*Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 1–30, *2025*
DOI:10.1017/prm.2025.12

# $SU(2)$-bundles over highly connected 8-manifolds

**Samik Basu**
Stat Math Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata, 700108, India (samik.basu2@gmail.com) (corresponding author)

**Aloke Kr Ghosh**
Stat Math Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700108, India (alokekrghosh005@gmail.com)

**Subhankar Sau**
Stat Math Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata 700108, India (subhankarsau18@gmail.com)

(Received 21 May 2024; revised 11 January 2025; accepted 29 January 2025)

In this paper, we analyse the possible homotopy types of the total space of a principal $SU(2)$-bundle over a 3-connected 8-dimensional Poincaré duality complex. Along the way, we also classify the 3-connected 11-dimensional complexes $E$ formed from a wedge of $S^4$'s and $S^7$'s by attaching a 11-cell.

*Keywords:* loop spaces; Poincaré duality complexes; principal bundles; sphere fibrations

2020 *Mathematics Subject Classification:* Primary: 55R25, 57P10
Secondary: 57R19, 55P35

## 1. Introduction

This paper explores $SU(2)$-bundles over 8-manifolds, aiming for results akin to those about circle bundles over 4-manifolds [4, 8]. In the case of simply connected 4-manifolds, the results are established by leveraging the classification of simply connected 5-manifolds achieved by Smale [12] and Barden [3].

A circle bundle $S^1 \to X \to M$ over a simply connected 4-manifold $M$ is classified by $\alpha \in H^2(M)$, the total space $X(\alpha)$ is simply connected if $\alpha$ is primitive, and there are only two possibilities of $X(\alpha)$ via the classification of simply connected 5-manifolds. Explicitly, we have [8, Theorem 2]

(i) For every simply connected 4-manifold $M$, there is a circle bundle $\alpha$, such that $X(\alpha)$ is homotopy equivalent to a connected sum of $S^2 \times S^3$. If $M$ is spin, among primitive $\alpha$, this is the only possibility.

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh



https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

(ii) For a simply connected 4-manifold $M$ which is not spin and a circle bundle $\alpha$ over it, $X(\alpha)$ is either homotopy equivalent to a connected sum of $S^2 \times S^3$ or to a connected sum of $S^2 \times S^3$ and another manifold $B$. The manifold $B$ is (up to diffeomorphism unique) a non-spin simply connected 5-manifold whose homology is torsion-free, and $\mathrm{Rank}(H_2(B)) = 1$.

The results of Smale and Barden are geometric in nature and do not generalize easily to higher dimensions. Using homotopy theoretic methods, it was possible to construct sphere fibrations [7] over highly connected Poincaré duality complexes possibly by inverting a few primes or in high enough rank. Among these sphere fibrations, the only case where they could be principal bundles was in dimension 8, and the question whether they may be realized as such was left unresolved.

In this paper, we consider principal $SU(2)$-bundles which are classified up to homotopy by maps to $\mathbb{H}P^\infty \simeq BSU(2)$, noting that $SU(2) = S^3$ is the only case apart from the circle where the sphere is a Lie group. The base space of the $SU(2)$-bundle which is appropriate for making a similar analysis is a highly connected 8-manifold. More precisely, we consider Poincaré duality complexes $M$ (8-dimensional) that are 3-connected. These are obtained by attaching a single 8-cell to a bouquet of 4-spheres. We denote

$$\mathcal{PD}_3^8 = \text{the collection of 3-connected 8-dimensional Poincaré duality complexes.}$$

The notation $M_k \in \mathcal{PD}_3^8$ assumes that $\mathrm{Rank}(H_4(M_k)) = k$. The attaching map of the 8-cell is denoted by $L(M_k) \in \pi_7\big((S^4)^{\vee k}\big) \cong \pi_7(S^4)^{\oplus k} \oplus \pi_7(S^7)^{\oplus \binom{k}{2}}$ and is of the form (once we have chosen a basis $\{\alpha_1, \ldots, \alpha_k\}$ of $\pi_4(M_k) \cong \mathbb{Z}^k$)

$$L(M_k) = \sum_{1 \le i < j \le k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{i=1}^{k} g_{i,i}\nu_i + \sum_{i=1}^{k} l_i \nu_i'. \tag{1.1}$$

The matrix $\big((g_{i,j})\big)$ is the matrix of the intersection form and, hence, is invertible. The notation $\nu_i$ stands for $\alpha_i \circ \nu$ and $\nu_i'$ for $\alpha_i \circ \nu'$. Here $\nu \in \pi_7(S^4)$ is the Hopf map, and $\nu' \in \pi_7(S^4)$ is the generator for the $\mathbb{Z}/(12)$ factor satisfying $[\iota_4, \iota_4] = 2\nu + \nu'$. For such complexes, we consider

$$\mathcal{P}(M_k) = \text{the set of principal } SU(2)\text{-bundles } E(\psi) \xrightarrow{\psi} M_k \text{ such that } E(\psi) \text{ is 3-connected.}$$

As in the simply connected 4-manifold case, we expect to construct $SU(2)$-bundles over $M_k$ for which the total space is relatively simple in terms of its homotopy type. In this case, the assumption that $E(\psi)$ is 3-connected implies that its homology is torsion-free. The bundle $\psi$ is classified by a primitive element $\psi \in H^4(M_k)$, which satisfies a criterion (see proposition 4.4). In this context, we first encounter the question of whether $\mathcal{P}(M_k)$ is non-empty. We prove the following theorem (see proposition 4.1 and proposition 5.1):

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

THEOREM A. *For $k \geq 3$, the set $\mathcal{P}(M_k)$ is non-empty.*

For $k=2$, there are examples where $\mathcal{P}(M_k)$ is empty (see the discussion before proposition 4.1 and example 5.1). This means that for every principal $SU(2)$-bundle over such complexes, the total space has non-trivial $\pi_3$. The idea here is that the existence of $\psi$ is given by a certain equation in $k$ variables, and solutions exist once $k$ is large enough.

In the case of simply connected 4-manifolds, the first kind of classification of circle bundles is the result of Giblin [9], which states that,

$$\text{If } X = S^2 \times S^2, \text{then } X(\alpha) \simeq S^2 \times S^3 \text{ for any primitive } \alpha.$$

We also have an analogous result in the 8-dimensional case

$$\text{If } \psi \in \mathcal{P}(S^4 \times S^4), \text{then } E(\psi) \simeq S^4 \times S^7.$$

In fact, this fits into a more general framework. We call a manifold $M_k \in \mathcal{PD}_3^8$ *stably trivial* if $L(M_k)$ is stably null-homotopic (that is the stable homotopy class of $L(M_k) : S^7 \to \left(S^4\right)^{\vee k}$ is 0). In terms of (1.1), this means for every $i$, $g_{i,i} - 2l_i \equiv 0 \pmod{24}$. We have the following theorem (see proposition 4.3):

THEOREM B. *Suppose* $\mathrm{M_k}$ *is stably trivial. Then, for every $\psi \in \mathcal{P}(M_k)$, $E(\psi) \simeq \#^{k-1} S^4 \times S^7$, a connected sum of* $\mathrm{k-1}$ *copies of $S^4 \times S^7$.*

This directly generalizes the result for circle bundles over simply connected 4-manifolds that are spin (identifying the spin manifolds as those whose attaching map is stably null).

We proceed towards a more general classification of the homotopy type of the space $E(\psi)$ for $\psi \in \mathcal{P}(M_k)$. Let $\mathcal{PD}_{4,7}^{11}$ be the class of 3-connected 11-dimensional Poincaré duality complexes $E$ such that $E \setminus \{pt\} \simeq$ a wedge of equal number of copies of $S^4$ and $S^7$. We first observe that $E(\psi) \in \mathcal{PD}_{4,7}^{11}$ (see proposition 3.1), and we try to address the question of the classification of complexes in $\mathcal{PD}_{4,7}^{11}$ up to homotopy equivalence. The homology of such complexes $E$ is given by

$$H_m(E) \cong \begin{cases} \mathbb{Z} & m = 0, 11 \\ \mathbb{Z}^r & m = 4, 7 \\ 0 & \text{otherwise.} \end{cases}$$

We denote the number $r$ by $\mathrm{Rank}(E)$. The classification works differently for $r=1$, and for $r \geq 2$. Table 1 lists the various possibilities for $r=1$. For $r \geq 2$, $E$ is a connected sum of copies of $S^4 \times S^7$, and the complexes $E_{\lambda,\epsilon,\delta}$ are defined below. Note that

$$\begin{aligned} \pi_{10}(S^4 \vee S^7) &\cong \pi_{10}(S^4) \oplus \pi_{10}(S^7) \oplus \pi_{10}(S^{10}) \\ &\cong \left(\mathbb{Z}/(24)\{x\} \oplus \mathbb{Z}/(3)\{y\}\right) \oplus \mathbb{Z}/(24)\{\nu_{(7)}\} \oplus \mathbb{Z}\{[\iota_4, \iota_7]\}. \end{aligned}$$

Here, $x = \nu \circ \nu_{(7)}$ and $y = \nu' \circ \nu_{(7)}$. The notation $\nu_{(7)}$ stands for the 3-fold suspension of $\nu$ explained in the following convention: for a class $\xi \in \pi_r(S^n)$, we write $\xi_{(s)} = \Sigma^{s-n}\xi \in \pi_{r+s-n}(S^s)$. Let

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

Table 1. *Homotopy equivalence classes of* $E_{\lambda,\epsilon,\delta}$.

| $\lambda$ | $\#E_{\lambda,\epsilon,\delta}$'s | $E_{\lambda,\epsilon,\delta}$'s |
|---|---|---|
| 0 | 2 | $E_{0,0,0}$, $E_{0,1,0}$ |
| 1 | 3 | $E_{1,0,0}$, $E_{1,1,0}$, $E_{1,2,0}$ |
| 2 | 12 | $E_{2,0,0}$, $E_{2,1,0}$, $E_{2,2,0}$, $E_{2,3,0}$, $E_{2,0,1}$, $E_{2,1,1}$, $E_{2,2,1}$, $E_{2,3,1}$, $E_{2,0,2}$, $E_{2,1,2}$, $E_{2,2,2}$, $E_{2,3,2}$ |
| 3 | 1 | $E_{3,0,0}$ |
| 4 | 6 | $E_{4,0,0}$, $E_{4,1,0}$, $E_{4,2,0}$, $E_{4,3,0}$, $E_{4,4,0}$, $E_{4,5,0}$ |
| 5 | 3 | $E_{5,0,0}$, $E_{5,0,1}$, $E_{5,0,2}$ |
| 6 | 4 | $E_{6,0,0}$, $E_{6,1,0}$, $E_{6,2,0}$, $E_{6,3,0}$ |
| 7 | 3 | $E_{7,0,0}$, $E_{7,1,0}$, $E_{7,2,0}$ |
| 8 | 6 | $E_{8,0,0}$, $E_{8,1,0}$, $E_{8,0,1}$, $E_{8,1,1}$, $E_{8,0,2}$, $E_{8,1,2}$ |
| 9 | 1 | $E_{9,0,0}$ |
| 10 | 12 | $E_{10,0,0}$, $E_{10,1,0}$, $E_{10,2,0}$, $E_{10,3,0}$, $E_{10,4,0}$, $E_{10,5,0}$, $E_{10,6,0}$, $E_{10,7,0}$, $E_{10,8,0}$, $E_{10,9,0}$, $E_{10,10,0}$, $E_{10,11,0}$ |
| 11 | 3 | $E_{11,0,0}$, $E_{11,0,1}$, $E_{11,0,2}$ |
| 12 | 2 | $E_{12,0,0}$, $E_{12,1,0}$ |

$$\phi_{\lambda,\epsilon,\delta} = [\iota_4, \iota_7] + \lambda(\iota_7 \circ \nu_{(7)}) + \epsilon(\iota_4 \circ x) + \delta(\iota_4 \circ y),$$

and define,

$$E_{\lambda,\epsilon,\delta} = (S^4 \vee S^7) \cup_{\phi_{\lambda,\epsilon,\delta}} D^{11}.$$

The attaching map of the top cell of $E$ takes the form

$$L(E) : S^{10} \to \left(S^4 \vee S^7\right)^{\vee r}.$$

The stable homotopy class of $L(E)$ lies in

$$\pi^s_{10}\left(\left(S^4 \vee S^7\right)^{\vee r}\right) \cong \left(\mathbb{Z}/(24)\{\nu\} \oplus \mathbb{Z}/(2)\{\nu^2\}\right)^{\oplus r}.$$

This takes the form $\lambda_s\beta \circ \nu + \epsilon_s\alpha \circ \nu^2$ for some $\beta \in \pi_7\left(\left(S^4 \vee S^7\right)^{\vee r}\right)$ and $\alpha \in \pi_4\left(\left(S^4 \vee S^7\right)^{\vee r}\right)$. Up to a change of basis, we may assume that $\lambda_s \mid 24$, and if $\lambda_s$ is even, $\epsilon_s \in \mathbb{Z}/(2)$. These numbers are invariant over the homotopy equivalence

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

class of $E$ and are denoted by $\lambda_s(E)$ and $\epsilon_s(E)$ (defined only if $\lambda_s(E)$ is even). We use these invariants to classify the homotopy types of elements in $\mathcal{PD}^{11}_{4,7}$ (see theorem 2.6).

THEOREM C. *Let $E \in \mathcal{PD}^{11}_{4,7}$. Then the homotopy type of* E *is determined by the following.*

(i) *If $\lambda_s(E)$ is even and $\epsilon_s(E) = 0$, then*

$$E \simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \text{where } \epsilon \equiv 0 \pmod 2.$$

(ii) *If $\lambda_s(E)$ is even and $\epsilon_s(E) = 1$, then*

$$\begin{aligned} E &\simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 1 \pmod 2 \\ \text{or } E &\simeq \#^{r-2} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 0 \pmod 2. \end{aligned}$$

(iii) *If $\lambda_s(E)$ is odd, then*

$$E \simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \text{or} \quad E \simeq \#^{r-2} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta}.$$

*Further given* $\lambda_s$*, the choices of $\epsilon$ and $\delta$ are those that are mentioned in table 1.*

We see that in the list given in table 1, for certain cases the homotopy type of $E$ is determined by $\lambda_s(E)$ and $\epsilon_s(E)$. This happens if $\lambda_s(E) = 0$ or 12. We also observe that the homotopy type of $\Omega E$ depends only on the rank $r$. Now, we look at $M_k \in \mathcal{PD}^8_3$ and try to determine the set of homotopy equivalence classes of $E(\psi)$ for $\psi \in \mathcal{P}(M_k)$. In this process, we determine a formula for $\lambda(\psi) := \lambda_s(E(\psi))$ (proposition 3.5), and using this we determine the set of possible values of $\lambda_s(\psi)$ for $\psi \in \mathcal{P}(M_k)$. The stable homotopy class of $L(M_k)$ lies in

$$\pi^s_7\Big((S^4)^{\vee k}\Big) \cong \big(\mathbb{Z}/(24)\{\nu\}\big)^{\oplus k}.$$

This takes the form $\sigma_s\alpha \circ \nu$ for some $\alpha \in \pi_4\big((S^4)^{\vee k}\big)$, and up to a change of basis for $k \geq 2$, $\sigma(M_k) := \gcd(\sigma_s, 24)$ is an invariant of the stable homotopy type of $M_k$. Other than $k$ and $\sigma(M_k)$, the explicit stable homotopy class of $\alpha$ above yields a linear map $\tau : H^4(M_k) \to \mathbb{Z}/(24)$ given by $\tau(\psi) = \psi(\sigma_s\alpha)$. We use the invariants $k$, $\sigma(M_k)$, $\tau$, and the intersection form to completely determine the possibilities of $\lambda(\psi)$ for $\psi \in \mathcal{P}(M_k)$ (see theorem 3.6, proposition 4.5, theorem 4.6, theorem 4.9, and theorem 5.2).

THEOREM D. *For any $\psi \in \mathcal{P}(M_k)$, $\lambda(\psi)$ is a multiple of $\sigma(M_k) \pmod{24}$. Conversely, the multiples of $\sigma(M_k)$ that may be achieved are described as follows:*

(i) *If the intersection form of* $M_k$ *is odd and $k \geq 7$, then $\{\lambda(\psi) \mid \psi \in \mathcal{P}(M_k)\}$ equals the set of multiples of $\sigma(M_k) \pmod{24}$.*
(ii) *If the intersection form of* $M_k$ *is even, each $\psi \in \mathcal{P}(M_k)$ satisfies $\epsilon_s(\psi) \equiv 0 \pmod 2$.*

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

(iii) *If $k \geq 7$, there are $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) = \sigma(M_k)$, and also there are $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) = 3\sigma(M_k)$.*

(iv) *If $\sigma(M_k) \equiv 2$, or $4 \pmod 8$ for $k \geq 5$, there is a $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) \equiv 0 \pmod 8$ if and only if the complex satisfies hypothesis $(H_8)$.*

(v) *If $\sigma(M_k) \equiv 2 \pmod 8$ for $k \geq 5$, there is a $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) \equiv 4 \pmod 8$ if and only if the complex satisfies hypothesis $(H_4)$.*

For lower values of $k$, we do not get systematic results like the above. That is, the set $\{\lambda(\psi) \mid \psi \in \mathcal{P}(M_k)\}$ is not completely determined by $\sigma(M_k)$, $k$, $\tau$, and the intersection form. Theorem D implies that there are certain $M_k$ whose intersection form is even and there is no $\psi \in \mathcal{P}(M_k)$ such that $E(\psi) \simeq \#^{k-1} S^4 \times S^7$, however, if the intersection form is odd, then for $k \geq 7$, there is a principal bundle $SU(2) \to \#^{k-1}(S^4 \times S^7) \to M_k$.

### 1.1. Organization

In §2, we prove a classification result for certain 3-connected 11-dimensional complexes which proves Theorem C. In §3, we prove formulas relating the stable homotopy type of the total space to that of the base using the Chern character. The results for manifolds with even intersection form are proved in §4, and those with odd intersection form in §5.

## 2. Homotopy classification of certain 3-connected 11-complexes

We study 3-connected 11-dimensional Poincaré duality complexes $E$ such that $E \setminus \{pt\}$ is homotopic to a wedge of copies of $S^4$ and $S^7$. We write $\mathcal{PD}^{11}_{4,7}$ for the collection of such complexes. Our target in this section is to analyse them up to homotopy equivalence. We show that these are classified by numbers $\lambda$, $\epsilon$, and $\delta$ which are explained in detail below.

### 2.1. The rank one case

Let $E \in \mathcal{PD}^{11}_{4,7}$ be such that $E \setminus \{pt\} \simeq S^4 \vee S^7$, that is, $\mathrm{Rank}(H_4(E)) = 1$. The homotopy type of $E$ is determined by the attaching map of the top cell, which is an element of

$$\pi_{10}(S^4 \vee S^7) \cong \pi_{10}(S^4) \oplus \pi_{10}(S^7) \oplus \pi_{10}(S^{10}).$$

This must be of the form

$$\phi_{\lambda,\epsilon,\delta} = [\iota_4, \iota_7] + \lambda(\iota_7 \circ \nu_{(7)}) + \epsilon(\iota_4 \circ x) + \delta(\iota_4 \circ y). \tag{2.1}$$

The total space associated with $\phi_{\lambda,\epsilon,\delta}$ is denoted by

$$E_{\lambda,\epsilon,\delta} = (S^4 \vee S^7) \cup_{\phi_{\lambda,\epsilon,\delta}} D^{11}.$$

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

Note that as $(-\iota_4)\circ\nu = \nu + \nu'$, we have $(-\iota_4)\circ x = x + y$. For given any $\lambda, \epsilon$, and $\delta$, we observe the effect of the self-homotopy equivalences on $E_{\lambda,\epsilon,\delta}$ as follows:

$$\begin{aligned}
&\iota_4 \mapsto -\iota_4, \quad \iota_7 \mapsto -\iota_7 && \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{-\lambda,\epsilon,\epsilon-\delta},\\
&\iota_4 \mapsto \iota_4, \quad \iota_7 \mapsto \iota_7 + a\iota_4\circ\nu && \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{\lambda,\epsilon+(\lambda+2)a,\delta},\\
&\iota_4 \mapsto \iota_4, \quad \iota_7 \mapsto \iota_7 + b\iota_4\circ\nu' && \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{\lambda,\epsilon-4b,\delta+(\lambda+1)b}.
\end{aligned}$$

This leads us to homotopy equivalences between $E_{\lambda,\epsilon,\delta}$'s depending on the choice of $\lambda \in \pi_{10}(S^7) \cong \mathbb{Z}/24$. Table 1 lists the different homotopy types in $\mathcal{PD}^{11}_{4,7}$ of rank 1.

### 2.2. A simplification of the ataching map

We simplify and reduce the attaching map of the top cell of $E$.

PROPOSITION 2.1. *Let $E \in \mathcal{PD}^{11}_{4,7}$ with Rank$(E) = k-1$. The attaching map $\phi$ of the top cell of* E *as in (2.1) can be reduced, up to homotopy, to the following form*

$$\phi = \sum_{i=1}^{k-1}[\iota_4^i, \iota_7^i] + \sum_{i=1}^{k-1}\lambda_i\iota_7^i\circ\nu_{(7)} + \sum_{i=1}^{k-1}s_i\nu_i\circ\nu_{(7)} + \sum_{i=1}^{k-1}r_i\nu_i'\circ\nu_{(7)}.$$

*Proof.* By Hilton–Milnor decomposition, we have the following equivalence

$$\begin{aligned}\pi_{10}((S^4\vee S^7)^{\vee k-1}) \cong \pi_{10}(S^4)^{\oplus(k-1)} \oplus \pi_{10}(S^7)^{\oplus(k-1)} \oplus \pi_{10}(S^7)^{\oplus\binom{k-1}{2}}\oplus\\ \pi_{10}(S^{10})^{\oplus(k-1)\times(k-1)} \oplus \pi_{10}(S^{10})^{\oplus\binom{k-1}{3}}.\end{aligned}$$

We choose $\eta_1,\dots,\eta_{k-1} \in \pi_4(E)$ and $\gamma_1,\dots,\gamma_{k-1} \in \pi_7(E)$ such that they correspond to the homology generators, say $\tilde\eta_1,\dots,\tilde\eta_{k-1} \in H_4(E)$ and $\tilde\gamma_1,\dots,\tilde\gamma_{k-1} \in H_7(E)$ such that

$$\tilde\eta_i^* \cup \tilde\gamma_j^* = \begin{cases} 1 & \text{if } i = j,\\ 0 & \text{if } i \neq j.\end{cases} \tag{2.2}$$

Let $\tilde f\colon (S^4\vee S^7)^{\vee k-1} \to E$ be the inclusion that sends $\iota_4^i \mapsto \eta_i$ and $\iota_7^i \mapsto \gamma_i$ for $1 \le i \le k-1$. Then $\tilde f\circ\phi \in \pi_{10}(E)$ whose image under the map $\rho\colon \pi_{10}(E) \to \pi_9(\Omega E) \to H_9(\Omega E)$ is 0 in the tensor algebra $T(\tilde\eta_1,\dots,\tilde\eta_{k-1},\tilde\gamma_1,\dots,\tilde\gamma_{k-1})/(\sum_{i=1}^{k-1}[\tilde\eta_i,\tilde\gamma_i])$.

The attaching map $\phi$ may contain triple Whitehead products as $[\iota_i^4,[\iota_j^4,\iota_\ell^4]]$, the Whitehead products of the form $[\iota_i^4,\iota_j^7]$ for $i \neq j$, the terms involving $[\iota_i^4,\iota_j^4]\circ\nu_{(7)}$ and $[\iota_i^4,\iota_j^4]\circ\nu'_{(7)}$. The triple Whitehead product maps injectively to the loop homology and hence they cannot occur in the attaching map. The cup product formula in (2.2) implies that there is no Whitehead product of the form $[\iota_i^4,\iota_j^7]$ for $i \neq j$. If $[\iota_i^4,\iota_j^4]\circ\nu_{(7)}$ and $[\iota_i^4,\iota_j^4]\circ\nu'_{(7)}$ appear in the attaching map, we update the map $\tilde f$ by appropriately sending $\iota_i^7 \mapsto \gamma_i - \eta_j\circ\nu'$, $\iota_i^7 \mapsto \gamma_i - \eta_j\circ\nu$ and $\iota_j^7 \mapsto \gamma_j - [\eta_i,\eta_j]$ to get the desired form of the attaching map. □

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

We note that the composition is given by

$$S^{10} \to (S^4 \vee S^7)^{\vee k-1} \to (S^7)^{\vee k-1}$$

which is an element of $(\pi_{10}S^7)^{\oplus k-1} \cong (\mathbb{Z}/24\{\nu\})^{\oplus k-1}$. This can be calculated using the real $e$-invariant, see [2]. We use this to reduce proposition 2.1 to the case $\lambda_i = 0$ for $i \leq k-2$.

PROPOSITION 2.2. *Let $E \in \mathcal{PD}^{11}_{4,7}$ and Rank$(E) = k-1$. Then the attaching map $\phi$ of the top cell of* E *can be reduced to the following form*

$$\phi = \sum_{i=1}^{k-1}[\iota_4^i, \iota_7^i] + \lambda\iota_7^{k-1} \circ \nu_{(7)} + \sum_{i=1}^{k-1} \epsilon_i \nu_i \circ \nu_{(7)} + \sum_{i=1}^{k-1} \delta_i \nu_i' \circ \nu_{(7)}. \tag{2.3}$$

*As a consequence, $E \simeq \#_{i=1}^{k-2} E_{0,\epsilon_i,\delta_i} \# E_{\lambda,\epsilon_{k-1},\delta_{k-1}}$.*

*Proof.* Let $\tau\colon H^7(E) \cong \mathbb{Z}\{\tilde{\gamma}_1^*, \tilde{\gamma}_2^*, \ldots, \tilde{\gamma}_{k-1}^*\} \to \mathbb{Z}/24$ be the linear map defined by

$$\tau(\tilde{\gamma}_i^*) = e(r_i \circ \phi), \qquad \text{for } 1 \leq i \leq k-1,$$

where $e$ denotes for the real $e$-invariant and $r_i : (S^4 \vee S^7)^{\vee k-1} \to S^7$ is the retraction onto the $i$-th factor. Let $\tilde{\tau}\colon H^7(E) \to \mathbb{Z}$ be the lift of $\tau$ and $\lambda = \gcd\big((\tilde{\tau}(\tilde{\gamma}_1^*), \ldots, \tilde{\tau}(\tilde{\gamma}_{k-1}^*)\big)$. Then we can change the generators $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{k-1}$ such that $\tilde{\tau}(\tilde{\gamma}_i) = 0$ for $1 \leq i < k-1$ and $\tilde{\tau}(\tilde{\gamma}_{k-1}) = \lambda$. So, for a suitable choice of dual bases, we can have $\phi$ as in (2.3). □

### 2.3. A general classification up to homotopy

We now proceed to the classification of elements in $\mathcal{PD}^{11}_{4,7}$. In the connected sum $E_{\lambda_1,\epsilon_1,\delta_1} \# E_{\lambda_2,\epsilon_2,\delta_2}$, we transform $\alpha_1 = \alpha_1' + \alpha_2'$ and $\alpha_2 = \alpha_2'$ by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If $2|\lambda_2$, then we transform

$$\beta_1 = \beta_1' - \epsilon_1\nu_2 - \frac{\lambda_2\epsilon_1}{2}\nu_2', \quad \beta_2 = -\beta_1' + \beta_2' - \epsilon_1[\alpha_1', \alpha_2'].$$

Hence by the following expression

$$\begin{aligned} &[\alpha_1, \beta_1] + [\alpha_2, \beta_2] + \lambda_1\beta_1 \circ \nu_{(7)} + \lambda_2\beta_2 \circ \nu_{(7)} + \epsilon_1 x_1 + \epsilon_2 x_2 + \delta_1 y_1 + \delta_2 y_2 \\ =&[\alpha_1', \beta_1'] + [\alpha_2', \beta_2'] + (\lambda_1 - \lambda_2)\beta_1' \circ \nu_{(7)} + \lambda_2\beta_2' \circ \nu_{(7)} + \epsilon_1 x_1 \\ &+ (-\epsilon_1 + \epsilon_2 + 2\lambda_2\epsilon_1 - \lambda_1\epsilon_1)x_2 \\ &\qquad \delta_1 y_1 + (\delta_1 + \delta_2 + \lambda_2\epsilon_1(1 + \lambda_1))y_2, \end{aligned}$$

we conclude

$$E_{\lambda_1,\epsilon_1,\delta_1} \# E_{\lambda_2,\epsilon_2,\delta_2} \simeq E_{\lambda_1-\lambda_2,\epsilon_1,\delta_1} \# E_{\lambda_2,\epsilon_2+\epsilon_1(2\lambda_2-\lambda_1-1),\delta_1+\delta_2+(1+\lambda_1)\epsilon_1\lambda_2} \quad \text{when } 2|\lambda_2. \tag{2.4}$$

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

PROPOSITION 2.3. *For any unit $a \in \mathbb{Z}/24$, we have homotopy equivalence*

$$E_{\lambda,\epsilon,\delta}\#E_{0,0,0} \simeq \begin{cases} E_{a\lambda,\epsilon,\delta}\#E_{0,0,0} & \text{if } a \equiv 1 \pmod 3 \\ E_{-a\lambda,\epsilon,\delta}\#E_{0,0,0} & \text{if } a \equiv 2 \pmod 3. \end{cases}$$

*Proof.* We transform

$$\begin{aligned} \alpha_1 &= a\alpha_1' + b\alpha_2' & \beta_1 &= a\beta_1' - 24\beta_2' - b\epsilon\nu_2 - 24b\epsilon\nu_1 \\ \alpha_2 &= 24\alpha_1' + a\alpha_2' & \beta_2 &= -b\beta_1' + a\beta_2' - \epsilon b[\alpha_1', \alpha_2'] \end{aligned}$$

with $a^2 - 24b = 1$ and calculate $[\alpha_1, \beta_1] + [\alpha_2, \beta_2] + \lambda\beta \circ \nu_{(7)} + \epsilon x_1 + \delta y_2$. This gives the homotopy equivalence

$$E_{\lambda,\epsilon,\delta}\#E_{0,0,0} \simeq E_{a\lambda,a^2\epsilon,a\delta+\binom{a}{2}\epsilon}\#E_{0,-b^2\epsilon-b\lambda\epsilon,b\delta+\binom{b}{2}\epsilon}. \tag{2.5}$$

This proves the proposition except for when $\lambda \equiv 0 \pmod 2$ and $\epsilon, b \equiv 1 \pmod 2$. In that case, we have $E_{0,1,0}$ instead of $E_{0,0,0}$ in the second component of the connected sum. From (2.4), we get

$$E_{\lambda,\epsilon,\delta}\#E_{0,1,0} \simeq E_{\lambda,\epsilon,\delta}\#E_{0,1+\epsilon(-\lambda-1),\delta} \tag{2.6}$$

which we apply following the equivalence in (2.5) for $\lambda$ even and $\epsilon$, $b$ odd. This concludes the proof. □

The transformations above further simplify the possibilities of $E \in \mathcal{PD}^{11}_{4,7}$ listed in proposition 2.2.

PROPOSITION 2.4. *Let $E \in \mathcal{PD}^{11}_{4,7}$ and Rank$(E) = k-1$. Then*

$$E \simeq \#^{k-3}E_{0,0,0}\#E_{0,\hat{\epsilon},0}\#E_{\lambda,\epsilon,\delta}$$

*for some $\lambda, \epsilon \in \mathbb{Z}/24$, $\delta \in \mathbb{Z}/3$, $\hat{\epsilon} \in \mathbb{Z}/2$.*

*Proof.* From proposition 2.2, we have the attaching map $\phi$ as in (2.3). The repeated use of the homotopy equivalences in (2.2) and (2.6) gives the reduced form (2.3) as follows

$$\phi = \sum_{i=1}^{k-1}[\iota_4^i, \iota_7^i] + \lambda\iota_7^{k-1} \circ \nu_{(7)} + \hat{\epsilon}x_{k-2} + \epsilon x_{k-1} + \delta y_{k-1},$$

where $\lambda, \epsilon \in \mathbb{Z}/24$, $\delta \in \mathbb{Z}/3$, $\hat{\epsilon} \in \mathbb{Z}/2$. □

COROLLARY 2.5. *Let $E \in \mathcal{PD}^{11}_{4,7}$ and Rank$(E) = k-1$. Then stably we have*

$$\Sigma^\infty E \simeq \Sigma^\infty(S^4 \vee S^7)^{\vee k-2} \vee \Sigma^\infty Cone(\lambda_s(E)\nu_{(11)} + \epsilon_s(E)x), \tag{2.7}$$

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

*where* x *is the generator of the stable homotopy group* $\pi_{14}(S^8) \cong \mathbb{Z}/2$ *and* $\lambda_s(E) \in \mathbb{Z}/24$, $\epsilon_s(E) \in \mathbb{Z}/2$. *Moreover, for* $k-1 \geq 2$ *if* $\lambda_s(E) \equiv 1 \pmod 2$ *in* (2.7), *then* $\epsilon_s(E) = 0 \in \mathbb{Z}/2$.

*Proof.* From (2.3) and proposition 2.4,

$$\Sigma^4\phi = \lambda\iota_{11}^{k-1} \circ \nu_{(11)} + \hat{\epsilon}\iota_8^{k-2} \circ \nu_{(11)} + \epsilon\iota_8^{k-1} \circ \nu_{(11)}.$$

Note that $\nu_{(11)} \in \pi_3^s(S^0) \simeq \mathbb{Z}/24$ and $\Sigma^4(\nu \circ \nu_{(7)}) \in \pi_6^s(S^0) \simeq \mathbb{Z}/2$ are the generators. If $\hat{\epsilon} = 0 \in \mathbb{Z}/2$, the result readily follows. Otherwise, if $\epsilon = 1, \hat{\epsilon} = 1 \in \mathbb{Z}/2$, we apply the transformation $\iota_8^{k-2} + \iota_8^{k-1} \mapsto \iota_8^{k-1}$. If $\epsilon = 0, \hat{\epsilon} = 1 \in \mathbb{Z}/2$, we interchange $\iota_8^{k-2}$ and $\iota_8^{k-1}$ to deduce the result. □

We see that the stable homotopy type of $E \in \mathcal{PD}_{4,7}^{11}$ is determined by $\lambda_s(E)$ which is a divisor of 24 and $\epsilon_s(E) \in \mathbb{Z}/2$. The following theorem classifies the different homotopy types of $E$ given the values of $\lambda_s$ and $\epsilon_s$.

THEOREM 2.6. *Let* $E \in \mathcal{PD}_{4,7}^{11}$ *and Rank*$(E) = k-1$. *Then depending on* $\lambda_s = \lambda_s(E)$ *and* $\epsilon_s = \epsilon_s(E)$, *the homotopy type of* E *is determined by the following.*

(i) *If* $\lambda_s$ *is even and* $\epsilon_s = 0$, *then*

$$E \simeq \#^{k-2} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \text{where } \epsilon \equiv \epsilon_s \pmod 2.$$

(ii) *If* $\lambda_s$ *is even and* $\epsilon_s = 1$, *then*

$$\begin{aligned} E &\simeq \#^{k-2} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 1 \pmod 2 \\ \text{or } E &\simeq \#^{k-3} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 0 \pmod 2. \end{aligned}$$

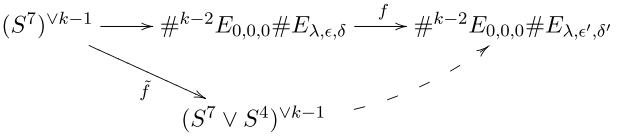
(iii) *If* $\lambda_s$ *is odd, then*

$$E \simeq \#^{k-2} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \text{or} \quad E \simeq \#^{k-3} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta}.$$

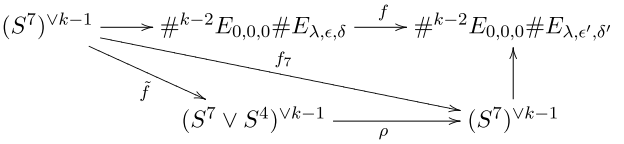
*Further given* $\lambda_s$, *the choices of* $\epsilon$ *and* $\delta$ *are those that are mentioned in table* 1.

*Proof.* We write $Y_1 = \#^{k-2} E_{0,0,0} \# E_{\lambda,\epsilon,\delta}$ and $Y_2 = \#^{k-2} E_{0,0,0} \# E_{\lambda,\epsilon',\delta'}$, and let $Y_1 \xrightarrow{f} Y_2$ be a homotopy equivalence with homotopy inverse $g$. We show that in this case, the pair $(\epsilon, \delta)$ is related to $(\epsilon', \delta')$ by the transformations (2.2). There exists a unique (up to homotopy) factorization $(S^7)^{\vee k-1} \to (S^7 \vee S^4)^{\vee k-1} \to Y_2$ through cellular approximation $\tilde{f}$ as in the following diagram.

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press




Now we consider the composition $f_7 \colon = \rho \circ \tilde{f}$, where $\rho$ is the projection map.




From the stable homotopy type, we get an isomorphism

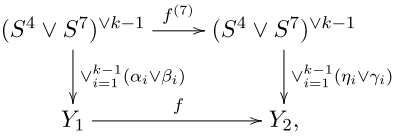
$$\begin{array}{ccccc} f_7 \colon & \pi_7((S^7)^{\vee k-1}) & \xrightarrow{\cong} & \pi_7((S^7)^{\vee k-1}) \\ & \wr\| & & \wr\| \\ & \mathbb{Z}\{\beta_1, \ldots, \beta_{k-1}\} & & \mathbb{Z}\{\gamma_1, \ldots, \gamma_{k-1}\} \end{array}$$

where $f_7(\beta_{k-1}) \equiv \gamma_{k-1} \pmod{24}$. Hence, the corresponding matrix of $f_7$ is

$$\equiv \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ * & \ldots & * & 1 \end{pmatrix} \pmod{24}$$

for some $A \in GL_{k-2}(\mathbb{Z})$. From the inverse homotopy equivalence $g \colon Y_2 \to Y_1$, we can construct a corresponding block matrix of $g_7$ similar to that of $f_7$ for some $B \in GL_{k-2}(\mathbb{Z}) \pmod{24}$. Through suitable pre-composition of $f_7$ and post-composition of $g_7$, we may assume that $A = B = I$ where $I$ is the identity matrix in $GL_{k-2}(\mathbb{Z})$. Thus both $f_7$ and $g_7$ are composition of shearing maps with $\beta_{k-1} \mapsto \gamma_{k-1}$, that is, they are composition of maps associated with $\beta_i \mapsto \gamma_i + c_i\gamma_{k-1}$ for some $c_i \in \mathbb{Z}$.

We now consider the map $f$ on the 7-skeleton




which takes the form

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

$$\beta_i \mapsto \gamma_i + c_i\gamma_{k-1} + \sum_{j=1}^{k-1} a_{i,j}\nu_j + \sum_{j=1}^{k-1} a'_{i,j}\nu'_j + \sum_{\substack{j=1\\ j\neq \ell}}^{\substack{j=k-1\\ \ell=k-1}} a_{i,j,\ell}[\eta_j,\eta_\ell],$$

$$\alpha_i \mapsto \eta_i, \quad \text{for } 1 \le i \le k-2,$$

$$\beta_{k-1} \mapsto \gamma_{k-1} + \sum_{j=1}^{k-1} b_j\nu_j + \sum_{j=1}^{k-1} b'_j\nu'_j + \sum_{\substack{j=1\\ j\neq \ell}}^{\substack{j=k-1\\ \ell=k-1}} b_{j,\ell}[\eta_j,\eta_\ell],$$

$$\alpha_{k-1} \mapsto \eta_{k-1} - \sum_{j=1}^{k-2} c_j\eta_j.$$

As $f$ is a homotopy equivalence, we must have that the attaching map of the 11-cell of $Y_1$ must be carried by $f^{(7)}$ to the attaching map of $Y_2$, that is, $f^{(7)} \circ L(Y_1) \simeq L(Y_2)$. We now look at the coefficients of $\eta_{k-1} \circ x$ and $\eta_{k-1} \circ y$ that arises in $f^{(7)} \circ L(Y_1)$ and note that the only terms which contribute to these coefficients are

$$f^{(7)}([\alpha_{k-1},\beta_{k-1}] + \lambda_s\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ x + \delta\alpha_{k-1} \circ y).$$

We now deduce

$$\epsilon' = (\lambda_s + 2)b_{k-1} - 4b'_{k-1} \qquad \text{and} \qquad \delta' = (\lambda_s + 1)b'_{k-1},$$

which verifies the result for complexes of the type $\#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon,\delta}$.

For the remaining cases, we follow the same argument with $Y_1 = \#^{k-3}E_{0,0,0}\#E_{0,1,0}\#E_{\lambda_s,\epsilon,\delta}$ with $\epsilon$ even if $\lambda_s$ is, and $Y_2 = \#^{k-3}E_{0,0,0}$ $\#E_{0,\hat{\epsilon},0}\#E_{\lambda_s,\epsilon',\delta'}$, where $\hat{\epsilon} = 0$ or 1. Note that the only terms which contribute to $\eta_{k-2} \circ x$ are

$$f^{(7)}([\alpha_{k-1},\beta_{k-1}] + [\alpha_{k-2},\beta_{k-2}] + \alpha_{k-2} \circ x + \lambda_s\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ x + \delta\alpha_{k-1} \circ y).$$

A direct computation implies

$$\lambda_s b_{k-2} + \epsilon c_{k-2}^2 + 1 \equiv \hat{\epsilon} \pmod 2. \tag{2.8}$$

First, let $\lambda_s$ be even and $\epsilon \equiv 0 \pmod 2$. This implies $\hat{\epsilon} = 1$. Finally, let $\lambda_s$ is odd. We look at the coefficients of $[\eta_{k-1},[\eta_{k-2},\eta_{k-1}]]$ and $[\eta_{k-2},\eta_{k-1}]\circ\nu_{(7)}$ in $f^{(7)}\circ L(Y_1)$ (mod 2), which are $b_{k-2,k-1} + c_{k-2}b_{k-1} - a_{k-2,k-1}$ and $\lambda_s b_{k-2,k-1} + a_{k-2,k-1} + b_{k-2} - c_{k-2}b_{k-1} - \epsilon c_{k-2}$. Since both these coefficients are zero, we have $b_{k-2} \equiv \epsilon c_{k-2}$ (mod 2). Using the relation (2.8), we observe that $\hat{\epsilon} = 1$. The conditions on $\epsilon'$ and $\delta'$ are verified analogously as in the previous case. This completes the proof of the various implications in the theorem. □

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

### 2.4. The loop space homotopy type

We study the loop space homotopy type of $E \in \mathcal{PD}^{11}_{4,7}$ with $E^{(7)} \simeq (S^4 \vee S^7)^{\vee k-1}$ and show that the loop space homotopy is independent of the $\lambda, \epsilon$, and $\delta$ occurring in the attaching map $\phi$ of $E$.

THEOREM 2.7. *The homotopy type of the loop space of* E *is a weak product of loop spaces on spheres and depends only on* $k-1 = Rank(H_4(E))$. *In particular,*

$$\Omega E \simeq \Omega(\#^{k-1}(S^4 \times S^7)).$$

*Proof.* This follows from the arguments in [5] and [6]. More explicitly, we first compute the homology of $\Omega E$ via cobar construction, see [1]. In this case, $H_*(E)$ is a coalgebra which is quasi-isomorphic to $C_*(E)$, and hence we may compute the cobar construction of $H_*(E)$ and deduce as in [6, Proposition 2.2]

$$H_*(\Omega E) \cong T(a_1, b_1, \ldots, a_{k-1}, b_{k-1})/(\sum [a_i, b_i])$$

where $\rho(\alpha_i) = a_i$ and $\rho(\beta_i) = b_i$ with $\rho$ defined as

$$\rho\colon \pi_r(E) \cong \pi_{r-1}(\Omega E) \xrightarrow{Hur} H_{r-1}(\Omega E).$$

We then note that $H_*(\Omega E)$ is the universal enveloping algebra of the graded Lie algebra $L(a_1, b_1, \ldots, a_{k-1}, b_{k-1})/(\sum [a_i, b_i])$, where $L$ is the free Lie algebra functor. Now we apply the Poincaré–Birkhoff–Witt theorem as in [6, Proposition 3.6] to deduce the result. □

## 3. Stable homotopy type of the total space

In this section, we examine the possible stable homotopy types of the total space $E$ for a principal $SU(2)$-fibration over $M_k \in \mathcal{PD}^8_3$. We relate this to the stable homotopy type of $M_k$.

Let $f\colon M_k \to \mathbb{H}P^\infty$ be a map such that $\pi_4(f)\colon \pi_4(M_k) \to \pi_4(\mathbb{H}P^\infty) \cong \mathbb{Z}$ is surjective. This ensures that the homotopy fibre $E(f)$ is 3-connected and is a Poincaré duality complex of dimension 11. One easily deduces

$$H_i(E(f)) = \begin{cases} \mathbb{Z} & i = 0, 11 \\ \mathbb{Z}^{\oplus k-1} & i = 4, 7 \\ 0 & \text{otherwise} \end{cases}$$

from the Serre spectral sequence associated with the fibration $S^3 \to E(f) \to M_k$. We may now consider a minimal CW-complex structure on $E := E(f)$ with $(k-1)$ 4-cells, $(k-1)$ 7-cells, and one 11-cell, see [10, Section 2.2]. The 7-th skeleton $E^{(7)}$ is, therefore, a pushout

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

$$\begin{array}{ccc} (S^6)^{\vee(k-1)} & \longrightarrow & (D^7)^{\vee(k-1)} \\ {\scriptstyle \vee_{i=1}^{k-1}\phi_i}\downarrow & & \downarrow \\ (S^4)^{\vee(k-1)} & \longrightarrow & E^{(7)}. \end{array} \tag{3.1}$$

We now observe that the $\phi_i$ are all 0.

PROPOSITION 3.1. *The maps $\phi_i \simeq 0$ for $1 \leq i \leq k-1$.*

*Proof.* Note that the homotopy class of each of the attaching maps is in $\pi_6(S^4)^{\oplus k-1}$ which lies in the stable range. Applying the $\Sigma^\infty$ functor on the diagram (3.1), we get the cofibre sequence

$$\Sigma^\infty (S^6)^{\vee(k-1)} \xrightarrow{\Sigma^\infty(\vee_{i=1}^{k-1}\phi_i)} \Sigma^\infty (S^4)^{\vee(k-1)} \to \Sigma^\infty E^{(7)}$$

which in turn induces a long exact sequence (on the stable homotopy groups)

$$\cdots \to \pi_7^s(E^{(7)}) \to \pi_6^s(S^6)^{\oplus k-1} \xrightarrow{\Phi} \pi_6^s(S^4)^{\oplus k-1} \to \pi_6^s(E^{(7)}) \to \ldots \tag{3.2}$$

where $\Phi$ is the induced map of $\vee_{i=1}^{k-1}\phi_i$. We have the following commutative diagram

$$\begin{array}{ccccc} & & \pi_6^s(S^4)^{\oplus k-1} & \longrightarrow & \pi_6^s(E) \xleftarrow[\simeq]{} \pi_6^s(E^{(7)}) \\ & & \downarrow & & \downarrow \\ \pi_6^s(S^7) & \longrightarrow & \pi_6^s(S^4)^{\oplus k} & \longrightarrow & \pi_6^s(M_k). \\ \| & & & & \\ 0 & & & & \end{array}$$

The second row is a part of a long exact sequence and so, the map $\pi_6^s(S^4)^{\oplus k} \to \pi_6^s(M_k)$ is injective. Hence, the map $\pi_6^s(S^4)^{\oplus(k-1)} \to \pi_6^s(E)$ is injective, in (3.2) $\Phi$ is forced to be 0. The result follows. □

Proposition 3.1 implies that $E$ fits into the pushout

$$\begin{array}{ccc} S^{10} & \longrightarrow & D^{11} \\ {\scriptstyle L(E)}\downarrow & & \downarrow \\ (S^4 \vee S^7)^{\vee(k-1)} & \longrightarrow & E \end{array}$$

for some $[L(E)] \in \pi_{10}((S^4 \vee S^7)^{\vee(k-1)})$. Hence, $E$ belongs to $\mathcal{PD}^{11}_{4,7}$. We consider

$$S^{10} \xrightarrow{L(E)} (S^4 \vee S^7)^{\vee(k-1)} \to (S^7)^{\vee(k-1)}$$

which is of the form

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

$$\sum_{i=1}^{k-1} \lambda_i \iota_7^i \circ \nu_{(7)} \in \pi_{10}(S^7)^{\oplus k-1} \cong \bigoplus_{i=1}^{k-1} \mathbb{Z}/24\{\nu_{(7)}\}.$$

The coefficients $\lambda_i$ can be computed via the $e$-invariant, see [2]. Recall that the $e$-invariant of a map $g\colon S^{11} \to S^8$ can be computed using Chern character. The complex $K$-theoretic $e$-invariant $e_{\mathbb{C}}$ is computed via the diagram

$$\begin{array}{ccccccccc}
 & & \mathbb{Z}\{b_{12}\} & & & & \mathbb{Z}\{b_8\} & & \\
 & & \wr\| & & & & \wr\| & & \\
0 & \longrightarrow & \tilde{K}(S^{12}) & \longrightarrow & \tilde{K}(\Sigma C_g) & \longrightarrow & \tilde{K}(S^8) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\
0 & \longrightarrow & \tilde{H}^{ev}(S^{12};\mathbb{Q}) & \longrightarrow & \tilde{H}^{ev}(\Sigma C_g;\mathbb{Q}) & \longrightarrow & \tilde{H}^{ev}(S^8;\mathbb{Q}) & \longrightarrow & 0. \\
 & & \wr\| & & & & \wr\| & & \\
 & & \mathbb{Q}\{a_{12}\} & & & & \mathbb{Q}\{a_8\} & &
\end{array}$$

We obtain

$$ch(b_{12}) = a_{12}, \quad ch(b_8) = a_8 + r a_{12}.$$

If $g = \lambda\nu_{(7)}$, $e_{\mathbb{C}}(g) = r = \frac{\lambda}{12} \in \mathbb{Q}/\mathbb{Z}$, see [2]. We also have $e_{\mathbb{C}} = 2e$, where $e$ is computed using the Chern character of the complexification $c\colon KO \to K$. Therefore, from [2, Proposition 7.14]

$$b_8 \in Im(c) \implies e(g) = \frac{r}{2} \in \mathbb{Q}/\mathbb{Z}. \tag{3.3}$$

### 3.1. $K$-theory of $M_k$

Consider the Atiyah–Hirzebruch spectral sequence

$$E_2^{**} = H^*(M_k; \pi_* K) \implies K^*(M_k).$$

As $M$ has only even dimensional cells, this has no non-trivial differential for degree reasons. This gives the additive structure of $K^0(M_k)$. Let $H^4(M_k) \cong \mathbb{Z}\{\psi_1, \ldots, \psi_k\}$ and $H^8(M_k) \cong \mathbb{Z}\{z\}$. Note that if $\alpha_1, \ldots, \alpha_k \in \pi_4(M_k) \cong H_4(M_k) \cong \mathbb{Z}^k$ is dual to the basis $\psi_1, \ldots, \psi_k$, in the expression (1.1), the matrix $\big((g_{i,j})\big)$ of the intersection form is related to the cup product via the equation $\psi_i \cup \psi_j = g_{i,j} z$. Let $\tilde{\psi}_1, \ldots, \tilde{\psi}_k, and \tilde{z}$ be classes in $K(M_k)$ corresponding to $\psi_1, \ldots, \psi_k, and z$ respectively, in the $E^\infty$-page. We have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}^0(S^8) & \xrightarrow{q^*} & \tilde{K}^0(M_k) & \xrightarrow{i^*} & \tilde{K}^0((S^4)^{\vee k}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} & & \\
0 & \longrightarrow & \tilde{H}^{ev}(S^8;\mathbb{Q}) & \longrightarrow & \tilde{H}^{ev}(M_k;\mathbb{Q}) & \longrightarrow & \tilde{H}^{ev}((S^4)^{\vee k};\mathbb{Q}) & \longrightarrow & 0
\end{array}$$

where

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

$$ch(\tilde{z}) = z, \quad ch(i^*(\tilde{\psi_j})) = \psi_j( \implies ch(\tilde{\psi_j}) = \psi_j + \tau_j z), 1 \leq j \leq k. \tag{3.4}$$

Note that in terms of the formula (1.1), $\psi_i\psi_j = g_{i,j}z$. We now use the fact that $ch$ is a ring map to get

$$ch(\tilde{\psi_i}\tilde{\psi_j}) = (\psi_i + \tau_i z) \cup (\psi_j + \tau_j z) = \psi_i \cup \psi_j = g_{i,j}z.$$

As $ch\colon K(M_k) \to H^{ev}(M_k;\mathbb{Q})$ is injective, we deduce $\tilde{\psi_i}\tilde{\psi_j} = g_{i,j}\tilde{z}$.

Further let $q_i\colon (S^4)^{\vee k} \to S^4$ be the retraction onto the $i$-th factor, and note that $q_i \circ L(M)$ is stably equivalent to $(g_{i,i} - 2l_i)\alpha_i \circ \nu_{(7)}$. Thus the $e$-invariant

$$e_{\mathbb{C}}(q_i \circ L(M)) = (g_{i,i} - 2l_i)e_{\mathbb{C}}(\Sigma\nu_{(7)}) = \frac{g_{i,i} - 2l_i}{12} \in \mathbb{Q}/\mathbb{Z}.$$

We summarize these observations in the following proposition.

PROPOSITION 3.2. *Let $M_k \in \mathcal{PD}_3^8$ with $L(M_k)$ as in (1.1). Then, $K^0(M_k) \cong \mathbb{Z}\{1, \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_k, \tilde{z}\}$. The ring structure is given by*

$$\tilde{\psi_i}\tilde{\psi_j} = g_{i,j}\tilde{z} \quad \textit{for } 1 \leq i, j \leq k.$$

*and*

$$e_{\mathbb{C}}(q_i \circ L(M)) = \frac{g_{i,i} - 2l_i}{12} \in \mathbb{Q}/\mathbb{Z} \quad \textit{for } 1 \leq i \leq k.$$

### 3.2. $K$-theory of $E(f)$

The space $E := E(f)$ is the total space of the sphere bundle associated with the quaternionic line bundle classified by $f$. We note that the quaternionic line bundle has a complex structure, and therefore, has a $K$-orientation.

Let $\gamma_{\mathbb{H}}$ be the canonical $\mathbb{H}$-bundle over $\mathbb{H}P^\infty$. The $K$-theoretic Thom class of $\gamma_{\mathbb{H}}$ is given by $\gamma_{\mathbb{H}} - 2 \in \tilde{K}^0(\mathbb{H}P^\infty)$. As the total space of the sphere bundle is contractible, the Thom space $Th(\gamma_{\mathbb{H}}) \simeq \mathbb{H}P^\infty$. Consider the map $\pi\colon \mathbb{C}P^\infty \to \mathbb{H}P^\infty$. The pullback bundle $\pi^*(\gamma_{\mathbb{H}}) = \gamma_{\mathbb{C}} \oplus \bar{\gamma}_{\mathbb{C}}$, where $\gamma_{\mathbb{C}}$ is the canonical line bundle over $\mathbb{C}P^\infty$. Therefore,

$$\pi^* ch(\gamma_{\mathbb{H}} - 2) = ch(< pi_> < /pi_>^*(\gamma_{\mathbb{H}} - 2)) = ch(\gamma_{\mathbb{C}}) + ch(\bar{\gamma}_{\mathbb{C}}) - 2 = e^x + e^{-x} - 2,$$

where $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$ and $x = c_1(\gamma_{\mathbb{C}})$. Since $\pi^*$ is injective on $H^*$, we may use this formula to deduce

$$ch(\Phi_K(\gamma_{\mathbb{H}})) = \Phi_H(\gamma_{\mathbb{H}})(1 + \frac{y}{12} + \frac{y2}{360} + \ldots) \tag{3.5}$$

where $H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[y]$ and $\pi^*(y) = x^2$. We use the Thom isomorphism associated to $f^*(\gamma_{\mathbb{H}})$ to deduce the following.

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

PROPOSITION 3.3. *Assume that* $f^*(y) = \psi_k$. *Then,*

$$\tilde{K}^0(Th(f^*(\gamma_{\mathbb{H}}))) \cong \tilde{K}^0(M)\{\Phi_K(f^*\gamma_{\mathbb{H}})\}$$

*as a* $\tilde{K}^0(M)$*-module, and*

$$ch(\Phi_K(f^*\gamma_{\mathbb{H}})) = \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \frac{\psi_k^2}{360} + \dots).$$

*Proof.* From the naturality of the Chern character as well as the Thom class, we have

$$ch\Big(\Phi_K(f^*(\gamma_{\mathbb{H}}))\Big) = ch\Big(f^*(\Phi_K((\gamma_{\mathbb{H}}))\Big) = f^*ch(\Phi_K(\gamma_{\mathbb{H}})).$$

The result follows from (3.5) and the fact $f^*(y) = \psi_k$. □

NOTATION 3.1. Suppose we are in the situation of proposition 3.3, that is, $f^*(y) = \psi_k$. We now assume that the basis $\{\psi_1, \dots, \psi_k\}$ is such that one of the following cases occur

Case 1 If $\psi_k \cup \psi_k = \pm z$, then $\psi_j\psi_k = 0$ for $1 \leq j \leq k-1$.
Case 2 If $\psi_k \cup \psi_k = g_{k,k}z$ for some integer $g_{k,k} \neq \pm 1$, then assume $\psi_{k-1}\psi_k = 1$ and $\psi_j\psi_k = 0$ for $1 \leq j \leq k-2$.

In terms of these notations, we prove the following calculation. Here we consider the cofibre sequence

$$E \to M_k \to Th(f^*(\gamma_{\mathbb{H}})) \to \Sigma E \to \Sigma M \to \dots,$$

which demonstrates $K(\Sigma E)$ as a submodule of $K(Th(f^*\gamma_{\mathbb{H}}))$ because $K(\Sigma M_k) = 0$. The following proposition identifies this submodule.

PROPOSITION 3.4. (i) *Suppose we are in case 1 of notation 3.1, then we have*

$$\tilde{K}^0(\Sigma E) \cong \mathbb{Z}\{\Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_1, \dots, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_{k-1}, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{z}\}.$$

(ii) *Suppose we are in case* 2 *of notation 3.1, then*

$$\tilde{K}^0(\Sigma E) \cong \mathbb{Z}\Big\{\Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_1, \dots, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_{k-2}, \Phi_K(f^*(\gamma_{\mathbb{H}}))(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1}), \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{z}\Big\}.$$

*Proof.* We have the following short exact sequence

$$0 \longrightarrow \tilde{K}^0(\Sigma E) \longrightarrow \tilde{K}^0\Big(Th(f^*(\gamma_{\mathbb{H}}))\Big) \xrightarrow{s_0} \tilde{K}^0(M) \longrightarrow 0$$

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

which implies that

$$\tilde{K}^0(\Sigma E) = \mathrm{Ker}(s_0 \colon \tilde{K}^0\Big(Th(f^*(\gamma_{\mathbb{H}}))\Big) \to \tilde{K}^0(M))$$

where $s_0$ is the restriction along the zero section. Note that $s_0$ is a $\tilde{K}^0(M)$-module map. Hence,

$$\begin{aligned} s_0(\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{z}) &= e_K(f^*\gamma_{\mathbb{H}})\tilde{z} = 0 \\ s_0(\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_i) &= e_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_i = g_{ik}\tilde{z}, 1 \leq i \leq k \end{aligned}$$

since $e_K(f^*\gamma_{\mathbb{H}}) = \tilde{\psi}_k + m\tilde{z}$ for some $m$. The result follows from a direct calculation of the kernel and the assumptions in the respective cases. □

We now choose the various maps $\chi_j \colon S^7 \to E$ for $1 \leq j \leq k-1$ such that on $K$-theory they precisely represent the choice of the first $(k-1)$-elements in the basis of proposition 3.4. In these terms, we calculate the $e_{\mathbb{C}}$-value of the composite

$$r_j \circ L(E) \colon S^{10} \overset{L(E)}{\to} (S^4 \vee S^7)^{\vee k-1} \overset{r_j}{\to} S^7$$

where $r_j$ is the restriction onto the $j$-th factor, and $L(E)$ is the attaching map of the top cell of $E$.

Proposition 3.5. (i) *If we are in case* 1*, then*

$$e_{\mathbb{C}}(r_j \circ L(E)) = \tau_j = \frac{g_{j,j} - 2l_j}{12} \in \mathbb{Q}/\mathbb{Z}, 1 \leq j \leq k-1.$$

(ii) *If we are in case* 2*, then*

$$\begin{aligned} e_{\mathbb{C}}(r_j \circ L(E)) &= \tau_j = \frac{g_{j,j} - 2l_j}{12} \in \mathbb{Q}/\mathbb{Z}, 1 \leq j \leq k-2; \\ e_{\mathbb{C}}(r_{k-1} \circ L(E)) &= \tau_k - g_{k,k}\tau_{k-1} \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

*Proof.* For the $e$-invariant, we calculate the Chern character $ch \colon \tilde{K}^0(Th(f^*\gamma_{\mathbb{H}})) \to H^{ev}(Th(f^*\gamma_{\mathbb{H}}); \mathbb{Q})$ in terms of (3.4) as follows

$$\begin{aligned} ch(\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{z}) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)z = \Phi_H(f^*\gamma_{\mathbb{H}})z, \\ ch(\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_j) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)(\psi_j + \tau_j z) \\ &= \Phi_H(f^*\gamma_{\mathbb{H}})\psi_j + \tau_j\Phi_H(f^*\gamma_{\mathbb{H}})z, \\ ch\Big(\Phi_K(f^*\gamma_{\mathbb{H}})(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1})\Big) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)\,(\psi_k + \tau_k z - g_{k,k}\psi_{k-1} \\ &\quad - g_{k,k}\tau_{k-1}z) \\ &= \Phi_H(f^*\gamma_{\mathbb{H}})(\psi_k - g_{k,k}\psi_{k-1}) + \Phi_H(f^*\gamma_{\mathbb{H}}) \\ &\quad \times (\tau_k - g_{k,k}\tau_{k-1})z. \end{aligned}$$

□

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

We now turn our attention to the attaching map

$$S^{10} \xrightarrow{L(E)} (S^4 \vee S^7)^{\vee k-1} \to (S^7)^{\vee k-1}.$$

In order to identify the composite, we are required to compute the $KO$-theoretic $e$-invariant. We know that

$$KO^* = \mathbb{Z}[\eta, u][\mu^{\pm 1}]/(2\eta, \eta^3, \eta u, u^2 - 4\mu) \quad \text{and} \quad K^* = \mathbb{Z}[\beta^{\pm 1}],$$

with $|\eta| = -1$, $|u| = -4$, $|\mu| = -8$ and $|\beta| = -2$. The complexification map $c\colon KO \to K$ induces a graded ring homomorphism $c\colon KO^*(X) \to K^*(X)$ with

$$c(\eta) = 0, \quad c(u) = 2\beta^2, \quad c(\mu) = \beta^4.$$

THEOREM 3.6. *Let $S^3 \to E \to M_k$ be the principal* SU(2)*-fibration classified by a given map $f\colon M_k \to \mathbb{HP}^\infty$. Suppose that $\Sigma L(M_k) \equiv 0 \pmod{\lambda}$. Then $\Sigma(r_j \circ L(E)) \equiv 0 \pmod{\lambda}, 1 \leq j \leq k-1$ where $r_j : (S^4 \vee S^7)^{\vee k-1} \to S^7$ is the retraction onto the* j*-th factor.*

*Proof.* We consider the Atiyah–Hirzebruch spectral sequences for $\mathbb{H}P^\infty$ and $M_k$

$$E_2^{*,*} = H^*(\mathbb{H}P^\infty; \mathbb{Z}) \otimes KO^*(pt) \implies KO^*(\mathbb{HP}^\infty),$$
$$E_2^{*,*} = H^*(M_k; \mathbb{Z}) \otimes KO^*(pt) \implies KO^*(M_k).$$

The spectral sequences have no non-trivial differentials for degree reasons. Thus, $KO^*(\mathbb{HP}^\infty) \cong KO^*[\hat{y}]$, where $\hat{y} \in KO^4(\mathbb{H}P^\infty)$. The class $\hat{y}$ serves as $KO$-theoretic Thom class for $\gamma_{\mathbb{H}}$. We have

$$c(\Phi_{KO}(\gamma_{\mathbb{H}})) = c(\hat{y}) = \beta^{-2}\Phi_K(\gamma_{\mathbb{H}}).$$

Let $\hat{\psi}_j \in \tilde{KO}^4(M_k)$, $\hat{z} \in \tilde{KO}^8(M_k)$ be the class in the $E^\infty$-page represented by $\psi_j \in H^4(M_k)$ and $z \in H^8(M_k)$. Then we get

$$c(\hat{\psi}_i) = \beta^{-2}\tilde{\psi}_i, \quad c(\hat{z}) = \beta^{-4}\tilde{z} \quad \text{and} \quad \hat{\psi}_i\hat{\psi}_j = g_{i,j}\hat{z}.$$

It follows that the $K$-theoretic generators $\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_j$ and $\Phi_K(f^*\gamma_{\mathbb{H}})(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1})$ lie in the image of the map $c$. Therefore by (3.3), we get in case 2 that

$$e(r_j \circ L(E)) \equiv \begin{cases} \frac{g_{j,j}-2l_j}{24} & (\text{mod } \mathbb{Z}), \quad \text{for } j < k-1, \\ \frac{(g_{kk}-2l_k)-g_{k,k}(g_{k-1,k-1}-2l_{k-1})}{24} & (\text{mod } \mathbb{Z}), \quad \text{for } j = k-1; \end{cases}$$

and in case 1 that

$$e(r_j \circ L(E)) \equiv \frac{g_{j,j} - 2l_j}{24} \pmod{\mathbb{Z}}, \quad \text{for } j \leq k-1.$$

The result now follows from proposition 3.2. □

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

## 4. $SU(2)$-bundles over even complexes

We study the homotopy type of $E \in \mathcal{PD}^{11}_{4,7}$, the total space of a principal $SU(2)$-bundle over $M_k \in \mathcal{PD}^8_3$. In this section, we consider the case where the intersection pairing $\langle -, - \rangle \colon H_4(M_k) \times H_4(M_k) \to \mathbb{Z}$ is even i.e. $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in H_4(M_k)$. Note that in this case, $k$ must be even. We observe the following:

(i) If $k \geq 4$, $M_k$ supports a principal $SU(2)$-bundle whose total space $E$ is 3-connected.
(ii) The possible stable homotopy types of $E$ can be determined directly from the stable homotopy type of $M_k$ and the intersection form. In this regard, the formulas in theorem 3.6 are used to demonstrate this connection.

### 4.1. Existence of $SU(2)$-bundles

We discuss the existence of principal $SU(2)$-bundle over $M_k \in \mathcal{PD}^8_3$ with an attaching map as in (1.1) whose intersection form is even. If $\mathrm{Rank}(H_4(M)) = 2$, then up to isomorphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the only possible matrix for the intersection form. The attaching map $L(M) \in \pi_7(S^4 \vee S^4)$ of $M$ is of the form

$$L(M) = [\alpha_1, \alpha_2] + l_1 \nu_1' + l_2 \nu_2', \tag{4.1}$$

where $l_1, l_2 \in \mathbb{Z}/12$. For a principal bundle $SU(2) \to E \to M_k$ where $E$ is 3-connected, the classifying map $f_E \colon M_k \to \mathbb{H}P^\infty$ is such that $\pi_4(f)$ is surjective. Suppose $\alpha_i \mapsto n_i \iota_4, i = 1, 2$ such that $\gcd(n_1, n_2) = 1$ and $f \circ L(M) \simeq *$. If both $l_1$ and $l_2$ are odd, no such $n_1$ and $n_2$ exists. For, the equation $f \circ L(M) \simeq *$ translates to $n_1 n_2 + n_1 + n_2 \equiv 0 \pmod{12}$ [7, Example 4.13]. However, for $k \geq 4$, one may always construct a suitable map.

Proposition 4.1. *Suppose $M_k \in \mathcal{PD}^8_3$ such that Rank$(H_4(M_k)) = k \geq 4$ and the intersection form is even. Then there exists a map $\psi \colon M_k \to \mathbb{H}P^\infty$ such that hofib$(\psi)$ is 3-connected.*

*Proof.* From [7, Theorem 4.20], if $k \geq 6$, the attaching map of $M_k$ can be expressed as

$$L(M_k) = \sum_{1 \leq i < j \leq k} g_{i,j} [\alpha_i, \alpha_j] + \sum_{1=1}^{k} \frac{g_{i,i}}{2} [\alpha_i, \alpha_i] + \sum_{i=1}^{k} s_i \nu_i'$$

such that $s_i = 0$ for $i \geq 2$ for a choice of basis $\{\alpha_1, \ldots, \alpha_k\}$ of $H_4(M_k)$. Then the map $\tilde{\psi} \colon (S^4)^{\vee k} \xrightarrow{(0,0,\ldots,0,1)} \mathbb{H}P^\infty$ extends to a map $\psi \colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(\psi)$ is surjective.

Now if $k = 4$, by the classification of inner product spaces in low dimensions [11, Ch. II, Theorem (2.2)], the attaching map of $M_k$ can be expressed as

$$L(M_k) = [\alpha_1, \alpha_2] + [\alpha_3, \alpha_4] + \sum_{i=1}^{4} l_i \nu_i'$$

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

for a choice of basis of $H_4(M_k)$. Choose two positive integers $m, n$ such that $\gcd(m,n) = 1$ and $ml_1 + nl_3 \equiv 0 \pmod{12}$. Then the map $\tilde{\psi}\colon (S^4)^{\vee 4} \xrightarrow{(m,0,n,0)} \mathbb{H}P^\infty$ extends to a map $\psi\colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(\psi)$ is surjective. □

We now focus on the stable homotopy type of the total space $E(f)$ for $f\colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(f)$ is injective. From the attaching map of $M_k$ as in (1.1) and even intersection form, we have

$$\Sigma^\infty M_k \simeq \Sigma^\infty (S^4)^{\vee k-1} \vee \Sigma^\infty (Cone(\sigma(M_k)\nu_{(7)}) \tag{4.2}$$

for some even $\sigma(M_k)$. Hence the stable homotopy type of $M_k$ is determined by $\sigma(M_k)$.

Proposition 4.2. *Let $E(f_\psi)$ be the total space of a principal* SU(2)*-bundle over $M_k \in \mathcal{PD}_3^8$, classified by a map $f_\psi\colon M_k \to \mathbb{H}P^\infty$ for $k \geq 4$. Then*

$$\Sigma^\infty E(f_\psi) \simeq \Sigma^\infty (S^4 \vee S^7)^{\vee k-2} \vee \Sigma^\infty Cone(\lambda(\psi)\Sigma^4 \nu_{(7)}),$$

*where $\lambda(\psi) := \lambda_s(E(f_\psi))$, is even and a multiple of $\sigma(M_k)$.*

*Proof.* Note that $E(f_\psi) \in \mathcal{PD}_{4,7}^{11}$ and its stable homotopy type is given in the Corollary 2.5 where $\epsilon = \epsilon_s(E(f_\psi)) \in \mathbb{Z}/2$ and $\lambda(\psi) \in \mathbb{Z}/24$. So, it suffices to show that $2|\lambda(\psi)$ and $\epsilon = 0$. The fact $2|\lambda(\psi)$ follows from (4.2) and theorem 3.6.

Now, the cofibre sequence obtained from cell structure of $E$ and $M_k$ induces the following commutative diagram

$$\begin{array}{ccccc}
\pi_{10}^s(S^{10}) & \xrightarrow{\phi_*} & \pi_{10}^s(S^4 \vee S^7)^{\oplus k-1} & \longrightarrow & \pi_{10}^s(E) \\
 & & \downarrow & & \downarrow \\
\pi_{10}^s(S^7) & \xrightarrow[0]{\pi_{10}^s(\sum^\infty L(M))} & \pi_{10}^s(S^4)^{\oplus k} & \longrightarrow & \pi_{10}^s(M_k)
\end{array}$$

of stable homotopy groups, where $\phi$ is the attaching map of top cell in $E$.

Since $\lambda(\psi)\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ x = 0$ in $\pi_{10}^s(E)$, its image in $\pi_{10}^s(M)$ is 0. Note that bottom left map $\pi_{10}^s(\sum^\infty L(M)) = 0$ because $2|\sigma(M_k)$ in (4.2) and hence the bottom right map is injective, where $\pi_{10}^s(S^4)^{\oplus k} \cong \mathbb{Z}/2\{\alpha_1 \circ \nu^2, \ldots, \alpha_k \circ \nu^2\}$. Since $2|\lambda(\psi)$, the middle vertical arrow sends $\Sigma^\infty\phi = \lambda(\psi)\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ \nu^2$ to $\epsilon\alpha_{k-1} \circ \nu^2$ which is in turn mapped to 0 via the bottom right map as $\Sigma^\infty\phi = 0 \in \pi_{10}^s(E)$. Hence $\epsilon = 0 \in \mathbb{Z}/2$. □

### 4.2. Stably trivial manifolds

The following result states that the total space of a principal $SU(2)$-bundle over stably trivial $M_k \in \mathcal{PD}_3^8$ ( i.e., $\sigma(M_k) \equiv 0 \pmod{24}$) is itself a connected sum of copies of $S^4 \times S^7$.

Proposition 4.3. *Let $E \in \mathcal{PD}_{4,7}^{11}$ be the total space of a principal* SU(2)*-bundle over stably trivial $M_k \in \mathcal{PD}_3^8$. Then $E \simeq \#^{k-1}(S^4 \times S^7)$.*

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

*Proof.* From theorem 2.6, we have $E \simeq \#^{k-2} E_{0,0,0} \# E_{0,\epsilon,\delta}$. Then the result follows from proposition 4.2 for $k \geq 4$. For $k = 2$, the attaching map of $M_2$ is of the form (4.1). Stably trivial condition implies $M_2 = S^4 \times S^4$ which further implies $E = S^4 \times S^7$. □

### 4.3. Possible stable homotopy types of the total space

Let $\psi \in H^4(M_k; \mathbb{Z})$ be a cohomology class represented by a map $\overline{\psi}\colon M_k \to K(\mathbb{Z}, 4)$, which has a unique lift $\tilde{\psi}\colon M_k \to S^4$ up to homotopy if $\overline{\psi}|_{(S^4)^{\vee k-1}} \circ L(M_k) \in \pi_7(S^4)$ is 0. As the inclusion $(S^4)^{\vee k} \to M_k$ induces an isomorphism on $H_4$ and $H^4$, a cohomology class $\psi \in H^4(M_k)$ always induces a map $\tilde{\psi}\colon (S^4)^{\vee k} \to S^4$. We consider the following diagram

$$\begin{array}{ccc} (S^4)^{\vee k} & \longrightarrow & M_k \\ \downarrow{\scriptstyle \tilde{\psi}} & & \vdots \\ S^4 & \longrightarrow & \mathbb{H}P^2 \end{array}$$

and formulate when the map $M_k \to \mathbb{H}P^2$ exists. Note that if the map exists then its homotopy fibre will be 3-connected.

For $\psi \in H^4(M_k)$, consider the composite

$$S^7 \xrightarrow{L(M_k)} (S^4)^{\vee k} \xrightarrow{\tilde{\psi}} S^4$$

and define $\tau(\psi) = [\tilde{\psi} \circ L(M)] \in \pi_7^s(S^4)$. Thus

$$\tau\colon H^4(M_k) \to \mathbb{Z}/24$$

and one can check that it is a linear map.

Proposition 4.4. *Suppose $\psi \in H^4(M_k)$ is primitive. The map $\tilde{\psi}\colon (S^4)^{\vee k} \to S^4$ extends to a map $f_\psi : M_k \to \mathbb{H}P^2$ if and only if $\psi \cup \psi \equiv \tau(\psi) z \pmod{24}$ for some $z \in H^8(M_k)$.*

*Proof.* Consider a primitive element $\psi \in H^4(M_k)$. We extend $\psi$ to a basis of $H^4(M_k)$ and use the dual basis of $\pi_4(M_k)$ to write down the attaching map of $M_k$ is as in (1.1). In this notation, we have $\psi^2 = g_{k,k} z$, where $z \in H^8(M_k)$ is the chosen generator and $\tau(\psi) = g_{k,k} - 2l_k$. Thus, $\tilde{\psi} \circ L(M)$ maps to $0 \in \pi_7(\mathbb{H}P^2)$ if and only if $l_k = 0$, that is $\psi \cup \psi \equiv \tau(\psi) z \pmod{24}$. Hence, the result follows. □

Proposition 4.4 gives us criteria for constructing the maps $f_\psi$ out of cohomology classes $\psi$. Using this we determine which multiples of $\sigma(M_k)$ may occur as $\lambda_s(E)$ for $E \to M_k$ a principal $SU(2)$-bundle where $E$ is 3-connected. We first show that there exists $f_\psi$ such that $\lambda(\psi) = \sigma(M_k)$ if $k$ is large enough.

Proposition 4.5. *Suppose the stable homotopy type of* $\mathrm{M_k}$ *is determined by $\sigma(M_k)$. Then there exists $f_\psi\colon M_k \to \mathbb{H}P^\infty$ such that*

(1) $\lambda(\psi) \equiv \sigma(M_k) \pmod{3}$ *for $k \geq 5$,*

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

(2) $\lambda(\psi) \equiv \sigma(M_k) \pmod 8$ *for* $k \geq 7$.

*Proof.* We begin the proof with the first case. If $\tau \equiv 0 \pmod 3$, the proof follows from theorem 3.6. Let $\tau \not\equiv 0 \pmod 3$ and $\psi_0$ a primitive cohomology class such that $\tau(\psi_0) \equiv \sigma(M_k) \not\equiv 0 \pmod 3$. We need to choose a $\psi \in \ker(- \cup \psi_0) \cap \ker(\tau)$ such that $\psi^2 \equiv \tau(\psi) \equiv 0 \pmod 3$. This we can do for $k-2 \geq 3$, see [11, Chapter II, (3.2)–(3.4)]. By Poincaré duality, we get $\psi'$ such that $\psi \cup \psi' = z$. We write $\psi' = \psi'_{\ker(\tau)} + t\psi_0$ for some $t$ where $\psi'_{\ker(\tau)} \in \ker(\tau)$. Since $z = \psi' \cup \psi = \psi'_{\ker(\tau)} \cup \psi$, we may assume $\psi' = \psi'_{\ker \tau}$. Thus by assigning $\tau_1 = \tau(\psi_0) \equiv \sigma(M_k) \pmod 3$, $\tau_{k-1} = \tau(\psi') \equiv 0 \pmod 3$ and $\tau_k = \tau(\psi) \equiv 0 \pmod 3$; and choosing other $\psi_i$'s such that $\tau_i = 0$ for $i = 2, \ldots, k-2$, we have

$$\lambda(\psi) \equiv \sigma(M_k) \pmod 3 \qquad \text{for } k \geq 5.$$

Now we look into the second case. The proof goes similarly to that of the above, except when we choose $\psi \in \ker(-\cup\psi_0)\cap\ker(\tau)$ such that $\psi^2 \equiv \tau(\psi) \equiv 0 \pmod 8$. We can choose such $\psi$ for $k-2 \geq 5$, see [11, Chapter II, (3.2)–(3.4)]. Then following similar arguments, one can deduce

$$\lambda(\psi) \equiv \sigma(M_k) \pmod 8 \qquad \text{for } k \geq 7.$$

□

The following theorem constructs $f_\psi$ with $\lambda(\psi) = 3\sigma(M_k)$ if $\sigma(M_k)$ is not divisible by 3.

THEOREM 4.6. *Suppose* $3 \nmid \sigma(M_k)$. *Then for* $k \geq 7$, *there exists* $f_\psi \colon M_k \to \mathbb{H}P^2$ *such that* $\lambda(\psi) = 3\sigma(M_k)$.

*Proof.* Let

$$\tau^{(3)} \colon H^4(M_k, \mathbb{F}_3) \to \mathbb{F}_3$$

be the restriction of $\tau$ in modulo 3. As $\tau^{(3)}$ is surjective, there exists $\psi_0 \in H^4(M_k, \mathbb{F}_3)$ such that for all cohomology class $\psi$, $\tau^{(3)}(\psi)z \equiv \psi \cup \psi_0 \pmod 3$. In particular, $\tau^{(3)}(\psi_0)z \equiv \psi_0 \cup \psi_0 \pmod 3$. We consider two cases, $\psi_0^2 = 0$ or $\psi_0^2$ is unit.

First let $\psi_0^2$ is unit. Then we can choose $\psi_1, \ldots, \psi_{k-1}$ such that $\psi_0 \cup \psi_i = 0$. We take the dual basis $\alpha_i$ corresponding to $\psi_i$ and $\alpha_k$ corresponding to $\psi_0$. Thus $\psi_0(\alpha_i) = 0$ for $i = 1, \ldots, k-1$ and $\psi_0(\alpha_k) = 1$. Hence

$$\tau_1 \equiv \cdots \equiv \tau_{k-1} \equiv 0 \pmod 3, \quad \text{and} \quad \tau_k \equiv g_{k,k} \equiv 1 \pmod 3,$$

which implies $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_{k-1}) \equiv 0 \pmod 3$.

Now let $\psi_0^2 = 0$. Then $\tau^{(3)}(\psi_0)z = 0$. We choose $\psi_1, \ldots, \psi_{k-1}$ such that $\psi_{k-1} \cup \psi_0 = 1$ and $\psi_i \cup \psi_0 = 0$ for $i = 1, \ldots, k-2$. After taking the dual basis, with similar argument, we have

$$\tau_1 \equiv \cdots \equiv \tau_{k-2} \equiv 0 \pmod 3, \quad \tau_{k-1} \equiv \sigma(M_k) \pmod 3, \quad \text{and}$$
$$\tau_k \equiv g_{k,k} \equiv 0 \pmod 3.$$

Hence $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_k - g_{k,k}\tau_{k-1}) \equiv 0 \pmod 3$. □

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

REMARK 4.7. We note that proposition 4.4, proposition 4.5, and theorem 4.6 do not use the fact that the intersection form is even and also holds in the case where the intersection form is odd.

Now we look to prove similar results modulo 8, which in turn provide us desired construction $\psi$ as in proposition 4.4 using the Chinese reminder theorem. However, in this case, certain conditions are required for obtaining analogous $f_\psi$.

DEFINITION 4.8.

- *A complex* $\mathrm{M_k}$ *with* $\sigma(M_k) = 2$ *or* $4$ *is said to satisfy hypothesis* $(H_8)$ *if* $(\ker \tau)^\perp = (\sigma(M_k)\psi) \pmod 8$, *where* $\psi \in H^4(M_k)$ *(which is unique* $\pmod{\frac{8}{\sigma(M_k)}}$*) satisfies*

$$\begin{cases} \psi^2 \equiv 0 \pmod 8 & \text{if } \sigma(M_k) = 2 \\ \psi^2 \equiv 0 \pmod 4 & \text{if } \sigma(M_k) = 4. \end{cases}$$

- *A complex* $\mathrm{M_k}$ *with* $\sigma(M_k) = 2$ *is said to satisfy hypothesis* $(H_4)$ *if* $(\ker \tau)^\perp = (2\psi) \pmod 4$, *where* $\psi \in H^4(M_k)$ *(which is unique* $\pmod 2$*) satisfies*

$$\psi^2 \equiv \tau(\psi) \equiv 0 \text{or } 4 \pmod 8.$$

Note that the hypotheses $(H_8)$ and $(H_4)$ depend only on the intersection form and $\tau$ and not on the choice of $\psi$. We now prove the existence of $f_\psi$ under the hypothesis defined above.

THEOREM 4.9.

(i) *Suppose* $8 \nmid \sigma(M_k)$. *For* $k \geq 5$, *there exists* $f_\psi \colon M_k \to \mathbb{H}P^2$ *such that* $\lambda(\psi) \equiv 0 \pmod 8$ *if and only if the complex satisfies hypothesis* $(H_8)$.
(ii) *Suppose* $\sigma(M_k) = 2$. *Then for* $k \geq 5$, *there exists* $f_\psi \colon M_k \to \mathbb{H}P^2$ *such that* $\lambda(\psi) \equiv 4 \pmod 8$ *if and only if the complex satisfies hypothesis* $(H_4)$.

*Proof.* The condition $k \geq 5$ comes from the fact that we are required to make certain choices modulo 3 using proposition 4.5. First suppose in case 1, a $f_\psi$ exists such that $\lambda(\psi) = 8\sigma(M_k)$. Then there exists a basis $\{\psi_1, \ldots, \psi_{k-2}, \psi', \psi\}$ satisfying (working $\pmod 8$)

$$\begin{aligned} &\psi \cup \psi_i = 0 \text{for } 1 \leq i \leq k-2, \qquad &&\psi \cup \psi' = 1, \\ &\tau(\psi_i) = 0 \text{for } 1 \leq i \leq k-2, \qquad &&\tau(\psi') = \sigma(M_k) \quad \text{and} \quad \tau(\psi) = \psi^2 = 0. \end{aligned} \tag{4.3}$$

Note that $\langle\psi\rangle^\perp = \langle\psi_1, \ldots, \psi_{k-2}, \psi\rangle \subset \ker(\tau) = \langle\psi_1, \ldots, \psi_{k-2}, \psi, \frac{8}{\sigma(M_k)}\psi'\rangle$. This implies $(\ker(\tau))^\perp = (\sigma(M_k)\psi)$, and thus the hypothesis $(H_8)$ is satisfied.

For the converse part if the complex satisfies hypothesis $(H_8)$, one can check that there is a choice of $\psi$ such that (4.3) is satisfied. We look into the cases $\sigma(M_k) = 2, 4$.

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

First, let $\sigma(M_k) = 2$. Then $(\ker(\tau))^{\perp} = (2\psi)$ and $\psi$ is well-defined modulo 4. We note that

$$\chi \cup (2\psi) \equiv \tau(\chi)z \pmod 8 \ \forall \chi \in H^4(M_k). \tag{4.4}$$

This implies

$$\tau(\psi) \equiv 2\psi^2 \equiv 0 \pmod 8.$$

For any $\chi \in \ker(\tau)$, we have $(2\psi)\chi = \tau(\chi) \equiv 0 \pmod 8$. In particular, $2\psi^2 = \tau(\psi) \equiv 0 \pmod 8$. Together these two imply $\psi^2 \equiv 0 \pmod 8$. Eq. (4.4) implies that if $\chi \cup \psi = 0$, $\tau(\chi) = 0$. Now choosing a basis as in case 2 of proposition 3.5, we obtain the conditions in (4.3).

Now let $\sigma(M_k) = 4$. Then $(\ker(\tau))^{\perp} = (4\psi)$ and $\psi$ is determined modulo 2. We proceed analogously observing that

$$\chi \cup (4\psi) \equiv \tau(\chi)z \pmod 8 \ \forall \chi \in H^4(M_k),$$

which implies

$$\tau(\psi) \equiv 4\psi^2 \equiv 0 \pmod 8.$$

Let $\psi'$ be such that $\tau(\psi') = 4$, and so we have that $\psi \cup \psi'$ is an odd multiple of $z$. If $\psi^2$ is 4 (mod 8), we change $\psi$ to $\psi + 2\psi'$ to ensure $\psi^2 = \tau(\psi) \equiv 0 \pmod 8$. Now choosing a basis as in case 2 of proposition 3.5, we obtain the conditions in (4.3).

Case 2 also proceeds along analogous lines. For the existence (of $f_\psi$ for some $\psi$) question, we need a basis satisfying

$$\begin{aligned}
&\psi \cup \psi_i = 0 \text{ for } 1 \le i \le k-2, \quad \psi \cup \psi' = 1,\\
&\tau(\psi_i) \equiv 0 \pmod 4 \text{ for } 1 \le i \le k-2, \quad \tau(\psi') = \sigma(M_k), \quad \tau(\psi) = \psi^2 \equiv 0 \text{ or } 4 \pmod 8,\\
&\text{such that at least one of } \tau(\psi_i) \text{ for } 1 \le i \le k-2, \text{ or } \tau(\psi) \equiv 4 \pmod 8.
\end{aligned} \tag{4.5}$$

Note that $\langle\psi\rangle^{\perp} = \langle\psi_1, \ldots, \psi_{k-2}, \psi\rangle \subset \ker(\tau) = \langle\psi_1, \ldots, \psi_{k-2}, \psi, 2\psi'\rangle \pmod 4$ and $(\ker(\tau))^{\perp} = (2\psi) \pmod 4$. Hence, the hypothesis $(H_4)$ is satisfied.

Conversely, if $(H_4)$ is satisfied, we obtain a $\psi$ such that $\psi^2 \equiv \tau(\psi) \equiv 0 \text{or } 4 \pmod 8$. This $\psi$ also satisfies

$$\chi \cup (2\psi) \equiv \tau(\chi)z \pmod 4 \ \forall \chi \in H^4(M_k).$$

Let $\psi'$ be such that $\tau(\psi') = 2$, which implies that $\psi \cup \psi'$ is an odd multiple of $z$. Now replacing $\psi$ by $\psi + 2\psi'$ if required we may assume that $\psi^2 \equiv \tau(\psi) \equiv 4 \pmod 8$. Now choosing a basis as in case 2 of proposition 3.5, we obtain the conditions in (4.5). □

If $\text{Rank}(H_4(M_k)) = k \ge 5$, the above results indicate a systematic computation of possible stable homotopy types of the total space depending on $k$, $\sigma(M_k)$, and the intersection form. In lower rank cases, the results depend on the explicit formula for the attachment $L(M_k)$, and not just on these variables. Hence the systematic

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

description turns out to be cumbersome. We demonstrate some observations on the $\mathrm{Rank}(H_4(M)) = 2$ case.

EXAMPLE 4.1. Recall that the attaching map $L(M) \in \pi_7(S^4 \vee S^4)$ of $M_2$ is of the form

$$L(M) = [\alpha_1, \alpha_2] + l_1 v_1' + l_2 v_2'$$

where $l_1, l_2 \in \mathbb{Z}/12$. We already have

- If both $l_1$ and $l_2$ are odd, there does not exist $f\colon M \to \mathbb{H}P^\infty$ such that $E = \mathrm{hofib}(f)$ is 3-connected.

If one of $l_1$ and $l_2$ is even, or both are even, there exists $f\colon M \to \mathbb{H}P^\infty$ such that $E = \mathrm{hofib}(f)$ is 3-connected. Via an explicit calculation using proposition 3.5, we observe the following:

(i) If none of $l_1$ and $l_2$ are divisible by 3, then we obtain $\lambda(E) \equiv 0 \pmod 3$.
(ii) If either of $l_1$ or $l_2$ but not both are divisible by 3, then $\lambda(E) \not\equiv 0 \pmod 3$.
(iii) If $\sigma(M) \equiv 4 \pmod 8$ and $l_1 l_2 \equiv 0 \pmod 8$, where none of $l_1$ and $l_2$ are divisible by 3, then $\lambda(E) \equiv 0 \pmod 8$.
(iv) If $\sigma(M) \equiv 2 \pmod 8$, we can never obtain $\lambda(E) \equiv 0, 4 \pmod 8$.

## 5. $SU(2)$-bundles over odd complexes

We now work out the case of $M_k \in \mathcal{PD}_3^8$ for which the intersection form is odd. Recall that the notation $M_k$ means that $\mathrm{Rank}(H_4(M_k)) = k$. The intersection form being odd implies that there are two possibilities of $\sigma(M_k)$, namely, 1 and 3 among the divisors of 24. Here, we prove

(i) For $k \geq 3$, it is possible to obtain a $SU(2)$-bundle whose total space is 3-connected.

(ii) Further if $k \geq 7$, it is possible to obtain maps $\psi(j) : M_k \to \mathbb{H}P^\infty$ with $\lambda(\psi(j)) = j$ for every multiple $j \pmod{24}$ of $\sigma(M_k)$ which is also a divisor of 24.

### 5.1. Existence of $SU(2)$-bundles

Through an explicit computation, we demonstrate the existence of principal $SU(2)$-bundles over $M_k \in \mathcal{PD}_3^8$ if $k \geq 3$ when the intersection form is odd.

PROPOSITION 5.1. *Suppose $M_k \in \mathcal{PD}_3^8$ such that Rank$(H_4(M_k)) = k \geq 3$ and the intersection form is odd. Then there exists a map $\psi\colon M_k \to \mathbb{H}P^\infty$ such that hofib$(\psi)$ is 3-connected.*

*Proof.* Recall that the attaching map of $M_k$ can be expressed as (1.1). Using proposition 4.4, we are required to find a primitive element $\psi$ such that $\tau(\psi)z \equiv \psi^2 \pmod{24}$, which is equivalent to checking that the coefficient of $\nu'$ in $\tilde{\psi} \circ L(M_k)$ is $0 \pmod{12}$. It suffices to find $\psi \pmod 8$ and $\psi \pmod 3$ separately.

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

We first work out the $\pmod 3$ case, where the base ring is a field of characteristic $\neq 2$ so that the form is diagonalizable. Considering the map $(S^4)^{\vee k} \xrightarrow{(0,\ldots,0,n_1,n_2,n_3)} \mathbb{H}P^\infty$ which sends $L(M_k)$ to

$$\left( \pm \binom{n_1}{2} \pm \binom{n_2}{2} \pm \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k \right) \cdot \nu' + \text{multiple of } \nu,$$

where the $\pm$ correspond to the diagonal entries $\pmod 3$. We observe through a direct calculation that for every fixed choice $\epsilon_1, \epsilon_2, \epsilon_3$ of $\pm 1$s, there is a $n_1, n_2, n_3$ with $\gcd(n_1, n_2, n_3) = 1$ such that

$$\left( \epsilon_1 \binom{n_1}{2} + \epsilon_2 \binom{n_2}{2} + \epsilon_3 \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k \right) \equiv 0 \pmod 3.$$

This completes the argument $\pmod 3$.

Working $\pmod 8$, the fact that the intersection form is odd implies that we may choose a basis such that $g_{k,k} \equiv \pm 1 \pmod 8$ and $g_{k,k-1} \equiv 0 \pmod 8$, see [11, Chapter II, (4.3)]. Also the intersection form can be written as the block matrix $\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$.

If $A$ is an even intersection form, then the result follows from proposition 4.1 for $k \geq 5$. Now let $A$ is not even, then the intersection form is $\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$ where $B^{'}$ is a diagonal matrix of order 2 with diagonal entries $\pm 1$. If $A^{'}$ is an even intersection form, then for $k \geq 6$ the result follows from proposition 4.1. If $A^{'}$ is not even, the intersection form updates to

$$\begin{pmatrix} A'' & 0 \\ 0 & B'' \end{pmatrix}$$

where $B^{''}$ is a diagonal matrix of order 3 with diagonal entries $\pm 1$. For $B'' = I_3$, the map $(S^4)^{\vee k} \xrightarrow{(0,\ldots,0,n_1,n_2,n_3)} \mathbb{H}P^\infty$ sends $L(M_k)$ to

$$\left( \binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k \right) \cdot \nu' + \text{multiple of } \nu,$$

where $\gcd(n_1, n_2, n_3) = 1$ ensures that the corresponding $\psi$ is primitive. We may directly compute and observe that the equations

$$\left( \binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k \right) \equiv 0 \pmod 8, \gcd(n_1, n_2, n_3) = 1,$$

have a common solution. A similar argument works for the other diagonal $\pm 1$ matrix choices for $B'''$. This proves the result for $k \geq 6$.

If $3 \leq k \leq 5$, we know from [11, Chapter II, (3.2)–(3.4)] that the form is a direct sum of $\pm 1$ and the hyperbolic form. The argument in the even case in proposition 4.1

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

implies the result for a sum of two hyperbolic forms, and the above argument implies the result for a sum of 3 $\pm 1$s. The remaining cases are taken care of if we show the result for the intersection form

$$\begin{pmatrix} H & 0 \\ 0 & \pm 1 \end{pmatrix}$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic matrix. We consider the map $(S^4)^{\vee 3} \xrightarrow{(n_1,n_2,n_3)} \mathbb{H}P^\infty$, and as mentioned above we need to find a common solution of

$$\binom{n_3}{2} + n_1 n_2 + \sum_{i=1}^{3} n_i l_i \equiv 0 \pmod 8, \gcd(n_1, n_2, n_3) = 1.$$

Through a direct calculation, we check that such solutions always exist. This completes the proof. □

The following example shows that proposition 5.1 does not extend to the $k = 2$ case.

Example 5.1. Consider $M_2$ such that

$$L(M_2) = \nu_1 + \nu_2 + l_1 \nu_1' + l_2 \nu_2', \quad l_1 \equiv l_2 \equiv 2 \pmod 3.$$

A map $M_2 \to \mathbb{H}P^\infty$ which restricts to $(n_1, n_2)$ on the 4-skeleton sends $L(M_2)$ to

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + n_1 l_1 + n_2 l_2\right) \cdot \nu' + \text{multiple of } \nu.$$

We may check directly that

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + n_1 l_1 + n_2 l_2\right) \equiv 0 \pmod 3 \implies n_1 \equiv n_2 \equiv 0 \pmod 3.$$

Therefore, there is no map $M_2 \to \mathbb{H}P^\infty$ whose homotopy fibre is 3-connected.

### 5.2. Possible stable homotopy type of the total space

We note from §4 that propositions 4.4, 4.5, and theorem 4.6 are also valid when the intersection form is odd. We check that all stable homotopy types are achievable in the odd case if the rank $k$ of $H_4(M)$ is $\geq 7$. Applying the results from §4, it only remains to check that the different possibilities $\pmod 8$ are achievable. We do this in the theorem below.

Theorem 5.2. *Suppose $\sigma(M_k)$ is odd. Then for $k \geq 5$ and $j \in \{0, 2, 4\}$, there exists $\psi\colon M_k \to \mathbb{H}P^2$ such that $\lambda(\psi) \equiv j \pmod 8$.*

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

*Proof.* We work $\pmod 8$, knowing that if $k \geq 5$, proposition 4.5 allows us to make a choice of $\psi$ so that $\lambda(\psi)$ is as required $\pmod 3$. The proof is very similar to the proof of theorem 4.6. If $\sigma(M_k)$ is odd, the linear map $\tau\colon (\mathbb{Z}/8)^k \to \mathbb{Z}/8$ is represented by some primitive class $\psi$ (that is, $\tau(\chi) = \langle \chi, \psi \rangle \pmod 8$, and $\mathbb{Z}\{\psi\}$ is a summand of $H^4(M_k)$). In particular, $\psi^2 = \tau(\psi)$. We have two cases.

First let $\tau(\psi)$ be odd, i.e. $\psi^2$ is unit in modulo 8. Then we can extend $\psi$ to a basis $\psi, \psi_1, \dots, \psi_{k-1}$ such that $\psi \cup \psi_i = 0$ for $1 \leq i \leq k-1$. We take the dual basis $\alpha_i$ corresponding to $\psi_i$ and $\alpha_k$ corresponding to $\psi$. Thus $\psi(\alpha_i) = 0$ for $i = 1, \dots, k-1$ and $\psi(\alpha_k) = 1$. Hence

$$\tau_1 \equiv \cdots \equiv \tau_{k-1} \equiv 0 \pmod 8, \quad \text{and} \quad \tau_k \equiv g_{k,k} \pmod 8,$$

which implies $\lambda(\psi) = \gcd(\tau_1, \dots, \tau_{k-2}, \tau_{k-1}) \equiv 0 \pmod 8$ by case 1 of proposition 3.5.

Now let $\psi^2$ be even. Extend $\psi$ to a basis $\psi, \psi_1, \dots, \psi_{k-1}$ such that $\psi_{k-1} \cup \psi = 1$ and $\psi_i \cup \psi = 0$ for $i = 1, \dots, k-2$. After taking the dual basis, with similar argument we have

$$\tau_1 \equiv \cdots \equiv \tau_{k-2} \equiv 0 \pmod 8, \quad \tau_{k-1} \equiv 1 \pmod 8, \quad \text{and} \quad \tau_k \equiv g_{k,k} \pmod 8.$$

Hence $\lambda(\psi) = \gcd(\tau_1, \dots, \tau_{k-2}, \tau_k - g_{k,k}\tau_{k-1}) \equiv 0 \pmod 8$ by case 2 of proposition 3.5. This proves the result for $j = 0$.

For $j = 2$, or 4, we can use $\psi(j) = \psi + j\psi_1$, and note that

$$\tau(\psi(j)) = \tau(\psi),\ \psi(j)^2 = \psi^2 + j^2\psi_1^2 \equiv \psi^2 \pmod 8.$$

The last equivalence comes from the fact that $\tau(\psi_1)z = \psi \cup \psi_1 \equiv 0 \pmod 8$, and so $\psi_1^2$ is forced to be an even multiple of $z$. We may now compute using the formulas of proposition 3.5 to conclude that $\lambda(\psi(j)) \equiv j \pmod 8$. □

As in the even case, when the rank is high enough we have a systematic idea of the possibilities of the total space. However, in the low rank cases ($k \leq 6$), the results are not systematic and may depend on individual cases rather than only on $\sigma(M_k), k$, and the intersection form.

## References


1 J. F. Adams. On the cobar construction. *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 409–412.
2 J. F. Adams. On the groups $J(X)$. IV. *Topology.* **5** (1966), 21–71.
3 D. Barden. Simply connected five-manifolds. *Ann. Math. (2)* **82** (1965), s365–s385.
4 S. Basu and S. Basu. Homotopy groups and periodic geodesics of closed 4-manifolds. *Int. J. Math.* **26** (2015), 1550059, 34.
5 S. Basu and S. Basu. Homotopy groups of highly connected manifolds. *Adv. Math.* **337** (2018), 363–416.
6 S. Basu and S. Basu. Homotopy groups of certain highly connected manifolds via loop space homology. *Osaka J. Math.* **56** (2019), 417–430.
7 S. Basu and A. K. Ghosh. Sphere fibrations over highly connected manifolds. *J. Lond. Math. Soc. (2).* **110** (2024), e70002, 38.
8 H. Duan and C. Liang. Circle bundles over 4-manifolds. *Arch. Math. (Basel).* **85** (2005), 278–282.


https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press

9 P. J. Giblin. Circle bundles over a complex quadric. *J. London Math. Soc.* **43** (1968), 323–324.
10 A. Hatcher. *Algebraic topology* (Cambridge University Press, Cambridge, 2002).
11 J. Milnor and D. Husemoller. Symmetric bilinear forms. *Of Ergebnisse der Mathematik und Ihrer Grenzgebiete [Results in mathematics and related areas]*, Vol. 73 (Springer-Verlag, New York-Heidelberg, 1973).
12 S. Smale. On the structure of 5-manifolds. *Ann. Math. (2)*. **75** (1962), 38–46.

https://doi.org/10.1017/prm.2025.12 Published online by Cambridge University Press