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Topological entropy of transitive dendrite maps

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Abstract. We show that every dendrite X satisfying the condition that no subtree of X contains all free arcs admits a transitive, even exactly Devaney chaotic map with arbitrarily small entropy. This gives a partial answer to a question of Baldwin from 2001.

1. *Introduction*

Let X be a compact metric space. By a *dynamical system* on X we mean a pair (X, f), where $f : X \to X$ is a continuous (not necessarily injective) map. The topological entropy of f, defined in [**1**], will be denoted by $h(f)$. This is a fundamental quantitative characteristic of a dynamical system. On the other hand, a dynamical system can possess some qualitative dynamical properties, such as transitivity, weak and strong mixing and exactness (see §2 for definitions). It is natural to study the connection between these qualitative properties and entropy. The first result in this direction, due to Blokh [**11**], says that the topological entropy of any transitive dynamical system on the interval is at least $(1/2)\log 2$; moreover, he showed that this bound is the best possible. Hence we have $I^{\mathcal{T}}([0, 1]) = (1/2)\log 2$, where $I^{\mathcal{T}}(X)$ is defined by

$$I^{\mathcal{T}}(X) = \inf\{h(f) : \ f : X \to X \text{ is transitive}\}.$$

Here, the infimum is *attainable* since there is a transitive interval map with the entropy $(1/2)\log 2$. In 1987, Blokh [**12**] showed that transitivity implies positiveness of entropy for maps on any graph which is not a circle. Later Alsedà *et al* [**4**] showed that for an n-star S we have $I^{\mathcal{T}}(S) = (1/n)\log 2$. For arbitrary tree T, Alsedà *et al* [**3**] showed that $I^{\mathcal{T}}(T) \geq (1/n)\log 2$, where n is the number of end points of the tree. For some classes of trees this bound is the best possible (see [**7, 25**]) but in general it is still an open problem to determine the value of $I^{\mathcal{T}}(T)$ for arbitrary tree T. By Alsedà *et al* [**5**], for any graph X which is not a tree, $I^{\mathcal{T}}(X) = 0$; they even showed that $I^{\mathcal{ED}}(X) = 0$, where

$$I^{\mathcal{ED}}(X) = \inf\{h(f) : \ f : X \to X \text{ is exactly Devaney chaotic}\}.$$

By [**15**, Lemma 8.5], every tree admitting a transitive P-linear Markov map also admits, for every $\varepsilon > 0$, an exact P'-linear Markov map f' with $h(f') < h(f) + \varepsilon$. Thus, for

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trees T, the upper bounds for $I^{\mathcal{T}}(T)$ from [**25**] are true also for $I^{\mathcal{ED}}(T)$; in particular, $I^{\mathcal{ED}}(T) = (1/n)\log 2$ for every $T \in \mathcal{T}_0$ (for the definitions of Ye's classes $\mathcal{T}_i$ of trees, see, for example, §2.3).

A natural class of compact metric spaces containing all trees is the class of dendrites. Recall that a *dendrite* is a locally connected continuum containing no simple closed curve. Dendrites possess many properties of trees: they have the fixed point property, they are absolute retracts, they are embeddable in the plane, etc. On the other hand, dendrites can be much more complicated than trees; for example, there is an uncountable system of pairwise non-homeomorphic dendrites. Recently, the study of dynamics on dendrites has attracted many authors; see, for instance, [**2, 8, 16–18, 20, 24**] and other references therein. If X is a dendrite we denote by $E(X)$ the set of all *end points* of X (that is, points $x \in X$ such that $X\backslash\{x\}$ is connected) and by $B(X)$ the set of all *branch points* of X (that is, points $x \in X$ such that $X\backslash\{x\}$ has at least three components). Notice that, for a non-degenerate dendrite X, $B(X)$ is always countable and it is empty if and only if X is an arc. Further, the set of end points $E(X)$ is always non-empty and it is finite if and only if X is a tree.

In this paper we study infima $I^{\mathcal{T}}(X)$, $I^{\mathcal{ED}}(X)$ for non-degenerate dendrites which are not trees. Every such dendrite X has infinitely many end points and so, for every n, it contains an n-star S_n or an n-comb C_n (see Corollary 4). Since $I^{\mathcal{T}}(S_n) = (1/n)\log 2$ by [**4**] and $I^{\mathcal{T}}(C_n) = (1/n)\log 2$ for $n = 2^r$ by [**25**], the dendrite X contains subtrees admitting transitive maps with arbitrarily small entropy. So it is natural to expect that the answer to the following question due to Baldwin is affirmative.

Problem. Is it true that $I^{\mathcal{T}}(X) = 0$ for every non-degenerate dendrite X which is not a tree?

The purpose of the present paper is to give a partial affirmative answer to this question. Roughly speaking, we show that $I^{\mathcal{T}}(X) = I^{\mathcal{ED}}(X) = 0$ provided X contains 'sufficiently many' free arcs. Recall that an *arc* is a homeomorphic image of $[0, 1]$ and a *free arc* is an arc A such that every non-end point of A is an interior point of A in the topology of X.

THEOREM A. *If X is a non-degenerate dendrite such that no subtree of it contains all free arcs of X then X admits a transitive, even exactly Devaney chaotic map with arbitrarily small entropy. Hence*

$$I^{\mathcal{T}}(X) = I^{\mathcal{ED}}(X) = 0,$$

where the infima are not attainable.

The class of dendrites satisfying the condition from Theorem A has not been studied in the literature, but it appears to be natural for our construction. Note that in Lemma 5 we will show that the condition is satisfied, for example, for dendrites which are not trees and which have nowhere dense branch points.

Combining Theorem A with the main result of [**23**], we obtain the following corollary. Recall that a continuum X is *completely regular* provided the only nowhere dense subcontinua of X are degenerate. Since graphs are completely regular, the corollary generalizes [**5**, Theorem 4.1].

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COROLLARY B. *If X is a non-degenerate completely regular continuum which is not a tree then $I^{\mathcal{T}}(X) = I^{\mathcal{ED}}(X) = 0$.*

As a by-product, by combining our results with the constructions from [**25**] we obtain the following upper estimate of $I^{\mathcal{ED}}(X)$ for dendrites X based on the number of end points of special subtrees of X. Here we say that an edge of a tree T is *terminal* if it contains an end point of T. For the definition of Ye's class $\mathcal{T}_0$, see the end of §2.3.

PROPOSITION C. *If a dendrite X contains a tree T from Ye's class $\mathcal{T}_0$ with n end points and with all terminal edges having non-empty interiors in X, then*

$$I^{\mathcal{T}}(X) \le I^{\mathcal{ED}}(X) \le \frac{\log 2}{n}.$$

In particular, if X contains a free arc then $I^{\mathcal{ED}}(X) \le (1/2)\log 2$.

The paper is organized as follows. In the next section we summarize all the definitions and facts which will be needed later and we also prove some elementary facts about dendrites. In §3 we introduce the so-called (P, S)-linear Markov maps on trees and we show how they can be used to define exactly Devaney chaotic tree maps with small entropy. In §4 we prove the main results of the paper. Finally, in two appendices we give simple constructions of zero-entropy (P, S)-linear Markov maps on stars and combs, and we show, for trees T, an upper bound for $I^{\mathcal{ED}}(T)$ which depends only on the number of end points of T.

2. *Preliminaries*

In this section we recall the basic notation and definitions which are used in the paper. The cardinality of a set A is denoted by $\#A$. By $\mathbb{Z}$ ($\mathbb{N}$, $\mathbb{N}_0$) we denote the set of all (positive, non-negative) integers, by $\mathbb{R}$ we denote the set of all reals and by I we denote the unit interval $[0, 1]$. For an interval J, $|J|$ is its length (that is, its Lebesgue measure). The symbol $\sqcup$ denotes a disjoint union.

A *space* is any topological space; it is *degenerate* if it consists of one point, otherwise it is *non-degenerate*. A *Cantor space* is a totally disconnected compact metric space without isolated points. A set is said to be *regular closed* if it is the closure of its interior or, equivalently, if it is the closure of an open set. The Hausdorff one-dimensional measure on a separable metric space (X, d) is denoted by $\mathcal{H}^1_d$.

A *dynamical system* is a pair (X, f), where X is a compact metric space and $f : X \to X$ is a continuous selfmap of X. For non-negative integers n we define the *nth iterate* f^n of f recursively as follows: f^0 is the identity map on X and $f^n = f \circ f^{n-1}$ for any $n \in \mathbb{N}$. A subset M of X is called *f-invariant* (*strongly f-invariant*) provided $f(M) \subseteq M$ ($f(M) = M$). A *subsystem* of (X, f) is any dynamical system $(M, f|_M)$ where M is a non-empty closed f-invariant subset of X. If (X, f) is a dynamical system, a point x of X is called *fixed* if $f(x) = x$ and *periodic* if $f^p(x) = x$ for some $p \in \mathbb{N}$.

The topological entropy of a system (X, f) will be denoted by $h(f)$. A system (X, f) is called *transitive* if for all non-empty open sets U, V there is $n \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset$. It is *exact* if for every non-empty open set U there is $n \in \mathbb{N}$ with $f^n(U) = X$. A system (X, f) is *(exactly) Devaney chaotic* if X is infinite and f is transitive (exact) and has dense periodic points.

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2.1. *Perron–Frobenius theory.* Let $M = (m_{ij})_{ij=1}^n$ be an $n \times n$ matrix with real entries. A sequence $\pi = (p_j)_{j=0}^k$ ($k \geq 1$) of elements of $\{1, \dots, n\}$ is called a *path* if its *width* $w(\pi) = \prod_{j=1}^k m_{p_{j-1}p_j}$ is non-zero; the *length* of π is $|\pi| = k$. A path $(p_j)_{j=0}^k$ is called a *loop* if $p_0 = p_k$. The matrix M is called *irreducible* if for every i, j there is a path from i to j. If M is irreducible and the greatest common divisor of lengths of all loops is equal to 1, M is said to be *primitive*.

If M is a non-negative square matrix, its spectral radius λ_M (which is equal to the maximal non-negative eigenvalue of M) will be called the *Perron eigenvalue* of M. Notice that if M, N are non-negative and $M \leq N$ (that is, $m_{ij} \leq n_{ij}$ for every i, j), then $\lambda_M \leq \lambda_N$. If $M \geq 0$ is an irreducible 0–1 matrix, we have $\lambda_M \geq 1$ and there are strictly positive left and right eigenvectors of M associated to λ_M. Moreover, $\lambda_M = 1$ if and only if M is a *permutation matrix* (that is, it has exactly one entry 1 in each row and each column and 0s elsewhere); see, for example, [**21**, p. 8]. In particular, if M is a primitive 0–1 matrix, then $\lambda_M > 1$.

The so-called *rome method* for calculating the characteristic polynomial of square matrices was proposed in [**10**, pp. 20–21]. Let $M = (m_{ij})_{ij=1}^n$ be an $n \times n$ matrix. A non-empty subset $\mathcal{R} \subseteq \{1, \dots, n\}$ is called a *rome* if for every loop $(p_j)_{j=0}^k$ there is j with $p_j \in \mathcal{R}$. Let $\mathcal{R} = \{r_1, \dots, r_k\}$ ($r_i \neq r_j$ for $i \neq j$) be a rome. A path $(p_j)_{j=0}^k$ is said to be *simple* (or $\mathcal{R}$-*simple*) if p_0, $p_k \in \mathcal{R}$ and $p_j \notin \mathcal{R}$ for $1 \leq j < k$. Define a matrix function $R_{\mathcal{R}}$ by

$$R_{\mathcal{R}} = (r_{ij})_{ij=1}^k, \quad r_{ij}(x) = \sum_{\pi} w(\pi) x^{-|\pi|} \quad (x \in \mathbb{R}),$$

where the summation is over all simple paths π from r_i to r_j. Let E denote the unit matrix (of an appropriate size). By [**10**, Theorem 1.7], the characteristic polynomial $\chi_M(x) = \det(M - xE)$ of M satisfies

$$\chi_M(x) = (-1)^{n-k} x^n \cdot \det(R_{\mathcal{R}}(x) - E) \quad \text{for every rome } \mathcal{R} \text{ and } x \in \mathbb{R}. \tag{2.1}$$

Thus, if M is non-negative with positive spectral radius λ_M, then λ_M is equal to the maximal positive root of $\det(R_{\mathcal{R}}(x) - E)$.

2.2. *Continua.* A *continuum* X is any connected compactum. For the definition of the *order* of a point $x \in X$, see, for example, [**19**, Definition 9.3]. If X is a continuum, a point $x \in X$ is called an *end (ordinary, branch) point* of X if its order is one (two, at least three). The sets of all end, ordinary and branch points of X are denoted by $E(X)$, $O(X)$ and $B(X)$, respectively.

An *arc* is a homeomorphic image of the unit interval I; the images of 0, 1 are called the *end points* of the arc. An arc A with end points $\{a, b\}$ in a space X is *free* if $A \backslash \{a, b\}$ is open in X. A free arc A in a connected space X is said to *disconnect* X if $X \backslash \{x\}$ is not connected for every non-end point x of A. A *circle* or a *simple closed curve* is a homeomorphic image of the unit circle $\mathbb{S}^1$.

Let X be a continuum. A metric d on X is said to be *convex* provided for every distinct x, $y \in X$ there is $z \in X$ such that $d(x, z) = d(z, y) = d(x, y)/2$. By [**9**, Theorem 8] every locally connected continuum admits a compatible convex metric. If X is endowed with a convex metric d, then for every $a \neq b$ there is an arc A with end points a, b, the length $\mathcal{H}_d^1(A)$ of which is equal to $d(a, b)$; every such arc will be called *geodesic*.

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2.3. *Trees.* A *graph* is a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points. A *tree* is a graph containing no simple closed curve. The *order* of a point x of a tree T is the number of connected components of $T\backslash\{x\}$. We assume that every tree has associated with it finitely many points of it, called *vertices*, in such a way that every end point as well as every branch point is a vertex (so we allow also vertices of order two). The edges of a tree are the free arcs both end points of which are vertices. An edge E of a tree T is said to be *terminal* provided it contains an end point of T.

Let $n \geq 3$. Then a tree T is an *n-star* if T has exactly one branch point and this branch point has order n. Further, a tree T is an *n-comb* if it has exactly $n-2$ branch points, every branch point has order three and all the branch points belong to an arc. We will also say that a tree T is a 2-*star* or a 2-*comb* if it is an arc. Notice that n-stars and n-combs have exactly n end points and that every subtree of a comb (star) is a comb (star).

The following simple lemma will be used later.

LEMMA 1. *Let $k \geq 3$ and $p \geq 0$ be integers. Let T be a tree with at least k^p branch points and such that every branch point of T has order at most k. Then T contains a $(p+3)$-comb.*

Proof. We prove the lemma by induction. For $p = 0$ it is trivial. Assume now that the assertion of the lemma holds for some $p \geq 0$; we are going to prove that it holds also for $p+1$. Let T be any tree with at least k^{p+1} branch points and such that the order of every branch point is at most k. Take an end point e of X and let b be the branch point of X nearest to e; then $[b, e]$ is a free arc. The open set $X\backslash\{b\}$ has at most k components the closures of which are trees with the total number of branch points at least $k^{p+1}-1$. Hence there is a component U of $X\backslash\{b\}$ the closure of which has at least k^p branch points. Since every branch point of $\overline{U}$ has order at most k, by the induction hypothesis $\overline{U}$ contains a $(p+3)$-comb S; we may assume that b is an end point of S. Let e' be an end point of X contained in a component of $X\backslash\{b\}$ different from both U and $(b, e]$. Then $S \cup [b, e] \cup [b, e']$ is a $(p+4)$-comb. □

In the following we recall Ye's classification of trees [**25**, pp. 298–299]. Let $T_0, \ldots, T_n$ be trees, let $e_1, \ldots, e_n$ be the end points of T_0 and e'_j be a non-end point of T_j $(j = 1, \ldots, n)$. We say that a tree T is an *extension* of $T_1, \ldots, T_n$ over T_0 if T is homeomorphic to the free union $\bigoplus_{j=0}^{n} T_j$ with the pairs (e_j, e'_j) $(j = 1, \ldots, n)$ identified.

Let $r \geq 1$, $n_1, \ldots, n_r \geq 2$ be integers and let T be a tree. We say that T belongs to $\mathcal{T}_0^{n_1,\ldots,n_r}$ if either T is an n_1-star (if $r = 1$) or T is an extension of some $T_1, \ldots, T_{n_r} \in \mathcal{T}_0^{n_1,\ldots,n_{r-1}}$ over an n_r-star (if $r \geq 2$). If $i > 0$ we say that T belongs to $\mathcal{T}_i^{n_1,\ldots,n_r}$ if T is a free union of some $T' \in \mathcal{T}_{i-1}^{n_1,\ldots,n_r}$ with an arc A, where a non-end point of T' is identified with an end point of A. Notice that a tree T from $\mathcal{T}_i^{n_1,\ldots,n_r}$ has $n = i + \prod_{l=1}^{r} n_l$ end points.

Denote by $\mathcal{T}_0$ the union of all $\mathcal{T}_0^{n_1,\ldots,n_r}$ and, for $i \geq 1$, define $\mathcal{T}_i$ inductively as the union of all $\mathcal{T}_i^{n_1,\ldots,n_r}$ minus $\bigcup_{0\leq j<i} \mathcal{T}_j$. Notice that every n-star as well as every 2^r-comb $(r \geq 1)$ belongs to $\mathcal{T}_0$; indeed, for n-stars choose $r = 1$, $n_1 = n$ and for 2^r-combs put $n_1 = \cdots = n_r = 2$.

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2.4. *Dendrites.* A *dendrite* is a (possibly degenerate) locally connected continuum containing no simple closed curve. A dendrite is a tree if and only if it is non-degenerate and the set of end points is finite. If $a \neq b$ are points of a dendrite X, the unique arc with end points a, b will be denoted by $[a, b]$. The set $[a, b]\backslash\{a, b\}$ will be denoted by (a, b); we define $(a, b]$ and $[a, b)$ analogously. For a subcontinuum Y of a dendrite X, the *first point retraction* from X onto Y (see, for example, [**19**, Lemma 10.25]) will be denoted by r_Y.

The following two lemmas are simple statements about dendrites. We prove just the second one since the first one is trivial. Recall that an ω-star is a dendrite with unique branch point, the order of which is infinite.

LEMMA 2. *If X is a non-degenerate dendrite with finitely many branch points then either X is a tree or X contains an ω-star S which is a union of free arcs.*

LEMMA 3. *If X is a dendrite with infinitely many branch points then at least one of the following holds:*

(a) *X contains an arc containing infinitely many branch points;*

(b) *X contains an ω-star which is a union of free arcs.*

Proof. Let b be a limit point of $B(X)$. If no component of $X\backslash\{b\}$ contains a sequence of branch points converging to b then we clearly have (b). Otherwise there is a component U of $X\backslash\{b\}$ and a sequence $(b_n)_n$ of branch points in U with $b_n \to b$. Take some $a \in U \cap E(X)$ and put $A = [a, b]$. Since, for every n, $b'_n = r_A(b_n) \in (a, b)$ is a branch point and $\lim_n b'_n = b$, we have that $A \cap B(X)$ is infinite, that is, X satisfies (a). □

COROLLARY 4. *Let X be a non-degenerate dendrite which is not a tree. Then for every n at least one of the following holds:*

(a) *X contains an n-comb;*

(b) *X contains an n-star which is a union of free arcs.*

LEMMA 5. *For a non-degenerate dendrite X the following are equivalent:*

(a) *no subtree T of X contains all free arcs of X;*

(b) *for every $n \geq 2$ there is a subtree T of X with n end points, the terminal edges of which have non-empty interiors in X;*

(c) *for every $n \geq 2$ there is an n-star or an n-comb T in X, the terminal edges of which have non-empty interiors in X.*

Moreover, the following properties (e), (f) are equivalent, they imply (d) and, if X is not a tree, (d) implies (a)–(c):

(d) *$B(X)$ is nowhere dense (that is, free arcs are dense in X);*

(e) *every arc in X has non-empty interior;*

(f) *X is completely regular.*

Notice that, in assertions (b) and (c), instead of the statement 'the terminal edges of which have non-empty interiors in X' we can write 'the terminal edges of which are free arcs in X' (recall that in trees we allow vertices of order two). Notice also that (d) is not equivalent to (e)–(f); for example, any dendrite such that all branch points of it belong to an arc A and are dense in A, satisfies (d) but not (e)–(f).

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Proof. We only prove that (a) is equivalent to (b), since it follows from Lemma 1 that (b) implies (c), and the rest of the proof is easy.

Let $\mathcal{S}$ denote the system of all subtrees T of X such that every terminal edge of T has non-empty interior in X. If T, S are subtrees of X, $S \in \mathcal{S}$ and $\#E(T) < \#E(S)$, then there is a terminal edge of S not overlapping with T and so S contains a free arc not contained in T. Hence (b) implies (a). To finish, it suffices to show that (a) implies (b).

Assume, to the contrary, that (a) is true but (b) is not; put $n = \sup_{T\in\mathcal{S}} \#E(T) < \infty$. Take $T \in \mathcal{S}$ with $n = \#E(T)$; we may assume that $E(T) \subseteq E(X)$. Let S be the minimal subtree of T containing all free arcs of T; then $\#E(S) = n$ and $S \in \mathcal{S}$. Denote by $x_1, \ldots, x_n$ the end points of S. Let $(V_k)_{k\in K}$ be the (countable) system of (open) components of $X\setminus E(S)$ disjoint from S and let $x_{m_k} \in E(S)$ be the boundary point of V_k ($k \in K$). Let $K' \subseteq K$ be the set of all k such that $Y_k = \overline{V}_k$ has a free arc. Since, by (a), S does not contain all free arcs of X, the set K' is non-empty. Moreover, if $k \neq k'$ belong to K' then $x_{m_k} \neq x_{m_{k'}}$, since otherwise we can easily construct a tree $S' \supseteq S$ from $\mathcal{S}$ with more than n end points. Hence K' is finite. An analogous argument shows that for every $k \in K'$ there is an arc A_k with $x_{m_k} \in E(A_k)$ containing all free arcs of Y_k. But then $S' = S \cup \bigcup_{k\in K'} A_k$ is a subtree of X containing all free arcs of X, contradicting (a). □

2.5. *Totally regular continua.* A continuum X is *totally regular* if for every $x \in X$ and every countable set $P \subseteq X$ there is a basis of neighborhoods of x with finite boundary not intersecting P. Every dendrite, as well as every *completely regular* continuum (that is, continuum containing no non-degenerate nowhere dense subcontinuum) is totally regular. Totally regular continua are locally connected one-dimensional continua. By [**14**], every totally regular continuum admits a compatible convex metric d such that (X, d) has finite one-dimensional Hausdorff measure $\mathcal{H}^1_d(X)$.

2.6. *Length-expanding Lipschitz maps.* Here we prove a slight refinement of the main result of [**22**] valid for dendrites. First we recall the definition of length-expanding Lipschitz maps. If X is a non-degenerate totally regular continuum and $\mathcal{C}$ is a family of non-degenerate subcontinua of X, we say that $\mathcal{C}$ is *dense* if every non-empty open set in X contains a member of $\mathcal{C}$. By $\mathcal{C}_I$ we denote the system of all non-degenerate closed subintervals of I; the Euclidean metric on I is denoted by d_I. A map f is called *Lipschitz-L* if $\mathrm{Lip}(f) \leq L$.

Definition. Let $X = (X, d)$, $X' = (X', d')$ be non-degenerate (totally regular) continua of finite length and let $\mathcal{C}$, $\mathcal{C}'$ be dense systems of subcontinua of X, X', respectively. We say that a continuous map $f : X \to X'$ is *length-expanding* with respect to $\mathcal{C}$, $\mathcal{C}'$ if there exists $\varrho > 1$ such that for every $C \in \mathcal{C}$ we have $f(C) \in \mathcal{C}'$ and

$$\text{if } f(C) \neq X' \quad \text{then } \mathcal{H}^1_{d'}(f(C)) \geq \varrho \cdot \mathcal{H}^1_d(C).$$

Moreover, if f is surjective and Lipschitz-L we say that $f : (X, d, \mathcal{C}) \to (X', d', \mathcal{C}')$ is *(ϱ, L)-length-expanding Lipschitz*. We sometimes abbreviate this by saying that f is *(ϱ, L)-LEL* or just *LEL*.

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In this paper we will use the above notions also in the case where $\varrho \in (0, 1]$; to distinguish them, in such a case we will say that f is *length-expanding** or (ϱ, L)-*LEL**.

Let X be a non-degenerate dendrite. By [**19**, Theorem 10.27], for example, we may write X in the form

$$X = \text{closure}\left(\bigcup_{n\in N} X_n\right) \quad \text{where } X_n = A_1 \cup \cdots \cup A_n, \\ A_n = [a_n, b_n] \text{ are arcs with } A_n \cap X_{n-1} = \{a_n\} \text{ for } n \geq 2 \tag{2.2}$$

(here $N = \mathbb{N}$ provided $X \neq A_1$ and $N = \{1\}$ otherwise). Define a linear ordering on every A_n in such a way that $a_n < b_n$. We will need the following refinement of [**22**, Theorem C]. Henceforth we fix a constant $q \in (0, 1/3]$.

PROPOSITION 6. *Let X be a dendrite and $a \neq b$ be points of it. Write X in the form (2.2) with $a_1 = a$, $b_1 = b$ and fix convex metrics d_n on A_n $(n \in \mathbb{N})$. Then there are a convex metric $d = d_{X,a,b}$ on X and Lipschitz surjections $\varphi = \varphi_{X,a,b} : I \to X$, $\psi = \psi_{X,a,b} : X \to I$ with the following properties:*

(a) $\mathcal{H}^1_d(X) = 1$, $d(a, b) > 1 - q$;

(b) *the system*

$$\mathcal{C} = \mathcal{C}_{X,a,b} = \{\varphi(J) : J \text{ is a non-degenerate closed subinterval of } I\}$$

of subcontinua of X is dense;

(c) *the maps*

$$\varphi : (I, d_I, \mathcal{C}_I) \to (X, d, \mathcal{C}) \quad \text{and} \quad \psi : (X, d, \mathcal{C}) \to (I, d_I, \mathcal{C}_I)$$

are (ϱ, L)-LEL*, where $0 < \varrho < 1 < L$ are constants depending only on q, and $\varphi(0) = a$, $\varphi(1) = b$, $\psi(a) = 0$, $\psi(b) = 1$;

(d) *there is $c > 0$ such that $\psi(x) = c \cdot d(a, x)$ for every $x \in X$;*

(e) *for every n there is $c_n > 0$ such that $d(x, y) = c_n \cdot d_n(x, y)$ whenever $x, y \in A_n$;*

(f) $\mathcal{H}^1_d(A_{n+1}) \leq q \cdot \mathcal{H}^1_d(A_n)$ *for every n;*

(g) *if $r = r_{A_1} : X \to A_1$ is the first point retraction of X onto A_1, then*

$$r \circ \varphi(s) \leq r \circ \varphi(t) \quad \text{for every } s \leq t \text{ from } I.$$

Notice that, if $X = [a, b]$ is an arc, then $\mathcal{C}_{X,a,b}$ is the system of all subarcs of X.

Proof. If $X = [a, b]$ the statement is trivial; so assume that $N = \mathbb{N}$. For every n let $f_n : X_{n+1} \to X_n$ be the first point retraction (that is, $f_n(x) = x$ for $x \in X_n$ and $f_n(x) = a_{n+1}$ for $x \in A_{n+1}$). By the Anderson–Choquet embedding theorem (see, for example, [**19**, Theorem 2.10]), X is homeomorphic to the inverse limit $\varprojlim(X_n, f_n)$. Repeating the construction from [**22**, §5.2] (see also the remark after [**22**, Corollary 26]) word for word, with $\tilde{d}_n = c_n \cdot d_n$ where the c_n are sufficiently small, we obtain $d_{X,a,b}$, $\varphi_{X,a,b}$ and $\psi_{X,a,b}$ such that all the properties (a)–(g) are satisfied. □

The following can be proved in the same way as [**22**, Theorem D]. For $k \in \mathbb{N}$, let $f_k : I \to I$ denote the continuous map which fixes 0 and maps every $[(i-1)/k, i/k]$ linearly onto I.

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PROPOSITION 7. *In the notation of Proposition 6, for every $\rho > 1$, all non-degenerate dendrites X, X' and all pairs of distinct points $a, b \in X$, $a', b' \in X'$ there are a constant L_ρ (depending only on ρ) and a (ρ, L_ρ)-LEL map*

$$f : (X, d_{X,a,b}, \mathcal{C}_{X,a,b}) \to (X', d_{X',a',b'}, \mathcal{C}_{X',a',b'})$$

with $f(a) = a'$ and $f(b) = b'$. Moreover, f is equal to the composition $\varphi_{X',a',b'} \circ f_k \circ \psi_{X,a,b}$ with some odd $k \geq 3$.

Finally, we prove the next lemma which is a consequence of Proposition 6.

LEMMA 8. *In the notation of Proposition 6, there is a constant $\gamma > 0$ with the following property: for every dendrite X and every pair a, b of distinct points in it, there exists a*

$$(\gamma, 1)\text{-LEL}^* \text{ retraction } r = r_{X,a,b} : (X, d_{X,a,b}, \mathcal{C}_{X,a,b}) \to (A, d_A, \mathcal{C}_A),$$

where $A = [a, b]$, d_A is the restriction of $d_{X,a,b}$ onto $A \times A$ and $\mathcal{C}_A$ is the system of all subarcs of A.

Proof. Let ϱ be the constant from (c) of Proposition 6 and put $\gamma = (1 - q)^2\varrho/4$; we prove that this γ satisfies the above stated property. To this end, let X be a dendrite, $a \neq b$ be points in X and $A = [a, b]$. Write X in the form (2.2) with $A_1 = A$. If $X = A$, the assertion is trivial (take $r = \mathrm{id}_X$); so assume that $X \neq A$. Proposition 6 gives us a convex metric $d = d_{X,a,b}$ and maps $\varphi_{X,a,b}$, $\psi_{X,a,b}$ satisfying conditions (a)–(g); put $\mathcal{C} = \mathcal{C}_{X,a,b}$ and let $r_A : X \to A$ be the first point retraction. Let y_k $(k \in K)$ be all points y of A with non-degenerate preimage $r_A^{-1}(y)$. Put $Y_k = r_A^{-1}(y_k)$ $(k \in K)$; then the Y_k are pairwise disjoint non-degenerate subdendrites of X and $X = A \cup \bigcup_k Y_k$. Put $t_0 = d(a, b)$, $t_k = d(a, y_k)$ $(k \geq 1)$ and for any $t \in [0, t_0]$ denote by a_t the unique point of A satisfying $d(a_t, a) = t$; hence, for example, $a_{t_k} = y_k$ and $\mathcal{H}^1_d([a_s, a_t]) = t - s$ for any $s < t$. Now define $r = r_{X,a,b} : X \to A$ by

$$r(x) = \begin{cases} x & \text{if } x \in A, \\ a_{t_k + d(x, y_k)} & \text{if } x \in Y_k \text{ with } t_k \leq t_0/2, \\ a_{t_k - d(x, y_k)} & \text{if } x \in Y_k \text{ with } t_k > t_0/2. \end{cases}$$

By Proposition 6(a) and the choice of q, $t_0 > 2/3$ and so $d(x, y_k) < 1/3 < t_0/2$ for every $x \in Y_k$. Hence $r(x)$ is well defined for every x. Trivially, r is a continuous retraction of X onto A. Using convexity of d, one can easily show that $\mathrm{Lip}(r) = 1$.

It remains to prove that r is γ-length-expanding*. To this end, fix $C \in \mathcal{C}$, $C = \varphi_{X,a,b}([s_0, s_1])$ and put $u_i = r_A \circ \varphi_{X,a,b}(s_i)$ $(i = 0, 1)$. By Proposition 6(g) we may write

$$C = \left[C_0 \sqcup \left(\bigsqcup_{k : y_k \in (u_0, u_1)} Y_k \right) \sqcup C_1 \right] \cup [u_0, u_1]$$

with C_i $(i = 0, 1)$ being connected (either degenerate or belonging to $\mathcal{C}$). We will assume that there is k with $y_k \in (u_0, u_1)$; the other (simpler) case can be described analogously. Put $n_0 = \min\{n : A_n \subseteq Y_k,\ u_0 < y_k < u_1\} \geq 2$. Then, using (f) from Proposition 6,

$$\begin{aligned} \mathcal{H}^1_d(C) &= \mathcal{H}^1_d(C_0) + \sum_{k : y_k \in (u_0, u_1)} \mathcal{H}^1_d(Y_k) + \mathcal{H}^1_d(C_1) + \mathcal{H}^1_d([u_0, u_1]) \\ &\leq \frac{4}{1 - q} \cdot \max\{\mathcal{H}^1_d(C_0), \mathcal{H}^1_d(A_{n_0}), \mathcal{H}^1_d(C_1), \mathcal{H}^1_d([u_0, u_1])\}. \end{aligned}$$

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On the other hand, by Proposition 6(d) we have $\psi(x) = c \cdot d(a, x)$ for every $x \in X$, where $c = 1/d(a, b) \in (1, 1/(1-q))$. For given k and a continuum $Y \subseteq Y_k$ from $\mathcal{C}$ containing y_k we have $\psi(Y) = [\psi(y_k), \psi(z_k)]$, where z_k is such that $d(z_k, y_k) = \max_{y \in Y_k} d(z, y_k)$. Thus, by convexity of d,

$$|\psi(Y)| = c \cdot (d(a, z_k) - d(a, y_k)) = c \cdot d(y_k, z_k) = c \cdot \mathcal{H}^1_d(r(Y)).$$

By Proposition 6(c), for every such continuum Y we have

$$\mathcal{H}^1_d(r(Y)) \geq (\varrho/c) \cdot \mathcal{H}^1_d(Y) \geq \varrho(1-q) \cdot \mathcal{H}^1_d(Y).$$

Hence, using the fact that $r|_A$ is the identity,

$$\begin{aligned}\mathcal{H}^1_d(r(C)) &\geq \max\{\mathcal{H}^1_d(r(C_0)), \mathcal{H}^1_d(r(A_{n_0})), \mathcal{H}^1_d(r(C_1)), \mathcal{H}^1_d(r([u_0, u_1]))\} \\ &\geq \varrho(1-q) \cdot \max\{\mathcal{H}^1_d(C_0), \mathcal{H}^1_d(A_{n_0}), \mathcal{H}^1_d(C_1), \mathcal{H}^1_d([u_0, u_1])\}.\end{aligned}$$

Thus, by the choice of γ, $\mathcal{H}^1_d(r(C)) \geq \gamma \cdot \mathcal{H}^1_d(C)$ for every $C \in \mathcal{C}$. The lemma is proved. □

2.7. *P-Lipschitz maps.* In [**23**] we introduced the so-called P-Lipschitz maps (where P is a finite invariant set) and we gave an upper bound for their entropy [**23**, Proposition 3.3]. We recall the definition as well as the corresponding result here. If X is a continuum, a *splitting* of X is any system $\mathcal{A} = \{X_1, \ldots, X_n\}$ of non-degenerate subcontinua covering X such that $P_{\mathcal{A}} = \bigcup_{i \neq j} X_i \cap X_j$ is finite.

Definition. Let X be a non-degenerate continuum and $f : X \to X$ be a continuous map. Let P be a finite f-invariant subset of X, $\mathcal{A}$ be a splitting of X with $P_{\mathcal{A}} \subseteq P$ and $(L_A)_{A \in \mathcal{A}}$ be positive constants. Then we say that f is *P-Lipschitz with respect to the splitting $\mathcal{A}$ and the constants $(L_A)_{A \in \mathcal{A}}$* if, for every $A \in \mathcal{A}$, $f(A)$ is non-degenerate and $\mathrm{Lip}(f|_A) \leq L_A$.

Let f be a P-Lipschitz map with respect to $\mathcal{A}$. For $A, B \in \mathcal{A}$ we write $A \to B$ or, more precisely, $A \xrightarrow{f} B$ provided $f(A)$ intersects the interior of B. The *P-transition graph G_f of f* is the directed graph the vertices of which are the sets $A \in \mathcal{A}$ and the edges of which correspond to $A \to B$. The corresponding 0–1 transition matrix will be called the *P-transition matrix* and will be denoted by M_f. For an integer $n \geq 1$ put

$$\mathcal{A}^n = \{\mathbb{A} = (A_0, A_1, \ldots, A_{n-1}) : A_i \in \mathcal{A},\ A_0 \to A_1 \to \cdots \to A_{n-1}\};$$

that is, $\mathcal{A}^n$ is the set of all paths of length $n-1$ in G_f. Finally, for non-empty $\mathcal{B} \subseteq \mathcal{A}$ put

$$\theta_{\mathcal{B}} = \limsup_{n \to \infty} \frac{k_n^{\mathcal{B}}}{n} \quad \text{where } k_n^{\mathcal{B}} = \max_{(A_0, \ldots, A_{n-1}) \in \mathcal{A}^n} \#\{j : A_j \notin \mathcal{B}\}, \tag{2.3}$$

which is a quantity measuring the maximal 'asymptotic frequency' of occurrences of $A \in \mathcal{A} \backslash \mathcal{B}$ in paths of f. The following proposition gives an upper bound for the entropy of P-Lipschitz maps. There and below, $\log^+ x$ denotes $\max\{\log x, 0\}$.

PROPOSITION 9. [**23**] *Let X be a non-degenerate totally regular continuum endowed with a convex metric d such that $\mathcal{H}^1_d(X) < \infty$. Let $f : X \to X$ be P-Lipschitz with respect to $\mathcal{A}$ and $(L_A)_{A \in \mathcal{A}}$. Then, for every non-empty subsystem $\mathcal{B}$ of $\mathcal{A}$,*

$$h(f) \leq \log^+ L_{\mathcal{B}} + 2\theta_{\mathcal{B}} \log^+ L_{\mathcal{A}},$$

where $L_{\mathcal{B}} = \max_{A \in \mathcal{B}} L_A$ and $L_{\mathcal{A}} = \max_{A \in \mathcal{A}} L_A$.

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3. *Markov maps on trees*

Let T be a tree and $f : T \to T$ be a continuous map. Assume that a finite set $P \subset T$ is f-invariant and contains all vertices of T. We say that f is *P-Markov* if it is monotone on each component of $T \setminus P$. The closures of the components of $T \setminus P$ are called *P-basic arcs* of f. The *P-transition graph* of f with respect to P is the directed graph having P-basic arcs as vertices, with an edge from A to B (we write $A \to B$) if and only if $B \subseteq f(A)$. The corresponding 0–1 matrix will be denoted by M_f and called the *P-transition matrix*. For the following results see, for example, [**7**, Proposition 1.4, Corollary 1.11].

LEMMA 10. *Let T be a tree and let $f : T \to T$ be a P-Markov map with the transition matrix M and the Perron–Frobenius value λ_M. Then $h(f) = \log^+ \lambda_M$.*

LEMMA 11. *Let T be a tree and let $f : T \to T$ be a P-Markov map with the transition matrix M. Then:*

(a) *f is transitive if and only if M is irreducible and not a permutation matrix;*

(b) *f is exact if and only if M is primitive.*

A special class of Markov maps consists of the so-called P-linear Markov maps. Let T be a tree endowed with a convex metric d. Let $f : T \to T$ be a P-Markov map. We say that f is *P-linear* if for every component C of $T \setminus P$ there is a positive constant λ_C such that

$$d(f(x), f(y)) = \lambda_C \cdot d(x, y) \quad \text{for every } x, y \in C.$$

Moreover, if there is λ such that every λ_C is equal to λ, we say that f is *P-linear with constant slope λ*.

By [**6**], every P-Markov tree map $f : T \to T$ is semiconjugate to a P'-linear tree map $f' : T' \to T'$ with constant slope $\lambda' = e^{h(f)}$. The following trivial fact shows that if f is P-linear and transitive then the 'constant slope' can be achieved by a simple change of the metric on T.

LEMMA 12. *Let T be a tree with a convex metric d and let $f : T \to T$ be a transitive P-linear Markov map on T with the entropy $h(f) = \log \lambda$. Then there is an equivalent convex metric d' on T such that $f : (T, d') \to (T, d')$ is a P-linear Markov map with constant slope λ.*

Proof. Denote the P-basic arcs by $A_i = [a_i, b_i]$, $i = 1, \ldots, m$. Since T is uniquely arcwise connected, every A_i is geodesic and so its length is $\mathcal{H}^1_d(A_i) = d(a_i, b_i)$. Since f is transitive, its P-transition matrix M is irreducible. Thus λ is the Perron eigenvalue of M with a strictly positive right eigenvector $\alpha = (\alpha_1, \ldots, \alpha_m)$; that is, $M \cdot \alpha = \lambda\alpha$. Let d' be the unique convex metric on T such that

$$d'(x, y) = \frac{\alpha_i}{d(a_i, b_i)} \cdot d(x, y) \quad \text{for every } i \text{ and } x, y \in A_i.$$

Trivially, f is P'-linear on (T, d') and $\mathcal{H}^1_{d'}(A_i) = \alpha_i$ for every i. By the choice of α we have that

$$\lambda \cdot \mathcal{H}^1_{d'}(A_i) = \sum_{j : A_i \to A_j} \mathcal{H}^1_{d'}(A_j) = \mathcal{H}^1_{d'}(f(A_i))$$

for every i. Hence f has constant slope λ on (T, d'). □

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3.1. *(P, S)-linear Markov maps.* Here we introduce the so-called (P, S)-linear Markov maps, where $S = (s_0, s_1, \dots, s_n)$ is a tuple of points from P. Maps with this property can be used to define exactly Devaney chaotic tree maps with 'controlled' entropy; see Proposition 16.

Definition. Let T be a tree with a convex metric, $P \subseteq T$ be finite and $f : T \to T$ be a P-linear Markov map. Let $n \geq 1$ and $S = (s_0, s_1, \dots, s_n)$ be a tuple of distinct points from P. We say that f is (P, S)-*linear* if:

(a) $f(s_i) = s_{i+1}$ for every $0 \leq i < n$;

(b) $A_S = [s_0, s_n]$ is a P-basic arc and s_n is an end point of T;

(c) for every P-basic arc A different from A_S there is a path from A to A_S.

Tree maps with this property were constructed, for example, in the proof of [**4**, Theorem 1.2] and also in the proofs of [**25**, Lemmas 4.6, 4.7]. Let us formulate the latter fact as a lemma. (For Ye's classification of trees, see §2.3.)

LEMMA 13. *Let T be a tree and let i be such that $T \in \mathcal{T}_i$. Then, for every $\varepsilon > 0$, there are a finite set P, a tuple $S = (s_0, \dots, s_m)$ with $\{s_1, \dots, s_m\} \subseteq E(T)$ and a transitive (P, S)-linear map $f : T \to T$ with*

$$h(f) < \frac{\log 2}{m} + \varepsilon \quad \text{where } m = \#E(T) - i.$$

Proof. We will retain the notation from the proofs of [**25**, Lemmas 4.6, 4.7]. Let $T \in \mathcal{T}_i^{n_1,\dots,n_r}$, where $i \geq 0$, $r \geq 1$ and $n_1, \dots, n_r \geq 2$. Put $m = m_0 = \prod_{l=1}^{r} n_l = \#E(T) - i$ and $n_0 = r + i$. The end points of T are t_q^1 $(1 \leq q \leq m_0)$ and y^j $(1 \leq j \leq i)$. The maps $f_n = f_{i,r,n}$ constructed in the proofs are P_n-linear Markov maps (we put $P_n = Q_n$ if $i \geq 1$). If we put $S_n = (s_0, \dots, s_{m_0})$ with $s_q = t_q^1$ for $q = 1, \dots, m_0$, and $s_0 = x^1_{(n+n_0)m_0}$, then the first two conditions from the definition are satisfied; see [**25**, p. 301]. The third one is satisfied trivially since the maps f_n are transitive. Thus every f_n is (P_n, S_n)-linear. Since $\lim h(f_n) = (1/m_0)\log 2$, the proof is finished. □

Let us note that in the proofs of Propositions 16 and 17 we do not need to assume that the (P, S)-linear maps considered are transitive. In Appendix A we give simple constructions of (non-transitive) zero entropy (P, S)-linear Markov maps on stars and combs; this makes the proof of Theorem A not dependent on the results of [**25**].

3.2. *Exact P-linear Markov maps.* In Proposition 16 we show that every (P, S)-linear Markov map f on a tree R can be used to define an exact map g on a tree $T \supseteq R$ such that the entropy of g is small provided the entropy of f is small and the cardinality of S is large. Such maps g will be used in §4 in the construction of exact small entropy maps on dendrites; see Proposition 17. First we introduce some notation which will be used for the entropy estimation.

Let $\mathcal{P}$ denote the system of maps of two variables x, N of the type

$$\psi : (1, \infty) \times \mathbb{N} \to \mathbb{R}, \quad \psi(x, N) = \frac{b_{1,N}}{x} + \frac{b_{2,N}}{x^2} + \cdots + \frac{b_{i,N}}{x^i} + \cdots$$

such that for some $b = b_\psi > 0$ and $k = k_\psi \in \mathbb{N}$,

$$|b_{i,N}| \leq b \cdot N^k \cdot i^k \quad \text{for every } i, N \in \mathbb{N}. \tag{3.1}$$

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Notice that (3.1) guarantees that $\psi(x, N)$ is always finite, since

$$|\psi(x, N)| \le b \cdot N^k \cdot \nu_k(x) \quad \text{where } \nu_k(x) = \sum_{i=1}^{\infty} \frac{i^k}{x^i} < \infty.$$

LEMMA 14. *The system $\mathcal{P}$ is closed under multiplication and linear combinations, that is, $c\psi + c'\psi' \in \mathcal{P}$ and $\psi \cdot \psi' \in \mathcal{P}$ for every $\psi, \psi' \in \mathcal{P}$ and $c, c' \in \mathbb{R}$.*

Proof. Let $c, c' \in \mathbb{R}$, $\psi, \psi' \in \mathcal{P}$ and let $b_{i,N}, b'_{i,N}$ be the coefficients of ψ, ψ', respectively. Put $b = b_\psi$, $k = k_\psi$, $b' = b_{\psi'}$ and $k' = k_{\psi'}$.

The fact that $c\psi + c'\psi' \in \mathcal{P}$ is trivial. To show that $\varphi = \psi \cdot \psi'$ belongs to $\mathcal{P}$, realize that $\varphi(x, N) = \sum_{i=2}^{\infty} c_{i,N}/x^i$, where the coefficients $c_{i,N}$ satisfy

$$|c_{i,N}| = \left| \sum_{j=1}^{i-1} b_{j,N} \cdot b'_{i-j,N} \right| \le b \cdot b' \cdot N^{k+k'} \cdot \sum_{j=1}^{i-1} j^k \cdot (i-j)^{k'}.$$

The sum on the right-hand side can be bounded from above by $i^{k+k'} \cdot (i-1) < i^{k+k'+1}$. So we can take $b_\varphi = b \cdot b'$ and $k_\varphi = k + k' + 1$. □

Denote by $\mathcal{P}_{\text{small}}$ the subsystem of $\mathcal{P}$ consisting of those $\psi \in \mathcal{P}$ for which there is $l \in \mathbb{N}$ (not depending on N) with $b_{1,N} = b_{2,N} = \cdots = b_{N-l,N} = 0$ for every $N \ge l$.

LEMMA 15. *The following are true:*

(a) $\psi \in \mathcal{P}_{\text{small}}$ *if and only if there exist $\varphi \in \mathcal{P}$ and $l \in \mathbb{N}$ such that*

$$\psi(x, N) = \frac{1}{x^{N-l}} \cdot \varphi(x, N);$$

(b) *if $\delta > 0$ and $\psi \in \mathcal{P}_{\text{small}}$ then, for $N \to \infty$, $x^{N/2} \cdot \psi(x, N)$ converges to 0 uniformly on $[1 + \delta, \infty)$;*

(c) *$c\psi + c'\psi' \in \mathcal{P}_{\text{small}}$ and $\psi \cdot \psi'' \in \mathcal{P}_{\text{small}}$ for every $\psi, \psi' \in \mathcal{P}_{\text{small}}$, $\psi'' \in \mathcal{P}$ and $c, c' \in \mathbb{R}$.*

Proof. (a) If $\psi \in \mathcal{P}_{\text{small}}$, there is $l \in \mathbb{N}$ such that $b_{i,N} = 0$ for every N and every $i \le N - l$. Put $\varphi(x, N) = \sum_i c_{i,N} x^{-i}$, where $c_{i,N} = b_{N-l+i,N}$. Then $\psi(x, N) = x^{-(N-l)}\varphi(x, N)$ and $|c_{i,N}| \le b \cdot N^k \cdot (N - l + i)^k \le b \cdot N^{2k} \cdot i^k$, so $\varphi \in \mathcal{P}$. The reverse implication is obvious.

(b) This follows from the inequality $|x^{N/2} \cdot \psi(x, N)| \le x^{-(N/2-l)} \cdot b \cdot N^k \cdot \nu_k(x)$, where $k = k_\varphi$, $b = b_\varphi$ with φ given by (a) and $\nu_k(x) = \sum_{i=1}^{\infty} i^k x^{-i}$.

(c) This follows from (a) and Lemma 14. □

PROPOSITION 16. *Let R be a tree and let $f : R \to R$ be a (P, S)-linear map, where $S = (s_0, s_1, \ldots, s_n)$ with $n \ge 2$. Let $T \supseteq R$ be a tree such that*

$$T = R \sqcup \bigsqcup_{i=1}^{n} [s_i, t_i]. \tag{3.2}$$

Then for every $\varepsilon > 0$ there is a map $g : T \to T$ such that:

(a) *g is a Q-linear Markov map, where $Q \cap R = P$;*

(b) *g is exactly Devaney chaotic;*

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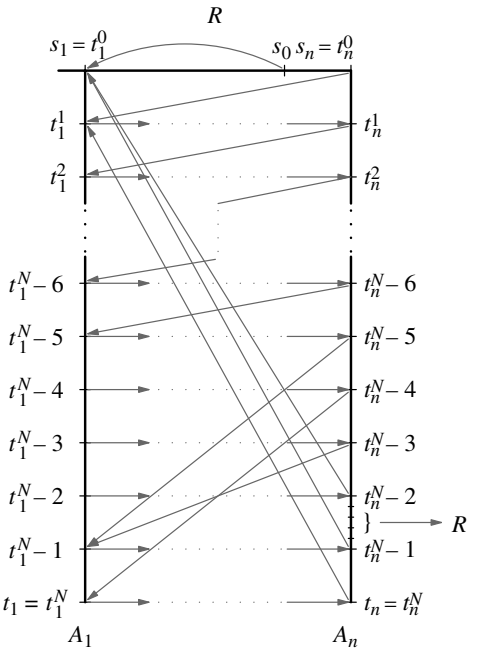


FIGURE 1. The map $g = g_N : T \to T$ (here R is an arc).

(c) $(1/n)\log 2 - \varepsilon < h(g) < \max\{h(f), (1/n)\log 2\} + \varepsilon$;
(d) $g(x) = f(x)$ *for every* $x \in R \backslash (s_0, s_n]$.

Proof. Endow T with a convex metric. For $i = 1, \ldots, n$ put $A_i = [s_i, t_i]$ and define an order on A_i in such a way that $s_i < t_i$. Denote the P-basic arcs of f by $B_1, \ldots, B_p$, where $B_p = [s_0, s_n]$. Let $\varphi : I \to R$ be a continuous surjection such that $\varphi(0) = \varphi(1) = s_1$ and, for some $m \in \mathbb{N}$, φ maps linearly every $[(l-1)/m, l/m]$ onto some B_{j_l}.

Fix an integer $N > 6$. In every $[s_i, t_i]$ choose points t_i^j $(j = 0, 1, \ldots, N)$ in such a way that $s_i = t_i^0 < t_i^1 < \cdots < t_i^N = t_i$; for $j = 1, \ldots, N$ put $A_i^j = [t_i^{j-1}, t_i^j]$. Moreover, take points $t_n^{N-2} = t_n^{N-1,0} < t_n^{N-1,1} < \cdots < t_n^{N-1,m} = t_n^{N-1}$ in A_n^{N-1} and put $A_n^{N-1,l} = [t_n^{N-1,l-1}, t_n^{N-1,l}]$ for $l = 1, \ldots, m$.

Put

$$Q = P \sqcup \{t_i^j : 1 \le i \le n,\ 1 \le j \le N\} \sqcup \{t_n^{N-1,l} : 1 \le l < m\}$$

and define a map $g = g_N : T \to T$ as follows (see Figure 1):

- $g(x) = f(x)$ for $x \in R \backslash (s_0, s_n]$;
- $g(t_i^j) = t_{i+1}^j$ for every $i < n$ and $j \ge 1$;

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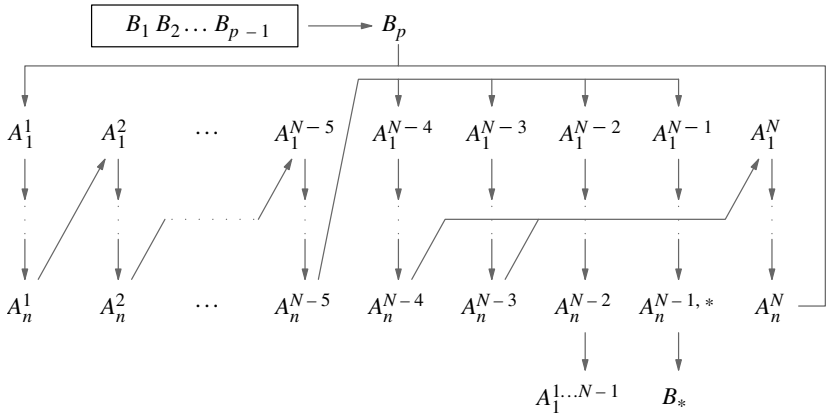


FIGURE 2. The transition graph of g.

- $g(t_n^j) = \begin{cases} t_1^{j+1} & \text{if } j = 0, 1, \dots, N-6, \\ t_1^{N-1} & \text{if } j = N-5, N-3, \\ t_1^N & \text{if } j = N-4, \\ t_1^0 = s_1 & \text{if } j = N-2, N-1, \\ t_1^1 & \text{if } j = N; \end{cases}$

  in particular, $g(s_n) = g(t_n^0) = t_1^1$;
- $g(t_n^{N-1,l}) = \varphi(l/m)$ for every $1 \le l < m$;
- g is Q-linear.

Obviously, the map g satisfies (d), it is a Q-linear Markov map and $Q \cap R = P$, so (a) is also satisfied. To prove (b) and (c) we need to describe the transition graph of g. The Q-basic arcs are:

$$\begin{aligned} &B_i \quad (1 \le i \le p), \quad A_i^j \quad (1 \le i \le n,\ 1 \le j \le N,\ (i, j) \ne (n, N-1)) \\ &\qquad\qquad \text{and} \quad A_n^{N-1,l} \quad (1 \le l \le m). \end{aligned} \tag{3.3}$$

The transition graph G_g of g is as follows (see Figure 2):

- $B_i \xrightarrow{g} B_j$ if and only if $B_i \xrightarrow{f} B_j$ for every $i < p$ and every j;
- $B_p \xrightarrow{g} A_1^1$;
- $A_i^j \xrightarrow{g} A_{i+1}^j$ if $i < n$ and $(i, j) \ne (n-1, N-1)$;
- $A_{n-1}^{N-1} \xrightarrow{g} A_n^{N-1,l}$ for every $l = 1, \dots, m$;
- $A_n^j \xrightarrow{g} \begin{cases} A_1^{j+1} & \text{if } j \le N-6, \\ A_1^{N-k}\ (k = 1, \dots, 4) & \text{if } j = N-5, \\ A_1^N & \text{if } j = N-4, N-3, \\ A_1^k\ (k = 1, \dots, N-1) & \text{if } j = N-2, \\ A_1^1 & \text{if } j = N; \end{cases}$
- $A_n^{N-1,l} \xrightarrow{g} B_{j_l}$ for every $l = 1, \dots, m$.

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Now we prove (b). Since there is a path from every B_i to B_p (f is (P, S)-linear), φ is a surjection (so $\{j_1, \dots, j_m\} = \{1, \dots, p\}$) and there is a path from B_p to any A_i^j, $A_n^{N-1,l}$, we have that the transition graph G_g is irreducible. But we even have that G_g is primitive since there are two loops over B_p the lengths of which differ by 1. Indeed, take a loop through some $A_n^{N-1,l}$ such that $j_l = p$; then take l' such that $B_{j_{l'}} \xrightarrow{g} B_p$, $j_{l'} < p$ (notice that $p \geq 2$ since $n \geq 2$) and construct another loop by replacing the subpath $A_{n-1}^{N-1} \xrightarrow{g} A_n^{N-1,l} \xrightarrow{g} B_p$ by $A_{n-1}^{N-1} \xrightarrow{g} A_n^{N-1,l'} \xrightarrow{g} B_{j_{l'}} \xrightarrow{g} B_p$. So g is an exact map; since on trees transitivity implies dense periodicity we have that g is exactly Devaney chaotic.

The rest of the proof is devoted to (c). Recall that $g = g_N$ depends on N. To prove the upper bound in (c) it suffices to show that $\limsup_{N\to\infty} h(g_N) \leq \max\{h(f), (1/n)\log 2\}$. Let λ_f be the Perron eigenvalue of the transition matrix of f. Let M_{g_N} be the transition matrix of g_N and λ_{g_N} be the Perron eigenvalue of M_{g_N}. Since M_{g_N} is primitive we have that $\lambda_{g_N} > 1$. Let M_N be the 0–1 matrix created from M_{g_N} by replacing 0s with 1s in the positions corresponding to the edges from B_p to B_i provided $B_p \xrightarrow{f} B_i$. If λ_N denotes the Perron eigenvalue of M_N then, since $M_{g_N} \leq M_N$,

$$h(g_N) = \log \lambda_{g_N} \leq \log \lambda_N. \tag{3.4}$$

Since every loop in G_g which is not a subset of $\{B_1, \dots, B_p\}$ passes A_n^1 or A_n^{N-2}, the set

$$\mathcal{R} = \{B_1, \dots, B_p;\ B_{p+1} = A_n^1, B_{p+2} = A_n^{N-2}\}$$

of Q-basic arcs is a rome for M_N. Notice that trivially $\{B_1, \dots, B_p\}$ is a rome for the transition matrix M_f of f. For $i, j = 1, \dots, p+2$ put

$$r_{ij} = r_{ij}(x, N) = \sum_{\pi} \frac{1}{x^{|\pi|}},$$

where the summation is over all $\mathcal{R}$-simple paths π from B_i to B_j. Since for $i, j \leq p$ any simple path from B_i to B_j visits only arcs $B_1, \dots, B_p$, for $i, j \leq p$ the quantity r_{ij} does not depend on N. Let $R_N(x)$ and $R_f(x)$ denote the matrices $(r_{ij}(x, N))_{i,j=1}^{p+2}$ and $(r_{ij}(x))_{i,j=1}^{p}$, respectively. Define also

$$\Phi_N(x) = \det(R_N(x) - E) \quad \text{and} \quad \Phi_f(x) = \det(R_f(x) - E),$$

where E denotes the identity matrix of the corresponding dimension. Then λ_N and λ_f are the largest positive roots of Φ_N and Φ_f, respectively (see §2.1). Our aim is to show that

$$\limsup_{N\to\infty} \lambda_N \leq \max\{\lambda_f, \sqrt[n]{2}\}. \tag{3.5}$$

To this end we must first estimate $r_{ij} = r_{ij}(x, N)$.

Realize that

$$r_{i,p+1} = 0 \quad \text{for } i < p, \quad r_{p,p+1} = x^{-n} \quad \text{and} \quad r_{i,p+2} = 0 \quad \text{for } i \leq p, \tag{3.6}$$

since every path from B_i ($i \leq p$) to $B_{p+1} = A_n^1$ or to $B_{p+2} = A_n^{N-1}$ must pass both B_p and B_{p+1}; hence the only such path which is simple is the path $B_p \to A_1^1 \to A_2^1 \to \cdots \to A_n^1 = B_{p+1}$ and its length is n.

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The simple paths starting from $B_{p+1} = A_n^1$ to any member of $\mathcal{R}$ are 'long' (their lengths are at least $n(N-5)$) and their number does not depend on N. Indeed, there are m simple paths from B_{p+1} to some of the B_i ($i \le p$), two simple loops over B_{p+1} and one simple path from B_{p+1} to B_{p+2}. Hence

$$r_{p+1,j} \in \mathcal{P}_{\text{small}} \quad \text{for every } j = 1, \dots, p+2. \tag{3.7}$$

There are exactly $N-3$ simple loops over B_{p+2}; indeed, for any $j = 2, \dots, N-2$ there is exactly one which starts with $B_{p+2} \to A_1^j$ and its length is $n(N-1-j)$. Hence, for $x > 1$,

$$r_{p+2,p+2} = \frac{1}{x^n} + \frac{1}{x^{2n}} + \cdots + \frac{1}{x^{(N-3)n}} = \frac{1}{x^n - 1} + o, \tag{3.8}$$

where $o \in \mathcal{P}_{\text{small}}$. Finally,

$$r_{p+2,i} \in \mathcal{P} \quad \text{for every } i \le p+1. \tag{3.9}$$

Now we are ready to estimate $\Phi_N(x) = \det(R_N(x) - E)$. Expanding the determinant along the $(p+1)$th row, then along the last column and using (3.6)–(3.9) and Lemma 15, we obtain (with o, o', $o'' \in \mathcal{P}_{\text{small}}$)

$$\Phi_N(x) = (-1) \cdot \left[-1 + \frac{1}{x^n - 1} + o \right] \cdot \det(R_f(x) - E) + o' = \frac{x^n - 2}{x^n - 1} \cdot \Phi_f(x) + o''.$$

Recall that λ_N denotes the largest positive root of $\Phi_N(x)$. Put $\lambda = \limsup \lambda_N$; without loss of generality we may assume that $\lambda_N \to \lambda$ for $N \to \infty$. We want to show that $\lambda \le \tilde{\lambda} := \max\{\lambda_f, \sqrt[n]{2}\}$. Assume, to the contrary, that $\lambda > \tilde{\lambda}$; so there exist $N_0 \in \mathbb{N}$ and $\delta > 0$ such that $\lambda_N \ge \tilde{\lambda} + \delta$ for every $N \ge N_0$. Put $\psi(x, N) = \Phi_N(x) - (x^n - 2)/(x^n - 1) \cdot \Phi_f(x)$ and take $\varepsilon > 0$. Since $\psi \in \mathcal{P}_{\text{small}}$, by Lemma 15 there is $N_1 \in \mathbb{N}$ such that

$$x^{N/2} \cdot |\psi(x, N)| < \varepsilon \quad \text{for every } N \ge N_1 \text{ and every } x \ge \tilde{\lambda}. \tag{3.10}$$

On the other hand, $\Phi_f(x) \ne 0$ for $x > \lambda_f$; since $x^p \Phi_f(x)$ is a polynomial by (2.1), there is $c > 0$ with $x^p |\Phi_f(x)| \ge c$ for every $x \ge \tilde{\lambda} + \delta$. Thus for every $N \ge N_0$,

$$\lambda_N^{N/2} \cdot |\psi(\lambda_N, N)| = \lambda_N^{N/2} \cdot \left| 0 - \frac{\lambda_N^n - 2}{\lambda_N^n - 1} \cdot \Phi_f(\lambda_N) \right| \ge c \lambda_N^{N/2-p} \cdot \frac{\lambda_N^n - 2}{\lambda_N^n - 1},$$

which goes to infinity for $N \to \infty$; this contradicts (3.10). Hence we have proved (3.5). Since $h(f) = \log^+ \lambda_f$, the upper bound in (c) for $g = g_N$ with large N follows from (3.4).

To prove the lower bound in (c), let $\tilde{M}_N \le M_{g_N}$ be the 0–1 matrix obtained from M_{g_N} by keeping only those 1s which correspond to paths $A \to B$ with $A, B \in \{A_i^j : i \le n, j \le N-2\}$; all other entries of M_{g_N} are replaced by 0. Then $\tilde{\mathcal{R}} = \{A_n^{N-2}\}$ is a rome for $\tilde{M}_N$ and the corresponding matrix function is

$$R_{\tilde{\mathcal{R}}} = (\tilde{r}_{11}), \quad \tilde{r}_{11}(x, N) = \frac{1}{x^n} + \frac{1}{x^{2n}} + \cdots + \frac{1}{x^{(N-4)n}}.$$

By a simple analysis we can show that the largest positive root $\tilde{\lambda}_N$ of $\det(R_{\tilde{\mathcal{R}}}(x) - E) = \tilde{r}_{11}(x) - 1$ converges to $\sqrt[n]{2}$. Thus $\liminf \lambda_{g_N} \ge \lim \tilde{\lambda}_N = \sqrt[n]{2}$. Now the lower bound in (c) for $g = g_N$ with large N also follows. □

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4. *Exact small entropy maps on dendrites*

Here we prove the main results of the paper stated in the introduction. They will follow from the next proposition.

PROPOSITION 17. *Let X be a non-degenerate dendrite and let $T \subseteq X$ be a tree with end points $t_1, \dots, t_n$ and with every terminal edge having non-empty interior in X. If there exists a map $f : T \to T$ which is (P, S)-linear for some P and $S = (s_0, t_1, \dots, t_n)$ (where $s_0 \in P$), then*

$$I^{\mathcal{ED}}(X) \leq \max\{h(f), (1/n)\log 2\}.$$

Before going into the details of the proof, let us outline its main steps. We write T in the form $T = R \sqcup \bigsqcup_{i=1}^{n} (s_i, t_i]$, where R is homeomorphic to T, arcs $[s_i, t_i]$ have non-empty interiors and s_i, t_i are end points of R, T, respectively. We may assume that the (P, S)-linear map f is defined on R and $S = (s_0, s_1, \dots, s_n)$ for some $s_0 \in P$. Using Proposition 16, we construct an exactly Devaney chaotic Q-linear map $g : T \to T$ with $h(g) < \max\{h(f), (1/n)\log 2\} + \varepsilon$.

Using the Q-basic arcs A of T we construct a splitting $\tilde{\mathcal{A}}$ of X into subcontinua $\tilde{A}$ in such a way that most of the members of $\tilde{\mathcal{A}}$ are free arcs; only subcontinua $\tilde{A}$ corresponding to Q-basic arcs A for which either $A \subseteq R$ or $A \ni t_i$, may be 'complicated' (that is, they need not be free arcs).

Then we construct a map $F : X \to X$ in three steps; see (4.10). First, on most of the subcontinua $\tilde{A}$ which are free arcs we put $F(x) = g(x)$. Second, for subcontinua $\tilde{A}$ which either contain an end point of T or are not subsets of R and $g(A)$ is a subset of some 'complicated' $\tilde{B}$ (see the definition of $\mathcal{D}$ in (4.2)), we use Propositions 6 and 7 to define F in such a way that the restrictions $F|_{\tilde{A}}$ are LEL maps. Finally, on the remaining 'complicated' subcontinua $\tilde{A}$ (that is, for those $\tilde{A}$ with A being a Q-basic arc of R), we define F to be the composition of g with a Lipschitz-1 retraction of $\tilde{A}$ onto A, obtained from Lemma 8.

During the construction of F we also define a convex metric d on X (see (4.7)), mainly using Lemma 12 (to obtain a convex metric d_T on T such that g has constant slope) and Proposition 6 (to obtain a convex metric on 'complicated' subcontinua $\tilde{A}$). Then we prove that the map F is Q-Lipschitz with respect to this metric and the splitting $\tilde{\mathcal{A}}$, see (4.11). Using properties of g, mainly inequality (4.3), and the fact that the transition matrices M_F and M_g of F and g coincide, Proposition 9 gives us that the entropy of F can be only slightly larger than that of g; see (4.12). Finally, we prove that F is exactly Devaney chaotic.

Proof of Proposition 17. Let X be a dendrite and let $T \subseteq X$ be a tree with end points $t_1, \dots, t_n$ and with every terminal edge having non-empty interior in X. So we may write

$$T = R \sqcup \bigsqcup_{i=1}^{n} (s_i, t_i],$$

where R is homeomorphic to T, $E(R) = \{s_1, \dots, s_n\}$ and for every i there is $u_i \in (s_i, t_i)$ such that $[s_i, u_i]$ is a free arc in X. By the assumption there are $P \subseteq R$ and a (P, S)-linear Markov map $f : R \to R$, where $S = (s_0, s_{\pi(1)}, \dots, s_{\pi(n)})$ for some $s_0 \in R$ and a

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permutation π of $\{1, \dots, n\}$. By reordering the indices, if necessary, we may assume that π is the identity, that is, $S = (s_0, s_1, \dots, s_n)$.

Let $N > 6$, let $g = g_N : T \to T$ be the map constructed in the proof of Proposition 16 and $\lambda = \lambda_N$ be such that $h(g) = \log \lambda$; then

$$\limsup_{N\to\infty} \lambda_N \le \exp(\max\{h(f), (1/n)\log 2\}). \tag{4.1}$$

We will retain the notation from the proof of Proposition 16 (see, for example, Figures 1 and 2); without loss of generality we may assume that $u_i = t_i^{N-1}$ for every i. Let $\mathcal{A}$ be the set of all Q-basic arcs of g; see (3.3). Put

$$\mathcal{D} = \{A_i^N, A_n^{N-4}, A_n^{N-3}, A_n^{N-1,l} : i \le n, l \le m\}, \quad \mathcal{B} = \mathcal{A}\backslash\mathcal{D}. \tag{4.2}$$

We first prove that

$$\theta_{\mathcal{B}} \le \frac{2}{N-5}, \tag{4.3}$$

where $\theta_{\mathcal{B}} = \limsup_{h\to\infty}(k_h/h)$, $k_h = \max_{\mathbb{C}} \#\{i < h : C_i \notin \mathcal{B}\}$ and the maximum is taken over all $\mathbb{C} = (C_0, \dots, C_{h-1}) \in \mathcal{A}^h$. To this end take $h \in \mathbb{N}$ and $(C_0, \dots, C_{h-1}) \in \mathcal{A}^h$. If $i < h$ is such that $C_i \in \mathcal{D}$ then there are $l \in \{i, \dots, i+n\}$ and $l' \ge \min\{h-1, l + n(N-5)\}$ such that

$$C_j \in \mathcal{B} \quad \text{for every } l < j \le l'.$$

(Indeed, to define l, distinguish two cases: if $g(C_i) \subseteq R$ put $l = i$; otherwise $g^{n'}(C_i) = A_1^1$ for some $1 \le n' \le n+1$ and put $l = i + n' - 1$. If there is no $h > i$ with $C_h \in \mathcal{D}$, put $l' = h - 1$. Otherwise let $l' \ge i$ be the smallest integer such that $C_{l'+1} \in \mathcal{D}$. Then the path $(C_{l+1}, C_{l+2}, \dots, C_{l'})$ contains subpath $(A_1^1, \dots, A_n^1; A_1^2, \dots, A_n^2; \cdots; A_1^{N-5}, \dots, A_n^{N-5})$; hence $l' - l \ge n(N-5)$.)

Denote by $i_1 < \cdots < i_k$ those indices i for which $C_i \in \mathcal{D}$. Then the previous observation easily implies that $i_{j+n+1} - i_j > n(N-5)$ for every $j < k - n$. Hence $h > i_{1+r(n+1)} - i_1 > rn(N-5)$, where $r = \lfloor (k-1)/(n+1) \rfloor$. Thus $(k-1)/(n+1) - 1 < r < m/(n(N-5))$ and so

$$\frac{k_h}{h} < \frac{2}{N-5} + \frac{n+2}{h}. \tag{4.4}$$

From this (4.3) follows.

To construct our Q-Lipschitz map $F : X \to X$, we first define a splitting $\tilde{\mathcal{A}} = \{\tilde{A} : A \in \mathcal{A}\}$ of X. For every $t \in T$ we denote by X_t the union of $\{t\}$ and of all components of $X\backslash\{t\}$ disjoint from T. For any subset M of T put

$$X_M = \bigsqcup_{t\in M} X_t.$$

For every arc $A \in \mathcal{A}$ we define a subcontinuum $\tilde{A}$ of X in such a way that:

- the system $\tilde{\mathcal{A}} = \{\tilde{A} : A \in \mathcal{A}\}$ is a splitting of X into continua;
- $\tilde{A} \supseteq A$, and $\tilde{A} = A$ for every $A = A_i^j$ with $j \ne N$ and for every $A = A_n^{N-1,l}$;
- $X_{\mathrm{int}_T(A)} \subseteq \tilde{A} \subseteq X_A$ for every $A \in \mathcal{A}$;
- for every $t \in T$ there is $A \in \mathcal{A}$ such that $X_t \subseteq \tilde{A}$;
- if $A \ne B$ then $\tilde{A} \cap \tilde{B} = A \cap B$; hence $P_{\tilde{\mathcal{A}}} = \bigcup_{\tilde{A}\ne\tilde{B}} (\tilde{A} \cap \tilde{B}) \subseteq Q$.

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Notice that if $A = [t, u] \in \mathcal{A}$ then $\tilde{A}$ is one of the following four sets: $X_{(t,u)} \sqcup \{t\} \sqcup \{u\}$, $X_{(t,u)} \sqcup X_t \sqcup \{u\}$, $X_{(t,u)} \sqcup \{t\} \sqcup X_u$, $X_A = X_{(t,u)} \sqcup X_t \sqcup X_u$.

Now we define a convex metric $d = d_N$ on X. Let d_T be a convex metric on T such that

$$g : (T, d_T) \to (T, d_T) \text{ is } Q\text{-linear with constant slope } \lambda = \lambda_N > 1; \tag{4.5}$$

see Lemma 12. Put $l_A = \mathcal{H}^1_{d_T}(A)$ for $A \in \mathcal{A}$; then

$$\mathcal{H}^1_{d_T}(g(A)) = \sum_{B : A \xrightarrow{g} B} l_B = \lambda \cdot \mathcal{H}^1_{d_T}(A) = \lambda \cdot l_A \quad \text{for every } A \in \mathcal{A}. \tag{4.6}$$

For $A = [a_0, a_1] \in \mathcal{A}$ let $d_{\tilde{A}} = d_{\tilde{A}, a_0, a_1}$ and $\mathcal{C}_{\tilde{A}} = \mathcal{C}_{\tilde{A}, a_0, a_1}$ be a convex metric on $\tilde{A}$ and a dense system of subcontinua of $\tilde{A}$ obtained from Proposition 6 (with $A_1 = A$); by (e) we may assume that $d_T|_{A\times A} = c_A \cdot d_{\tilde{A}}|_{A\times A}$ for some $c_A > 0$. If we put $\tilde{l}_A = \mathcal{H}^1_{d_{\tilde{A}}}(A)$, then $c_A = l_A/\tilde{l}_A$ and $c_A \in [l_A, 2l_A]$ by Proposition 6(a) and the choice of q. Let d be the unique convex metric on X with

$$d|_{\tilde{A}\times\tilde{A}} = c_A \cdot d_{\tilde{A}} \quad \text{for every } A \in \mathcal{A}. \tag{4.7}$$

Notice that X has finite length:

$$\mathcal{H}^1_d(X) = \sum_{A\in\mathcal{A}} c_A < \infty. \tag{4.8}$$

Now we can define the map $F : X \to X$. First, for $A \in \mathcal{B}$, $A \not\subseteq R$ we have that $\tilde{A} = A$ and $\tilde{B} = B$ for every B with $A \xrightarrow{g} B$; in this case we put $F_A = g|_A$. Second, if $A \in \mathcal{B}$ and $A \subseteq R$, we put $F_A = g \circ r_A$, where $r_A : \tilde{A} \to A$ is a $(\gamma, 1)$-LEL* retraction from Lemma 8. Third, if $A = [a_0, a_1] \in \mathcal{D}$, take the unique Q-basic arc $B = [b_0, b_1] \in \mathcal{A}$ ($b_i = g(a_i)$ for $i = 0, 1$) with $A \xrightarrow{g} B$. Let $f_A : (\tilde{A}, d_{\tilde{A}}) \to (\tilde{B}, d_{\tilde{B}})$ be a $(2, L_2)$-LEL map obtained from Proposition 7 and define

$$F_A : (\tilde{A}, d) \to (\tilde{B}, d), \quad F_A(x) = f_A(x) \quad \text{for } x \in \tilde{A}$$

(here d denotes the corresponding restriction of the metric). By (4.6), $l_B = \lambda l_A$; thus $\mathrm{Lip}(F_A) = (l_B/l_A) \cdot (\tilde{l}_A/\tilde{l}_B) \cdot \mathrm{Lip}(f_A) \le 2\lambda L_2$ and F_A is length-expanding with respect to the constant $(l_B/l_A) \cdot (\tilde{l}_A/\tilde{l}_B) \cdot 2 \ge \lambda$. Thus,

$$F_A \text{ is } (\lambda, 2\lambda L_2)\text{-LEL}. \tag{4.9}$$

Finally, define $F = F_N : X \to X$ by

$$F(x) = F_A(x) \quad \text{for } x \in \tilde{A},\ A \in \mathcal{A}. \tag{4.10}$$

For $A \in \mathcal{A}$ put $L_{\tilde{A}} = \lambda$ if $A \in \mathcal{B}$ and $L_{\tilde{A}} = 2\lambda L_2$ if $A \in \mathcal{D}$. Using (4.9) and (4.5) we have that

$$F \text{ is a } Q\text{-Lipschitz map with respect to } \tilde{\mathcal{A}}, (L_{\tilde{A}})_{\tilde{A}\in\tilde{\mathcal{A}}} \quad \text{and} \quad M_F = M_g. \tag{4.11}$$

Concerning the entropy of $F = F_N$ we use Proposition 9. Since $M_{F_N} = M_{g_N}$ and $\lambda = \lambda_N$ is the maximal eigenvalue of M_{g_N}, Proposition 9 and (4.3) give

$$h(F_N) \le \log \lambda_N + \frac{4 \log(2\lambda_N L_2)}{N - 5}.$$

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Since L_2 does not depend on N, (4.1) gives

$$\limsup_{N\to\infty} h(F_N) \le \limsup_{N\to\infty} \log \lambda_N \le \max\{h(f), (1/n)\log 2\}. \tag{4.12}$$

We now show that $F = F_N$ is exactly Devaney chaotic provided N is sufficiently large. Put $\mathcal{C} = \bigcup_{\tilde{A}\in\mathcal{A}} \mathcal{C}_{\tilde{A}}$ and fix any $C \in \mathcal{C}$. For $k \in \mathbb{N}_0$ put $C_k = F^k(C)$ and take $A_k \in \mathcal{A}$ such that $C_k \cap \tilde{A}_k$ is non-degenerate. Let $h \ge 0$ be such that

$$C_k \subseteq \tilde{A}_k \quad \text{for every } k < h. \tag{4.13}$$

Then, by (4.5), (4.9) and the choice of r_A ($A \in \mathcal{B}$, $A \subseteq R$),

$$\mathcal{H}^1(C_h) \ge \gamma^{p_h} \cdot \lambda^h \cdot \mathcal{H}^1(C), \quad p_h = \#\{k < h : A_k \subseteq R \text{ and } C_k \not\subseteq A_k\}. \tag{4.14}$$

By the definition of F_A for $A \subseteq R$, if $A_k \subseteq R$ and $C_k \not\subseteq A_k$ for some $0 < k < h$, then A_{k-1} is equal to some $A_n^{N-1,l}$. Hence, by (4.4),

$$\frac{p_h}{h} \le \frac{k_h + 1}{h} \le \frac{2}{N-5} + \frac{n+3}{h}.$$

Fix a constant $\theta \in (1/\lambda, 1)$ and assume that N is so large that $\gamma^{2/(N-5)} > \theta$. Using (4.14), we have that $\mathcal{H}^1(C_h) \ge (\lambda\theta)^h \cdot \gamma^{n+3} \cdot \mathcal{H}^1(C)$. Since $\mathcal{H}^1(C) > 0$, $\lambda\theta > 1$ and X has finite length by (4.8), we have that the h satisfying (4.13) are bounded from above. Hence there is h_0 such that $C_{h_0} \not\subseteq \tilde{A}_{h_0}$ and thus

$$C_{h_0} \text{ intersects } Q \text{ for some } h_0 \in \mathbb{N}. \tag{4.15}$$

Take a subcontinuum C' of C_{h_0} such that $C' \in \mathcal{C}$, $C' \subseteq \tilde{A}_{h_0}$ and $C' \cap Q \ne \emptyset$ (this is possible due to (4.15) and the definition of $\mathcal{C}$). By analogous arguments we can find $h' > 0$ such that $C'' = F^{h'}(C')$ is not a subset of any $\tilde{A}$. We may assume that h' is the smallest integer with this property, hence $F^{h'-1}(C') \subseteq \tilde{A}$ for some $A \in \mathcal{A}$. By inspecting the transition graph of g we see that either $A \subseteq R$ or $A \in \{A_n^{N-5}, A_n^{N-2}, A_{n-1}^{N-1,l} : l = 1, \dots, m\}$. Since $F^{h'-1}(C')$ intersects Q, the set C'' contains a Q-basic arc B such that either $B \subseteq R$ or B is a free arc in X. In the former case we have that $F^k(B) \supseteq A_1^1$ for some k; so in both cases

$$\text{there is } h \in \mathbb{N} \text{ such that the set } C_h \text{ contains some } A \in \mathcal{A} \text{ with } \tilde{A} = A.$$

But now $C_{h+k} \supseteq \bigcup_{g^k(A)\supseteq B} \tilde{B}$ for every $k > 0$. Since g is exact we have that $C_{h+k} = X$ for every sufficiently large k. Hence F is exact. Finally, since X has a free arc which disconnects X, by [**4**, Theorem 1.1] the map F, being transitive, has dense periodic points.

To summarize, we have proved that F_N is exactly Devaney chaotic for every sufficiently large N and that $\limsup_{N\to\infty} h(F_N) \le \max\{h(f), (1/n)\log 2\}$. Thus the proposition is proved. □

We are now in a position to prove the results stated in the introduction.

Proof of Theorem A. Let X be a non-degenerate dendrite such that no subtree of it contains all free arcs of X. Fix $n = 2^r$, $r \ge 1$ and $\varepsilon > 0$. By Lemma 5, X contains a subtree T which is either an n-star or an n-comb such that every terminal edge of T has non-empty interior

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in X. Since $T \in \mathcal{T}_0$, by Lemma 13 there is a (P, S)-linear Markov map $f : R \to R$ with $h(f) < (1/n)\log 2 + \varepsilon$, where $P \subseteq T$ and $S = (s_0, s_1, \ldots, s_n)$ for some $s_0 \in T$ and the end points $s_1, \ldots, s_n$ of T. Now Proposition 17 gives that $I^{\mathcal{ED}}(X) < (1/n)\log 2 + \varepsilon$. Since $n = 2^r$ and ε are arbitrary, $I^{\mathcal{ED}}(X) = 0$.

The fact that the infima $I^{\mathcal{T}}(X)$, $I^{\mathcal{ED}}(X)$ are not attainable follows from [**13**]. Indeed, the space X, being a compact metric space which is not a finite union of disjoint simple closed curves and which contains a free arc, admits no zero-entropy transitive map. □

In the proof of Theorem A we have used Lemma 13, which is a reformulation of [**25**, Lemmas 4.6 and 4.7]. The use of Lemma 13 can be replaced by Lemmas A.1 and A.2 which we prove in Appendix A. This makes the proof of Theorem A independent of Ye's results.

Proof of Corollary B. Let X be a non-degenerate completely regular continuum which is not a tree. If X is a dendrite then $I^{\mathcal{ED}}(X) = 0$ by Theorem A, since free arcs are dense in X by Lemma 5. Otherwise X contains a simple closed curve S. Since X is completely regular, S has non-empty interior and thus S contains a free arc; of course, this arc does not disconnect X. So $I^{\mathcal{ED}}(X) = 0$ by [**23**]. □

Proof of Proposition C. Just use Proposition 17 and Lemma 13. □

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A. *Appendix. Zero-entropy (P, S)-linear Markov maps on stars and combs*

The purpose of this appendix is to give a simple construction of (P, S)-linear Markov maps with zero entropy on stars and combs. This makes the proof of Theorem A independent of the construction of [**25**].

LEMMA A.1. *Let T be an n-star with end points $s_1, \ldots, s_n$ ($n \geq 2$). Then there exist a map $f : T \to T$, a finite subset P of T and $s_0 \in P$ such that f is a (P, S)-linear Markov map with zero entropy, where $S = (s_0, s_1, \ldots, s_n)$.*

Proof. Let b be the only branch point of T; so $T = \bigcup_{i=1}^{n} [b, s_i]$. For every i take a point $s_i' \in (b, s_i)$. Put $P = \{b;\ s_1', \ldots, s_n';\ s_1, \ldots, s_n\}$, $s_0 = s_n'$ and $S = (s_0, s_1, \ldots, s_n)$. Define the P-linear Markov map f on T by $f(b) = s_1'$, $f(s_n') = s_1$, $f(s_n) = s_1'$ and, for every $1 \leq i < n$, $f(s_i') = s_{i+1}'$ and $f(s_i) = s_{i+1}$ (see Figure A.1 for $n = 4$).

Obviously, f is (P, S)-linear. It is an easy exercise to show that $h(f) = 0$. Indeed, one can use, for example, the rome method with the rome consisting of two arcs: $A = [b, s_n']$, $B = [s_n', s_n]$. Since there is no path from B to A and the length of every simple path from A to A and from B to B is n, the characteristic polynomial of the transition matrix of f is (see (2.1))

$$(-1)^{2n-2} \cdot x^{2n} \cdot \left(\frac{1}{x^n} - 1\right)^2 = (x^n - 1)^2,$$

the largest positive root of which is $\lambda = 1$. Hence the entropy of f is $h(f) = 0$. □

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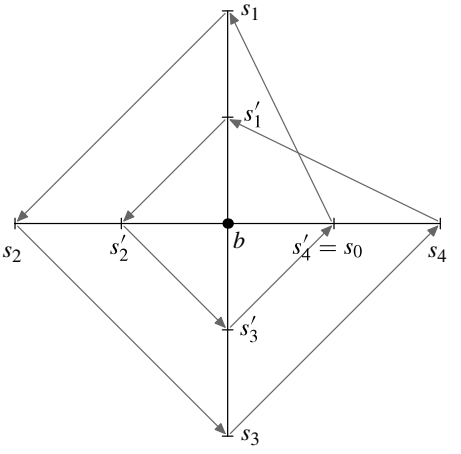


FIGURE A.1. The map f on a 4-star.

LEMMA A.2. *Let $n = 2^r$ ($r \geq 1$) and let T be an n-comb. Then there are a finite set P and a tuple $S = (s_0, s_1, \ldots, s_n)$, where $s_0 \in P$ and $\{s_1, \ldots, s_n\} = E(T)$, such that T admits a (P, S)-linear map $f : T \to T$ with zero entropy.*

Proof. First we introduce some notation for distinguished points of T, which will form P. Put $\Sigma = \{0, 1\}$ and $\Sigma_k^l = \bigcup_{i=k}^{l} \Sigma^i$ for $0 \leq k \leq l \leq r$, where Σ^0 is the singleton $\{\theta\}$ (θ denotes the empty word); that is, Σ_k^l is the set of all words $\alpha = \alpha_0 \cdots \alpha_{i-1}$ over the alphabet Σ of length $|\alpha| = i \in \{k, \ldots, l\}$. The concatenation $\alpha\beta$ and the powers α^k for words α, β and integers $k \geq 1$ are defined in a natural way. For every k let $<$ denote the lexicographical ordering on Σ^k. Notice that Σ^k with addition from left to right is a group; for $\alpha \in \Sigma^k$ and $n \in \mathbb{Z}$ we define $\alpha + n$ as usual.

Fix $r \geq 1$ and $n = 2^r$. Put $a_\theta = 1/2$ and

$$a_\alpha = \frac{\alpha_0}{2} + \frac{\alpha_1}{2^2} + \cdots + \frac{\alpha_{k-1}}{2^k} + \frac{1}{2^{k+1}} \quad \text{for } \alpha = \alpha_0 \cdots \alpha_{k-1} \in \Sigma_1^{r-1}.$$

For $\alpha \in \Sigma_0^{r-1}$ put $b_{\alpha 0} = a_\alpha - 2^{-(r+2)}$ and $b_{\alpha 1} = a_\alpha + 2^{-(r+2)}$. Notice that the following are true:

(i) $b_{\alpha 0} < a_\alpha < b_{\alpha 1}$ for $\alpha \in \Sigma_0^{r-1}$;

(ii) $b_\beta < b_\gamma$ for every $\beta < \gamma$ from Σ_1^r of the same length;

(iii) $b_{\alpha 0 \beta} < b_{\alpha 0} < b_{\alpha 1} < b_{\alpha 1 \beta}$ for every $\alpha \in \Sigma_0^{r-2}$ and $\beta \in \Sigma_1^{r-|\alpha|-1}$.

Identify the points $a_\alpha, b_\beta \in (0, 1)$ with the points $(a_\alpha, 0)$, $(b_\beta, 0)$ of the Euclidean plane and define $c_\gamma = (b_\gamma, 1)$ for $\gamma \in \Sigma^r$. Without loss of generality we may assume that the n-comb T is given by

$$T = A \cup \bigcup_{\gamma \in \Sigma^r} B_\gamma,$$

where $A = [b_{0^r}, b_{1^r}]$ and $B_\gamma = [b_\gamma, c_\gamma]$ ($\gamma \in \Sigma^r$) are segments; see Figure A.2 for an illustration. Put

$$P = \{a_\alpha : \alpha \in \Sigma_0^{r-1}\} \cup \{b_\beta : \beta \in \Sigma_1^r\} \cup \{c_\gamma : \gamma \in \Sigma^r\}.$$

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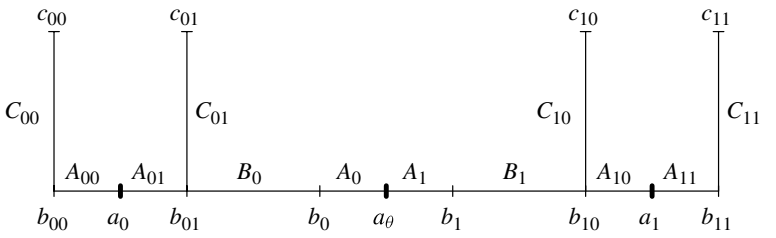


FIGURE A.2. The n-comb T for $n = 2^2$.

Obviously, $P \supseteq E(T) \cup B(T)$ since

$$E(T) = \{c_\gamma : \gamma \in \Sigma^r\} \quad \text{and} \quad B(T) = \{b_\gamma : \gamma \in \Sigma^r,\ \gamma \neq 0^r, 1^r\}.$$

The P-basic arcs are:

- $A_{\alpha 0} = [b_{\alpha 0}, a_\alpha]$, $A_{\alpha 1} = [a_\alpha, b_{\alpha 1}]$ for $\alpha \in \Sigma_0^{r-1}$;
- $B_{\alpha 0} = [b_{\alpha 01^k}, b_{\alpha 0}]$, $B_{\alpha 1} = [b_{\alpha 1}, b_{\alpha 10^k}]$ for $\alpha \in \Sigma_0^{r-2}$ and $k = r - |\alpha| - 1$;
- $C_\gamma = [b_\gamma, c_\gamma]$ for $\gamma \in \Sigma^r$.

For a P-basic arc D, we define its *level*, denoted by $\mathrm{lev}(D)$, as the length of its index; for example, $\mathrm{lev}(B_{\alpha 0}) = |\alpha 0|$ and $\mathrm{lev}(C_\gamma) = r$. For $1 \le k \le r$ denote by $\mathcal{S}_k$ the system of all P-basic arcs D with $\mathrm{lev}(D) \ge k$; put $\mathcal{S}_{r+1} = \emptyset$.

Define the map $f : T \to T$ by:

(a) $f(a_\alpha) = a_{\alpha+1}$ for $\alpha \in \Sigma_0^{r-1}$;

(b) $f(b_{1^r}) = c_{0^r}$, $f(b_{1^k}) = b_{0^{k+1}}$ for $k < r$ and $f(b_\beta) = b_{\beta+1}$ for $\beta \in \Sigma_1^r \setminus \{1^k : 1 \le k \le r\}$;

(c) $f(c_{1^r}) = b_{0^r}$ and $f(c_\gamma) = c_{\gamma+1}$ for $\gamma \in \Sigma^r$, $\gamma \neq 1^r$;

(d) f is P-linear.

A simple analysis shows that the edges of the transition graph G of f are as follows (here we write $D \to \mathcal{S}_k$ if D f-covers zero or more arcs from $\mathcal{S}_k$, $1 \le k \le r+1$):

- $A_\alpha \to A_{\alpha+1}$ for $\alpha \notin \{1^k : k \le r\}$;
- $A_{1^k} \to A_{0^k}$, B_{0^k}, $A_{0^{k+1}}$, $\mathcal{S}_{k+1}$ for $k < r$;
- $A_{1^r} \to A_{0^r}$, C_{0^r};
- $B_\beta \to B_{\beta+1}$, $\mathcal{S}_{|\beta|+1}$ for $\beta \notin \{1^k : k < r\}$;
- $B_{1^k} \to A_{0^{k+1}}$, $\mathcal{S}_{k+1}$ for $k < r$;
- $C_\gamma \to C_{\gamma+1}$ for every γ.

Hence the loops of G are:

- $A_{0^k} \to A_{0^k+1} \to \cdots \to A_{1^k} \to A_{0^k}$, $k = 1, \ldots, r$;
- $C_{0^r} \to C_{0^r+1} \to \cdots \to C_{1^r} \to C_{0^r}$.

Since every P-basic arc is contained in at most one loop of G, the entropy of f is zero.

Put $S = (s_0, s_1, \ldots, s_n)$, where $s_0 = b_{1^r}$ and $s_i = c_{0^r+(i-1)}$ $(i = 1, \ldots, n)$. To finish the proof we need to show that f is a (P, S)-linear Markov map. Indeed, for $0 \le i < n$ we have $f(s_i) = s_{i+1}$. Moreover, $s_n = c_{1^r}$ is an end point of T and $[s_0, s_n] = C_{1^r}$ is a P-basic arc. Finally, from the description of the transition graph of f we see that there is a path from any P-basic arc D to $C_{1^r} = [s_0, s_n]$. □

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B. *Appendix. Trees with large number of end points*

The lower and upper bounds for the infimum of entropies of transitive systems on a given tree T (see [**3, 7, 25**]) suggest that if the number of end points of T is 'large' then T admits a transitive map with 'small' entropy. In this appendix we show that this is in fact true, even with transitivity replaced by exactness. Moreover, we give an upper bound for the infimum depending only on the number of end points of the tree. This bound is in no sense optimal but it is very simple. (In what follows, log means the natural logarithm.)

PROPOSITION B.3. *Let T be a tree with n end points. Then*

$$\frac{\log 2}{n} \le I^{\mathcal{T}}(T) \le I^{\mathcal{ED}}(T) \le \frac{\log 2}{\sqrt{\log n}}. \tag{B.1}$$

Hence, if $(T_n)_n$ is a sequence of trees with the number of end points going to infinity, then

$$\lim_{n\to\infty} I^{\mathcal{ED}}(T_n) = 0.$$

Before proving Proposition B.3 we prove the following simple lemma.

LEMMA B.4. *Let T be a tree with $n \ge 3$ end points and such that every branch point of it has order at most k. Then T has at least n/k branch points.*

Proof. We prove the lemma by induction. If $n = 3$ then T is a 3-star and the assertion is trivial ($k \ge 3$, hence $n/k \le 1$). Assume that for some $n > 3$ the assertion of the lemma holds for every tree (which is not an arc) with the number of end points strictly smaller than n. Let T be a tree with n end points. Take a branch point b of T (there are some since T is not an arc). Let $U_1, \dots, U_m$ ($m \le k$) be the components of $X\backslash\{b\}$; without loss of generality we may assume that there is an integer $0 \le m' \le m$ such that U_i is not an arc for every $i \le m'$ and U_i is an arc for every $i > m'$. For every $i \le m'$ put $n_i = \#E(\overline{U_i})$; then $n_i < n$ and so the tree $\overline{U_i}$ has at least n_i/k branch points by the induction hypothesis. We easily have that $n = \sum_{i=1}^{m'}(n_i - 1) + (m - m')$, so $\sum_{i=1}^{m'} n_i \ge n - (m - 2m') \ge n - k$. The number of branch points of T is at least $1 + \sum_{i=1}^{m'} n_i/k \ge n/k$. □

Proof of Proposition B.3. The lower bound from (B.1) was shown in [**3**]. To show the upper bound take any tree T with n end points. If $n = 2$ we have $I^{\mathcal{ED}}(T) = (1/2)\log 2$, so we may assume that $n \ge 3$. Denote by $k \ge 1$ the only integer such that $\sqrt{\log n} - 1 \le k < \sqrt{\log n}$. By standard techniques one can show that $2x \cdot \log x < x^2$ for $x \ge 1$ (indeed, $f(x) = x^2 - 2x \cdot \log x$ is strictly increasing on $[1, \infty)$ and $f(1) = 1 > 0$), hence $k^{2k} < n$.

If T has a branch point of order at least $k + 1$ then it contains a $(k + 1)$-star, hence $I^{\mathcal{ED}}(T) \le \log 2/(k + 1) \le \log 2/\sqrt{\log n}$ by Proposition C and the choice of k. Otherwise every branch point of T has order at most k. By Lemma B.4 the tree T has at least $n/k > k^{2k-1}$ branch points, so by Lemma 1 it contains a $(2k + 2)$-comb. Let $r \ge 1$ be the largest integer such that $2^r \le 2k + 2$; hence $2^{r+1} > 2k + 2$, that is, $2^r > k + 1 \ge \sqrt{\log n}$. By Proposition C we have that $I^{\mathcal{ED}}(T) \le \log 2/2^r < \log 2/\sqrt{\log n}$. Hence the proof of (B.1) is finished. □

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