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List Total Colourings of Graphs

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We study the concept of list total colourings and prove that every multigraph of maximum degree 3 is 5-total-choosable. We also show that the total choice number of a graph of maximum degree 2 is equal to its total chromatic number.

1. Introduction

Graphs (and multigraphs) in this paper are undirected and finite. They have no loops, but multigraphs may contain multiple edges. For convenience, we also allow edges with only one end, called *halfedges*. Edges with two ends are also called *proper edges*. The maximum vertex-degree in G is denoted by $\Delta(G)$. By $VE(G)$ we denote the union $V(G) \cup E(G)$.

The *total colouring* of G is a colouring of $VE(G)$ such that any two adjacent or incident elements are coloured differently. The *total chromatic number* $\chi''(G)$ is the smallest number of colours needed for a total colouring of G. Total colourings were introduced by Vizing [14, 15] and Behzad [2]. They both, independently, conjectured that $\chi''(G) \leqslant \Delta(G) + \mu(G) + 1$ holds for multigraphs, where $\mu(G)$ denotes the (edge) multiplicity of G. This conjecture became known as the Total Colouring Conjecture (see [6, Problem 4.9]). The conjecture has been verified for multigraphs of sufficiently small maximum degree. Rosenfeld [10] and Vijayaditya [12] independently proved that the total chromatic number for multigraphs of maximum degree 3 is at most 5. Kostochka, in [8] and [9], proved that the total chromatic number of multigraphs of maximum degree 4 (respectively 5) is at most 6 (respectively 7). It is also easy to see that for a bipartite multigraph G we have $\chi''(G) \leqslant \Delta(G) + 2$. Let us remark that there exist infinitely many bipartite graphs with total chromatic number equal to $\Delta + 2$ (see, for instance, Theorem 2.2).

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A (*total*) *list assignment* of G is a function L which assigns to each element $t \in VE(G)$ a *list* $L(t) \subseteq \mathbb{N}$ of *admissible* colours for t. An *L-total-colouring* is a function $\lambda : VE(G) \to \mathbb{N}$ such that $\lambda(t) \in L(t)$ for each $t \in VE(G)$ and such that for any pair of adjacent or incident elements $t, t' \in VE(G)$, $\lambda(t) \neq \lambda(t')$. If G admits an L-total-colouring, it is *L-total-colourable*. For $k \in \mathbb{N}$, the multigraph is *k-total-choosable* if it is L-total-colourable for every list assignment L such that $|L(t)| \geqslant k$ for each $t \in VE(G)$. The smallest k for which G is k-total-choosable is called the *total choice number* of G and is denoted by $\chi''_\ell(G)$. *List vertex colourings* and *list edge colourings* are defined analogously. The corresponding choice numbers of a multigraph G are denoted by $\chi_\ell(G)$ and $\chi'_\ell(G)$, respectively.

List colourings were introduced by Vizing [13] and independently by Erdős, Rubin, and Taylor [4]. Probably the most well-known conjecture about list colourings is the List Colouring Conjecture, which states that every (multi)graph G is $\chi'(G)$-edge-choosable, where $\chi'(G)$ is the usual chromatic index of G; see [6, Problem 12.20] for additional information. Recently, Galvin [5] (see also [3] and [11]) proved the conjecture in the affirmative for bipartite multigraphs.

As far as list total colourings are concerned, we have no counterexample to the following generalization of the Total Colouring Conjecture.

Conjecture (A) $\chi''_\ell(G) \leqslant \Delta(G) + \mu + 1$, where μ is the multiplicity of G.

Although in general $\chi(G) \neq \chi_\ell(G)$, we also find it plausible to conjecture:

Conjecture (B) $\chi''_\ell(G) = \chi''(G)$.

It is not hard to see that

$$\chi''_\ell(G) \leqslant \chi'_\ell(G) + 2.$$

First colour the vertices of G. Since $\chi_\ell(G) \leqslant \Delta(G) + 1$ and $\chi'_\ell(G) \geqslant \Delta(G)$, such a colouring exists. Now, each edge retains at least $\chi'_\ell(G)$ admissible colours, and so, by the definition of χ'_ℓ, it is possible to colour the edges and get a total colouring of G. The above inequality, combined with the result of Galvin [5] that every bipartite multigraph is Δ-edge-choosable, implies that

$$\chi''_\ell(G) \leqslant \chi'_\ell(G) + 2 = \Delta(G) + 2 \tag{1.1}$$

for every bipartite multigraph G.

In Section 2 we show that Conjecture (B) is true for multigraphs of maximum degree 2. In particular, we determine total choice numbers of cycles. In Section 3 we show that every multigraph of maximum degree 3 is 5-total-choosable. This verifies Conjecture (A) for such multigraphs.

2. Graphs of maximum degree two

In this section we show that every multigraph G with $\Delta(G) \leqslant 2$ is $\chi''(G)$-total-choosable. Since it suffices to prove the claim only for connected multigraphs, we may assume that G is isomorphic to a path or to a cycle. Trivially, $\chi''(P_1) = \chi''_\ell(P_1) = 1$. If $G \cong P_n$, $n \geqslant 2$, then $\chi''(G) = 3$. It is easy to see that also $\chi''_\ell(G) = 3$. Colour a vertex of degree 1 and its incident edge arbitrarily. Then the colouring extends along the path without any trouble.

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The situation with cycles is slightly more complicated. Let us first show that cycles are 4-total-choosable. More precisely, we have the following lemma.

Lemma 2.1. *Let L be a list assignment for C_n ($n \geqslant 2$) such that*

$$|L(t)| \geqslant \begin{cases} 3, & t \in V(G), \\ 4, & t \in E(G). \end{cases} \tag{2.1}$$

Then C_n has an L-total-colouring.

Proof. If $n = 2$, the proof is straightforward. Therefore we assume $n \geqslant 3$. Denote by $v_1, v_2, \ldots, v_n$ the vertices of C_n in the order they appear in C_n. First we construct an L-vertex-colouring λ of C_n. After the vertices are coloured, each edge has at least two admissible colours left. If λ cannot be extended to an L-total-colouring of C_n, then n must be an odd number and each edge must have the same pair of admissible colours left. We may also assume that $L(v_i) = \{\lambda(v_{i-1}), \lambda(v_i), \lambda(v_{i+1})\}$ for $i = 1, \ldots, n$; otherwise we can recolour v_i so that we obtain an edge with three colours or two edges with distinct pairs of admissible colours left. In particular, $\lambda(v_1) \neq \lambda(v_3)$ and $\lambda(v_2) \neq \lambda(v_n)$. This allows us to exchange the colours of v_1 and v_2 in order to obtain an L-vertex-colouring that can be extended to an L-total-colouring of C_n. □

It is easy to see [12] that

$$\chi''(C_n) = \begin{cases} 3, & n = 3k, \ \ k \in \mathbb{N}, \\ 4, & \text{otherwise.} \end{cases} \tag{2.2}$$

In the sequel we show that $\chi''_\ell(C_n) = \chi''(C_n)$. By (2.2) and by Lemma 2.1, it suffices to verify that $\chi''_\ell(C_n) = 3$ for $n \equiv 0 \pmod 3$. In order to prove this, we apply a theorem of Alon and Tarsi [1] that relates orientations, Eulerian subgraphs, and vertex-choosability of graphs.

By an Eulerian directed graph we understand a (not necessarily connected) directed graph in which the outdegree of each vertex equals its indegree. An Eulerian digraph is called *even* (*odd*) if it has an even (odd) number of edges. For a digraph D, denote by $\mathscr{E}^e(D)$ and $\mathscr{E}^o(D)$ the set of even and odd spanning Eulerian subdigraphs in D, respectively. Let $\mathscr{E}(D) = \mathscr{E}^e(D) \cup \mathscr{E}^o(D)$, *i.e.*, the set of all spanning Eulerian subdigraphs in D. Denote by $d^+_D(v)$ the outdegree of v in D.

Theorem 2.1. (Alon and Tarsi) *If G is a graph that has an orientation D satisfying $|\mathscr{E}^e(D)| \neq |\mathscr{E}^o(D)|$ in which the maximum outdegree is d, then $\chi_\ell(G) \leqslant d + 1$.*

Our result follows from Theorem 2.1.

Theorem 2.2. *If G is a cycle of length $3n$, then $\chi''_\ell(G) = 3$.*

Proof. Denote by T the total graph of G, *i.e.*, a graph with vertex set $V(T) = VE(G)$ where two elements are adjacent if and only if they are incident or adjacent in G. It is enough to prove that $\chi_\ell(T) = 3$. Observe that T is isomorphic to the circulant $C(6n, 2)$, a graph composed of one $6n$-cycle C_1 and two pairwise interlacing $3n$-cycles C_2 and C_3. Let

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D be an orientation of T such that C_1, C_2, C_3 are directed cycles oriented consistently. Let $\mathscr{E} = \mathscr{E}(D)$ and define

$$\begin{aligned} \mathscr{E}_0 &= \left\{H \in \mathscr{E} \mid \forall v \in V(D) : d_H^+(v) \leqslant 1\right\}, \\ \mathscr{E}_1 &= \left\{H \in \mathscr{E} \mid \forall v \in V(D) : d_H^+(v) = 1\right\}, \\ \mathscr{E}_2 &= \left\{H \in \mathscr{E} \mid \forall v \in V(D) : d_H^+(v) \geqslant 1\right\}. \end{aligned}$$

Obviously, $\mathscr{E}_0 \cap \mathscr{E}_2 = \mathscr{E}_1$. Note also that $\mathscr{E}_1$ contains only two digraphs: C_1 and $C_2 \cup C_3$. Let $H \in \mathscr{E}$. If there is a vertex v with $d_H^+(v) = 2$, then it is not hard to see that for every $u \in V(D)$ we have $d_H^+(u) \geqslant 1$. This implies that $\mathscr{E} = \mathscr{E}_0 \cup \mathscr{E}_2$.

Let $\varphi : \mathscr{E} \to \mathscr{E}$ be a mapping defined by $\varphi(H) = D - E(H)$. Note that φ is a bijection and that $\varphi(H) \neq H$ for every $H \in \mathscr{E}$. Moreover, since $|E(D)|$ is even, $\varphi(\mathscr{E}^e(D)) = \mathscr{E}^e(D)$ and $\varphi(\mathscr{E}^o(D)) = \mathscr{E}^o(D)$. This implies that $|\mathscr{E}^e(D)|$ and $|\mathscr{E}^o(D)|$ are even. If $|\mathscr{E}^e(D)| = |\mathscr{E}^o(D)|$, then $|\mathscr{E}| = 4k$ for some integer k. To apply Theorem 2.1 it is therefore enough to show that $|\mathscr{E}| = 4k + 2$ for some integer k. Since $\varphi(\mathscr{E}_0) = \mathscr{E}_2$ and $\varphi(\mathscr{E}_2) = \mathscr{E}_0$, we have $|\mathscr{E}_0| = |\mathscr{E}_2|$. Hence it suffices to show only that $|\mathscr{E}_0|$ is even.

Every graph from $\mathscr{E}_0$ is composed of 0, 1, or 2 directed cycles and an appropriate number of isolated vertices. There is only one graph without a cycle, the empty graph, and only one containing two cycles, $C_2 \cup C_3$. To determine the number of graphs with exactly one cycle, we proceed as follows. Let f_i denote the number of ways in which we can write i as an ordered sum of numbers 1 and 2. It is easy to see that the sequence f_i is determined by the following recursion:

$$f_1 = 1, \quad f_2 = 2 \quad \text{and} \quad f_i = f_{i-1} + f_{i-2}, \quad i \geqslant 3.$$

Choose a vertex $v \in V(D)$. Since each directed cycle in D containing v can be uniquely represented by a sequence of numbers 1 and 2 with the sum $6n$, the number of such cycles is equal to f_{6n}. Each cycle that does not contain v contains the edge between the neighbours of v in C_1. Similarly, as before, the number of such cycles is equal to f_{6n-2}. Combining the above results, we have

$$|\mathscr{E}_0| = f_{6n} + f_{6n-2} + 2 = f_{6n-1} + 2f_{6n-2} + 2.$$

Since f_{6n-1} is even for every integer n, the claim follows. □

3. Graphs of maximum degree three

In this section we consider list total colourings of *subcubic* multigraphs, *i.e.*, multigraphs with $\Delta \leqslant 3$. To prove the main result of this section, Theorem 3.1, we need some auxiliary lemmas. Let us first present a result on list edge colourings of cycles with halfedges.

Lemma 3.1. *Let G be a cycle with a halfedge attached to each vertex. Suppose that L is an edge list assignment for G such that*

$$|L(e)| \geqslant \begin{cases} 3, & e \text{ is a proper edge,} \\ 2, & e \text{ is a halfedge.} \end{cases} \tag{3.1}$$

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Then G admits an L-edge-colouring unless $|V(G)|$ is odd and there exist colours a, b, c such that $L(e) = \{a, b, c\}$ for each proper edge e and $L(e) = \{a, b\}$ for each halfedge e.

Proof. We may assume that there are equalities in (3.1). Let $n = |V(G)|$ and denote by $v_1, \dots, v_n$ the vertices of G enumerated in the order they appear on G. Let s_i be the halfedge incident with v_i and let $e_i = v_i v_{i+1}$, $1 \leqslant i \leqslant n$. Choose the starting vertex $v_1 \in V(G)$ and the direction of G such that $L(s_2) \not\subseteq L(e_1)$. If such a choice is not possible, let v_1 be such that $L(s_1) \neq L(s_2)$ or $L(e_1) \neq L(e_n)$. (If this rule cannot be satisfied either, G is as excluded by our lemma except that its length may be even. However, in that case it can easily be L-coloured.)

Let us start colouring at the vertex v_1. Colour e_1 with a colour from $L(e_1) \setminus L(s_1)$ and proceed with the vertex v_2. At vertex v_i $(2 \leqslant i < n)$, we colour s_i with a colour from $L(s_i) \setminus \{\lambda(e_{i-1})\}$ and e_i with a colour from $L(e_i) \setminus \{\lambda(e_{i-1}), \lambda(s_i)\}$ and then proceed to the next vertex. Arriving to v_n, it remains to colour s_n, e_n, and s_1. By our choice of $\lambda(e_1)$, every L-edge-colouring of s_n and e_n can be extended to s_1. So, an obstruction can occur only when colouring the edge e_n.

Suppose that $L(s_n) = \{c, d\}, L(s_1) = \{a, b\}, x = \lambda(e_1), y = \lambda(s_2)$, and $z = \lambda(e_2)$. If we cannot colour e_n, we have $\lambda(e_{n-1}) \in \{c, d\}$ (say $\lambda(e_{n-1}) = c$) and $L(e_n) = \{c, d, x\}$. If $L(e_1) \neq \{x, y, z\}$, we recolour e_1 by using a colour in $L(e_1) \setminus \{x, y, z\}$, and set $\lambda(s_n) = d$, $\lambda(e_n) = x$. Since $x \notin L(s_1)$, there is also an available colour for s_1. Therefore $L(e_1) = \{x, y, z\}$. If $L(s_2) \neq \{y, z\}$, we recolour $\lambda(s_2) \in L(s_2) \setminus \{y, z\}$, $\lambda(e_1) = y$, $\lambda(e_n) = x$, $\lambda(s_n) = d$, and $\lambda(s_1) \in L(e_1) \setminus \{y\}$. Therefore $L(s_2) = \{y, z\} \subseteq L(e_1)$. Then v_1 was not selected according to the first choice, and hence also $L(s_1) \subseteq L(e_n)$ and $L(s_1) \subseteq L(e_1)$. This implies that $L(s_1) = \{y, z\}$ and $L(e_n) = \{x, y, z\}$, a contradiction to the choice of v_1. □

Let us remark that stronger results on list edge colourings of cycles with halfedges are proved in [7].

Next we prove a similar result about list total colourings of cycles with halfedges.

Lemma 3.2. *Let G be a cycle with a halfedge attached to each vertex. Suppose that L is a list assignment for G such that*

$$|L(t)| \geqslant \begin{cases} 5, & t \text{ is a proper edge,} \\ 4, & t \text{ is a vertex,} \\ 2, & t \text{ is a halfedge.} \end{cases} \tag{3.2}$$

Then G admits an L-total-colouring.

Proof. Set $n = |V(G)|$ and let $v_1, \dots, v_n$ be the vertices of G enumerated in the order they appear on G. Denote by s_i the halfedge incident with v_i and let $e_i = v_i v_{i+1}$, $1 \leqslant i \leqslant n$. We also assume that there are equalities in (3.2).

Suppose first that there exists a colour $c \in L(s_1) \cap \cdots \cap L(s_n)$. Colour all halfedges by c. After that, each edge has at least 4 and each vertex at least 3 admissible colours left. By Lemma 2.1 the colouring can be extended to G.

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Suppose now that there exists an L-colouring λ of $V(G)$ such that $\lambda(v_i) \notin L(s_i)$, $1 \leqslant i \leqslant n$. Note that now each edge of G has at least three admissible colours left. Since not all lists of colours on halfedges are the same, λ can be extended to G by Lemma 3.1.

Otherwise, G is an odd cycle and there are two colours x, y such that $L(v_i) = L(s_i) \cup \{x, y\}$, $1 \leqslant i \leqslant n$. To construct a colouring we distinguish two cases. Suppose first that there is an edge of G which does not contain both colours x and y. We may assume that $x \notin L(e_n)$. Colour $v_2, v_4, \ldots, v_{n-1}$ by y and $v_3, v_5, \ldots, v_n$ by x. Also colour v_1 and s_1 by the two colours of $L(s_1)$. Since $x \notin L(e_n)$, e_n still has at least three admissible colours. Choose $\lambda(e_1)$ such that e_n retains an admissible colour not in $L(s_n)$. The other possibility is that $\{x, y\} \subset L(e_i)$, $1 \leqslant i \leqslant n$. In this case, colour $v_1, \ldots, v_n, s_1$ as above. Moreover, set $\lambda(e_1) = x$. This ensures that $L(e_n) \setminus \{\lambda(v_1), \lambda(s_1), x\} \neq L(s_n)$ since y is still admissible for e_n.

In both cases, we colour the edges $s_2, e_2, \ldots, s_{n-1}, e_{n-1}$ in this order. Note that at each step we have at least one available colour. Since we ensured that $L(e_n) \setminus \{\lambda(v_1), \lambda(s_1), \lambda(e_1)\} \neq L(s_n)$, the obtained colouring can also be extended to s_n and e_n. □

We will also need to know how to colour three small multigraphs under an additional constraint.

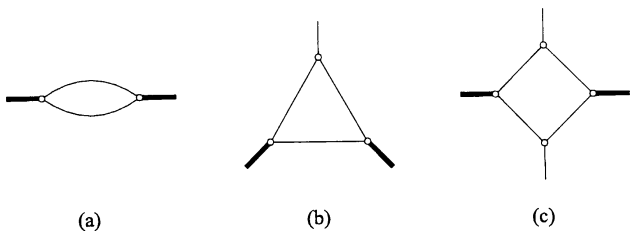


Figure 1 Small cycles with thick and thin halfedges.

Lemma 3.3. *Let G be isomorphic to one of the multigraphs from Figure 1. Suppose that L is a list assignment for G such that*

$$|L(t)| \geqslant \begin{cases} 5, & t \text{ is a proper edge,} \\ 4, & t \text{ is a vertex,} \\ 3, & t \text{ is a thick halfedge,} \\ 2, & t \text{ is a thin halfedge.} \end{cases} \tag{3.3}$$

Then G admits an L-total-colouring such that the two thick halfedges are coloured by distinct colours.

Proof. The case when G is isomorphic to the multigraph from Figure 1(a) is left to the reader as an easy exercise. When G is isomorphic to the graph from Figure 1(b), we first colour the halfedges so that the thick pair receives distinct colours. After that, all the elements of the 3-cycle have at least 3 admissible colours left. Since C_3 is 3-total-choosable, the colouring of the halfedges can be extended to the whole G.

The last case is when G is isomorphic to the graph from Figure 1(c). We may assume that we have equalities in (3.3). Let $v_1, \ldots, v_4$ be the vertices of G such that v_1 and v_3 are incident with the thick halfedges. For $i = 1, \ldots, 4$, denote by s_i the halfedge of G incident

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with v_i. We will show that it is possible to colour v_2, v_4, s_1, s_3 so that s_1 and s_3 are coloured differently and each of v_1, v_3, s_2, s_4 still has at least 2 admissible colours left. After that, the problem of extending this partial colouring to an admissible colouring of the whole G is equivalent to the problem of finding an edge colouring of a 4-cycle with a halfedge attached to each vertex where each proper edge has at least 3 and each halfedge at least 2 admissible colours. By Lemma 3.1 such an edge colouring of the 4-cycle always exists. Let $L_i = L(v_i)\backslash L(s_i)$, $i = 2,4$. To find a required colouring of v_2, v_4, s_1, s_3, we distinguish two cases. If there exists $c \in L_2 \cap L_4$, we colour v_2, v_4 by c and colour s_1, s_3 arbitrarily (by distinct colours). Otherwise, colour s_1 by an arbitrary colour $c_1 \in L(s_1)$. Since L_2 and L_4 are disjoint, there exists a colour $c_2 \in L_2 \cup L_4$, say $c_2 \in L_2$, such that $|L(v_1)\backslash\{c_1, c_2\}| \geqslant 3$. Colour v_2 by c_2. This ensures that v_1 will retain at least 2 admissible colours after any colouring of v_4. Similarly, we take care of v_3. If there exists a colour in $L_4 \cap (L(s_3)\backslash\{c_1\})$, we use it to colour v_4 and s_3. Otherwise, there is a colour $c_4 \in L_4 \cup (L(s_3)\backslash\{c_1\})$, say $c_4 \in L_4$, such that $|L(v_3)\backslash\{c_2, c_4\}| \geqslant 3$. Use c_4 to colour v_4. Finally, colour s_3 by a colour from $L(s_3)\backslash\{c_1\}$. □

To make the induction in the proof of Theorem 3.1 work, we also need the following result about the structure of subcubic multigraphs. By Θ we denote the cubic multigraph consisting of two vertices and three parallel edges joining them.

Lemma 3.4. *Let G be a connected subcubic multigraph that contains at least one cycle. Then, G is either isomorphic to Θ, K_4, or $K_{3,3}$, or it contains an induced cycle C with the following property. The cycle C contains at most one pair of vertices $u, v \in V(C)$ that has a common neighbour out of C. If such a pair exists, then either $|V(C)| \leqslant 3$, or $|V(C)| = 4$ and u, v are not adjacent in G.*

Proof. Let C be a shortest cycle in G. If $|V(C)| \leqslant 3$, then either G is isomorphic to Θ or K_4, or C has the required property. Similarly, if $|V(C)| > 4$, then no two vertices in $V(C)$ have a common neighbour out of C. Hence, C can be taken as the required cycle.

It remains to consider the case when $|V(C)| = 4$. The cycle C does not have the required property if and only if there are two vertices $v_1, v_2 \in V(G)\backslash V(C)$ both having two neighbours in C. If v_1 and v_2 are adjacent, then $G \cong K_{3,3}$. Otherwise, the subgraph of G induced by $(V(C) \cup \{v_1\})\backslash\{w\}$, where $w \in V(C)$ is not adjacent to v_1, is a cycle having the required property. □

By combining the above auxiliary results, we will now prove the main theorem.

Theorem 3.1. *Every subcubic multigraph G (without loops, but possibly containing halfedges) is 5-total-choosable.*

Proof. Since a halfedge is adjacent to at most 3 elements of $VE(G)$, it suffices to consider only multigraphs without halfedges. The proof is by induction on $|V(G)|$. Let L be a list assignment for G such that $|L(t)| \geqslant 5$ for each $t \in VE(G)$.

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Suppose first that G contains a vertex v of degree less than 3. By induction, $G-v$ admits an L-colouring, and this colouring can always be extended to G. If G is not connected, each component, and hence the whole multigraph, can be L-coloured by induction. If G is isomorphic to Θ or $K_{3,3}$, then G is bipartite and therefore can be L-coloured by (1.1). If G is isomorphic to K_4, first L-colour one of the vertices and all edges incident to it. Since C_3 is 3-total-choosable, the partial colouring can be extended to G. In the remaining possibility G contains an induced cycle C with the property stated in Lemma 3.4. Colour $G - V(C)$ by induction and consider the edges between $G - V(C)$ and C as halfedges attached to the vertices of C. Then Lemmas 3.2 and 3.3 guarantee that the colouring can be extended to an L-colouring of G. □

Since the proof of Theorem 3.1 is inductive, we also have the following algorithmic result.

Corollary 3.1. *There is a polynomial time algorithm for list total colouring subcubic multigraphs if each list of colours contains at least* 5 *elements.*

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