other greenhouse gases, namely ozone and methane. Methane is destroyed and this produces a cooling effect, while ozone is created and this has a warming effect. At a global level, these opposing effects almost cancel out. However, the methane imbalance appears in both the northern and southern hemispheres, while the ozone imbalance appears mainly in the northern hemisphere. The impact of an emission of NO_X on global warming largely dissipates after a few months, although the small residual perturbation to methane persists for a long time.

Perhaps a more significant fact is that NO_X is harmful to humans and there is already EU legislation in place that specifies the maximum safe exposure limits. Therefore, NO_X emissions near the ground are important, since they affect airport "local air quality". Consequently, NO_X generation and its local dispersion during ground operations and during the landing and take-off cycle need to be understood and quantified. There are also health concerns associated with particulate material (PM). So called primary PM comes directly from engine soot and tyre smoke, while secondary PM is formed in the atmosphere by the action of the emissions of NO_X and the oxides of sulphur (SO_X). Therefore, the sources and dispersion characteristics of PM also need to be understood and quantified.

In terms of the emission index, the amount of NO_X produced depends upon both the fuel and the design of the engine combustion chamber. Therefore, the precise relationship between the amount of fuel burned and the amount of NO_X produced depends upon the type of engine. However, typically,

$$EI_{NOX} \approx 0{\cdot}015 \quad \ldots (3)$$

Although water vapour is itself a greenhouse gas, the amount emitted by the gas turbine is relatively small and the resulting direct impact upon global warming is low. Similarly, soot particles have a small warming effect, while sulphur particles have a small cooling effect. However, when the atmospheric conditions are right i.e. when the air through which the aircraft is flying is supersaturated with respect to ice, the water efflux from the engine may produce a contrail[6]. Relative to the size of the aircraft, contrails are massive features and, although the water from the combustion of the fuel initiates the contrail, the vast majority (99%) of the visible ice comes from water in the atmosphere. The contrail influence on radiative forcing depends upon whether it is day or night, but on balance the net impact is a warming effect. If the contrail is persistent, it may evolve over time into 'Contrail Cirrus' cloud, or the particulates emitted from the engine may trigger the formation of a cloud without the need for a contrail. Either way, the formation of clouds produces a warming effect that may be very strong.

It is useful to note that contrail formation could be virtually eliminated by avoiding air that is supersaturated with respect to ice. At present, the optimum cruise altitudes for civil transport aircraft coincides almost exactly with the altitudes most likely to carry supersaturated air. Over the UK, this critical altitude is about 10km and the likelihood of creating a contrail is considerably reduced by flying significantly higher or lower.

4.0 ECONOMIC EFFICIENCY

The operation of airlines is dominated by economic considerations and the economics of airline operations is an extremely complex subject. However, there are some parameters that are guaranteed to be important. A clear case in point is the ratio of the revenue generated to the cost of energy expended during the flight i.e.

$$\frac{revenue\ generated}{cost\ of\ energy\ used} = \frac{A}{B}\left(\frac{MP.g.R}{MMF.LCV}\right) \quad \ldots (4)$$

with

A	= revenue/unit payload weight/unit distance travelled
MP	= payload mass
g	= acceleration due to gravity
R	= the great circle distance between departure point and destination
B	= the fuel cost/unit of energy released
MMF	= the mass of fuel consumed
LCV	= the fuel lower calorific value

The elements A and B are dependent upon market forces and will vary during the working life of the aircraft. However, the term in brackets is something that is fixed by the aircraft's designers and is constant throughout the life of the aircraft.

In general, the best economic operating condition is determined through a balance between the cost of fuel and the cost of time, as currently exemplified through the Boeing cost index (CI) approach[7]. However, in the future and for a whole variety of reasons, the likelihood is that the price of fuel will rise and so fuel will become an increasingly large element in the aircraft direct operating cost ($CI \rightarrow 0$). Therefore, achieving a high value of revenue generated over energy cost will become progressively more important and the market will demand aircraft with progressively smaller values of the energy liberated to revenue work ratio ($ETRW$), where

$$ETRW = \frac{MMF.LCV}{MP.g.R} \quad \ldots (5)$$

This parameter has already been identified by some authors, e.g. Hileman *et al*[8], as a key metric, not only for the investigation of airline economics but also for comparisons between different modes of transport.

5.0 ENVIRONMENTAL IMPACT

In the context of an aircraft's impact on the environment, we may define a coefficient of environmental performance, CEP, as the amount of emission released per unit of revenue work performed, i.e.

$$CEP = \frac{emissions\ mass.LCV}{useful\ work\ done} \quad \ldots (6)$$

In terms of the chemical species emitted, this becomes

$$CEP = \left(ETRW\right)\left(a.EI_{CO_2} + b.EI_{NO_X} + c.EI_{H_2O} + d.EI_{SO_X} + \ldots\ldots\right) \quad \ldots (7)$$

where the constants a, b, c, d, etc. and the Emission Indices depend upon both the fuel being used and the level of engine technology. For example, if the fuel is oil-derived kerosene, a, b, c and d will be unity, but, if biomass-derived kerosene is being used, a could be less than unity, since

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the net carbon dioxide addition to the atmosphere could be lower than that for oil-derived kerosene. Similarly, if the fuel was hydrogen then the Emission Indices for carbon dioxide and SO_X would be zero.

Notwithstanding the values of the constants and the Emission Indices, the CEP is lowest when the value of ETRW is as low as possible. However, in the context of aviation, it is sometimes necessary to consider both the amount of emission and the altitude at which it takes place. The altitude is an important parameter for NOX chemistry. It is also important for water vapour induced contrails, since the appropriate atmospheric conditions tend to occur at specific altitudes.

6.0 THE OPTIMUM AEROPLANE

From the very beginning of aviation, the design process of any aircraft has begun with a specification. For a given level of technology, one group of parameters that is sufficient to specify a complete civil aircraft configuration is

- a passenger payload
- a range at the specified payload
- the engine characteristics and number of engines; and
- the length of runway to be used

In addition, there will be a number of constraints that have to be satisfied. The most important, and the most comprehensive, of these are the regulator requirements for safety, i.e. compliance with the provisions of an appropriate design code. However, in certain instances there may be others e.g. a specific cruise Mach number, airport limits for wingspan, pavement weight or noise restrictions or en-route height limitations for particular customers.

Once these have been taken into account, the design process is an optimisation involving one, or more, target functions. Simple examples of possible target functions are maximum payload fraction, i.e. the smallest aircraft mass to carry a given load, or minimum fuel burn.

In the past, the target functions and the constraints have varied depending upon the economic and regulatory conditions in force at the time. However, as already explained, in the future economic and environmental forces are likely to become strongly aligned in favour of minimising the *ETRW* and flying in a way that avoids, or at least minimises, the formation of contrails.

7.0 WHAT DETERMINES THE VALUE OF THE *ETRW*?

The *ETRW* depends upon the amount of fuel burned to carry a given payload a given distance. As the flight progresses, the total mass of the aircraft decreases as fuel is burned, i.e.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\mathrm{d}m_f}{\mathrm{d}t} = -\dot{m}f \qquad \ldots (8)$$

where M and m_f are the instantaneous masses of the total aircraft and the fuel respectively. In addition, the overall thermodynamic efficiency, η_o, of the propulsion system is given by

$$\eta_0 = \frac{(T.V_\infty)}{(\dot{m}f.LCV)} \qquad \ldots (9)$$

where T is the thrust developed and V_∞ (= dS/dt) is the flight speed.

When the aircraft is in the cruise condition, the aerodynamic lift is equal to the total aircraft weight and the engine thrust is equal to the drag, D, i.e.

$$-\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}S}\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\mathrm{d}M}{\mathrm{d}S}V_\infty = \frac{(Mg.V_\infty)}{LCV.(\eta_0.L/D)} \qquad \ldots (10)$$

The value of $\eta_o(L/D)$ depends upon the Mach number and the altitude at which the aircraft is flown. For a given aircraft, there is a Mach number-altitude trajectory that gives a constant value of $\eta_o(L/D)$ in cruise. For a turbo-fan powered aircraft, this is a constant Mach number, constant lift coefficient (cruise climb) trajectory. With $\eta_o(L/D)$ constant, the equation may be integrated to obtain the total fuel consumed as the aircraft cruises between two points separated by a total great circle distance, R, —

$$\frac{(MF)_{cruise}}{M_1} = 1 - EXP\left(-\frac{(g.R/LCV)}{(\eta_0.L/D)}\right) \qquad \ldots (11)$$

where M_1 is the mass of the aircraft at the beginning of cruise.

However, any flight between two points on the surface of the Earth involves a take-off, a climb, a cruise, a descent and a landing. This means that, in reality, more fuel will be consumed in flying from the departure point to the destination than would be required to cruise the same distance. If this additional fuel is designated by Δmf, where

$$\frac{\Delta mf}{MTO} = \varepsilon = 1 - k \qquad \ldots (12)$$

and MTO is the mass of the aircraft at the beginning of the take-off run, the total (mission) fuel, MMF, consumed on a trip between two locations separated by a great circle distance, R, is

$$\frac{MMF}{MTO} = 1 - kEXP\left(-\frac{g.R}{LCV(\eta_o L/D)}\right) \qquad \ldots (13)$$

In practice, ε has a value of about 0·025. However, the performance of the aircraft is very sensitive to this parameter.

If we introduce a non-dimensional range, X, where

$$X = \frac{g.R}{LCV.(\eta_0.L/D)} \qquad \ldots (14)$$

then,

$$\frac{MMF}{MTO} = 1 - kEXP(-X) = \alpha \qquad \ldots (15)$$

This is a form of the classic range equation originally derived during the First World War and frequently attributed to Louis Bréguet, the French aviation pioneer, although other authors have been associated with it[8]. It is important to note that this equation is exact, for the conditions specified, and it applies to all aircraft irrespective of their configuration.

Turning now to the mass of the aircraft, the total mass at the beginning of the take-off run, MTO, is made up of the following components,

$$MTO = MOE + MP + MMF + MF_{NC} \qquad \ldots (16)$$

where MOE is the operational empty mass (mass of everything except payload and fuel), MP is the payload mass (passengers + 'belly' freight), MMF is the mass of mission fuel (fuel consumed during flight) and MF_{nc} is the mass of fuel carried, but not consumed, i.e. reserve fuel plus tankered fuel, where

$$MF_{NC} = \beta \cdot MTO \qquad \ldots (17)$$

The total mass at the end of the landing run, ML is given by

$$ML = MOE + MP + MF_{NC} = MTO(1-\alpha) \qquad \ldots (18)$$

Similarly, the zero fuel mass, MZF, is

$$MZF = MOE + MP = ML - MF_{NC} = MTO(1-\alpha-\beta) \qquad \ldots (19)$$

Hence, the fuel burn per unit payload per unit non-dimensional

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distance flown is given by

$$\frac{MMF}{MP.X}=\left(\frac{MZF}{MP}\right)\left(\frac{1}{X}\frac{\left(1-kEXP\left(-X\right)\right)}{\left(kEXP\left(-X\right)-\beta\right)}\right) \quad \ldots (20)$$

and this is related directly to the *ETRW* since

$$ETRW=\left(\frac{MMF}{MP.X}\right)\Big/\left(\eta_o.\frac{L}{D}\right) \quad \ldots (21)$$

These equations are also exact for the conditions specified. They do not depend upon the aircraft configuration, i.e. they apply to all flying machines, and it is possible to use them to derive a number of theorems that are relevant to the optimisation problem.

Theorem 1
Consider a given aircraft travelling a fixed distance and cruising at constant $\eta_o(L/D)$, i.e. α is constant. It follows that

$$\frac{MMF}{MZF+MF_{NC}}=\frac{MMF}{ML}=\frac{\mathrm{d}MMF}{\mathrm{d}ML}=\frac{\alpha}{\left(1-\alpha\right)} \quad \ldots (22)$$

and, hence,

$$\frac{\mathrm{d}MMF}{MMF}=\frac{\mathrm{d}\left(MZF+MF_{NC}\right)}{\left(MZF+MF_{NC}\right)}=\frac{\mathrm{d}ML}{ML} \quad \ldots (23)$$

Therefore, if the mass of the aircraft is changed in any way, the percentage change in the fuel actually burned to carry out the flight is equal to the percentage change in the sum of the aircraft zero fuel mass and the fuel carried but not burned.

Theorem 2
If the load factor, *LF*, is defined as the ratio of the actual payload mass to the maximum permitted payload mass, *MMP*, i.e.

$$LF=\frac{MP}{MMP} \quad \ldots (24)$$

then

$$\frac{MZF}{MP}=\frac{MOE+MP}{MP}=\left(\frac{MOE}{MMP}\right)\left(\frac{1}{LF}\right)+1 \quad \ldots (25)$$

Hence, for a given aircraft travelling a fixed distance, cruising at constant $\eta_o(L/D)$, the minimum value of *ETRW* occurs when the payload has its maximum possible value (*LF* is unity) and the fuel carried, but not consumed, has its minimum possible value i.e. the minimum reserve required by law.

Theorem 3
Consider what happens to the *ETRW* of an aircraft cruising at constant $\eta_o(L/D)$ with a fixed ratio of zero-fuel mass to payload mass, *MOE*/*MP*, i.e. constant *MOE*/*MP*, as more fuel is added and the aircraft flies further and further.

A typical result is shown in Fig. 1.

Under these conditions, the best value of *ETRW* occurs at a value of *X* that depends only upon ε (the fuel used over and above the cruise value for the same range) and β (the fuel carried, but not used).

Since ε is small (~0·025) and *X* is typically less that 0·4, using the expression derived in the Appendix –

$$\frac{MMF}{MP.X}\approx\frac{\left(MZF/MP\right)}{\left(1-\beta\right)}\left(\frac{\varepsilon}{X}+\left(1+\varepsilon\frac{\left(1+\beta\right)}{\left(1-\beta\right)}\right)+\frac{X}{2}\left(\left(\frac{1+\beta}{1-\beta}\right)+\varepsilon\left(\frac{1+4\beta+\beta^2}{\left(1-\beta\right)^2}\right)\right)+.....\right) \quad \ldots (26)$$

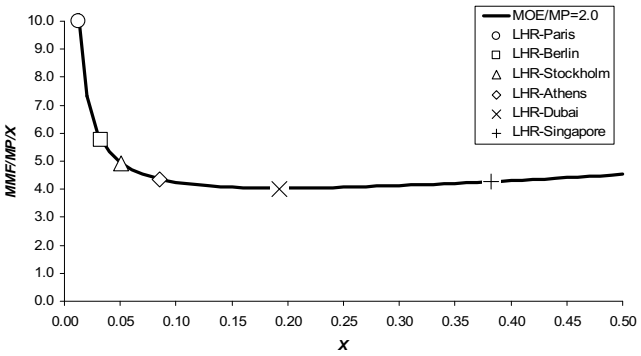


Figure 1. Variation of fuel burn per unit payload per unit non-dimensional distance with non-dimensional range, *MOE*/*MP* = 2, ε = 0·025 and β = 0·05.

Therefore, the range for best *ETRW* is

$$R\approx\left(\frac{LCV\left(\eta_o\,L/D\right)}{g}\right)\left(1-\beta\right)\left(\frac{2\varepsilon}{1-\beta^2}\right)^{1/2}\left(1-\left(\frac{1+4\beta+\beta^2}{4}\right)\left(\frac{2\varepsilon}{1-\beta^2}\right)\right) \quad \ldots (27)$$

and, if β is small (< 0·1),

$$R\approx\left(\frac{LCV\left(\eta_o\,L/D\right)}{g}\right)\left(2\varepsilon\right)^{1/2}\left(1-\frac{1}{2}\left(2\beta+\varepsilon\right)\right) \quad \ldots (28)$$

Theorem 4
For an aircraft cruising at constant $\eta_o(L/D)$ and carrying the maximum permissible payload, *MMP*, the best value of *ETRW* is given by

$$ETRW\approx\left(\frac{MMZF/MMP}{\eta_0\,L/D\left(1-\beta\right)}\right)\left(1+\left(1+\beta\right)\left(\left(\frac{2\varepsilon}{1-\beta^2}\right)^{1/2}+\frac{\left(1+\beta\right)}{2}\left(\frac{2\varepsilon}{1-\beta^2}\right)+\left(\frac{1+4\beta+\beta^2}{4}\right)\left(\frac{2\varepsilon}{1-\beta^2}\right)^{3/2}\right)\right) \quad \ldots (29)$$

or, if β is small,

$$ETRW=\left(\frac{MMZF}{MMP}\right)\left(\frac{1+\varepsilon+\beta}{\eta_0\,L/D}\right)\left(1+\left(2\varepsilon\right)^{1/2}\left(1+\frac{1}{2}\left(2\beta-\varepsilon\right)\right)\right) \quad \ldots (30)$$

For a typical optimum, long range flight

$$ETRW\approx1\cdot3\left(\frac{MMZF/MMP}{\eta_0\left(L/D\right)}\right) \quad \ldots (31)$$

Under these conditions, the influence of the aircraft characteristics and the operational characteristics are separated and the absolute minimum value of *ETRW* is obtained when the product of the structural efficiency (*MMP*/*MMZF*), the propulsion efficiency (η_∞) and the aerodynamic efficiency (*L*/*D*) is maximised.

In general, these three efficiencies are interrelated in a complex way.

Theorem 5
Consider what happens to the *ETRW* of a given aircraft operating at a fixed take-off mass, i.e. constant *MOE*/*MTO*, and cruising at constant $\eta_o(L/D)$, as more fuel is added and the aircraft flies further and further. However, since the take-off mass is fixed, as more fuel is added, the payload mass must be reduced. Hence,

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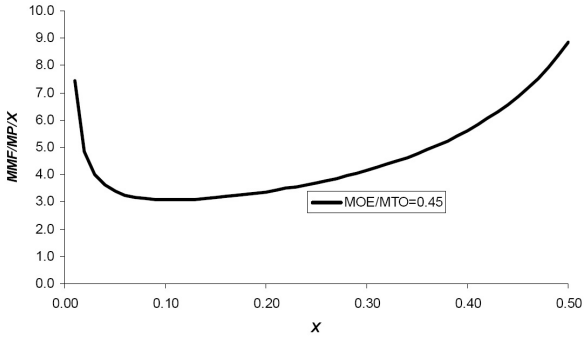


Figure 2. Variation of fuel burn per unit payload per unit non-dimensional distance with non-dimensional range, *MOE*/*MTO* = 0·45, ε = 0·025 and β = 0·05.

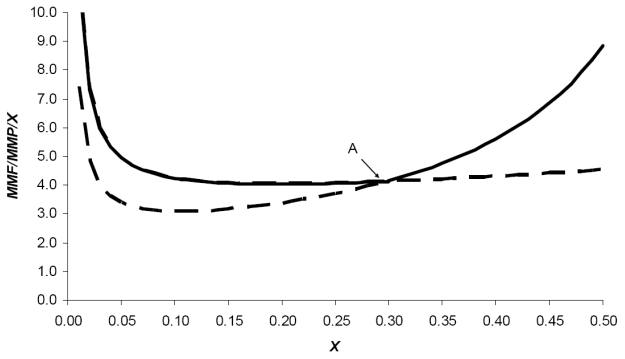


Figure 3. Variation of fuel burn per unit payload per unit non-dimensional distance with non-dimensional range, *MOE*/*MMP* = 2, *MOE*/*MMTO* = 0·45, ε = 0·025 and β = 0·05.

$$LF = \frac{MTO}{MMP}\left(1-\left(\beta+\alpha+\frac{MOE}{MTO}\right)\right) \quad \ldots (32)$$

Therefore,

$$\frac{MZF}{MP} = \frac{1-(\beta+\alpha)}{1-\left(\beta+\alpha+\frac{MOE}{MTO}\right)} \quad \ldots (33)$$

and

$$\frac{MMF}{MP.X} = \left(\frac{1}{X}\right)\left(\frac{1-kEXP(-X)}{\left(kEXP(-X)-\left(\beta+\frac{MOE}{MTO}\right)\right)}\right) \quad \ldots (34)$$

In this case, the best *ETRW* occurs when *MTO* has its largest possible value i.e. the maximum take-off mass, *MMTO*, and β has its minimum possible value.

Theorem 6

The variation of *ETRW* with range for an aircraft with a fixed value of *MOE*/*MTO* is given in Fig. 2. For a given aircraft with a fixed take-off mass, the best *ETRW* occurs at a value of *X* that depends upon ε (the fuel used over and above the cruise value for the same range), β (the fuel carried, but not used) and the ratio *MOE*/*MTO*

Using the result derived in the Appendix, the range for best *ETRW* is found to be

$$R \approx \left(\frac{LCV(\eta_0\, L/D)}{g}\right)\left(1-\left(\beta+\frac{MOE}{MTO}\right)\right) \quad \ldots (35)$$

$$\times\left[\ 2\varepsilon\Big/\left(1-\left(\beta+\frac{MOE}{MTO}\right)^2+\varepsilon\left(1+4\left(\beta+\frac{MOE}{MTO}\right)+\left(\beta+\frac{MOE}{MTO}\right)^2\right)\right)\right]^{1/2}$$

Theorem 7

For an aircraft operating at its maximum take-off mass and cruising at constant $\eta_o(L/D)$, the best value of *ETRW* is given by

$$ETRW \approx \left(\frac{1}{\eta_0\, L/D}\right)\left(\frac{\left(1-\left(\beta+\frac{MOE}{MMTO}\right)+\varepsilon\left(1+\frac{MOE}{MMTO}\right)\right)}{\left(1-\left(\beta+\frac{MOE}{MMTO}\right)\right)^2}+\right. \quad \ldots (36)$$

$$\left.\left(\frac{2\varepsilon\left(1-\left(\beta+\frac{MOE}{MMTO}\right)^2+\varepsilon\left(1+4\left(\beta+\frac{MOE}{MMTO}\right)+\left(\beta+\frac{MOE}{MMTO}\right)^2\right)\right)}{\left(1-\left(\beta+\frac{MOE}{MMTO}\right)\right)^4}\right)^{1/2}\right)$$

Under these conditions, the influence of the aircraft characteristics and the operational characteristics are not separated and the absolute minimum value of *ETRW* is obtained when the structural efficiency, *MMTO*/*MOE*, and the product of the propulsive and aerodynamic efficiencies, $\eta_o(L/D)$, have their largest values.

Theorem 8

As shown in Fig. 3, the complete ETRW variation with range for any real aircraft is determined by a combination of curves of the types shown in Figs 1 and 2. When the payload and the take-off masses have their maximum values, the solid line defines the operating boundary for the aircraft.

The point of intersection of the two curves is labelled point 'A'. Here,

$$\alpha_A = (1-\beta)-\frac{MMZF}{MMTO} \quad \ldots (37)$$

and

$$X_A = \ln\left(\frac{1-\varepsilon}{\beta+\frac{MMZF}{MMTO}}\right) \quad \ldots (38)$$

By expanding EXP(-*X*) as a power series in ascending powers of *X* and neglecting terms of order X^3 and above, an approximation to X_A is given by.

$$X_A \approx 1-\left(\frac{2}{(1-\varepsilon)}\left(\beta+\frac{MMZF}{MMTO}\right)-1\right)^{1/2} \quad \ldots (39)$$

Therefore, for an aircraft operating at fixed values of $\eta_o(L/D)$, ε and β the curves of *ETRW versus X* at maximum payload and maximum take-off mass cross only once. The point of intersection is determined by the ratio of the maximum zero fuel mass to the maximum take- off mass, *MMZF*/*MMTO*.

Theorem 9

When the distance flown is less than X_A, the minimum *ETRW* is determined by the curve for maximum permissible payload, *LF* = 1. While, for distances greater than X_A, the minimum *ETRW* is determined by the curve for maximum take-off mass, *LF* < 1. In the space above the solid line, both the payload mass and the take-off mass are always less than the maximum permitted values.

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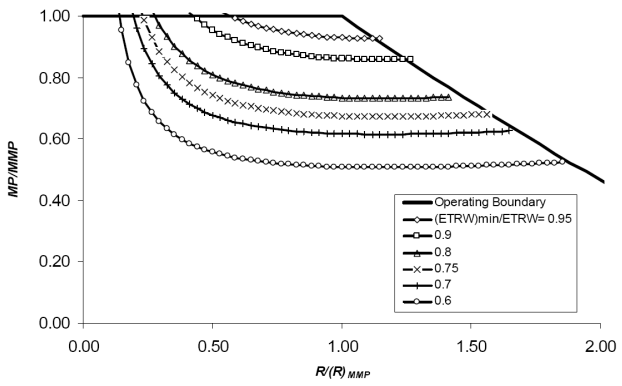


Figure 4. Variation of normalised *ETRW* with normalised range, *MOE*/*MMTO* = 0·55, ε = 0·025 and β = 0·05.

At point *A*,

$$\left(\frac{MMF}{MMP.X}\right)_A = \frac{MMTO}{MMP}\frac{1}{X_A}\left(1-\beta-\frac{MMZF}{MMTO}\right) \quad \ldots (40)$$

i.e.

$$(ETRW)_A \approx \frac{(MMTO/MMP)}{(\eta_0 L/D)}\left(\frac{1-\beta-(MMZF/MMTO)}{1-\left(\frac{2}{(1-\varepsilon)}\left(\beta+\frac{MMZF}{MMTO}\right)-1\right)^{1/2}}\right) \quad \ldots (41)$$

For an aircraft operating at fixed values of $\eta_o(L/D)$, ε and β, the value of *ETRW* at point *A* is determined by the ratio of maximum zero fuel mass to maximum take-off mass, *MMZF*/*MMTO*, and the ratio of maximum zero fuel mass to maximum payload mass, *MMZF*/*MMP*.

Theorem 10

For a given aircraft, the minimum achievable ETRW occurs at point *A* when the value of *X* for minimum ETRW at the maximum permissible payload mass (Theorem 3) is greater than the value of X at point *A* (Theorem 8). Therefore, the best possible value of ETRW occurs at point A when

$$(1-\beta)\left(\frac{2\varepsilon}{1-\beta^2+\varepsilon(1+4\beta+\beta^2)}\right)^{1/2} \geq 1-\left(\frac{2}{(1-\varepsilon)}\left(\beta+\frac{MMZF}{MMTO}\right)-1\right)^{1/2} \quad \ldots (42)$$

or, if ε is small compared to 1,

$$\frac{MMZF}{MMTO} \geq (1-\beta)\left(1-\left(\frac{2\varepsilon}{1-\beta^2}\right)^{1/2}-\beta\left(\frac{2\varepsilon}{1-\beta^2}\right)+\left(\frac{3+4\beta-\beta^2}{4}\right)\left(\frac{2\varepsilon}{1-\beta^2}\right)^{3/2}\right) \quad \ldots (43)$$

For a given mission, the location of the point of minimum ETRW on the payload range diagram is determined by the ratio of maximum zero fuel mass to maximum take-off mass, e (the fuel used over and above the cruise value for the same range) and β (the fuel carried, but not used).

8.0 OBSERVATIONS

While the *MOE* is a mass that can be determined by weighing the aircraft, the *MMZF* and *MMTO* are 'certification masses' whose values are agreed with the Regulator. These masses, $\eta_o L/D$ and the minimum value of ε are determined by the design process. The minimum permissible value of β is specified by the regulator, while the actual value of β and the value of *LF* are determined by the aircraft operator and the actual value of ε is determined by air traffic system that is managing the flight. Once the values of these parameters have been fixed the aircraft *ETRW* for a given flight is determined.

For a given aircraft, lines of constant *ETRW* (constant *MMF*/*MP*/*X*) can be plotted on a normalised payload range diagram, as shown in Fig. 4. Here the payload is normalised with the maximum payload and the range is normalised with the maximum range at maximum payload, *RMMP*.

The smallest value of *ETRW* occurs along the line *MP*/*MMP* = 1 (*LF* = 1) at a value of $R/(R)_{MMP}$ less than, or equal to, unity. When the aircraft is operated with payloads below the maximum permitted and over ranges above and below the range for best *ETRW*, the economic efficiency of the flight is reduced. The figure shows that load factor is a very important parameter for flight efficiency and that operating an aircraft over short ranges can result in very large efficiency penalties.

The data given in Fig. 1 show that, for current conventional kerosene powered aircraft, all flights conducted within Europe are 'short' i.e. less than the range for best *ETRW*.

It is also possible to determine the sensitivity of *ETRW* to variations in the governing parameters. In order of increasing importance, these are

$$\frac{\partial ETRW}{\partial(\eta_o L/D)} = -\frac{ETRW}{(\eta_o L/D)} \quad \ldots (44)$$

$$\frac{\partial ETRW}{\partial LF} = -\left(\frac{1}{1+LF\dfrac{MMP}{MOE}}\right)\frac{ETRW}{LF} \quad \ldots (45)$$

$$\frac{\partial ETRW}{\partial \varepsilon} = \left(\frac{\varepsilon(1-\beta)EXP(-X)}{(kEXP(-X)-\beta)(1-kEXP(-X))}\right)\frac{ETRW}{\varepsilon} \quad \ldots (46)$$

and

$$\frac{\partial ETRW}{\partial \beta} = \left(\frac{\beta}{(kEXP(-X)-\beta)}\right)\frac{ETRW}{\beta} \quad \ldots (47)$$

All aircraft have an operating condition at which the *ETRW* is a minimum. This corresponds to a particular value of $\eta_o L/D$ that is achieved at a particular combination of flight Mach number and altitude. If the aircraft cannot fly at this optimum condition, there is a fuel burn penalty. In percentage terms, this penalty is equal to the percentage increase in $\eta_o L/D$ that results from the changes to the flight Mach number and altitude.

The load factor has a maximum value of unity, but in normal operations it is likely to be closer to 0·7. If the load factor is increased above 0·7, e.g. by the addition of 'belly freight', the *ETRW* is decreased. Typically, a 10% increase in *LF* will reduce the *ETRW* by about 7·5%, making the flight more fuel efficient. This change is independent of all parameters except *MOE*/*MMP*.

When extra fuel is consumed due to non-optimum climb and descent profiles, or due to additional miles being flown for any reason e.g. air traffic directives or adverse weather, ε is greater that its minimum value and a fuel burn penalty is incurred. For a flight distance equal, or near, to the optimum given in Theorem 3, a 10% increase in ε relative to its optimum value ($\varepsilon_{opt} \approx 0{\cdot}02$) causes ETRW to increase by about 1·5%.

Finally, it is possible to identify the impact of reserve fuel. The minimum reserve fuel is equal to approximately 4·5% of the take-off mass, i.e β ≈ 0·045. Therefore, for flights at, or near, the optimum range given by Theorem 3, the addition of minimum reserve fuel increases the ETRW by about 6%. Therefore, the environmental cost

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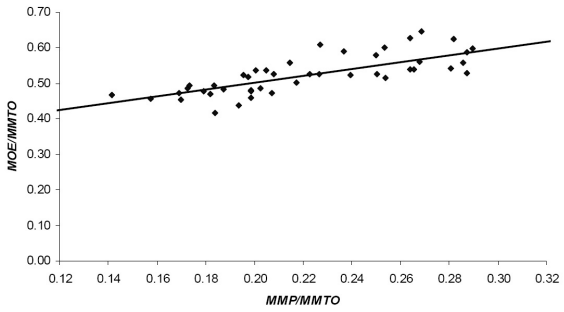


Figure 5. Comparison between the approximate relation between *MOE*/*MMTO* and *MMP*/*MMTO* and data for Boeing and Airbus aircraft.

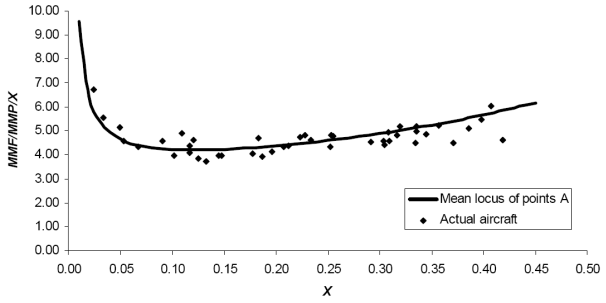


Figure 6. Comparison between the mean locus of points *A* (*MTO* = *MMTO* and *MP* = *MMP*) and the data for Boeing and Airbus aircraft.

of safety, in the form of reserve fuel, is a 6% increase in fuel burn per unit payload per unit distance. If the aircraft carries more fuel in the form of additional reserves or, if the aircraft is tankering fuel for the onward, or return, trip, a 10% increase in β relative to its minimum legal value will increase the *ETRW* by about 0·7%.

It is important to note that all the theorems and the observations apply to any flying machine using a propulsion system that consumes fuel as the flight proceeds.

9.0 HOW GOOD ARE CURRENT AIRCRAFT?

The economic and business considerations that determine the requirements of airlines are both complex and subject to variation. Therefore, the aircraft in service today have not, in general, been specifically designed for minimum *ETRW*, since, although fuel consumption has always been an important consideration, it has not been the dominant consideration. Therefore, the obvious questions are 'what are the values of *ETRW* for current aircraft?' and 'how much better could the values be in future?'. In order to provide answers, it is necessary to resort to empiricism. Therefore, it should be noted that the following analysis involves approximations.

As noted above, the relationship between the three key masses, *MOE*, *MMP* and *MMTO* is central to the determination of the *ETRW*. In reality, the relationship is complex and determined by many factors. However, it is possible to make some progress by recognising that the masses of some components of an aircraft are related to the maximum take-off mass, while the masses of others are related to the maximum payload mass. A rough breakdown is —

- lifting surfaces ≈ 0·14*MMTO*,
- propulsion system ≈ 0·08*MMTO*,

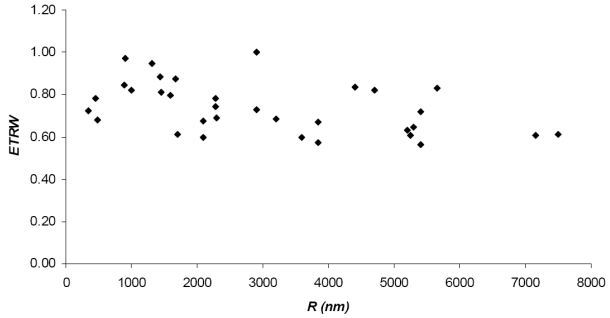


Figure 7. The variation of *ETRW* with range for current aircraft with *LF* = 1 and *MTO*/*MMTO* = 1.

- undercarriage ≈ 0·05*MMTO*,
- avionics, instruments and electronics ≈ 0·02*MMTO*; and
- flight controls and hydraulics ≈ 0·02*MMTO*; with
- fuselage ≈ 0·50*MMP*,
- furnishings ≈ 0·27*MMP*,
- air conditioning ≈ 0·03*MMP;* and
- auxiliary power unit ≈ 0·01*MMP*; plus
- operational items ≈ 0·13*MMP*; and
- crew ≈ 0·02*MMP*.

Therefore, summing the components gives

$$\frac{MOE}{MMTO} \approx 0\cdot31 + 0\cdot96\frac{MMP}{MMTO} \qquad \ldots (48)$$

In Fig. 5, this relation is compared to data from a wide range of Boeing and Airbus aircraft currently in service and, although there is some variation, it is clear that there is a correlation between *MOE*/*MMTO* and *MMP*/*MMTO*.

It follows that

$$\frac{MMP}{MMTO} \approx \frac{1}{1\cdot96}\left(\frac{MMZF}{MMTO} - 0\cdot31\right) \qquad \ldots (49)$$

and, from Theorems 8 and 9,

$$\left(\frac{MMF}{MMP.X}\right)_A \approx \frac{1\cdot96}{X_A}\left(\frac{\alpha_A}{(1-\beta-0\cdot31)-\alpha_A}\right) \qquad \ldots (50)$$

or

$$\left(\frac{MMF}{MMP.X}\right)_A \approx \frac{1\cdot96}{X_A}\left(\frac{1-kEXP(-X_A)}{kEXP(-X_A)-(\beta+0\cdot31)}\right) \qquad \ldots (51)$$

This last equation describes the approximate locus of 'points *A*' for current aircraft i.e. the 'state of the art' for aircraft performance. This locus is shown in Fig. 6, together with data for current Airbus and Boeing aircraft. These data have been derived from information contained in published payload-range diagrams. It can be seen that the aircraft follow the general trend of the mean line, but that the value for any individual aircraft may differ from the estimate by as much as 20%.

The payload-range diagrams can also be used to estimate the *ETRW* at point *A* and a set of values for aircraft currently in service is given in Fig. 7. It is immediately apparent that, in this form, the data exhibit considerable scatter and there is no clear minimum value for *ETRW*. The reason for this loss of coherence is that, while the masses *MOE*, *MMP* and *MMTO* are reasonably well correlated over a wide range of different aircraft, $\eta_o L/D$ can exhibit large varia-

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tions from aircraft to aircraft for a number of reasons. First, as already noted, aircraft are not usually optimised for minimum *ETRW*. Therefore, different specifications result in different values of $\eta_o L/D$. Secondly, $\eta_o L/D$ depends upon the design cruise Mach number and this varies from aircraft to aircraft and, thirdly, $\eta_o L/D$ has been the subject of continuous improvement over time due to advances in propulsion efficiency and high-speed aerofoil and wing design. Therefore, different versions of the same aircraft may have very different values of $\eta_o L/D$.

Nevertheless, it is safe to conclude from Fig. 7 that the *ETRW* for current aircraft is roughly constant, with the average value being about 0·75 (±0·15), while the best aircraft achieve values of around 0·6. Therefore, from Fig. 6,

$$ETRW \approx 0\cdot75 \approx \frac{5}{\eta_o\, L/D} \qquad \ldots (52)$$

giving $\eta_o L/D$ a current fleet average value of approximately 6·5.

Also from Fig. 6, the *ETRW* for the best aircraft is

$$ETRW \approx \frac{4}{\eta_0\, L/D} \qquad \ldots (53)$$

Provided that

$$0\cdot075 \le \left(\frac{g.R}{LCV.\eta_o\, L/D}\right) \le 0\cdot25 \qquad \ldots (54)$$

This aircraft has the following characteristics

- Operational empty weight — $MOE/MMTO \approx 0\cdot50$
- Disposable load (including reserve fuel) — $M_{disp}/MMTO \approx 0\cdot50$
- Payload – $MMP/MMTO \approx 0\cdot25$
- Mission fuel – $MMF/MMTO \approx 0\cdot20$

and it will deliver its best ETRW for a trip, with maximum permitted payload, of between 1,500 and 5,000 nautical miles. The longer range is equivalent to a flight from London to Los Angeles, Rio de Janeiro, Cape Town, Bangkok or Beijing, while the shorter range is equivalent to a flight from London to Reykjavik, Tromso, Moscow, Bucharest, Ankara, Crete, Casablanca or the Azores.

On these flights, the fuel consumption per unit payload per unit distance flown would be about 0·00028kg/kg/nm or, taking the average mass of a passenger plus baggage to be 95kg, 0·026kg/passenger/nm. By way of comparison, this is equivalent to a family car, with four passengers and luggage, i.e. a load factor of unity, achieving 45 statute miles per Imperial gallon.

Having obtained an estimate for the optimum performance of the aircraft currently in service, it in useful to compare this to the global fleet average fuel burn. According to Airbus[3], on average 0·45 litres of kerosene are burned to take 1,000kg of payload one kilometre. This translates to 0·00067kg/kg/nm, which is equivalent to an *ETRW* of 1·6.

Therefore, the global fleet average *ETRW* is over 2·5 times the optimum value for the best aircraft and over twice the average value for all aircraft.

The obvious questions that follow this observation is ‘where does it all go?’ and ‘can we reduce this ratio by better use of the fleet?’.

10.0 WHERE DOES IT ALL GO?

It has already been shown that *ETRW* is very sensitive to the load factor. The publicly available data, e.g. Refs 9 and 10, indicate that, the overall load factor, *LF*, is only about 60%. From Theorem 2, we can see that, with an *LF* of 60%, the *ETRW* is 50% greater than would be the case for a load factor of 100%. Therefore, half of the discrepancy is attributable directly to low average load factors.

The majority of the flights taking place inside Europe and inside the United States are less that 1,500nm and, therefore, are defined as ‘short’. Short flights are intrinsically less efficient than long flights and the *ETRW* is more sensitive to extra fuel burned due to air traffic requirements, though slightly less sensitive to variations in tankered fuel carried. Referring to Figs 1 and 6, the average *ETRW* for short flights could be 15-20% higher than the average *ETRW* for all flights. Therefore, operations probably account for 70 to 80% of the discrepancy.

The air traffic management system is in place to make sure that operations are conducted safely. However, if an aircraft is required to fly at an altitude that is not optimum from the point of fuel burn for any part of the journey, or, if the distance actually flown is larger than the great circle distance between the departure point and the destination, the fuel consumed will be greater that the minimum i.e. ε will be increased and, consequently, *ETRW* will be increased.

While the amount of fuel wasted in non-optimum climb profiles has not yet been quantified, the extra distance flown due to non-optimum routing is known. According to Reynolds[11], on the average an extra 9nm is flown in the departure local area, an extra 27nm is flown in the destination local area and an extra 20nm plus 2·5% of the great circle distance is flown en-route. The maximum deviations can be up to 2·5 times these values. For short flights (<1,500nm), the result is an increase in *ETRW* of approximately 10%, while, for long flights, the increase is about 5%. When the effects of non-optimum climb and descent are included, ATM is likely to be responsible for a 10-15% increase in *ETRW*.

The combination of the effects of low *LF* (factor 1·5), large numbers of ‘short’ flights (factor 1·2) and the wastages in the ATM (factor 1·125) fully account for the discrepancy between the average aircraft and the global fleet average *ETRW*, since

$$1\cdot5 \times 1\cdot175 \times 1\cdot125 \approx 2. \qquad \ldots (55)$$

11.0 WHAT CAN BE DONE TO IMPROVE THE SITUATION?

On the assumption that, in the future, there will be a requirement to reduce *ETRW* to the lowest practical value, the objectives will be to:

- operate with the largest possible passenger and cargo load
- reduce the mass of the aircraft structure for a given payload
- increase the propulsive system efficiency, η_o
- increase the airframe lift-to-drag, (L/D), ratio
- increase the energy per unit volume of the fuel; and
- optimise the combinations of the above to give minimum *ETRW*

In the context of payload, each passenger requires a certain amount of physical space. For economy class, the floor space per passenger is about 0·85m^2. An analysis of current civil aircraft reveals that the maximum permitted payload mass per unit area is about 140kg/m^2 i.e. about 118kg/passenger. For long haul flights, the standard passenger plus bags has a mass of 95kg, while for short haul, the value is closer to 82kg. Therefore, it is immediately apparent that the maximum number of passengers that can be accommodated in an aircraft is determined, not by the maximum permitted payload mass, but by the available cabin floor area.

If the passenger load factor, *LFP*, is the actual number of passengers divided by the total number of seats and the cargo load factor, *LFC*, is the mass of cargo divided by the maximum available cargo mass, then, for long haul

$$LF = LFP\left(0\cdot8\right) + LFC\left(1 - 0\cdot8LFP\right) \qquad \ldots (56)$$

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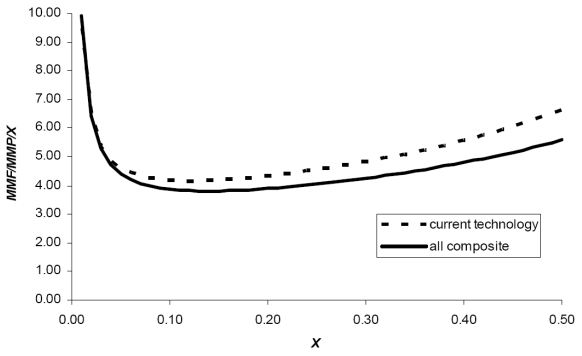


Figure 8. Comparison of the mean locus of points *A* (*MTO* = *MMTO* and *MP* = *MMP*) for current aircraft and future 'composite' aircraft.

and, for short haul,

$$LF = LFP(0{\cdot}7) + LFC(1 - 0{\cdot}7LFP) \quad \ldots (57)$$

Therefore, if no cargo is carried, but every seat on the aircraft is full, the maximum *LF* for long haul is 0·8 and the maximum *LF* for short haul is 0·7. Unfortunately, it is very unlikely, and probably undesirable, that all flights will carry the maximum number of passengers. A practical upper limit for *LFP* is likely to be closer to 0·8. Therefore, the load factors without cargo are probably closer to 0·65 and 0·55. Clearly, the only way to extract more revenue work from the fuel burned during a flight is to add as much cargo to the aircraft as possible. This is particularly important for short flights.

Aircraft structures have to be designed to meet strict codes defined by the regulator and given that aircraft design has always focused on weight minimisation, there would appear to be little scope for reducing structure mass through novel structural concepts. Therefore, the most effective approach is to adopt new, light weight materials.

The recent move towards carbon fibre composite for primary structure, i.e. wing boxes and fuselages, offers the opportunity for considerable weight saving. However, there are a number of challenges to be overcome before this benefit can be realised to its fullest extent. Amongst these are

- the availability of the raw material at an economically viable cost
- the achievement of a safe structural design
- the provision of reliable and affordable through life maintenance; and
- an environmentally acceptable disposal process for the structure at the end of its working life.

Unfortunately, none of these are trivial issues.

When all the structural design processes have matured, and this may take another decade, the maximum likely impact of carbon fibre composite is a reduction of about 25% in the mass of the wing and the fuselage of a conventional civil aircraft. However, this does not mean that the *ETRW* will decrease by the same percentage.

The impact of reduced component mass can be estimated by modifying the (very) approximate relationship between *MOE*/*MMTO* and *MMP*/*MMTO*. When a lighter fuselage and wing are introduced the relationship becomes —

$$\frac{MOE}{MMTO} \approx 0{\cdot}27 + 0{\cdot}84\frac{MMP}{MMTO} \quad \ldots (58)$$

The locus of possible 'all composite' aircraft is shown in Fig. 8, together with the corresponding result for current 'metal' aircraft. For *X* greater than 0·1, the 'all composite'" aircraft have values of *ETRW* that are between 8% and 15% lower than current 'metal' aircraft, with the largest improvement being achieved with the aircraft that are designed to fly the longest ranges. However, the observation that such a large change in the properties of two major components delivers a much reduced improvement in the *ETRW* is disappointing and it highlights the need to always assess the implications for the total system when a change is made at the component level.

The engine efficiency, η_o, can be subdivided into two parts –

$$\eta_0 = \eta_C.\eta_P = \frac{\Delta KE}{(\dot{m}f.LCV)}.\frac{(T.V_\infty)}{\Delta KE} \quad \ldots (59)$$

Here, η_c is the thermodynamic cycle efficiency, which is the ratio of the rate of mechanical energy generated to the rate of thermal energy liberated, and η_P is the propulsive efficiency, which is the ratio of the rate at which work is done on the aircraft to the rate of generation of mechanical energy.

The thermodynamic cycle efficiency can be improved by increasing the overall pressure ratio across the compressor stages or by increasing the gas temperature at the turbine entry station. Unfortunately, both these measures tend to increase the production of NO_X and an increase in the thermal efficiency of the engine also increases the likelihood of forming a contrail in a given set of meteorological conditions[6]. Therefore, an environmentally optimum balance needs to be struck between the emissions of CO_2 and NO_X and the formation of contrails. However, at present, the understanding of the environmental impacts of these three 'pollutants' is not yet sufficiently clear to be able to design aircraft for minimum environmental impact.

However, it is also possible to improve the propulsive efficiency of the engine by replacing the turbo-fan design with either an advanced turboprop, open-rotor or unducted fan design. The propulsive efficiency of a high bypass ratio turbo-fan engine cruising at a Mach number of 0·85 is about 75%. However, advanced turboprops and unducted fans offer propulsive efficiencies of about 85% at the same Mach number. Unfortunately, before this improvement can be realised, there are several severe technical challenges to be overcome. For example, without the bypass duct, there is nothing to contain the fragments in the event of a rotor burst. This is a major safety issue. Also, without the bypass duct, there is nothing to absorb the noise radiating from the rotor and nowhere to put acoustic shielding material. Therefore, noise is also an issue. Solutions to these problems will probably involve increased mass and this will reduce the overall performance benefit. Nevertheless, simple theoretical considerations suggest that the open rotor system has the potential to deliver a one off, improvement in engine efficiency of order 10% to 15%. Since the *ETRW* is directly proportional to engine efficiency, this would also improve by the same amount and there would be no secondary environmental penalties.

For a conventional aircraft with a classical drag polar, the aerodynamic efficiency, *L*/*D*, is largely governed by the wing aspect ratio, *AR*, and the zero-lift drag coefficient, Cd_0, since

$$\left(\frac{L}{D}\right)_{\max} \propto \left(\frac{AR}{Cd_0}\right)^{1/2} \quad \ldots (60)$$

Hence, the *ETRW* can be improved by increasing aspect ratio and decreasing Cd_0. However, the cruise lift coefficient, C_L, also depends upon these parameters, since

$$(C_L)_{cruise} \propto (AR.Cd_0)^{1/2} \quad \ldots (61)$$

Therefore, an increase in the aspect ratio will improve *L*/*D* but the required cruise C_L will also increase. This can lead to problems

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since, for safety reasons, there is an airworthiness requirement for the aircraft to be able to execute a buffet free, 1·3g manoeuvre in the cruise condition and, for a given wing, this sets an upper limit on C_L. In addition, with no other changes, an increase in cruise C_L will increase the wave drag and this will reduce L/D. Therefore, an increase in aspect ratio has to be accompanied either by an increase in the wing sweep angle so that the drag penalty is avoided or by the introduction of a new wing section that has the same, or lower, wave drag at the original sweep angle. Unfortunately, changing the sweep angle has structural implications and usually leads to a heavier wing. Moreover, there is a safety issue related to the pitch up characteristics when the wing stalls at low speed. This condition is such that, for a given sweep angle, there is a maximum aspect ratio below which the stalling characteristics are deemed to be acceptable. This aspect ratio versus sweep boundary limits the use of high aspect ratio wings ($AR > 10$) to situations in which the wing sweep is less than 20° and this poses challenges for aircraft with cruise Mach numbers in excess of 0·8. Therefore, the research challenge is to design high aspect ratio wings that have acceptable low-speed stalling and high-speed manoeuvre characteristics at sweep angles that are sufficiently large to eliminate cruise wave drag penalties.

A decrease in the zero-lift drag coefficient, Cd_0, is clearly beneficial, since L/D is increased and the cruise C_L is reduced. One obvious target is to reduce, or even eliminate, the interference drag between the various components e.g. the engine and the wing. However, thanks to the increasing sophistication of computer based design tools, modern aircraft have very low interference drag and there is probably not much more performance improvement to be extracted for the current configuration. At an operational level, more precise rigging of the aircraft to avoid misalignment of surfaces and, in an extreme case, polishing the outer surface could lead to measurable reductions in Cd_0. Beyond these simple measures, the technology that has the potential to deliver large performance improvements is laminar flow control. This has been studied continuously since the end of the Second World War. It is well understood and it has been demonstrated repeatedly in full scale flight. However, this is a 'high risk' technology with significant development costs and it will be difficult, and expensive, to integrate into routine airline operations. Finally, we should note that one way of changing the value of Cd_0 dramatically is to move from the conventional configuration to the flying wing, or blended wing body, layout. This reduces Cd_0 by a factor of two. However, the apparent advantage is offset by a great many complex design considerations, many of which lead to a much reduced aspect ratio and this lowers the potential benefit dramatically. Nevertheless, significant net performance improvements are expected and so a change of configuration must remain a possibility in the medium to long term, i.e. beyond 2020.

Recently, there has been considerable interest in the possibility of using fuels other than kerosene for aviation[12]. From an environmental point of view, there is much to be said for bio-fuels. The advantage is that, while it was growing, the material from which the fuel is made has absorbed carbon from the present day atmosphere. Therefore, when the fuel is burned the emitted CO_2 effectively returns that carbon to the atmosphere, thus forming a closed cycle in which the total amount of carbon is fixed. This contrasts with the use of fossil fuel which, when burned, releases CO_2 that has been trapped under ground for millions of years, thereby increasing the total amount of CO_2 in the present day atmosphere. Unfortunately, in the case of aviation, all of the easily obtainable bio-fuels e.g. the alcohols and the bio-esters, are deemed to be either unsatisfactory, or unsafe, for a variety of reasons, e.g. the basic energy density is too low, the freezing point is too high or they may be contaminated with traces of metallic elements that can seriously damage very hot engine components. However, it may be possible to use blends of oil-derived kerosene and small percentages of bio-ester fuel.

The most promising approach is to make synthetic kerosene from non-food, bio-mass using the Fischer-Tropsch process. This would be a so call 'drop-in' fuel that could be used in current aircraft engines without the need for any modifications. Synthetic kerosene also has the advantage that it produces less primary and secondary PM. Unfortunately, the production of synthetic kerosene requires a substantial chemical engineering plant that consumes a considerable amount of energy and this naturally casts doubt upon the economic feasibility of the process. Nevertheless, it has been suggested that, with an aggressive research and development programme, 30% of jet fuel requirement could be derived from biological sources by 2030.

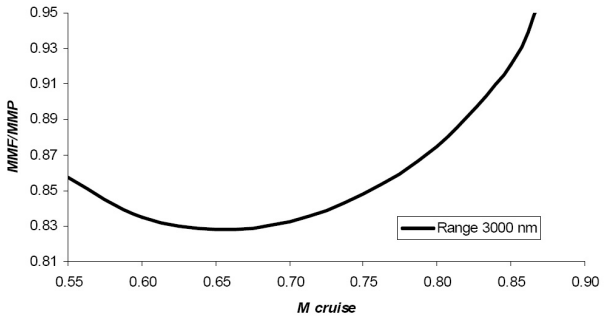


Figure 9. The variation of *MMF*/*MMP* with cruise Mach number for an aircraft optimised for minimum *ETRW* with 295 passengers (three class) and a design range of 3,000nm.

Hydrogen has frequently been proposed as an alternative to kerosene. It has some operational advantages over kerosene, the most notable being the complete absence of carbon. However, a hydrogen powered engine will still produce NO_X and, because the EI_{H2O}/LCV for hydrogen is 2·6 times the value for kerosene, contrails are more likely over a wider range of meteorological conditions[6]. Moreover, there are two enormous disadvantages. The first is that the density of liquid hydrogen in very low, approximately 11 times smaller than that of kerosene. This means that new aircraft, with a very substantial increase in the available fuel capacity, would be required. Secondly, hydrogen does not occur freely in nature and so it has to be made. Strictly speaking hydrogen is not a fuel, rather it is a medium for storing energy that can be generated in one place e.g. in a ground based power station, for use in another place e.g. in an aircraft. The making of hydrogen in the quantities required for aviation would be a very substantial engineering project in itself and it also has to be stored and delivered under cryogenic conditions, bringing enormous infrastructure costs, handling costs, transportation costs and safety issues. It is for these reasons that hydrogen is unlikely to be used for aircraft propulsion in either the short or medium terms.

In addition to all the possible technical advances, it is important to examine the potential for optimisation as a means of reducing the value of *ETRW*. This is relevant because current designs were not optimised for best *ETRW*. In order to gain an indication of the possible improvements, a simple optimisation exercise has been undertaken.

The requirements for the aircraft were

- 295 passengers in a three class configuration (440 in single class configuration) with a maximum permitted payload (*MMP*) of 52,000kg.
- a maximum range of 3,000nm at maximum zero fuel mass
- a specified cruise Mach number
- a twin engine configuration using a 'typical', current technology, power unit with a by-pass ratio of about 6; η_0 is a function of M_∞ only and the engine mass is proportional to sea level static thrust.
- a minimum of 1,750m of runway available; and
- a diversion distance of 200nm

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The airframe technology level was defined by specifying the following

- component mass models based upon the characteristics of current Boeing and Airbus aircraft with the option to vary the masses of the components individually.
- wing aerodynamics
- low-speed characteristics with, and without, high lift systems deployed
- transonic drag rise characteristics modelled
- low-speed, pitch-up boundary defined in terms of aspect ratio versus wing sweep ($AR.\text{Tan}(\Lambda) = 5{\cdot}8$)
- transonic maximum manoeuvre C_L without buffet
- option to vary boundary layer transition location on the wings to simulate laminar flow control technology
- brake characteristics
- glide slope angle; and
- fuel — *LCV* and density

The model was run for a range of cruise Mach numbers and the results for fuel burn per unit payload for the 3,000nm trip are given in Fig. 9. These indicate that there is a cruise Mach number at which fuel burn per unit payload and, hence, *ETRW* are minimised. Therefore, we may conclude that there is an *ETRW* penalty paid for cruising at higher Mach numbers and that, in this case, if the aircraft is designed to cruise at Mach 0·85 the *ETRW* is about 10% greater than the aircraft designed to cruise at Mach 0·65.

If minimum *ETRW* was the absolute requirement, the best aircraft with the current technology level would have the following characteristics —

- cruise Mach number 0·65
- initial cruise altitude 30,000ft
- *MMTO* ≈ 220,000kg
- wing span ≈ 67m
- aspect ratio ≈ 14
- sweep angle ≈ 15°
- *MMP*/*MMTO* ≈ 0·23
- *MMF*/*MMTO* ≈ 0·2
- *MOE*/*MMTO* ≈ 0·53; and
- *ETRW* ≈ 0·65

In this case, the aspect ratio is limited by the buffet free 1·3g cruise manoeuvre requirement. Not only does this aircraft have a significantly better *ETRW*, but the initial cruise altitude is only 30,000ft with maximum payload. Consequently, this aircraft will be less likely to encounter the atmospheric conditions necessary for the formation of contrails.

As an additional point, it was found that by moving boundary layer transition to 50% chord on both top and bottom surfaces of the wing, but with no allowance for increased system mass, the *ETRW* was reduced by a further 20%. This is an indication of the upper limit for the reduction in *ETRW* that could be obtained the application of laminar flow control technology.

Therefore, by re-optimising for minimum *ETRW*, engineering the full weight reduction benefit from composite materials and introducing the open–rotor propulsion system, the *ETRW* could be reduced by as much as 30% relative to current levels. This could be achieved in a relatively short time scale and would not require the use of any 'high risk' technologies such as laminar flow control.

12.0 WHAT COULD WE DO TO ADDRESS THE CLIMATE ISSUE?

In the short term, i.e. over the next ten years, the priorities should be to tackle the operational and ATM inefficiencies and to maintain the flow of new technology, particularly weight reduction though improved use of composite material on the Boeing 787 and Airbus A350 programmes and the development of the open-rotor concept.

In the medium term, 2020 to 2030, the highest priority is to minimise the *ETRW* of the Boeing 737 and Airbus A320 replacement aircraft. There will be a large number of these aircraft in service and they will be operating for over 20 years. They will be the principal source of aviation emissions during the period between 2020 and 2050 and it is vital that they are as environmentally friendly as possible. This will be achieved if the new technologies being developed in the ACARE programmes are brought through. Towards 2030, serious consideration must be given to the introduction of new configurations, such as the Blended Wing Body. Initial studies have suggested that the BWB could offer another 25% improvement in *ETRW* without the introduction of any high risk technologies.

Finally, looking to the long term, it is clear that the only way to deliver an air transport system that can grow without limits to meet the demands of global wealth creation is to eliminate the dependence on oil-derived kerosene completely. This is possible, but it is not easy. Bio-fuel containing carbon from the present day atmosphere is one option, but we should not forget nuclear power. Today, it is possible to travel across France in a nuclear powered train. This is because the train is powered by electricity that is generated by a nuclear power station. In the context of aviation, nuclear power could be used to generate hydrogen from sea water, with very little environmental impact. The cryogenic liquid hydrogen would need to be supplied to airports and loaded onto aircraft. This would require new infrastructure. Nevertheless, if it could be made available at about 200 major airports, this would probably be sufficient to address a large part of the problem. The gas turbines on the aircraft would then burn the hydrogen. However, as already noted, special aircraft would need to be designed to provide the high storage volume that hydrogen needs and the enhanced contrail formation issue would need to be eliminated by better weather prediction and an air traffic 'supersaturated air' avoidance system. Alternatively, a small airborne nuclear heat source could be developed for the gas turbine. In this case, the aircraft could take off using hydrogen fuel and the switch to the nuclear powered gas turbine for the cruise phase. Clearly, the mass of the radiation shielding would be a major issue, but, of course, this would be offset by the reduction in the mass of the fuel. This idea was studied extensively in the 1950s when it was taken very seriously. However, the concept raises many challenges and not all of them are technical.

13.0 CONCLUSIONS

A successful aviation industry has the potential to make a significant contribution to the level of carbon dioxide in the atmosphere and to introduce short term perturbations to other contributors to the Earth's radiation balance. These effects contribute directly to global warming. The consensus scientific view (IPCC) is that increasing global warming is leading to climate change and this is set to become an increasingly important political and economic issue over the next few decades.

Since the cost of kerosene is a large, and growing, element in aircraft direct operating cost, there is a direct link between the economics of aviation and its environmental impact. It has been argued that this link is characterised by *ETRW*, the thermal energy liberated per unit revenue work delivered for an individual flight.

By using the 'Breguet' range equation, ten theorems relating to aspects of an aircraft's *ETRW* have been derived. These theorems are

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valid for all flying machines. They allow the aircraft parameters that govern *ETRW* to be identified, the *ETRW* values to be calculated and the sensitivity of *ETRW* to changes in these parameters to be determined. In particular, it is demonstrated that *ETRW* becomes very large when the distances flown become small.

The *ETRW* for aircraft currently in service have been estimated and the best aircraft from an environmental impact point of view has been identified. This has half its maximum take-off mass available for payload and fuel and a maximum permissible payload mass equal to 25% of *MMTO*. It operates at close to minimum *ETRW* over ranges of between 1,500 and 5,000 nautical miles.

It has been shown that the global fleet average fuel burn is currently more than twice that of an average aircraft operating at maximum payload and maximum take-off mass and more than 2·5 times that of the best aircraft. This difference is attributed to the large number of flights over short routes in Europe and North America (factor ≈ 1·2), low load factors (factor ≈ 1·5) and inefficient ATM (factor ≈ 1·1).

Ways in which the environmental impact can be improved have been examined. It has been argued that, since passenger load factors are already relatively high, the most effective way to improve the *ETRW* is to carry more belly cargo. This is particularly important on short flights. Improvements in technology have also been considered. Structure weight reduction through the extensive used of carbon fibre composite could eventually deliver an 8% to 15% reduction in *ETRW*, while the use of the open rotor propulsor could provide a 10% to 15% reduction on appropriate aircraft and there are some small improvements available made through better aerodynamic design. In addition, it has been shown that *ETRW* is dependent upon the design cruise Mach number, with a minimum occurring at about 0·65-0·7. The *ETRW* at this condition is about 10% less than for an aircraft with a design cruise Mach number of 0·85 and it would cruise at altitudes of about 30,000ft i.e. below those associated with contrail formation.

If a new single aisle aircraft took advantage of all these improvements, the *ETRW* would be 35% lower than the aircraft that it would replace. With better load factors and more efficient ATM, this number could be closer to 50%. Similar levels of improvement are possible for other types of aircraft.

Looking beyond 2025, further reduction in *ETRW* will require either the introduction of high risk technologies or new configurations like the Blended Wing Body. However, while aviation is dependent upon oil-derived kerosene, it will always pose a threat to the environment. Sooner, or later, this dependency must be broken, either by the use of bio-fuels or through a radical change in propulsion technology. The inescapable requirement to transform aviation into a safe, clean and highly efficient transportation system growing, without constraint, to supply the needs of a sustainable global economy is the greatest challenge yet faced by the aerospace and aviation communities.

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APPENDIX

Starting from the equation

$$Y = \frac{\alpha}{(b-\alpha)} \qquad \ldots (A1)$$

where

$$\alpha = 1 - k\mathrm{EXP}(-X) \qquad \ldots (A2)$$

and

$$k = 1 - \varepsilon \qquad \ldots (A3)$$

If $(\alpha/b) < 1$, then

$$Y = \left(\frac{\alpha}{b}\right) + \left(\frac{\alpha}{b}\right)^2 + \left(\frac{\alpha}{b}\right)^3 + \left(\frac{\alpha}{b}\right)^4 + \left(\frac{\alpha}{b}\right)^5 + \ldots\ldots\ldots \qquad \ldots (A4)$$

Now

$$\left(\frac{\alpha}{b}\right) = \frac{1}{b}\left(1 - k\mathrm{EXP}\left(-X\right)\right)$$

$$\left(\frac{\alpha}{b}\right)^2 = \frac{1}{b^2}\left(1 - 2k\mathrm{EXP}\left(-X\right) + k^2\mathrm{EXP}\left(-2X\right)\right) \qquad \ldots (A5)$$

$$\left(\frac{\alpha}{b}\right)^3 = \frac{1}{b^3}\left(1 - 3k\mathrm{EXP}\left(-X\right) + 3k^2\mathrm{EXP}\left(-2X\right) - k^3\mathrm{EXP}\left(-3X\right)\right)$$

etc.

and, since, for any value of X,

$$\mathrm{EXP}\left(X\right) = 1 + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \ldots\ldots\frac{1}{n!}X^n \qquad \ldots (A6)$$

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Y may be expanded as a power series in ascending powers of X where each coefficient of X is a series in ascending powers of ε. To order ε^2, the result is

$$Y \approx \left(\left(\frac{\varepsilon}{b}\right)+\left(\frac{\varepsilon}{b}\right)^2\right)+\left(1+(2-b)\left(\frac{\varepsilon}{b}\right)+(3-2b)\left(\frac{\varepsilon}{b}\right)^2\right)\left(\frac{X}{b}\right)+$$
$$\left(\left(1-\frac{b}{2}\right)+\left(3-3b+\frac{b^2}{2}\right)\left(\frac{\varepsilon}{b}\right)+\left(6-\frac{15}{2}b-2b^2\right)\left(\frac{\varepsilon}{b}\right)^2\right)\left(\frac{X}{b}\right)^2+$$
$$\left(\left(1-b+\frac{b^2}{6}\right)+\left(4-6b+\frac{7}{3}b^2-\frac{b^3}{6}\right)\left(\frac{\varepsilon}{b}\right)+\left(10-18b+\frac{19}{2}b^2-\frac{4}{3}b^3\right)\left(\frac{\varepsilon}{b}\right)^2\right)\left(\frac{X}{b}\right)^3+\ldots. \quad \ldots (A7)$$

This series converges if (X/b) is less than one.

However, in this paper, an approximation retaining the first three terms in X and neglecting terms of order ε^2 and above is used, i.e.

$$Y \approx \left(\frac{\varepsilon}{b}\right)+\left(1+(2-b)\left(\frac{\varepsilon}{b}\right)\right)\left(\frac{X}{b}\right)$$
$$+\left(\left(1-\frac{b}{2}\right)+\left(3-3b+\frac{b^2}{2}\right)\left(\frac{\varepsilon}{b}\right)\right)\left(\frac{X}{b}\right)^2 \quad \ldots (A8)$$

This is within 5% of the exact result provided that $\varepsilon < 0{\cdot}05$ and $X < 0{\cdot}5(b)^{1/2}$.

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