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# PROBLEMS AND SOLUTIONS

## PROBLEMS

02.3.1. *Regression with an Evaporating Logarithmic Trend*, proposed by P.C.B. Phillips and Y. Sun.

Part A. The time series $X_t$ is generated by the model

$$X_t = \alpha + \frac{\beta}{\ln t} + u_t, \qquad t = 2, \ldots, n \tag{1}$$

where $\alpha$ and $\beta$ are unknown parameters whose least squares regression estimates are denoted by $\hat{\alpha}$ and $\hat{\beta}$, respectively. The error $u_t$ in (1) is assumed to be *iid* $(0, \sigma^2)$ with finite fourth moment.

1. Show that $\hat{\alpha}$ and $\hat{\beta}$ are strongly consistent for $\alpha$ and $\beta$ as $n \to \infty$.
2. Find the asymptotic distribution of $\hat{\alpha}$ and $\hat{\beta}$.

Part B. Suppose that $u_t$ in (1) is the linear process

$$u_t = \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}, \quad \text{with} \sum_{j=0}^{\infty} j|c_j| < \infty \tag{2}$$

and where $\varepsilon_t$ is *iid* $(0, \sigma^2)$ with finite fourth moment. Explain how you would modify your derivations in Part A to allow for such an error process in the regression model (1).

02.3.2. *Badly Weighted Least Squares*, proposed by R. Koenker and S. Portnoy.

Consider the classical linear regression model

$$y_i = x_i' \beta + u_i$$

with $V(u_i) = \sigma_i^2 > 0$, $u_i$ independent over $i = 1, 2, \ldots, n$, and $X = (x_i')_{i=1}^n$ of full column rank $p$.

Let $\Omega = \text{diag}(\sigma_1, \ldots, \sigma_n)$ and assume $\sigma_i \neq \sigma_j$ for some $i \neq j$. The asymptotic covariance matrix of the weighted least squares estimator

$$\hat{\beta}(a) = (X'\Omega^{-2a}X)^{-1}X'\Omega^{-2a}y$$

is the Eicker–White matrix, $(X'X)^{-1}X'\Omega^2X(X'X)^{-1}$ for $a = 0$, whereas for $a = 1$ we have $(X'\Omega^{-2}X)^{-1}$. For $a = \frac{1}{2}$ we obtain an expression for the asymptotic covariance matrix of the median regression estimator.

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This *suggests* that in particular,

$$(X'X)^{-1}X'\Omega^2X(X'X)^{-1} \geq (X'\Omega^{-1}X)^{-1}X'X(X'\Omega^{-1}X)^{-1}$$

and more generally that, for any $\delta \in \mathbb{R}^p$,

$$e_{a,\delta} = \delta'(X'\Omega^{-2a}X)^{-1}X'\Omega^{2-4a}X(X'\Omega^{-2a}X)^{-1}\delta$$

is monotone in $a \in [0,1]$. Prove or disprove the monotonicity claim.

## SOLUTIONS

01.3.1. *A Necessary and Sufficient Condition for the Equivalence of Affine Unbiased Estimators*—Solution,[1] proposed by Geert Dhaene, the poser of the problem. The affine unbiased estimators of $\beta$ can be written as $\hat{\beta} = Ay$ where $A$ satisfies $AX = I$. If the ordinary least squares (OLS) residuals $y - X(X'X)^{-1}X'y$ are zero, we have $\hat{\beta} = Ay = AX(X'X)^{-1}X'y = (X'X)^{-1}X'y$, which proves the "only if" part. If all affine unbiased estimators of $\beta$ coincide, their residuals $y - Ay$ also coincide. Now consider any set of $k$ residuals, where $k = \text{rank}(X)$. The weighted least squares estimator that minimizes the sum of squares of those residuals is affine and unbiased and makes those residuals zero. Hence all residuals of affine unbiased estimators are zero, and so are the OLS residuals. This proves the "if" part.

*NOTE*

1. Independent solutions have been proposed by Kai Carstensen, Heinz Neudecker, and Götz Trenkler.

01.3.2. *Conditional and Unconditional Correlatedness and Heteroskedasticity*—Solution, proposed by Stanislav Anatolyev, the poser of the problem.

The following example demonstrates all phenomena: Let $z_t$ be a strict white noise with unit variance. Let a standard bivariate normal white noise $(\epsilon_{vt}, \epsilon_{wt})'$ be independent of the process $z_t$, and generate $v_t = \epsilon_{vt}\sqrt{1 - \alpha + \alpha z_t^2}$ and $w_t = \epsilon_{wt} z_t$, where $0 < \alpha < 1$. Construct $e_t$ as $e_t = v_{t+1} + \theta_v v_t + w_{t+1} + \theta_w w_t$ for $\theta_v, \theta_w \neq 0$, $|\theta_v| < 1$, $|\theta_w| < 1$. Then one can find that $E[e_t|\Omega_t] = 0$ and $E[e_t e_{t-1}|\Omega_t] = \theta_v(1 - \alpha) + (\theta_v\alpha + \theta_w)z_t^2$, while $E[e_t^2|\Omega_t] = 2 + \theta_v^2(1 - \alpha) + (\theta_v^2\alpha + \theta_w^2)z_t^2$.

The phenomena in (a)–(c) are illustrated as follows:

(a) If we set $\theta_v + \theta_w = 0$, we have $E[e_t e_{t-1}] = \theta_v(1 - \alpha) + (\theta_v\alpha + \theta_w)E[z_t^2] = 0$, while $E[e_t e_{t-1}|\Omega_t] \neq 0$ almost everywhere.

(b) If $\theta_v\alpha + \theta_w = 0$ but $\theta_v^2\alpha + \theta_w^2 \neq 0$, we observe that $E[e_t e_{t-1}|\Omega_t]$ is almost surely constant, while $E[e_t^2|\Omega_t]$ is nonconstant. Conversely, we may have $\theta_v^2\alpha +$

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$\theta_w^2 = 0$ but $\theta_v\alpha + \theta_w \neq 0$ (this requires $\alpha$ to be negative, which is possible if $z_t$ has finite support[1]).

(c) Finally, if we set $\zeta_t = z_t^2$, both $E[e_t e_{t-1}|\Omega_t]$ and $E[e_t^2|\Omega_t]$ are $\sigma(\zeta_t)$-measurable. If $\theta_v\alpha + \theta_w$ and $\theta_v^2\alpha + \theta_w^2$ are nonzero and have opposite signs, we observe the phenomenon in (c).

## NOTE

1. For example, one can take $z_t \sim i.i.d.U[-\sqrt{3}, \sqrt{3}]$, $\alpha = -\frac{1}{2}$, $\theta_v = 1/\sqrt{2}$, $\theta_w = \frac{1}{2}$.

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