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Lipschitz classes and convolution approximation processes

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For a continuous function $f(x)$ on the reals or on the circle T (continuous and 2π periodic) we say that $f(x)$ belongs to the generalized Lipschitz class, denoted by $f\in \mathrm{Lip}^*\alpha$, if

$$\|\Delta_h^r f(x)\| \leqslant M(f)|h|^\alpha \quad \text{for some integer } r, \quad r>\alpha, \tag{1·1}$$

where $\Delta_h^r f(x)\equiv \Delta_h(\Delta_h^{r-1}f(x))$ and $\Delta_h f(x)=f(x+\frac{1}{2}h)-f(x-\frac{1}{2}h)$. For a convolution approximation process given by

$$A_n(f;x)=\int G_n(x-y)f(y)\,dy, \tag{1·2}$$

where

$$\int G_n(x)\,dx=1 \quad \text{and} \quad \|G_n\|_{L_1}\leqslant M_0, \tag{1·3}$$

we shall investigate equivalence relations between the asymptotic behaviour of $(d/dx)^r A_n(f,x)$ and $f\in \mathrm{Lip}^*\alpha$.

A well-known theorem of this type is the relation between the behaviour of $(\partial/\partial x)^r P_y * f(x)$ or $(\partial/\partial y)^r P_y * f$ where $P_y(x)$ is the Poisson kernel, and $f\in \mathrm{Lip}^*\alpha$ (see for example ((6), p. 141–146). This type of connection is also known for some other very well-behaved kernels. Conditions on the kernel G_n will be given so that results of the above type will be possible and examples will be produced to show the necessity of imposing the restrictions.

When the kernel is not smooth enough we will deal with its m-iterate, that is its m-convolution with itself:

$$\begin{aligned} A_n(A_n^{m-1}(f,\cdot),x) &\equiv A_n^m(f,x)\equiv G_n * \ldots * G_n * f(x) \\ &\equiv \int \ldots \int G_n(x-y_1)\ldots G_n(x-y_m)f(y_1+\ldots+y_m)\,dy_1\ldots dy_m. \end{aligned} \tag{1·4}$$

As the well-known relation for the Poisson kernel, mentioned already above, is valid for L_p and other Banach spaces, we shall investigate the present result in that direction too. We shall generalize our results to Banach spaces B that are contained and continuously embedded in the Schwartz space of tempered distributions $S'(R)$ (or $S'(T)$), for which translation $f(\cdot+a)$ which is defined by

$$\langle f(\cdot+a),\psi(\cdot)\rangle=\langle f(\cdot),\psi(\cdot-a)\rangle, \quad \forall\,\psi\in S$$

is an isometry, that is $\|f(\cdot+a)\|_B=\|f(\cdot)\|_B$ for $f\in B$, and, either for all $f\in B$ or at least for those in question, $\|\Delta_h f\|_B=o(1)$, $h\to 0$. Of course, the $\mathrm{Lip}^*\alpha$ class in (1·1) will now use the B norm rather than the C norm. Generalization to R^d and to T^d will also be achieved.

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To illustrate the result, one may look at the following application:
Let $F_n * f$ be the Cesàro summability of the Fourier series and F_n the Fejér kernel; then

$$\left\| \frac{d}{dx} F_n * f \right\| = O(n^{1-\alpha}) \quad \text{if and only if} \quad f \in \mathrm{Lip}^* \alpha \quad \text{for} \quad \alpha < 1.$$

Moreover

$$\left\| \left(\frac{d}{dx}\right)^2 F_n * F_n * f \right\| = O(n^{2-\alpha}) \quad \text{if and only if} \quad f \in \mathrm{Lip}^* \alpha \quad \text{for} \quad \alpha < 2.$$

We also observe that the result for the sequence A_n is slightly more general than the analogous result for $A_\rho(f,x) = G_\rho * f(x)$ with the continuous parameter ρ, since assumptions are made only on ρ_n, a subsequence of ρ.

2. *Some preliminary results*

We define the rth modulus of continuity by

$$(2\cdot1) \qquad \omega_r(f,h,B) = \sup_{0<t\leqslant h} \|\Delta_t^r f(\cdot)\|_B \quad \text{where} \quad \Delta_t f(\cdot) = f\left(\cdot + \frac{t}{2}\right) - f\left(\cdot - \frac{t}{2}\right)$$

and we can write the following elementary result.

LEMMA 2·1. *Let B be a Banach space of generalized functions on R or T such that $\|f(\cdot + a)\|_B = \|f(\cdot)\|_B$. Then there exists $f_{h,r} \in B$ such that $f_{h,r}^{(r)}$ exists in B as a strong derivative whenever $\|\Delta_t f\|_B = o(1), t \to 0$, for the f in question, and we have*

$$(2\cdot2) \qquad \|f - f_{h,r}\|_B \leqslant \omega_r(f,h;B) \quad \text{and} \quad \|f_{h,r}^{(r)}\| \leqslant C_r h^{-r} \omega_r(f,h;B),$$

where C_r does not depend on h, f or B.

Proof. We construct $f_{h,r}(\cdot)$,

$$(2\cdot3) \qquad f_{h,r}(\cdot) \equiv \frac{r^r}{h^r} \int_0^{h/r} \cdots \int_0^{h/r} \left(\sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} f(\cdot + l(u_1 + \ldots + u_r)) \right) du_1 \ldots du_r,$$

which obviously satisfies $\|f - f_{h,r}\|_B \leqslant \omega_r(f,h;B)$.

We can calculate $(d/dx)^r f_{h,r}(x)$ (we shall write the variable x instead of the dot where there is no danger of confusion) to be

$$(2\cdot4) \qquad (d/dx)^r f_{h,r}(x) = \frac{r^r}{h^r} \sum_{l=1}^{r} \frac{(-1)^{l-1}}{l^r} \binom{r}{l} \Delta_{lh}^r f(x + \tfrac{1}{2}lh).$$

This formula is known for a continuous function f (as a strong derivative). Assuming $\|\Delta_t f\|_B = o(1)$ as $t \to 0$, and using $g = f'$ if $\|(1/h)(f(\cdot + h) - f(\cdot)) - g(\cdot)\|_B = o(1), h \to 0$, we show

$$\frac{1}{h} \left\| \frac{1}{t} \left\{ \int_0^h f(x+t+u)\,du - \int_0^h f(x+u)\,du \right\} - f(x+h) + f(x) \right\|$$

$$\leqslant \frac{1}{h} \left\| \frac{1}{h} \int_0^t f(x+h+u)\,du - f(x+h) \right\|_B + \frac{1}{h} \left\| \frac{1}{t} \int_0^t f(x+u)\,du - f(x) \right\|_B = o(1) \quad \text{as} \quad t \to 0$$

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and (2·4) follows by repeating this argument r times. Using ((7), p. 103 (5)), we now have

$$\left\| \left(\frac{d}{dx}\right)^r f_{h,r}(\cdot) \right\|_B \leqslant \frac{r^r}{h^r} \sum_{l=1}^{r} l^{-r} \binom{r}{l} \| \Delta_{lh}^r f(\cdot) \|_B \leqslant \frac{r^r}{h^r} \sum_{l=1}^{r} \binom{r}{l} \| \Delta_h^r f(\cdot) \|_B$$

and

$$C_r = r^r \sum_{l=1}^{r} \binom{r}{l}.$$

If r is even, we can choose $r = 2r_1$ and write

(2·5) $$f_{h,2r_1,1}(x) = \left(\frac{r_1}{h}\right)^{2r_1} \int_{-h/2r_1}^{h/2r_1} \cdots \int_{-h/2r_1}^{h/2r_1} \sum_{l=r_1+1}^{2r_1} \binom{2r_1}{l} (-1)^{l+1} f(x+u_1+\ldots+u_{2r_1})\, du_1 \ldots du_{2r_1},$$

and (2·2) will follow with a somewhat more economical C_r.

From Lemma 2·1, we obtain Lemma 2·2 ($\in$ BV meaning *has bounded variation.*)

LEMMA 2·2. *Let* $A_n(f,.) \equiv \int G_n(y) f(.-y)\, dy$ *satisfying* $\int G_n(x)\, dx = 1$, $\|G_n\|_{L_1} \leqslant M_0$ *and* $G_n^{(r-1)} \in \mathrm{BV}$ *such that* $\|G_n^{(r-1)}(.)\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r}$ *(or alternatively* $G_n^{(r)} \in L_1$ *such that* $\|G_n^{(r)}\|_{L_1} \leqslant M_r \sigma_n^{-r}$*). Then for* $f \in B$ *given in Lemma* 2·1 *with* $\|\Delta_t f\| = o(1)$, $t \to 0$ *(for all* $f \in B$*) the derivative* $(d/dx)^r A_n(f,x)$ *exists in* B *in the strong sense and*

(2·6) $$\left\| \left(\frac{d}{dx}\right)^r A_n(f,x) \right\|_B \leqslant (M_r + M_0 C_r)\, \sigma_n^{-r}\, \omega_r(f, \sigma_n, B),$$

where C_r *is that for given* n *in Lemma* 2·1.

Remark 2·2*a*. To clarify the notation we should say that:

I. Derivatives of G_n are taken in the distributional sense, and therefore $G^{n-k} \in \mathrm{AC}$, where AC means absolute continuous;

II. $\int G_n(y) f(\cdot - y)\, dy$ is an integral of $f(\cdot - y)$ as elements of B for different y.

Proof. To prove existence, which we do only for $r = 1$ (the rest follows similarly), we write

$$\lim_{t\to 0} \frac{1}{t} [A_n(f,x+t) - A_n(f,x)] = \lim_{t\to 0} \frac{1}{t} \int (G_n(x+t-y) - G_n(x-y)) f(y)\, dy$$

$$= \lim_{t\to 0} \frac{1}{t} \int \left(\int_0^t dG_n(x-y+u) \right) f(y)\, dy = \lim_{t\to 0} \frac{1}{t} \int_0^t \left\{ \int f(y)\, dG_n(x+u-y) \right\} du,$$

and, since the inner integral exists for each u, it is a translate of

$$L_n(f,x) = \int f(y)\, dG_n\ (x-y)$$

by u which tends to $L_n(f,x) = (d/dx) A_n(f,x)$ as u tends to zero. We now write $f = f - f_{h,r} + f_{h,r}$ and therefore

$$\left\| \left(\frac{d}{dx}\right)^r A_n(f,x) \right\|_B \leqslant \left\| \left(\frac{d}{dx}\right)^r A_n(f - f_{h,r}, x) \right\|_B + \left\| \left(\frac{d}{dx}\right)^r A_n(f_{h,r}, x) \right\|_B$$

$$\leqslant M_r \sigma_n^{-r} \| f - f_{h,r} \| + \left\| A_n \left(\frac{d}{dx}\right)^r f_{h,r}, x \right\|$$

$$\leqslant M_r \sigma_n^{-r} \omega_r(f,h,B) + M_0 C_r h^{-r} \omega_r(f,h,B).$$

Choosing $h = \sigma_n$, we complete the proof.

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COROLLARY 2·3. *Under the assumptions of Lemma 2·2 if*

$$f \in \mathrm{Lip}^* \alpha \quad i.e. \quad \|\Delta_h^r f\|_B \leqslant M|h|^\alpha,$$

we have

$$\left\| \left(\frac{d}{dx}\right)^r A_n(f,x) \right\|_B = O(\sigma_n^{-r+\alpha}), \quad n \to \infty. \tag{2·7}$$

Remark 2·4. One would try to prove the converse of Corollary 2·3 because of the known special cases for which it is valid. However, without additional conditions on G_n it does not hold, as the following example shows. Let

$$G_n(y) = (n/\pi)^{\frac{1}{2}}(1-n^{\frac{1}{4}})\exp(-y^2 n) + n^{\frac{3}{4}}\pi^{-\frac{1}{2}}\exp(-y^2 n^2).$$

We have $\|G_n'(y)\|_{L_1} = -2n^{\frac{3}{4}}/\pi(1+o(1))$ as $n \to \infty$. By calculating for $f \in \mathrm{Lip}\ \alpha$, $0 < \alpha < 1$, we obtain

$$\|G_n' * f\| = O(n^{-(1-\alpha)/2}) + O(n^{-(1-\alpha)}n^{-\frac{1}{4}}) = O(n^{-(1-\alpha)/2}) \quad \text{as} \quad n \to \infty,$$

which is different from what we would expect from (2·7). In a later section a condition of G_n will be given under which (2·7) would be necessary and sufficient for $f \in \mathrm{Lip}^* \alpha$.

Remark 2·5. If in Lemma 2·2 and 2·3 instead of $\|\Delta_t f\|_B = o(1)$, $t \to 0$, for all $f \in B$ we had $\|\Delta_t g\|_{B^*} = o(1)$, $t \to 0$, for all $g \in B^*$ the dual of B we would have $(d/dx)^r A_n(f,x)$ exist in the weak sense and (2·6) and (2·7) still valid. The reason is that $f * g = F$ is continuous under this restriction and the rest follows our theorem for a continuous function and the techniques used in later sections of this paper repeatedly.

3. *Some direct theorems*

In this section we will state and prove some elementary direct theorems which will later help in giving equivalence relations between Lipschitz classes and conditions on $A_n(f,x)$.

THEOREM 3·1. *For B a Banach space, $B \subset S'(R)$ or $B \subset S'(T)$, satisfying*

$$\|f(\cdot + a)\|_B = \|f(\cdot)\|_B,$$

and G_n satisfying

$$\int G_n(y)\,dy = 1 \quad \text{and} \quad \|G_n\|_{L_1} \leqslant M_0, \tag{3·1}$$

we have the following results A, B, C.

(A) If $\int y^i G_n(y)\,dy = 0$, $i = 1, \ldots, k-1$, and $\int |y^k G_n(y)|\,dy \leqslant A\sigma_n^k$, then

$$\|A_n(f,\cdot) - f(\cdot)\|_B \leqslant (M_0 + 1 + A \cdot C_k)\,\omega_k(f, \sigma_n, B) \tag{3·2}$$

whenever $\omega_k(f,h,B) = o(1)$, $h \to 0$, and where C_k is that of Lemma 2·1.

(B) If $0 < \beta \leqslant 1$ and $\int |y|^\beta |G_n(y)|\,dy \leqslant A(\beta)\sigma_n^\beta$ and $k\beta > \alpha$, then for $f \in \mathrm{Lip}^* \alpha$, that is $\|\Delta_h^k f\|_B \leqslant M|h|^\alpha$,

$$\|(A_n - I)^k (f,\cdot)\|_B \leqslant M\sigma_n^\alpha \tag{3·3}$$

where

$$(A_n - I)^k f \equiv A_n(A_n - I)^{k-1} f - (A_n - I)^{k-1} f.$$

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(C) If $G_n(y)$ is even and, for some β with $0 < \beta \leqslant 2$, $\int |y|^\beta |G_n(y)|\, dy \leqslant A(\beta)\sigma_n^\beta$ and $k\beta > \alpha$, then for $f \in \mathrm{Lip}^* \alpha$, that is $\|\Delta_h^{2k} f\|_B \leqslant M|h|^\alpha$, (3·3) is valid.

Remark 3·1. We did not require here $\|\Delta_t f\|_B = o(1)$, $t \to 0$, for all $f \in B$.

Proof. We shall prove the theorem for the space $C(R)$ or $C(T)$ first. To prove (A) we use Lemma 2·1 and write

$$|A_n(f,x)-f(x)| = |A_n(f-f_{n,k},x)-(f(x)-f_{h,k}(x))| + |A_n(f_{h,k},x)-f_{h,k}(x)|$$
$$\leqslant M_0\omega_k(f,h,C)+\omega_k(f,h,C)+|A_n(f_{h,k},x)-f_{h,k}(x)|.$$

Using Taylor's formula on $f_{h,k}(x)$,

$$|A_n(f_{h,k},x) = f_{h,k}(x)| \leqslant \|f_{h,k}^{(k)}(\cdot)\| \int |y^k G(y)|\, dy \leqslant AC_k(\sigma_n/h)^k\, \omega_k(f,h,C).$$

We choose $h = \sigma_n$ to complete the proof of (A). To prove (B) and (C) we observe that after arrangement

$$(3\cdot4)\quad (A_n - I)^k (f,x)$$
$$= (-1)^k \int G_n(u_1) \int G_n(u_2) \dots \int G_n(u_k)\, \Delta_{u_1} \dots \Delta_{u_k} f(x-(u_1+\dots u_1)/2)\, du_1 \dots du_k,$$

and

$$(3\cdot5)\qquad 2^k(A_n - I)^k (f,x) = \int G_n(u_1) \int G_n(u_2) \dots \int G_n(u_k)\, \Delta_{u_1}^2 \dots \Delta_{u_k}^2 f(x)\, du_1 \dots du_k$$

respectively.

We can also show that for $f \in \mathrm{Lip}^* \alpha$ or $\|\Delta_h^k f\|_C \leqslant M|h|^\alpha$ $(\alpha < k)$ we have

$$(3\cdot6)\qquad \|\Delta_{u_1} \dots \Delta_{u_k} f\|_C \leqslant M(2^k + C_k)\, |u_1 \dots u_k|^{\alpha/k},$$

and for $\|\Delta_h^{2k} f\|_C \leqslant M|h|^\alpha$ we have

$$(3\cdot7)\qquad \|\Delta_{h_1}^2 \dots \Delta_{h_k}^2 f\|_C \leqslant M(4^k + C_{2k})\, |u_1 \dots u_k|^{\alpha/k}.$$

Inequalities (3·6) and (3·7) follow Lemma 2·1 substituting $f = f - f_{h,k} + f_{h,k}$ and $f = f - f_{h,2k} + f_{h,2k}$ for f in $\Delta_{u_1} \dots \Delta_{u_k} f$ and $\Delta_{u_1}^2 \dots \Delta_{u_k}^2 f$ respectively. Using repeatedly the mean value theorem we have for $g \in C^k$ or $g \in C^{2k}$, $\Delta_{u_1} \dots \Delta_{u_k} g(x) = u_1 \dots u_k g^{(k)}(\xi)$ or $\Delta_{u_1}^2 \dots \Delta_{u_k}^2 g(x) = u_1^2 \dots u_k^2 g^{(2k)}(\xi)$ respectively. We now choose $g(x)$ to be $f_{h,k}(x)$ or $f_{h,2k}(x)$, and substitute $h = (u_1 \dots u_k)^{\alpha/k}$ or $h = (u_1 \dots u_k)^{\alpha/2k}$ to complete the proof of (3·6) or (3·7) respectively.

We now prove the theorem for a Banach space B such that $B \subset S'(R)$ (or $B \subset S'(T)$), $\|f(\cdot + a)\| = \|f(\cdot)\|$. We choose $g \in B^*$, B^* the dual of B; obviously we have

$$\|g(\cdot + a)\|_{B^*} = \|g(\cdot)\|_{B^*}$$

where $\langle g(\cdot + a), f(\cdot)\rangle = \langle g(\cdot), f(\cdot - a)\rangle$, $\forall f \in B$. Writing $F(x) = \langle g(\cdot), f(\cdot + x)\rangle$, we note $|F(x)| \leqslant \|f\|_B \cdot \|g\|_{B^*}$ and therefore $F \in C$ (for f satisfying conditions of (A), (B) or (C)). Moreover for $\|g\|_{B^*} = 1$, $\|\Delta_h^k F(\cdot)\|_C \leqslant \|\Delta_h^k f\|_B$. Therefore (3·2) and (3·3) are valid with F replacing f but with a constant that depends on $\omega_k(F,h,C)$ and therefore $\omega_k(f,h,B)$ and remains the same for any choice $g \in B^*$ such that $\|g\|_{B^*} = 1$. We can now choose $g_\epsilon \in B^*$, $\|g_\epsilon\|_{B^*} = 1$ such that

$$|\langle A_n(f,\cdot) - f(\cdot), g_\epsilon(\cdot)\rangle| \geqslant (1-\epsilon)\, \|A_n(f,\cdot) - f(\cdot)\|_B$$

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or (for a different g_ϵ)

$$|\langle (A_n - I)^k f(\cdot), g_\epsilon(\cdot)\rangle| \geqslant (1-\epsilon)\,\|(A_n - I)^k f\|_B.$$

With $F_\epsilon(x) = \langle g_\epsilon(\cdot), f(\cdot + x)\rangle$ we recall, using $F_\epsilon \in C$,

$$\begin{aligned}|\langle A_n(f,\cdot) - f(\cdot), g_\epsilon(\cdot)\rangle| &= |A_n(F_\epsilon, 0) - F_\epsilon(0)| \\ &\leqslant \|A_n(F_\epsilon,\cdot) - F_\epsilon(\cdot)\|_C \leqslant K\omega_k(f, \sigma_n, B),\end{aligned}$$

or, when proving (B) and (C),

$$|\langle (A_n - I)^k f(\cdot), g_\epsilon(\cdot)\rangle| = |(A_n - I)^k (F_\epsilon, 0)| \leqslant K|h|^\alpha.$$

Combining the above, we complete the proof, since ϵ is arbitrary.

4. *Inverse result*

The following very simple result constitutes an inverse theorem for continuous functions.

THEOREM 4·1. *Suppose on R or T $\int G_n(x)\,dx = 1$, $\|G_n\|_{L_1} \leqslant M_0$ and, for some integer r, $r \geqslant 1$, $\|G_n^{(r-1)}\|_{BV} \leqslant M_r \sigma_n^{-r}$ (or alternatively $\|G_n^{(r)}\|_{L_1} \leqslant M_r \sigma_n^{-r}$) with $\sigma_n = o(1)$, $n \to \infty$, which satisfies $1 \leqslant \sigma_n/\sigma_{n+1} \leqslant M$, then*

$$\|A_n(f,\cdot) - f(\cdot)\|_C \equiv \|G_n * f - f\|_C \leqslant K\sigma_n^\alpha, \quad n \to \infty, \tag{4·1}$$

implies $f \in \mathrm{Lip}^*\,\alpha$ for $\alpha < r$.

Proof. Using (2·6) we can write

$$\begin{aligned}|\Delta_h^r f| &\leqslant |\Delta_h^r (G_n * f(x) - f(x))| + |\Delta_h^r A_n(f,x)| \\ &\leqslant 2^r K\sigma_n^\alpha + |h|^r \left|\frac{d^r}{dx^r} A_n(f,\xi)\right| \leqslant 2^r K\sigma_n^\alpha + (M_r + M_0 C_r)\,|h|^r \sigma_n^{-r}\omega_r(f,\sigma_n, C).\end{aligned}$$

Following the method of Berens and Lorentz ((2), pp. 696, 704), used also in (5) and (1), we have $\omega_r(f,h,C) \leqslant K|h|^\alpha$.

Remark 4·1*a*. For $r = 2$ Theorem 4·1 constitutes a result by Becker and Nessel (1). The conditions of this theorem are not sufficient for an equivalence between

$$\|G_n * f - f\|_C \leqslant K\sigma_h^\alpha \quad \text{and} \quad f \in \mathrm{Lip}^*\,\alpha$$

as the example of Remark 2·4 for $r = 1$ and $r = 2$ demonstrates. We will obtain such necessary and sufficient conditions in a later section.

As a corollary we have the theorem for other spaces:

THEOREM 4·2. *If we replace the space C on R or T by a Banach space $B \subset S'(R)$ (or $B \subset S'(T)$) such that $\|f(\cdot + a)\|_B = \|f(\cdot)\|_B$ and either $\|\Delta_n f\|_B = o(1)$, $h \to 0$, for all $f \in B$ or $\|\Delta_h g\|_{B^*} = o(1)$, $h \to 0$, for all $g \in B^*$, then $\|A_n(f,\cdot) - f(\cdot)\|_B \leqslant K\sigma_n^\alpha$ implies $f \in \mathrm{Lip}^*\,\alpha$ in B, that is $\|\Delta_h^r f\|_B \leqslant K|h|^\alpha$.*

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Proof. We choose again $g \in B^*$, $\|g\|_{B^*} = 1$ and $F(x) = g(\cdot), f(\cdot + x)$ and have $\|A_n(F, \cdot) - F(\cdot)\|_C \leqslant \|A_n(f, \cdot) - f(\cdot)\|_B$ and therefore, using Theorem 4·1,

$$\|\Delta_h^r F\|_C \leqslant K_1 |h|^\alpha.$$

We can complete the proof by choosing, for each h and $\epsilon > 0$, $g_{\epsilon,h}$ such that

$$\|\Delta_h^r F_{\epsilon,h}\|_C \geqslant |\Delta_h^r F_{\epsilon,h}(0)| = |\langle \Delta_h^r f, g_{\epsilon,h} \rangle| \geqslant (1-\epsilon)\|\Delta_h^r f\|_B$$

where $F_{\epsilon,h}(x) \equiv \langle g_{\epsilon,h}(\cdot), f(\cdot + x) \rangle$.

We can further generalize our result using the following lemma.

LEMMA 4·3. *Let* $\|G_n\|_{L_1} \leqslant M_0$ *and* $\|G_n^{(r-1)}\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r}$ *and let* $G_{n,k}(x) \equiv G_n * G_{n,k-1}(x)$ *and* $G_{n,1}(x) \equiv G_n(x)$, *then* $\|G_n^{(rk-1)}\|_{\mathrm{BV}} \leqslant M_r^k \sigma_n^{-r}$. *If* $\|G_n^{(r)}\|_{L_1} \leqslant M_r \sigma_n^{-r}$, *we have* $\|G_{n,k}^{(rk)}\|_{L_1} \leqslant M_r^k \sigma_n^{-rk}$, *where derivatives of* G_n *are taken in* S.

Proof. We shall show that

$$\|G_n^{(r-1)}\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r} \quad \text{is equivalent to} \quad \|h^{-r}\Delta_h^r G_n(\cdot)\|_{L_1} \leqslant M_r \sigma_n^{-r}.$$

But $\|h^{-r}\Delta_h^r G_n(\cdot)\|_{L_1} \leqslant M_r \sigma_n^{-r}$ implies $\|h^{-rk}\Delta_h^{rk} G_{n,k}(\cdot)\|_{L_1} \leqslant \|h^{-r}\Delta_h^r G_n\|_{L_1}^k$, which will prove the theorem when the first statement is proved. If $\|G_n^{(r-1)}\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r}$, then

$$\|h^{-r}\Delta_h^r G_n(\cdot)\|_{L_1} = \|h^{-1}\Delta_h G_n^{(r-1)}(\cdot)\|_{L_1} = \int \left| \frac{1}{h} \int_x^{x+h} dG_n^{(r-1)}(u) \right| dx$$

$$\leqslant \int \left| \frac{1}{h} \int_x^{x+h} \left| dG_n^{(r-1)}(u) \right| \right| dx \leqslant \int \left| dG_n^{(r-1)}(u) \right| = \|G_n^{(r-1)}(\cdot)\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r}.$$

On the other hand $\|h^{-r}\Delta_h^r G_n(\cdot)\|_{L_1} \leqslant M_r \sigma_n^{-r}$ implies that the $r-1$ derivative of G_n exists and $\|h^{-1}\Delta_h G_n^{(r-1)}\|_{L_1} \leqslant M \sigma_n^{-r}$. Looking at $\int h^{-1}\Delta_h G_n^{(r-1)}(u)\,du$ as a collection of functions in BV, one observes that it has a weak $*$ limit as $h_l \to 0$ which on the one hand has the same bound i.e. $M\sigma_n^{-r}$ and on the other hand is equal to $G_n^{(r-1)}$ as a distribution. It should be noted that for the case $\|G_n^{(r)}\|_{L_1} \leqslant M_r \sigma_n^{-r}$, the proof of the above result is substantially simpler.

THEOREM 4·3. *For* B *as in Theorem* 4·2, G_n *and* $A_n(f, x)$ *as in Theorem* 4·1 *and for any* α *we have*

$$\|(A_n - I)^l (f, \cdot)\|_B \leqslant M\sigma_n^\alpha \quad \textit{implies} \quad f \in \mathrm{Lip}^* \alpha \tag{4·2}$$

and

$$\|(A_n^k - I)^l (f, \cdot)\|_B \leqslant M\sigma_n^\alpha \quad \textit{implies} \quad f \in \mathrm{Lip}^* \alpha. \tag{4·3}$$

Proof. For $\alpha < r$ we have

$$(I - A_n)^l (f, x) = f(x) + \sum_{k=1}^{l} (-1)^{k-1} \binom{l}{k} A_n^k(f, x) \quad \text{and} \quad \sum_{k=1}^{l} (-1)^{k-1} \binom{l}{k} A_n^k(f, x)$$

satisfies the conditions of Theorem 4·1 with different constants since

$$\left\| \left(\frac{d}{dx} \right)^r G_n * \ldots * G_n(x) \right\|_{\mathrm{BV}} \leqslant M_0^{k-1} M_r \sigma_n^{-r} \quad \text{and} \quad \sum_{k=1}^{l} (-1)^{k-1} \binom{l}{k} = 1.$$

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The interesting case is $\alpha \geqslant r$. If $\|G_n^{(r-1)}(\cdot)\|_{\text{B.V.}} \leqslant M_r \sigma_n^{-r}$, then $G_{n,k} \equiv G_{n,k-1} * G_n$ satisfies $\|G_{n,k}^{(kr-1)}(\cdot)\|_{\text{B.V.}} \leqslant M_r^k \sigma_n^{-rk}$ (in case $\|G_n^{(r)}\|_{L_1} \leqslant M_r \sigma_n^{-r}$ we may write

$$\|G_{n,k}^{(kr)}(\cdot)\|_{L_1} \leqslant M_r^k \sigma_n^{-rk}).$$

If $\alpha \leqslant r$, there exists k such that $\alpha < kr$. We have

$$\|(A_n - I)^l (f, \cdot)\| \leqslant M \sigma_n^\alpha$$

implies

$$\|(A_n^k - I)^l (f, \cdot)\| \leqslant \|(A_n^k + A_n^{k-2} + \dots I)^l (A_n - I)^l (f, \cdot)\| \leqslant (M_0^{k-1} + M_0^{k-2} + \dots + 1)^l M \sigma_n^\alpha.$$

Theorem 4·1 is applicable to $\|(A_n^k - I)^l (f, \cdot)\| \leqslant M \sigma_n^\alpha$ and hence we conclude the proof.

5. *The equivalence relation*

We are now in a position to prove our main equivalence relation.

THEOREM 5·1. *Let A_n be given by* (1·2) *and G_n satisfy*

$$\int G_n(y) = 1, \quad \|G_n(y)\|_{L_1} \leqslant M_0 \|G_n^{(r-1)}\|_{\text{BV}} \leqslant M_r \sigma_n^{-r}$$

for some integer r and $\int |y|^\beta |G_n(y)|\, dy \leqslant M(\beta) \sigma_n^\beta$ for some positive β, and let $\sigma_n = o(1)$, $n \to \infty$, such that $1 \leqslant \sigma_n / \sigma_{n+1} \leqslant M$, then for $f \in B$ and B a Banach space for which $\|f(\cdot + a)\| = \|f(\cdot)\|$, $\|\Delta_h f\| = o(1)$, $h \to 0$, and $B \subset S'(R)$ or $B \subset S'(T)$ the following are equivalent:

(A) $f \in \text{Lip}^* \alpha$ *in* B.

(B) $\left\| \left(\frac{d}{dx} \right)^{rk} A_n^k(f, x) \right\|_B = O(\sigma_n^{-rk+\alpha})$ for $\alpha < rk$.

(C) $\|(A_n - I)^l f\|_B = O(\sigma_n^\alpha)$ *for* $l > \alpha / \min(\beta, 1)$, *for even* G_n, $l > \alpha / \min(\beta, 2)$ *is sufficient and when* $\beta = r > \alpha$ *and* $\int y^i G_n(y)\, dy = 0$ *for* $0 < i < r$, $l = 1$ *is sufficient.*

(D) $\|(A_n^m - I)^l f\|_B = O(\sigma_n^\alpha)$ *for* m, *an integer, and* l *as in* (C).

Remark 5·1*a*. The condition $\|\Delta_t f\|_B = o(1)$, $t \to 0$, for all $f \in B$ can be replaced in the implication (A) implies (B) by $\|\Delta_t g\|_{B^*} = o(1)$, $t \to 0$, in which case the expression in (B) is taken in the weak sense. Otherwise the condition is not needed. However, we are always interested in any case in a subspace of B for which $\|\Delta_t f\|_B = o(1)$, $t \to 0$.

Proof. That (A) implies (B) is shown in Lemma 2·2 if we realize that

$$\|(d/dx)^{rk-1} G_n * \dots * G_n\|_{\text{BV}} \leqslant M_r^k \sigma_n^{-rk}.$$

That (A) implies (C) is proved in Theorem 3·1. (C) implies (D) for any m, since

$$\|(A_n^m - I)^l f\| = \|(A_n^{m-1} + A_n^{m-2} + \dots + I)^l (A_n - I)^l f\| \leqslant (M_0^{m-1} + \dots + 1)^l \|(A_n - I)^l f\|.$$

For $\alpha < kr$ choose $m = k$ and let $O_n(f, x) = -\Sigma_{i=1}^l (-1)^i \binom{l}{i} A_n^{ki}(f, x)$.

The operator $O_n(f, x)$ satisfies the conditions of Theorem 4·1 with kr replacing r and therefore (D) implies (A). To complete the proof we have to show (B) implies (D), maybe with a bigger m and bigger than the minimal l. Those will suffice because in turn (A) will be implied, which will imply (C) and (D) with the l required. We first choose $m = k$, then we choose l such that l will be suitable for kr, that is $l > kr / \min(\beta, 1)$.

Let $O_n(f, x) = -\Sigma_{i=1}^l (-1)^i \binom{l}{i} A_n^{ki}(f, x)$. Obviously $\|d^{kr} O_n(f, x)/dx^{kr}\| \leqslant M \sigma_n^{-kr+\alpha}$.

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Let n_j be a subsequence of N such that $A_1\sigma_{n_{j+1}} \leqslant \sigma_{n_j} \leqslant A_2\sigma_{n_{j+1}}$ where $A_2 \geqslant A_1 > 1$. (We can choose such a subsequence since $1 \leqslant \sigma_n/\sigma_{n+1} \leqslant M$.) Using Theorem 3·1 (B) and recalling that $\|d^{kr}O_n(f,x)/dx^{kr}\| \leqslant M\sigma_n^{-kr+\alpha}$ implies $\|\Delta_h^l O_n(f,x)\| \leqslant 2^{l-kr}h^{kr}M\sigma_n^{-kr+\alpha}$, and that $(I-O_{n_j}) \equiv (I-A_{n_j}^k)^l$, we have

$$\{\|O_{n_j}(f,x)-O_{n_j}^2(f,x)\| \leqslant M(1)\,\sigma_{n_j}^{kr}\sigma_{n_j}^{-kr+\alpha} \leqslant M(1)\,\sigma_{n_j}^{\alpha}, \tag{5·1}$$

where $M(1)$ depends on earlier constants but not on f, h and n_j.

We have also

$$\begin{aligned}\|O_{n_j}^2(f,x)-O_{n_{j+1}}^2(f,x)\| &\leqslant \|O_{n_j}^2(f,x)-O_{n_{j+1}}O_{n_j}(f,x)\| + \|O_{n_j}O_{n_{j+1}}(f,x)-O_{n_{j+1}}^2(f,x)\| \\ &\leqslant \|O_{n_j}^2(f,x)-O_{n_j}(f,x)\| + \|O_{n_j}(f,x)-O_{n_{j+1}}O_{n_j}(f,x)\| \\ &\quad + \|O_{n_j}O_{n_{j+1}}(f,x)-O_{n_{j+1}}(f,x)\| + \|O_{n_{j+1}}^2(f,x)-O_{n_{j+1}}(f,x)\|,\end{aligned}$$

which, using an argument similar to that used in proving (5·1), implies

$$\|O_{n_j}^2(f,x)-O_{n_{j+1}}^2(f,x)\| \leqslant M(2)\,\sigma_{n_j}^{\alpha}. \tag{5·2}$$

Using $\|f-O_nf\| = o(1)$ and (5·1), we have $f(x) = O_{n_j}^2(f,x) + \Sigma_{s=j}^{\infty}(O_{n_{s+1}}^2(f,x)-O_{n_s}^2(f,x))$ and therefore

$$\|O_{n_j}^2(f,x)-f(x)\| \leqslant M(3)\,\sigma_{n_j}^{\alpha}. \tag{5·3}$$

Using (5·1), (5·2) and (5·3), we have $\|O_{n_j}(f,x)-f(x)\| \leqslant M(4)\,\sigma_{n_j}^{\alpha}$. We now complete the proof using the Theorem of section 4 with O_{n_j} as the operator.

COROLLARY 5·2. *In Theorem* 5·1 (B) *can be replaced by*

$$\left\|\frac{d^j}{dx^j}A_n^k(f,x)\right\|_B = O(\sigma_n^{-j+\alpha}) \quad \textit{where} \quad \alpha < j \leqslant rk. \tag{B_1}$$

Proof. It is obvious that (B) implies (B_1) using Kolmogorov's inequality

$$\|f^{(l)}\|_B \leqslant C_{n,l}\|f^{(n)}\|_B^{1-l/n}\|f\|_B^{l/n}$$

for $l < n$, which is valid for space B in our paper with the Kolmogorov constant for $C(R)$ (see (4)). That (B_1) implies (B) will follow through (D) in the same way it was shown for $j = kr$.

6. *Applications*

(1) As mentioned in the introduction, one can take the Fejér kernel

$$F_n(t) = \frac{1}{2\pi n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2.$$

Obviously $\int F_n(y)\,dy = 1$ and $F_n \geqslant 0$. Also clearly

$$\int|F_n'(y)|\,dy = O(n) \quad \text{and} \quad \int|y|^{\beta}F_n(y)\,dy = O(n^{-\beta})$$

for $\beta > 1$, and therefore $\|(d/dx)F_n * f(x)\| = O(n^{1-\alpha})$ if and only if $f \in \mathrm{Lip}\,\alpha$ for $\alpha > 1$.

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More generally, for $\alpha < r$

(a) $\left\| \left(\frac{d}{dx}\right)^r F_n * \ldots * F_n * f(x) \right\|_B = O(n^{r-\alpha})$ (r times repeated convolution),

(b) $\|(F_n - I) * \ldots * (F_n - I) * f(x)\|_B = O(n^{-\alpha})$ (r times repeated convolution),

(c) $f \in \mathrm{Lip}^*(\alpha, B)$

are equivalent.

(2) Taking the kernel $G_n(y) = An/(1+|ny|^{\frac{3}{2}})$, we have

$$\|G_n''(y)\|_{L_1} \sim n^2 \quad \text{and} \quad \int |y|^\beta G_n(y)\,dy \sim n^{-\beta}$$

for $\beta < \frac{1}{2}$, and therefore $\|(d/dx)^2 G_n * f\| = O(n^{2-\alpha})$ is equivalent to $f \in \mathrm{Lip}^*(\alpha, B)$ for $\alpha < 2$. ($\|G_n * f - f\| = O(n^{-\alpha})$ is equivalent to $f \in \mathrm{Lip}(\alpha, B)$ only for $\alpha < \frac{1}{2}$.)

(3) Let $G(x) \in L_1$ and $G(x)\,dx = 1$ and let

$$A_\rho(f,x) = \int G(x-y) f(\rho^{-1}x)\,dx, \tag{6·1}$$

then the condition $\int |t|^\beta |G_n(t)|\,dt = O(\sigma_n^\beta)$ is replaced by $x^\beta G(x) \in L_1$ and

$$\|G_n^{(r-1)}\|_{\mathrm{BV}} \leqslant M_r \sigma_n^{-r}$$

is replaced by $G^{(r-1)} \in \mathrm{BV}$ and therefore we obtain:

THEOREM. *For $A_\rho(f,x)$ defined by (6·1), $f \in B$, $|x|^\beta G(x) \in L_1$ and $G^{(r-1)} \in \mathrm{BV}$ the following are equivalent*

(A) $$f \in \mathrm{Lip}^* \alpha \quad \text{in } B,$$

(B) $$\|A_\rho^k(f,x)\|_B = O(\rho^{-rk+\alpha}), \quad \rho \to 0+, \quad \text{for} \quad \alpha < rk,$$

and

(C) $$\|(1 - A_\rho^m)^l (f,x)\|_B = O(\rho^\alpha), \ \rho \to 0+,$$

for any integer m and for $l > \alpha/\min(\beta, 1)$ ($l > \alpha/\min(\beta, 2)$ is sufficient if $G(t)$ is even).

7. *The theorem in several variables*

In many cases in analysis the generalization to d dimensions causes substantial difficulties and in many other cases it is straightforward. We shall show that with the right assumptions on the kernel our main theorem is still true and is straightforward in d dimensions.

For f defined on R^d or T^d $\Delta_h f(x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$, where h is a vector in R^d (or T^d). For a Banach space B over R^d or T^d $f \in \mathrm{Lip}^* \alpha$ if $\|\Delta_h^r f\|_B \leqslant K|h|^\alpha$ where $|h|$ is the length of h and $r > \alpha$. We shall again deal with Banach spaces for which

$$\|f(\cdot + h)\|_B = \|f(\cdot)\|_B, \quad \|\Delta_h f\|_B = o(1), \quad h \to 0, \quad \text{and} \quad B \subset S'(R^d) \quad \text{or} \quad B \subset S'(T^d).$$

THEOREM 7·1. *Suppose that $f \in B$ over R^d as specified above,*

$$\int_{R^d} G_n(x)\,dx = 1, \quad \|G_n\|_{L_1(R^d)} \leqslant M_0, \quad \left\| \frac{\partial^r}{\partial \xi^r} G_n(x) \right\|_{L_1(R^d)} \leqslant M_r \sigma_n^{-r}$$

for any $\xi = \Sigma_{i=1}^d \alpha_i x_i$, such that $\Sigma \alpha_i^2 = 1$. Suppose that $\int_{R^d} |x_i|^\beta |G_n(x)|\,dx \leqslant M(\beta)\sigma_n^\beta$ for some $\beta > 0$ and $1 \leqslant i \leqslant d$, where $\sigma_n = o(1)$, $n \to \infty$, such that $1 \leqslant \sigma_n/\sigma_{n+1} \leqslant M$ (or

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the same conditions with T^d replacing R^d are satisfied). Then the following are equivalent:

(A) $f \in \mathrm{Lip}^* \alpha$ *in* B,

(B) $\|(\partial/\partial\xi)^{rk} A_n^k(f,x)\|_B = O(\sigma_n^{-rk+\alpha})$ *for* $\alpha < rk$ *and all* ξ,

(C) $\|(A_n^m - I)^l f\|_B = O(\sigma_n^\alpha)$ *for any integer* $m \geqslant 1$ *and for* $l > \alpha/\min(\beta, 1)$;
$l > \alpha/\min(\beta, 2)$ *is sufficient when* G_n *is symmetric (around* O*).*

Proof. The parts in which (C) with $m = 1$ implies (c) with other m is exactly as in the proof of Theorem 4·1. That (C) implies (A) follows exactly Theorem 4·1, 4·2 and 4·3. (The restriction on l is not used in this direction.) To prove (A) implies (B) follows exactly the theorem of section 2 where the function $f_{h,r,\xi}$ replacing $f_{h,r}$ is given for $\xi = \Sigma\alpha_i x_i$, $\Sigma\alpha_i^2 = 1$ by

$$(7\cdot1) \qquad f_{h,r,\xi}(x) = \frac{r^r}{h^r}\int_0^{h/r} \dots \int_0^{h/r} \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} f(x + l\xi(u_1 + \dots + u_r))\, du_1 \dots du_r.$$

We obviously have

$$(7\cdot2) \qquad f_{h,r,\xi}(x) - f(x) = \frac{r^r}{h^r}\int_0^{h/r} \dots \int_0^{h/r} \{\Delta^r_{(u_1+\dots+u_r)\xi} f(x + \tfrac{1}{2}(u_1 + \dots + u_r)\,\xi)\}\, du_1 \dots du_r$$

and

$$(7\cdot3) \qquad \left(\frac{\partial}{\partial\xi}\right)^r f_{h,r,\xi}(x) = \frac{r^r}{h^r} \sum_{l=1}^{r} \frac{(-1)^{l-1}}{l^r} \binom{r}{l} \Delta^r_{lh\xi_0} f(x + \tfrac{1}{2}lh\xi_0).$$

But from (7·2) and (7·3) we see that (A) implies (B) for all ξ, repeating arguments of Lemma 2·2. The condition $\int |x_i|^\beta |G_n(x)|\, dx = O(\sigma_n^\beta)$ for all x_i implies

$$\int |\xi|^\beta |G_n(x)|\, dx \leqslant d \sum_{i=1}^{d} \int |x_i|^\beta |G_n(x)|\, dx = O(\sigma_n^\beta)$$

and therefore (A) implies (C) in the same way as we proved Theorem 3·1. The technique used to show (B) implies (C) for big enough m is not changed by using several variables.

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