Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 140 MAY 2006 PART 3

Math. Proc. Camb. Phil. Soc. (2006), **140**, 369 © 2006 *Cambridge Philosophical Society*
doi:10.1017/S0305004105009138 *Printed in the United Kingdom*

Recognising $\mathbb{Z}_p[[t]]$-analytic pro-p groups

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(*Received* 1 *November* 2004; *revised* 5 *May* 2005)

Abstract

We define a group theoretic criterion on a pro-p group that guarantees the existence of an analytic structure over $\mathbb{Z}_p[[t]]$. This is a one-way analogue of Lazard's succesful answer to the p-adic version of Hilbert's Fifth Problem. In the course of the proof we define a T-map, a T-uniform group and set up a category equivalence between T-uniform pro-p groups and powerful $\mathbb{Z}_p[[t]]$-Lie algebras.

1. *Introduction*

In this work we translate some of Lazard's work on $\mathbb{Z}_p$-analytic pro-p groups to $\mathbb{Z}_p[[t]]$-analytic pro-p groups. In particular, we (*i*) set up a category equivalence between a class of groups and a class of Lie algebras and (*ii*) identify a group theoretic criterion on a pro-p group that guarantees the existence of an analytic structure over $\mathbb{Z}_p[[t]]$, i.e. a partway analogue of Hilbert's 5th problem.

We begin with some definitions. A topological group G is called a $\mathbb{Z}_p[[t]]$-analytic group if it has a $\mathbb{Z}_p[[t]]$-analytic structure and both multiplication and inversion are analytic maps with respect to this structure. A $\mathbb{Z}_p[[t]]$-analytic group contains an open $\mathbb{Z}_p[[t]]$-standard group [**2**, 13·20], which has a rather nice structure: a $\mathbb{Z}_p[[t]]$-standard group of level h and dimension d can be homeomorphically identified with the set $(\langle p,t\rangle^h)^{(d)}$ (where $\langle p,t\rangle$ denotes the maximal ideal of $\mathbb{Z}_p[[t]]$) and the group

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operation is given by a formal group law **F(X, Y)**. For more detailed definitions and discussion see [**2**, chapter 13].

One of the main tools in proving the category equivalence between uniform pro-p groups and powerful $\mathbb{Z}_p$-Lie algebras is the use of the Baker–Campbell–Hausdorff formula (BCHF). We recall the following definitions. Let

$$\epsilon = \begin{cases} 1 & \text{if } p \neq 2 \\ 2 & \text{if } p = 2, \end{cases}$$

a pro-p group G is called powerful if $[G, G] \leqslant \overline{G^{p^\epsilon}}$. Let $P_i(G)$ denote the lower p-series of a pro-p group G, that is $P_1(G) = G$ and $P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]}$. We can now define a uniform pro-p group.

Definition. A pro-p group G is uniform if:
(i) G is finitely generated;
(ii) G is powerful; and
(iii) for all i, $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$.

Definition. A Lie algebra L is called powerful if $L \cong \mathbb{Z}_p^d$ additively for some positive integer d, and $(L, L) \subseteq p^\epsilon L$ (where (,) denotes the Lie bracket on L).

The BCHF can be used to define a group operation on the class of powerful Lie algebras. The powerfulness of the Lie algebras ensures that the coefficients that occur in the formula lie in $\mathbb{Z}_p$, this depends on ps in the denominators cancelling with ps in the numerators. We needed an equivalent definition for $\mathbb{Z}_p[[t]]$-Lie algebras.

Definition. A Lie algebra over $\mathbb{Z}_p[[t]]$ is called powerful and of dimension d if $L \cong \mathbb{Z}_p[[t]]^d$ additively and $(L, L) \subseteq p^\epsilon L$.

Lazard's succesful analysis of p-adic analytic pro-p groups depended on understanding the important role of the p-power map for these groups. In a $\mathbb{Z}_p[[t]]$-analytic group we have the additional structure of multiplication by t. We introduce a T-map which models this multiplication by t. The work is closely related to work of D. Cartwright [**1**] who was looking at $\mathbb{F}_p[[t]]$-analytic groups. Because this was his focus he did not consider the connection between his T-map and the p-power map, hence our definition of a T-map is slightly different.

Definition. Let G be a pro-p group. We say G has a d-dimensional $\mathbb{Z}_p$-filtration if there exists a series of normal subgroups

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots$$

such that G/G_i is uniform, there exists a bijection $G_i/G_{i+1} \to \mathbb{Z}_p^d$, and $\bigcap G_i = 1$.

Note that the definition of a $\mathbb{Z}_p$-filtration implies that $G \cong \lim_{\leftarrow} G/G_i$, i.e. that G is an inverse limit of uniform pro-p groups. Also since G/G_i is uniform for all i it follows that G has a nice p-power structure, in particular G is powerful.

For a pro-p group with a $\mathbb{Z}_p$-filtration we can now define a T-map.

Definition. Suppose G is a pro-p group with a $\mathbb{Z}_p$-filtration. A T-map on G is a map $T: G \to G$ which induces a bijection

$$T: G_{i-1}/G_i \longrightarrow G_i/G_{i+1}.$$

Further, it is required that for $g \in G^{p^i}$, $h \in G^{p^j}$ and $\lambda \in \mathbb{Z}_p$, T satisfy the following:
(i) $T(g^\lambda) = (Tg)^\lambda$;

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(ii) $T(gh) = (Tg)(Th) \bmod T(G^{p^{i+j}})$;
(iii) $T[g,h] = [Tg,h] \bmod T(G^{p^{2i+j}})$.

Property (i) tells us that the T-map commutes with the p-power map. Consequently in general for a $\mathbb{Z}_p[[t]]$-standard group multiplying coordinates by t does not define a T-map. However a $\mathbb{Z}_p[[t]]$-standard group does have a T-map as we see in Section 5. Also note that if $g \in G^{p^n}$ then $T(g) \in G^{p^n}$.

Definition. We call a pro-p group G with a d-dimensional $\mathbb{Z}_p$-filtration and a T-map a d-dimensional T-uniform pro-p group, or simply a T-uniform pro-p group.

We prove the following theorems.

THEOREM A. *There is a category equivalence between the category of powerful d-dimensional $\mathbb{Z}_p[[t]]$-Lie algebras and the category of d-dimensional T-uniform pro-p groups.*

THEOREM B. *A T-uniform pro-p group has a $\mathbb{Z}_p[[t]]$-analytic structure.*

As already mentioned, this work follows closely ideas of Lazard [**3**]; our notation and terminology follows [**2**].

2. *Defining a Lie algebra*

Let H be a uniform pro-p group. Lazard suggested defining an 'intrinsic' Lie algebra on the set H as follows [**3**, 2·1·10] (for details see [**2**, sections 4·3, 4·5]). Let $g,h \in H$ and $\lambda \in \mathbb{Z}_p$ then

$$g +_H h \coloneqq \lim_n \left(g^{p^n} h^{p^n}\right)^{p^{-n}}$$
$$(g,h)_H \coloneqq \lim_n \left[g^{p^n}, h^{p^n}\right]^{p^{-2n}}$$
$$\lambda g \coloneqq g^{\lambda}.$$

We denote this $\mathbb{Z}_p$-Lie algebra by L_H. As $\overline{[H,H]} \subseteq \overline{H^{p^c}}$ it follows that $(L_H, L_H) \subseteq p^c L_H$, i.e. L_H is a powerful $\mathbb{Z}_p$-Lie algebra.

Suppose G is a pro-p group with a $\mathbb{Z}_p$-filtration. Then for $g,h \in G$ we can define the element $g +_i h$ to be the sum in the uniform quotient G/G_i as above, and since

$$g +_{i+1} h \equiv g +_i h \bmod G_i,$$

we can define addition in G:

$$g +_G h \coloneqq \lim g +_i h.$$

Similarly we can define a Lie bracket and multiplication by $\mathbb{Z}_p$. Thus we have defined an 'intrinisic' Lie algebra L_G on the set G and since G is powerful it follows that $(L_G, L_G) \subseteq p^c L_G$.

Further if we now suppose G is a d-dimensional T-uniform pro-p group we can define a T-action on the Lie algebra L_G. Let $g \in G^{p^i}, h \in G^{p^j}$, then for some $k \in T(G^{p^{n\,i+n\,j}}) \leqslant G^{p^{n\,i+n\,j}}$ we have

$$\begin{aligned} T\left(\left(g^{p^n} h^{p^n}\right)^{p^{-n}}\right) &= \left(T\left(g^{p^n} h^{p^n}\right)\right)^{p^{-n}} \\ &= \left(T\left(g^{p^n}\right) T\left(h^{p^n}\right) k\right)^{p^{-n}} \\ &= \left((Tg)^{p^n} (Th)^{p^n} k\right)^{p^{-n}}. \end{aligned}$$

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Since $\lim_n((Tg)^{p^n}(Th)^{p^n}k)^{p^{-n}}) = \lim_n((Tg)^{p^n}(Th)^{p^n})^{p^{-n}}$ it follows that

$$T(g +_G h) = Tg +_G Th.$$

Similarly, for some $k \in T(G^{p^{2n\,i+n\,j}}) \leqslant G^{p^{2n\,i+n\,j}}$,

$$\begin{aligned} T\big[g^{p^n}, h^{p^n}\big]^{p^{-2n}} &= \big(T\big[g^{p^n}, h^{p^n}\big]\big)^{p^{2n}} \\ &= \big(\big[T\big(g^{p^n}\big), h^{p^n}\big]k\big)^{p^{-2n}} \\ &= \big(\big[(Tg)^{p^n}, h^{p^n}\big]k\big)^{p^{-2n}}. \end{aligned}$$

Since $\lim_n([(Tg)^{p^n}, h^{p^n}]k)^{p^{-2n}} = \lim_n([(Tg)^{p^n}, h^{p^n}])^{p^{-2n}}$ it follows that

$$T(g,h)_G = (Tg,h)_G.$$

Thus we have a T-action on L_G which commutes with the action of $\mathbb{Z}_p$. Consequently we can define a $\mathbb{Z}_p[[t]]$-action on L_G by letting $tl := Tl$ for $l \in L_G$. Furthermore, since G_j/G_{j+1} is bijective with $\mathbb{Z}_p^d$ and $T\colon G_{i-1}/G_i \to G_i/G_{i+1}$ is a bijection it follows that $L \cong \mathbb{Z}_p[[t]]^d$. Thus L_G is a d-dimensional powerful $\mathbb{Z}_p[[t]]$-Lie algebra.

Remark. Equivalently we could have defined L_G as the inverse limit $\lim_{\leftarrow} L_{G/G_i}$, i.e.

$$L_{\lim_{\leftarrow} G/G_i} = \lim_{\leftarrow} L_{G/G_i}.$$

This fact will be used in the proof of Theorem A.

3. *Defining a group operation*

In this section we define a group operation on a powerful d-dimensional $\mathbb{Z}_p[[t]]$-Lie algebra, L. The ideas, proofs and notation are essentially the same as in [**2**, section 6·5]. We record them here for completeness.

Let (X,Y) denote a Lie bracket and for a vector $\mathbf{e} = (e_1, \ldots, e_n)$ of positive integers we write

$$\begin{aligned} \langle \mathbf{e} \rangle &= e_1 + \cdots + e_n \\ (X,Y)_{\mathbf{e}} &= (X, \underbrace{Y, \ldots, Y}_{e_1}, \underbrace{X, \ldots, X}_{e_2}, \ldots). \end{aligned}$$

Recall the Baker–Campbell–Hausdorff formula:

$$\Phi(X,Y) = \sum_{n \in \mathbb{N}} u_n(X,Y)$$

where $u_n(X,Y)$ is the sum of terms of degree n. Further

$$u_n(X,Y) = \sum_{\langle \mathbf{e} \rangle = n-1} q_{\mathbf{e}}(X,Y)_{\mathbf{e}}$$

where each $q_{\mathbf{e}}$ is a rational coefficient satisfying

$$p^{\epsilon\langle \mathbf{e} \rangle} q_{\mathbf{e}} \in p^{\epsilon}\mathbb{Z}_p.$$

Moreover,

$$\lim_{\langle \mathbf{e} \rangle \to \infty} \left| p^{\epsilon\langle \mathbf{e} \rangle} q_{\mathbf{e}} \right| = 0.$$

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Let $x, y \in L$. We claim that

$$\tilde{\Phi}(x, y) = \sum_{n \in \mathbb{N}} u_n(x, y)$$

defines a group operation on L. The following Lemma is [**2**, 6·38].

LEMMA. (i) $u_n(x, y) \in L$ *for all* $n \in \mathbb{N}$.

(ii) *The series*

$$\tilde{\Phi}(x, y) = \sum_{n \in N} u_n(x, y)$$

converges in L.

Proof. (i) Consider $q_{\mathbf{e}}(x, y)_{\mathbf{e}}$ where $\langle \mathbf{e} \rangle = n - 1$. Since L is powerful it follows that

$$(x, y)_{\mathbf{e}} \in p^{\epsilon \langle \mathbf{e} \rangle} L.$$

By above $p^{\epsilon(n-1)} q_{\mathbf{e}} \in \mathbb{Z}_p$. Thus

$$p^{\epsilon(n-1)} q_{\mathbf{e}}(x, y)_{\mathbf{e}} \in p^{\epsilon(n-1)} L.$$

Therefore $q_{\mathbf{e}}(x, y)_{\mathbf{e}} \in L$ giving (i).

(ii) We show that the partial sums of the given series form a Cauchy sequence in L (thinking of L as an additive pro-p group). Let $N \in \mathbb{N}$. By above, for all sufficiently large n we have $p^{\epsilon(n-1)} q_{\mathbf{e}} \in p^N \mathbb{Z}_p$. This implies that

$$p^{\epsilon(n-1)} q_{\mathbf{e}}(x, y)_{\mathbf{e}} \in p^{(N + \epsilon(n-1))} L,$$

which in turn implies that $u_n(x, y) \in p^N L$ for all sufficiently large n.

Thus, we have that $x * y = \tilde{\Phi}(x, y)$ defines a binary operation on L. It is clear from the definition that

$$x * 0 = 0 * x = x$$

$$x * (-x) = 0$$

for all $x \in L$. So we just need to prove associativity. This follows from [**2**, lemma 9·9].

Furthermore we have the following proposition.

PROPOSITION. $(L, *)$ *is a* d*-dimensional* T*-uniform pro-p group.*

Proof. Let L_n be the quotients $L/t^n L$. Then $L \cong \lim_{\leftarrow} L_n$. Since $L \cong \mathbb{Z}_p[[t]]^d$ the Lie algebras L_n can be considered as nd-dimensional $\mathbb{Z}_p$-Lie algebras. They are powerful since L is. Thus $x * y = \tilde{\Phi}(x, y)$ defines a group operation on L_n and this group is uniform [**2**, 9·8]. So $(L, *)$ has a d-dimensional $\mathbb{Z}_p$-filtration.

Now we prove that $(L, *)$ has a T-map. Clearly L has a natural T-action which is just multiplying coordinates by t. Does this define a T-map on $(L, *)$? Denote $(L, *)$ by H and $t^i L$ by H_i. Note that $\overline{[H^{p^i}, H^{p^j}]} \leqslant \overline{H^{p^{i+j}}}$, since this holds for each uniform quotient H/H_i.

Now the BCHF is of the following form

$$\Phi(X, Y) = X + Y + \frac{1}{2}(X, Y) + O'(3).$$

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So, in particular, the p-power map becomes multiplication by p which clearly commutes with the action of t. Also $t : t^{i-1}L/t^iL \to t^iL/t^{i+1}L$ is a bijection.

Consider the t-action on $\Phi(X, Y)$

$$\begin{aligned} t(\Phi(X,Y)) &= tX + tY + \frac{1}{2}t(X,Y) + tO'(3) \\ &= \Phi(tX,tY) + \frac{1}{2}(t(X,Y) - (tX,tY)) + tO'(3). \end{aligned}$$

Let $g \in H^{p^i}$ and $h \in H^{p^j}$. Then we have

$$t(g * h) \equiv tg * th \bmod tH^{p^{i+j}}.$$

Similarly, the commutator BCHF is given by

$$\Psi(X,Y) = (X,Y) - \frac{1}{2}((X,Y,X) + (X,Y,Y)) + O'(4).$$

So

$$\begin{aligned} t(\Psi(X,Y)) &= t(X,Y) - \frac{1}{2}t((X,Y,X) + (X,Y,Y)) + tO'(4) \\ &= \Psi(tX,Y) + \frac{1}{2}((tX,Y,tX) - t(X,Y,X)) + tO'(4). \end{aligned}$$

It follows that

$$t[g,h] \equiv [tg,h] \bmod tH^{p^{2i+j}}.$$

Thus $(L, *)$ has a T-map.

Remark. Equivalently the pro-p group $(L, *)$ could have been defined as the inverse limit $\lim_{\leftarrow}(L/t^iL, *)$, i.e.

$$(\lim_{\leftarrow} L/t^iL, *) = \lim_{\leftarrow}(L/t^iL, *).$$

This fact will be used in the proof of Theorem A.

4. *Category equivalence*

In this section we prove the following theorem.

THEOREM A. *The assignments*

$$G \longmapsto L_G, \quad L \longmapsto (L, *)$$

are mutually inverse isomorphisms between the category of d-dimensional T-uniform pro-p groups and powerful d-dimensional $\mathbb{Z}_p[[t]]$-Lie algebras.

The proof of this theorem relies on the result linking uniform pro-p groups with powerful $\mathbb{Z}_p$-Lie algebras, see [**2**, theorem 9·10].

Proof of Theorem A. Let G be a d-dimensional T-uniform pro-p group. Then $G = \lim_{\leftarrow} G/G_i$, an inverse limit of uniform pro-p groups. By [**2**, theorem 9·10] $G/G_i = (L_{G/G_i}, *)$ where L_{G/G_i} is the intrinsic Lie algebra associated with G/G_i as defined in Section 2. Then, as noted, the Lie algebra L_G defined in Section 2 can be viewed as the inverse limit $\lim_{\leftarrow} L_{G/G_i}$. Consequently, as a pro-p group,

$$G = \lim_{\leftarrow} G/G_i = \lim_{\leftarrow} \left(L_{G/G_i}, *\right) = \left(\lim_{\leftarrow} L_{G/G_i}, *\right) = \left(L_{\lim_{\leftarrow} G/G_i}, *\right) = (L_G, *).$$

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We now need to consider the T-map. Note that as we move from the group to the Lie algebra and back to the group we are always considering the same underlying set. Now G has a T-map and, as explained in Section 2, this T-map defines a $\mathbb{Z}_p[[t]]$-action on L_G. Furthermore this t-action, which is essentially just the original T-map on G, then defines a T-map on $(L_G, *)$. Consequently $G = (L_G, *)$ as a T-uniform pro-p group.

Now we start with a powerful $\mathbb{Z}_p[[t]]$-Lie algebra L. We consider the powerful $\mathbb{Z}_p$ quotients $L_i := L/t^i L$. Then $L = \lim_{\leftarrow} L_i$. Again by [**2**, theorem 9·10] $(L_i, *)$ defines a uniform pro-p group and $L_i = L_{(L_i,*)}$. By Section 3, $(L, *)$ is a T-uniform pro-p group. So, as a $\mathbb{Z}_p$-Lie algebra,

$$L_{(L,*)} = L_{(\lim_{\leftarrow} L_i, *)} = L_{\lim_{\leftarrow}(L_i,*)} = \lim_{\leftarrow} L_{(L_i,*)} = \lim_{\leftarrow} L_i = L.$$

Now we need to consider L as a $\mathbb{Z}_p[[t]]$-Lie algebra. As such it has an underlying $\mathbb{Z}_p[[t]]$-coordinate structure with a natural T-action which is just multiplying coordinates by t. Our group $(L, *)$ has the same underlying structure, and we have shown in the previous section that multiplying coordinates by t does indeed define a T-map on $(L, *)$. Thus $(L, *)$ is a T-uniform pro-p group. From Section 2 we see that this T-map on $(L, *)$ defines a T-action on $L_{(L,*)}$ and this T-action is again multiplying coordinates by t. Thus $L = L_{(L,*)}$ as a $\mathbb{Z}_p[[t]]$-Lie algebra.

5. *An example of a T-uniform group*

Let H be a $\mathbb{Z}_p[[t]]$-standard group of level 1 and dimension d. Then H is powerful but not uniform as it is not finitely generated. Let I be an ideal of $\langle p, t\rangle$, then we denote by $H(I)$ the closed subgroup of H given by the formal group law restricted to $I^{(d)}$. It follows that $H(I)$ is a normal subgroup of H, [**2**, 13·21]. We denote the normal subgroup $H(\langle p^\epsilon\rangle)$ of H by H_1. Then H_1 has a $\mathbb{Z}_p$-filtration given by the subgroups $H_1(\langle t^n\rangle)$ and we can define the intrinsic Lie algebra L_{H_1}. However we need to be careful about defining a T-map for H_1 since the 'obvious' T-map, i.e. multiplying coordinates by t, does not necessarily commute with the p-power map. We use the results of Section 3 and 4 to give us a coordinate structure for H_1 for which multiplication by t is a T-map.

We can associate with H_1 a formal Lie algebra $L(H_1)$. Denote the formal group law of H_1 as follows.

$$\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \mathbf{B}(\mathbf{X}, \mathbf{Y}) + O'(3)$$

where $\mathbf{B}(\mathbf{X}, \mathbf{Y}) = (B_i(\mathbf{X}, \mathbf{Y}))_{i=1}^d$ and $B_i(\mathbf{X}, \mathbf{Y})$ is bilinear in $\mathbf{X}$ and $\mathbf{Y}$. We can now define a Lie bracket by setting

$$(\mathbf{X}, \mathbf{Y})_{\mathbf{F}} = \mathbf{B}(\mathbf{X}, \mathbf{Y}) - \mathbf{B}(\mathbf{Y}, \mathbf{X}),$$

a d-tuple of bilinear forms. Then we have a Lie algebra structure on $\langle p^\epsilon\rangle^{(d)}$ and we denote this Lie algebra by $L(H_1)$.

Furthermore, $L(H_1)$ is a powerful $\mathbb{Z}_p[[t]]$ Lie algebra. This follows from the fact that $[H_1, H_1] = [H(\langle p^\epsilon\rangle), H(\langle p^\epsilon\rangle)] \leqslant H(\langle p^{2\epsilon}\rangle) \leqslant H_1^{p^\epsilon}$ and the commutator formula for a formal group law takes the following form

$$\mathbf{C}(\mathbf{X}, \mathbf{Y}) = \mathbf{B}(\mathbf{X}, \mathbf{Y}) - \mathbf{B}(\mathbf{Y}, \mathbf{X}) + O'(3).$$

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Thus using the results of Section 3, $(L(H_1), *)$ is a group in our category, in particular it has a T-map which is given by multiplying coordinates by t.

Although we appear to have constructed a new Lie algebra $L(H_1)$ in fact the formal Lie algebra $L(H_1)$ is isomorphic to the intrinsic Lie algebra L_{H_1}. The argument is similar to that of [**2**, exercise 9·13]. The idea is to note that if the group operation is a formal group law then when the addition and Lie bracket are defined in the intrinsic Lie algebra the majority of terms vanish resulting in the operations defined for the formal Lie algebra. Thus $L(H_1)$ induces a T-map on L_{H_1} and in turn this T-map on L_{H_1} induces a T-map on H_1, which is given by multiplication by t with respect to the coordinate structure given by $(L_{H_1}, *)$. So H_1 is a T-uniform pro-p group as required.

Although our example above considered a $\mathbb{Z}_p[[t]]$-standard group of level 1 we could equally well have considered a $\mathbb{Z}_p[[t]]$-standard group of level $h > 1$. If H were such a group then the subgroup $H(\langle p^h \rangle)$ would be T-uniform. Consequently since any $\mathbb{Z}_p[[t]]$-analytic group contains, as an open submanifold, a $\mathbb{Z}_p[[t]]$-standard group of level h for some h [**2**, theorem 13·20], it follows that any $\mathbb{Z}_p[[t]]$-analytic group contains a subgroup in our category.

6. *Hilbert's 5th or thereabouts*

In this section we prove the following.

Theorem B. *A T-uniform pro-p group has a $\mathbb{Z}_p[[t]]$-analytic structure.*

Let G be a T-uniform pro-p group. We have seen previously that we can define an 'intrinsic' Lie algebra L_G. Further, the BCHF defines a group operation on L_G and $(L_G, *) = G$. The following lemma is analogous to [**2**, lemma 9·12] and is proved similarly.

LEMMA. *Let L be a powerful $\mathbb{Z}_p[[t]]$-Lie algebra. Then the function $\tilde{\Phi}: L \times L \to L$ is analytic.*

Since L_G is a powerful $\mathbb{Z}_p[[t]]$-Lie algebra we have a coordinate structure for G with respect to which the group operation is analytic. Thus the theorem is proved.

Acknowledgements. This work was funded by an EPSRC grant whilst both authors were members of the Department of Pure Mathematics and Mathematical Statistics at the University of Cambridge.

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