THE BULLETIN OF SYMBOLIC LOGIC
Volume 32, Number 2, June 2026

ÉTALE STRUCTURES AND THE JOYAL–TIERNEY REPRESENTATION THEOREM IN COUNTABLE MODEL THEORY

RUIYUAN CHEN

Abstract. An étale structure over a topological space X is a continuous family of structures (in some first-order language) indexed over X. We give an exposition of this fundamental concept from sheaf theory and its relevance to countable model theory and invariant descriptive set theory. We show that many classical aspects of spaces of countable models can be naturally framed and generalized in the context of étale structures, including the Lopez-Escobar theorem on invariant Borel sets, an omitting types theorem, and various characterizations of Scott rank. We also present and prove the countable version of the Joyal–Tierney representation theorem, which states that the isomorphism groupoid of an étale structure determines its theory up to bi-interpretability; and we explain how special cases of this theorem recover several recent results in the literature on groupoids of models and functors between them.

§1. Introduction. A standard technique in countable model theory associates, to each (possibly infinitary) first-order theory $\mathcal{T}$ in some language $\mathcal{L}$, a topological space X parametrizing all of its countable models up to isomorphism. This enables the application to model theory of powerful tools from topology, dynamics, descriptive set theory, and computability theory, such as Baire category techniques and Borel complexity theory. See [7], [25, Sections 3.6 and 11.4], [33, Section 16.C], [49].

In fact, there are several variants of such a "topological space of models" X. Perhaps the best-known consists of all models on a *fixed* countably infinite set such as $\mathbb{N}$, regarded as a subspace of a Cantor space $X \subseteq \prod_i 2^{\mathbb{N}^{n_i}}$ specifying the interpretations of each relation and function symbol in the language $\mathcal{L}$; see [25, Section 3.6]. By taking instead models on (certain) subsets of $\mathbb{N}$, one can also encompass finite models up to isomorphism; see [14]. One can also broaden (e.g., to all finitary first-order) or narrow (e.g.,

Received August 6, 2024.
2020 *Mathematics Subject Classification*. 03E15, 03C15, 03G30, 22A22, 18B25.
Key words and phrases. infinitary logic, étale, sheaf, Polish groupoid, Joyal–Tierney, interpretation, Lopez-Escobar.

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

1079-8986/26/3202-0003
DOI:10.1017/bsl.2025.10086

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

to only atomic) the class of formulas which define an open subset of X; see [5], [25, Section 11.4]. Alternatively, one can consider *marked models* over a fixed generating set, e.g., groups generated by $\mathbb{N}$, hence quotients of the free group $\langle \mathbb{N} \rangle$ by a normal subgroup, represented as the space of all normal subgroups of $\langle \mathbb{N} \rangle$; see, e.g., [54].

This landscape is clarified by the observation that such X, as a space of *codes* of countable structures, may be meaningfully distinguished from the structures themselves. That is, rather than a "space of models" X, one really has a topological space X together with a "continuous map"

$$\begin{aligned} X &\longrightarrow \{\text{all } \mathcal{L}\text{-structures}\} \\ x &\longmapsto \mathcal{M}_x \end{aligned}$$

where the right-hand side is *not* a topological space, but a higher-order analog thereof, crucially differing in that two "points" (i.e., structures) may be "equal" (i.e., isomorphic) in more than one way. Such a generalized "space" is made precise by the concept of the *classifying topos* of $\mathcal{L}$-structures. Classifying toposes have been widely influential in such areas as algebraic geometry, topology, and category theory; see [2], [30], [40]. However, to our knowledge, topos-theoretic ideas are not very well-known or used in the countable model theory literature, perhaps due to the substantial amount of category theory needed to define and work with them.[1]

1.1. Étale structures. The goal of this article is to give a self-contained development of the above perspective from a classical model-theoretic and descriptive set-theoretic angle, minimizing the category theory needed. The central concept of a "continuous family of structures" $(\mathcal{M}_x)_{x \in X}$ parametrized over a topological space X, as above, may be represented more concretely as an **étale bundle of structures** $\mathcal{M} \to X$, where $\mathcal{M}$ is the disjoint union of the structures $\mathcal{M}_x$ equipped with a global topology that captures the "continuity" of the fibers $\mathcal{M}_x$ over $x \in X$. This concept is well-known in topos theory, perhaps more commonly in the equivalent form of a *sheaf of structures*. However, by taking an étale structure $\mathcal{M} \to X$ as the formal meaning of a "continuous map" $X \to \{\mathcal{L}\text{-structures}\}$ as above, one may develop much of classifying topos theory (for countable structures and theories) in a point-set topological manner, without reference to toposes, sheaves, or other categorical notions.

A recurring theme of our account is that many standard concepts and constructions in point-set topology and descriptive set theory have *precise* analogs for étale structures, thought of as "continuous families of structures"

[1] There are of course exceptions, such as [12], [22], [36], as well as parts of model theory further from the countable realm where categorical tools are more routinely used, e.g., [41], [10], [50], and [37].

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

Table 1. Correspondence between topological and étale model-theoretic notions, when Y is replaced with the "space of all structures"

Topology	Étale model theory
Continuous map $X \xrightarrow{f} Y$	Étale structure $\mathcal{M} \to X$ (Definition 4.8)
Continuous open map (onto image)	Étale structure with Σ_1 saturations (5.1)
Borel ($\mathbf{\Sigma}^0_\alpha$) set $V \subseteq Y$	$\mathcal{L}_{\omega_1\omega}$ (Σ_α) formula ϕ (3.1)
Preimage $f^{-1}(V)$ of $V \subseteq Y$	Interpretation $\phi^{\mathcal{M}}$ of formula ϕ (4.9)
Image $f(U)$ of $U \subseteq X$	Theory/type of $U \subseteq M$
Saturation $[U]_{\ker(f)} = f^{-1}(f(U))$	Saturation $\mathrm{Iso}_X(\mathcal{M}) \cdot U$
Baire quantifier $\exists^*_f(U)$	Vaught transform $U^{\triangle \mathrm{Iso}_X(\mathcal{M})}$ (10.1)
Kernel $\ker(f) \subseteq X^2$	Isomorphism groupoid $\mathrm{Iso}_X(\mathcal{M})$ (4.15)
Composition $Z \to X \xrightarrow{f} Y$	Pullback $Z \times_X \mathcal{M}$ (4.11)
Change of topology on Y	Morleyization (4.12)

as above. Often, these analogies yield more precise formulations of well-known folklore connections. For instance, it is well-known that the classical Baire category theorem is closely related to the omitting types theorem in (infinitary) first-order logic, and in fact these two results may be used to prove each other; see, e.g., [23]. In Section 7, we explain how the omitting types theorem is the literal first-order generalization of the Baire category theorem; and we show how the former may be reduced to the latter via an étale structure, as a higher-order instance of the fact that a continuous open surjection (is category-preserving, hence) may be used to transfer Baire category from its domain to its codomain. Table 1 depicts various other topological concepts with an étale model-theoretic analog that we will discuss.

In the course of developing this dictionary, we also prove generalizations to arbitrary étale parametrizations of classical descriptive set-theoretic results known for *specific* parametrizations. For instance, the Lopez-Escobar theorem [38] and its strengthening by Vaught [55] show that every Borel isomorphism-invariant set of models, in the classical space of countably infinite models on $\mathbb{N}$, is axiomatizable by an infinitary formula, of the same quantifier complexity as the Borel complexity of the given set. Recently in [5], the authors prove a similar result, for a different parametrizing space, that takes positive (i.e., negation-free) formulas into account.[2] In [14], we proved such a result that also takes finite models into account. One could

[2]The papers [5], [26], and [27] are also largely concerned with tracking the effective (lightface) complexity of formulas and models, which we do not consider at all in this article; we therefore recover only the boldface versions of the mentioned results from these papers.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

also ask whether a Lopez-Escobar theorem holds for the space of marked structures. We prove a result encompassing all of these as special cases.

THEOREM 1.1 (Lopez-Escobar for étale structures; see Theorem 10.2). *For any étale parametrization of countable structures $\mathcal{M} \to X$ with Σ_1 saturations, every isomorphism-invariant $\mathbf{\Sigma}^0_\alpha$ set of tuples $A \subseteq M^n_X$ is defined by a Σ_α formula.*

Here Σ_α refers to the αth level of the complexity hierarchy of the countably infinitary logic $\mathcal{L}_{\omega_1\omega}$. To say that the étale structure $\mathcal{M} \to X$ has Σ_1 **saturations** means that the isomorphism saturation of each open set in M (or a finite power) is Σ_1-definable; see Definition 5.1. This is the key property on an étale structure that implies it is a "nice" parametrization of countable models obeying the usual results, and holds for all of the aforementioned "spaces of countable models" commonly considered in practice (subject to minor conditions on the theory); see Section 6.

As shown in Table 1, this condition is analogous to a continuous map between topological spaces $f : X \to Y$ being open, or more precisely, being *open onto its image* $f(X) \subseteq Y$, since that is equivalent to each open $U \subseteq X$ having open saturation by the equivalence relation $\ker(f) \subseteq X^2$ induced by f. Such maps play an important role in descriptive set theory: for nice (e.g., Polish) spaces X, Y, the image $f(X) \subseteq Y$ must be $\mathbf{\Pi}^0_2$, by a combination of classical results of Sierpiński [33, Theorem 8.19] (that an open T_3 quotient of a Polish space is Polish) and Alexandrov [33, Theorem 3.11] (that Polish subspaces are $\mathbf{\Pi}^0_2$). We prove the analogous result for étale structures.

THEOREM 1.2 (see Theorem 5.4). *For any étale parametrization of countable structures $\mathcal{M} \to X$ with Σ_1 saturations, over a (quasi-)Polish base X, the class of parametrized models (i.e., those isomorphic to a fiber of $\mathcal{M}$) is axiomatizable among countable structures by a Π_2 sentence.*

Here *quasi-Polish* refers to a generalization [19] of Polish spaces that we find more convenient; one definition is that they are precisely the open T_0 quotients of Polish spaces. See Definition 2.4.

In fact, Theorem 1.2 generalizes not only the aforementioned classical results of Sierpiński and Alexandrov in topology but also (by taking $X = 1$) the classical model-theoretic fact that an atomic model has a Π_2 Scott sentence (relative to a countable fragment of $\mathcal{L}_{\omega_1\omega}$; see [25, Theorem 11.5.7]). That these three classical results are connected at all may come as a surprise, and is an example of the conceptual clarification we believe is provided by the perspective of étale structures.

1.2. The Joyal–Tierney and Moerdijk representation theorems. Classically, given a continuous open surjection $f : X \twoheadrightarrow Y$ between topological spaces, the topology on Y is necessarily the quotient topology induced by the topology on X. This fact means that if one is interested in studying a class of

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

spaces Y, which can all be parametrized as open quotients of a restricted class of spaces X, then in some sense it "suffices" to study the spaces X together with the equivalence relations $\ker(f) \subseteq X^2$. For example, this provides one formal explanation of the idea that to do descriptive set theory, one really only needs to consider the Baire space $\mathbb{N}^{\mathbb{N}}$, whose open quotients yield all Polish spaces.

While this topological fact is rather easy, the analogous fact for étale structures with Σ_1 saturations, i.e., "continuous open maps $X \to$ {all structures}", is much deeper, since the "space" we are parametrizing is a much more complicated object than the genuine space of parameters X. The analog of the equivalence relation $\ker(f) \subseteq X^2$, which consists of pairs of points $x, y \in X$ such that $f(x) = f(y)$, is the **isomorphism groupoid** $\mathrm{Iso}_X(\mathcal{M})$ of an étale structure $\mathcal{M} \to X$, consisting of $x, y \in X$ *together with an isomorphism* $g : \mathcal{M}_x \cong \mathcal{M}_y$. There is a natural "pointwise convergence" topology on $\mathrm{Iso}_X(\mathcal{M})$ (generalizing the pointwise convergence topology on $\mathrm{Aut}(\mathcal{M})$ for a single countable structure $\mathcal{M}$) that turns it into a topological groupoid with space of objects X; see Definition 4.15. We now have one of the central results of topos theory, which we state and prove for the restricted context of countable structures and $\mathcal{L}_{\omega_1\omega}$ in Section 8:

THEOREM 1.3 (Joyal–Tierney [31]; see Theorem 8.4 and Remark 8.6). *Let $\mathcal{M} \to X$ be an étale parametrization of countable structures with Σ_1 saturations, and let $\mathcal{T}$ be the Π_2 theory of its fibers from Theorem 1.2. Then the Σ_1 imaginaries over $\mathcal{T}$ are canonically equivalent (as a category) to the continuous étale actions of the groupoid $\mathrm{Iso}_X(\mathcal{M})$.*

Here by Σ_1 **imaginaries**, we mean the usual model-theoretic concept [28, Chapter 5] adapted to the infinitary setting: namely, a Σ_1-definable quotient of a countable disjoint union of Σ_1-definable sets (represented syntactically as families of Σ_1 formulas); see Definition 8.2. Analogously to the topological setting, this result really says that étale structures in some sense "fully parametrize" countable models, since the theory $\mathcal{T}$ of the parametrized models may be "recovered" from the topological groupoid $\mathrm{Iso}_X(\mathcal{M})$. More precisely, we recover $\mathcal{T}$ up to Σ_1 bi-interpretability, since a theory is determined up to bi-interpretability by its imaginaries.

While Theorem 1.3 concerns the topological context, meaning on the model-theoretic side that we restrict to Σ_1 formulas, there is an analogous result in the Borel context, which shows that a theory may be recovered up to $\mathcal{L}_{\omega_1\omega}$ bi-interpretability from its *Borel* groupoid of isomorphisms. The proof combines Theorem 1.3 with a (Becker–Kechris-type) topological realization theorem for groupoid actions from [18] in order to translate between the topological and Borel settings. (As explained in Section 1.3 below, this generalizes results from [14] and [27].)

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

THEOREM 1.4 (see Theorem 10.7 and Remark 10.8). *Let $\mathcal{M} \to X$ be an étale parametrization of countable structures with Σ_1 saturations, and let $\mathcal{T}$ be the theory of its fibers. Then the $\mathcal{L}_{\omega_1\omega}$ imaginaries over $\mathcal{T}$ are canonically equivalent (as a category) to the fiberwise countable Borel actions of the groupoid* $\mathrm{Iso}_X(\mathcal{M})$.

As we explain in Section 9, it is in fact possible to make this recovery of a theory from its groupoid of models fully precise, provided that one is willing to make full use of category-theoretic language (which is why we postpone this discussion until Section 9). We are saying that

$$\{\text{theories}\} \longrightarrow \{\text{groupoids}\}$$
$$\mathcal{T} \longmapsto \text{isomorphism groupoid of some parametrization of } \mathcal{T}$$

is an "embedding": we can recover a theory $\mathcal{T}$ from its isomorphism groupoid. To make this precise, we should specify the kind of structure on the domain and codomain of this map. Namely, they are *2-categories*: between theories $\mathcal{T}_1, \mathcal{T}_2$, we have interpretations $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$ as morphisms; and between parallel interpretations, we have *definable isomorphisms as 2-cells* (all either Σ_1 or $\mathcal{L}_{\omega_1\omega}$):

$$\mathcal{T}_1 \underset{\mathcal{G}}{\overset{\mathcal{F}}{\Downarrow h}} \mathcal{T}_2 \xrightarrow{\mathcal{H}} \mathcal{T}_3.$$

Similarly, between groupoids, we have functors, between which we have natural isomorphisms (continuous or Borel, respectively). See Definition 9.2 and Remark 9.7 for details.

THEOREM 1.5 (see Corollaries 11.13 and 11.14). *We have contravariant equivalences of* 2*-categories*

$$\left\{\begin{array}{c}\textit{countable } \Pi_2 \textit{ theories}\\ \Sigma_1 \textit{ interpretations}\\ \Sigma_1\textit{-definable isomorphisms}\end{array}\right\} \simeq \left\{\begin{array}{c}\textit{open non-Archimedean quasi-Polish groupoids}\\ \textit{continuous functors}\\ \textit{continuous natural transformations}\end{array}\right\},$$

$$\left\{\begin{array}{c}\textit{countable } \mathcal{L}_{\omega_1\omega} \textit{ theories}\\ \mathcal{L}_{\omega_1\omega} \textit{ interpretations}\\ \mathcal{L}_{\omega_1\omega}\textit{-definable isomorphisms}\end{array}\right\} \simeq \left\{\begin{array}{c}\textit{open non-Archimedean quasi-Polish groupoids}\\ \textit{Borel functors}\\ \textit{Borel natural transformations}\end{array}\right\},$$

taking a theory to the isomorphism groupoid of some parametrization of it.

As alluded to above, the "injectivity" of these maps (more precisely, that they are fully faithful 2-functors) boils down to the Joyal–Tierney Theorem 1.3 and its Borel analog, Theorem 1.4.

The "surjectivity," that every topological groupoid satisfying certain conditions listed on the right-hand side above (see Definition 11.1 for details) can be represented as an isomorphism groupoid, boils down to a topos-theoretic argument of Moerdijk [47], which constructs from an *abstract*

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

topological groupoid G (satisfying certain conditions) an étale structure $\mathcal{M}$ over the objects of G. We present a self-contained proof in Theorem 11.5, and show that in the countable case, via a Baire category argument, we get a full representation of the groupoid G as $\mathrm{Iso}(\mathcal{M})$; see Theorem 11.11. This result simultaneously generalizes the Yoneda lemma from category theory (which shows that every *discrete* groupoid is canonically an isomorphism groupoid, via the left action on itself), and the classical result that non-Archimedean Polish groups are precisely the automorphism groups of countable structures (see, e.g., [25, Theorem 2.4.4]).

1.3. Connections to other work. This article was largely motivated by several recent works in the countable model theory literature showing connections between the syntax and semantics of infinitary logic. The earliest inspiration for this line of work seems to be [1], showing that the automorphism group of an $\aleph_0$-categorical (in the usual finitary first-order sense) structure determines it up to (finitary) bi-interpretability. Analogously, [27] showed that the automorphism group of an arbitrary countable structure determines it up to $\mathcal{L}_{\omega_1\omega}$ bi-interpretability. In that paper, respectively [26], the authors also showed that Borel, respectively continuous, functors between the groupoids of all isomorphic copies of two structures on underlying set $\mathbb{N}$ correspond to $\mathcal{L}_{\omega_1\omega}$, respectively Σ_1, interpretations between those structures.

From the étale perspective, these results can be seen as special cases of the Joyal–Tierney Theorem 1.3 and its Borel version, Theorem 1.4, for specific examples of étale parametrizations $\mathcal{M} \to X$ (whose fibers are all isomorphic).[2] The special case of bi-interpretations follows from general categorical considerations, once Joyal–Tierney is recast into its 2-categorical "injectivity" form of Theorem 1.5. In [14], whose main result was yet another special case of Theorem 1.4 (again for a specific parametrization), we made this categorical perspective explicit, including the connections to the Joyal–Tierney and Becker–Kechris theorems (as noted above Theorem 1.4).

We should also note that, in the categorical literature, there have been previous point-set topological presentations of the Joyal–Tierney theorem, such as [11] and [4]. The latter paper also used the Joyal–Tierney machinery to give a 2-categorical "duality" between theories and groupoids, as in Theorem 1.5, but for finitary first-order logic, and again considering only a specific étale parametrization (or rather, its sheaf-theoretic formulation).

We view this article as a sequel to [14], that is informed by the realization that working with an abstract étale structure, rather than a specific "space of countable models" as was done in that paper, yields a cleaner and more general presentation. In the preprint [15], we proved the Borel version of Theorem 1.5 using [14] and the aforementioned Moerdijk groupoid representation, in the process developing the needed parts of countable

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

étale model theory in a rather dense fashion; the present article is a more comprehensive and (hopefully) readable rewrite of the first half of [15]. The second half of [15], dealing with continuous logic for metric structures [8], will be subsumed in a future work that also includes a metric version of the Joyal–Tierney theorem and [14].

Finally, as mentioned in Footnote 2, several of the aforementioned works that inspired this article were rooted in computable structure theory, and proved effective strengthenings of the Lopez-Escobar theorem, Joyal–Tierney theorem, and related results, albeit for specific parametrizations; see [49] for a detailed survey. While we work entirely in the boldface setting in this article, it would be interesting to develop an effective version of the theory of étale structures presented here.

§2. Preliminaries on topology. Recall that a **Polish space** is a second-countable, completely metrizable topological space. We will be working in a more general non-Hausdorff setting, that obeys better closure properties.

DEFINITION 2.1 (Selivanov [52]). In an arbitrary topological space X, possibly non-metrizable (hence closed sets may not be G_δ), we define the **Borel hierarchy** as follows:

- A $\mathbf{\Sigma}^0_1$ set is an open set.
- For $\alpha \geq 2$, a $\mathbf{\Sigma}^0_\alpha$ set is a countable union $\bigcup_i (A_i \setminus B_i)$ where each A_i, B_i is $\mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$.
- A $\mathbf{\Pi}^0_\alpha$ set is the complement of a $\mathbf{\Sigma}^0_\alpha$ set, thus for $\alpha \geq 2$ is of the form $\bigcap_i (A_i \Rightarrow B_i)$, where $A_i, B_i \in \mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$, and $(A_i \Rightarrow B_i) := \neg A_i \cup B_i$. (Here $\neg$ denotes set complement.)
- As usual, a $\mathbf{\Delta}^0_\alpha$ set is a set that is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.

EXAMPLE 2.2. In a second-countable T_0 space X, the equality relation is $\mathbf{\Pi}^0_2$ in X^2:

$$x = y \iff \forall \text{ basic open } U\, (x \in U \iff y \in U).$$

DEFINITION 2.3. **Sierpiński space** is the topological space $\mathbb{S} := 2$ with $\{1\}$ open but not closed.

DEFINITION 2.4 (de Brecht [19][3]). A **quasi-Polish space** is a topological space homeomorphic to a $\mathbf{\Pi}^0_2$ subspace of $\mathbb{S}^{\mathbb{N}}$.

In [19], de Brecht introduced quasi-Polish spaces and proved that they satisfy almost all of the usual descriptive set-theoretic properties of Polish

[3]This was not the original definition used in [19], but was proved to be equivalent in that paper.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

spaces, including the following which we will freely use. For proofs, see [19] or [13].

(2.5) All quasi-Polish spaces are T_0, second-countable, and obey the Baire category theorem (which automatically then holds also for intersections of dense $\mathbf{\Pi}^0_2$ sets, not just dense G_δ).
(2.6) A topological space is Polish iff it is quasi-Polish and regular (T_3).
(2.7) If X is a quasi-Polish space, and $A_i \subseteq X$ are countably many $\mathbf{\Sigma}^0_\alpha$ sets, then there is a finer quasi-Polish topology containing each A_i and contained in $\mathbf{\Sigma}^0_\alpha(X)$. In more detail,
  (a) adjoining a single $\mathbf{\Delta}^0_2$ set to the topology of X preserves quasi-Polishness;
  (b) if the intersection of countably many quasi-Polish topologies contains a quasi-Polish topology, then their union generates a quasi-Polish topology.
(2.8) A quasi-Polish space can be made zero-dimensional Polish by adjoining countably many closed sets to the topology.
(2.9) Countable products of quasi-Polish spaces are quasi-Polish.
(2.10) A space with a countable cover by open quasi-Polish subspaces is quasi-Polish. In particular, a countable disjoint union of quasi-Polish spaces is quasi-Polish.
(2.11) A subspace of a quasi-Polish space is quasi-Polish iff it is $\mathbf{\Pi}^0_2$.
(2.12) A continuous open T_0 quotient of a quasi-Polish space is quasi-Polish. Conversely, every quasi-Polish space is an open quotient of a zero-dimensional Polish space.

DEFINITION 2.13. For a topological space X, its **lower powerspace** $\mathcal{F}(X)$ is the space of all closed subsets $F \subseteq X$ equipped with the topology generated by the open subbasis consisting of

$$\Diamond U := \{F \in \mathcal{F}(X) \mid F \cap U \neq \varnothing\}$$

for each open $U \subseteq X$. Equivalently, fix any open basis $\mathcal{U}$ for X; then the topology on $\mathcal{F}(X)$ is induced via the embedding

$$\begin{aligned} \mathcal{F}(X) &\longrightarrow \mathbb{S}^{\mathcal{U}} \\ F &\longmapsto \{U \in \mathcal{U} \mid F \cap U \neq \varnothing\}. \end{aligned} \tag{2.14}$$

If X is quasi-Polish, then so is $\mathcal{F}(X)$ [20] (see also (5.5) below).

We will also need the **fiberwise lower powerspace** $\mathcal{F}_X(Y)$ of a continuous map $p : Y \to X$ (thought of as a continuous bundle of spaces $Y_x := p^{-1}(x)$ indexed over X), which is the disjoint union of the lower powerspaces of the fibers $\mathcal{F}(Y_x)$, i.e.,

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\mathcal{F}_X(Y) := \{(x, F) \mid x \in X \ \&\ F \in \mathcal{F}(Y_x)\},$$

equipped with the topology generated by the first coordinate projection $\mathcal{F}_X(Y) \to X$ as well as

$$\Diamond_X U := \{(x, F) \in \mathcal{F}_X(Y) \mid F \cap U \neq \varnothing\}$$

for each open $U \subseteq Y$. When X, Y are quasi-Polish, so is $\mathcal{F}_X(Y)$ [18, Proposition 2.5.9].

§3. Preliminaries on infinitary logic. We now review some conventions on infinitary first-order logic. Throughout, $\mathcal{L}$ will denote a countable first-order language. We note that we allow both nullary function symbols (i.e., constants) and nullary relation symbols (i.e., atomic propositions). In particular, $\mathcal{L}$ could consist entirely of nullary relation symbols, in which case we are working essentially with propositional logic. Everything we do will also generalize straightforwardly to multi-sorted languages, although we will rarely bother to spell out the details of such a generalization.

The **countably infinitary first-order logic** $\mathcal{L}_{\omega_1\omega}$ is constructed like the usual finitary first-order logic, but allowing countable conjunctions $\bigwedge_i \phi_i(\vec{x})$ and disjunctions $\bigvee_i \phi_i(\vec{x})$ of formulas. We will use the symbols $\top, \bot$ for nullary conjunction and disjunction. We adopt the convention that each $\mathcal{L}_{\omega_1\omega}$ formula $\phi(\vec{x})$ may only have finitely many free variables $\vec{x}$. For general background on $\mathcal{L}_{\omega_1\omega}$, see [3, Chapter 6], [25, Chapter 11], [42].

For a first-order $\mathcal{L}$-structure $\mathcal{M}$ and formula $\phi(x_0, \dots, x_{n-1})$, we write $\phi^{\mathcal{M}} \subseteq M^n$ for the interpretation of ϕ in $\mathcal{M}$. We always allow structures to be empty.

There are completeness theorems for various deductive systems for $\mathcal{L}_{\omega_1\omega}$ [32], [38], [24], which say that an $\mathcal{L}_{\omega_1\omega}$ sentence has a proof iff it is true in every countable $\mathcal{L}$-structure.

DEFINITION 3.1 (complexity classes of $\mathcal{L}_{\omega_1\omega}$ formulas).

- A **basic formula** is a finite conjunction of atomic formulas (including equality, but no negations, not even $\neq$).
- A **positive-primitive formula** is one built from atomic formulas using $\wedge, \top, \exists$. Up to logical equivalence, these may all be written in the prenex normal form $\exists \vec{y}\, \phi(\vec{x}, \vec{y})$ where ϕ is basic; we will also call this a $\exists\wedge$ formula.
- A Σ_1 **formula** is built from atomic formulas using $\wedge, \top, \exists, \bigvee$. These have a $\bigvee\exists\wedge$ normal form.
- For $\alpha \geq 2$, a Σ_α **formula** is built from Σ_β formulas and their negations, for $\beta < \alpha$, using $\wedge, \top, \exists, \bigvee$. These have a normal form

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\bigvee_i \exists \vec{y}_i \left(\phi_i(\vec{x}, \vec{y}_i) \wedge \neg \psi_i(\vec{x}, \vec{y}_i)\right)$$

where each ϕ_i, ψ_i is Σ_β for some $\beta < \alpha$.

- For $\alpha \geq 2$, a Π_α **formula** is the dual of a Σ_α formula, or equivalently, for ϕ_i, ψ_i as above,

$$\bigwedge_i \forall \vec{y}_i \left(\phi_i(\vec{x}, \vec{y}_i) \to \psi_i(\vec{x}, \vec{y}_i)\right).$$

CONVENTION 3.2. We will assume whenever convenient that every $\mathcal{L}_{\omega_1\omega}$ formula is built using $\wedge, \top, \exists, \bigvee, \neg$ only, with $\bigwedge, \forall$ treated as abbreviations for $\neg \bigvee \neg, \neg\exists\neg$, respectively. Thus, every formula is Σ_α in the above sense for some α.

The above definitions of $\Sigma_\alpha, \Pi_\alpha$ are not the standard ones found in e.g., [3], which count all finitary quantifier-free formulas as Σ_1. (They are *almost* the "positive Σ^p_α, Π^p_α formulas" of [5], except that we do not even admit $\neq$ as Σ_1.) While the standard definitions interact well with the Borel hierarchy of the usual zero-dimensional Polish space parametrizing $\mathcal{L}$-structures on $\mathbb{N}$ (see [25, Chapter 11], [33, Section 16.C], and Example 6.1 below), the above definitions are the natural counterpart to Selivanov's non-Hausdorff Borel hierarchy (Definition 2.1) for quasi-Polish spaces.

Just as the non-Hausdorff Borel hierarchy generalizes the usual Borel hierarchy for Polish spaces, so too can the above "non-Hausdorff" hierarchy of formulas be regarded as generalizing the usual hierarchy. For given a theory $\mathcal{T}$, we may replace it with another $\mathcal{T}'$ whose $\Sigma_\alpha, \Pi_\alpha$ formulas, in the above sense, correspond to the usual $\Sigma_\alpha, \Pi_\alpha$ formulas of $\mathcal{T}'$, via the following standard device.

DEFINITION 3.3. A **fragment** of $\mathcal{L}_{\omega_1\omega}$ is a set $\mathcal{F}$ of formulas which contains all atomic formulas and is closed under variable substitutions and subformulas. (Note that this generalizes Sami's notion of fragment from [51] which requires closure under $\neg$, which in turn generalizes the more common notion requiring closure under all finitary operations [25, Definition 11.2.3].)

The **Morleyization** of a fragment $\mathcal{F}$ of $\mathcal{L}_{\omega_1\omega}$ consists of an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ with a new n-ary relation symbol R_ϕ for each $\phi(x_0, \dots, x_{n-1}) \in \mathcal{F}$, together with the following $\Pi_2\mathcal{L}'$-theory $\mathcal{T}'$ which says that this is an expansion by definitions:

$$\begin{aligned}
&\forall \vec{x} \left(R_\phi(\vec{x}) \leftrightarrow \phi(\vec{x})\right) \quad \text{for atomic } \phi,\\
&\forall \vec{x} \left(R_{\phi\wedge\psi}(\vec{x}) \leftrightarrow R_\phi(\vec{x}) \wedge R_\psi(\vec{x})\right),\\
&\forall \vec{x} \left(R_\top(\vec{x}) \leftrightarrow \top\right),\\
&\forall \vec{x} \left(R_{\bigvee_i \phi}(\vec{x}) \leftrightarrow \bigvee_i R_\phi(\vec{x})\right),
\end{aligned}$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\forall \vec{x}\,\big(R_{\exists y\,\phi}(\vec{x}) \leftrightarrow \exists y\, R_\phi(\vec{x}, y)\big),$$
$$\forall \vec{x}\,\big(R_\phi(\vec{x}) \wedge R_{\neg\phi}(\vec{x}) \to \bot\big),$$
$$\forall \vec{x}\,\big(\top \to R_\phi(\vec{x}) \vee R_{\neg\phi}(\vec{x})\big),$$

whenever the formulas in the subscripts are in $\mathcal{F}$, and adopting here Convention 3.2. Then each $\mathcal{L}$-structure $\mathcal{M}$ has a unique expansion to a model $\mathcal{M}'$ of $\mathcal{T}'$, where $R_\phi^{\mathcal{M}'} := \phi^{\mathcal{M}}$; and Σ_α $\mathcal{L}'$-formulas ϕ' up to equivalence mod $\mathcal{T}'$ are in bijection with $\mathcal{L}$-formulas ϕ which are "**$\boldsymbol{\Sigma}_\alpha$ over $\boldsymbol{\mathcal{F}}$**," meaning constructed as in Σ_α-formulas but replacing atomic formulas with formulas in $\mathcal{F}$. We also call such an expansion $\mathcal{M}'$ a **Morleyization of the structure $\boldsymbol{\mathcal{M}}$** (over the fragment $\mathcal{F}$).

If we start with an $\mathcal{L}$-theory $\mathcal{T}$, then we may instead add the above axioms to $\mathcal{T}$ to obtain a new theory $\mathcal{T}' \supseteq \mathcal{T}$, whose models correspond bijectively to models of $\mathcal{T}$. We may furthermore replace each original axiom $\phi \in \mathcal{T}$ which is "Π_α over $\mathcal{F}$" with the equivalent Π_α $\mathcal{L}'$-axiom ϕ'.

Example 3.4. If we Morleyize the fragment $\mathcal{F}$ consisting of all atomic and negations of atomic formulas, we obtain a theory $\mathcal{T}'$ modulo which the $\Sigma_\alpha, \Pi_\alpha$ $\mathcal{L}'$-formulas, in our "non-Hausdorff" sense, are equivalently the $\Sigma_\alpha, \Pi_\alpha$ $\mathcal{L}$-formulas in the traditional sense.

Example 3.5. Suppose $\mathcal{L}$ is a countable *propositional* language, consisting only of nullary relation symbols. It is best to regard such a language as being 0-sorted, so that an "$\mathcal{L}$-structure" has no underlying set, and is simply a truth assignment $\mathcal{M} : \mathcal{L} \to 2$. The Σ_1 formulas define the open sets of a topology, namely, the Sierpiński power topology on $\mathbb{S}^{\mathcal{L}}$ (also known as the *Scott topology*). The $\Sigma_\alpha, \Pi_\alpha$ formulas define precisely the $\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha$ subsets of $\mathbb{S}^{\mathcal{L}}$. Thus, quasi-Polish spaces are up to homeomorphism precisely the spaces of models of countable Π_2 propositional theories.

The Morleyization of the preceding remark corresponds in this case to changing topology on $\mathbb{S}^{\mathbb{N}}$ to turn it into $2^{\mathbb{N}}$, as in (2.8). More generally, Morleyization of an arbitrary countable fragment $\mathcal{F}$ corresponds to changing topology to make countably many Borel sets open as in (2.7).

(In Definition 4.12, we relate Morleyization of *non-propositional* languages to change of topology.)

Remark 3.6. More generally, given an arbitrary countable language $\mathcal{L}$ and countable $\Pi_2\mathcal{L}$-theory $\mathcal{T}$, and $n \in \mathbb{N}$, the Σ_1 formulas $\phi(x_0, \dots, x_{n-1})$ up to $\mathcal{T}$-provable equivalence yield a quasi-Polish topology on the space of Σ_1 **$\boldsymbol{n}$-types of $\boldsymbol{\mathcal{T}}$**, meaning countably prime (i.e., complement closed under countable disjunctions) filters of such formulas, or equivalently (by the completeness theorem) complete Σ_1 theories of n-tuples in countable models of $\mathcal{T}$. When $n = 0$, this space has equivalent names in other contexts: the *localic reflection* of a topos (see [30, Proposition A4.6.12]), and the

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

topological ergodic decomposition of a Polish group(oid) action (see [35, Corollary 10.3], [16, Section 2]).

§4. Étale spaces and structures.

DEFINITION 4.1. Let X be a topological space. An **étale space over X** is another topological space Y equipped with a continuous map $p : Y \to X$ which is a local homeomorphism, i.e., Y has an open cover consisting of open sets $S \subseteq Y$ such that $p|S : S \to X$ is an open embedding. We will refer to such an $S \subseteq Y$ as an **open section**, and to p as the **projection map**.

We may think of an étale space as a bundle of discrete sets indexed over X, namely, the fibers $Y_x := p^{-1}(x)$, with the discreteness witnessed uniformly across all fibers by the open sections. For this reason, when discussing étale spaces, we usually emphasize the space Y rather than the map p. For general background on étale spaces (and their close relatives, sheaves), see [53].

REMARK 4.2. By (2.10), for a quasi-Polish base space X, an étale space Y over it is quasi-Polish iff it is second-countable, in which case it has a countable basis of open sections.

DEFINITION 4.3. If $p : Y \to X$ is an étale space, and $f : Z \to X$ is a continuous map, we write $f^*(Y) = Z \times_X Y$ for the **pullback** or **fiber product**

$$f^*(Y) = Z \times_X Y := \{(z, y) \in Z \times Y \mid f(z) = p(y)\}$$

fitting into a commutative square:

$$\begin{array}{ccc} f^*(Y) & \longrightarrow & Y \\ {\scriptstyle f^*(p)}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ f\ } & X \end{array}$$

Here $f^*(p)$ is the first coordinate projection, and turns $f^*(Y)$ into an étale space, with a cover by open sections $f^*(S)$ for open sections $S \subseteq Y$. The second projection yields canonical bijections

$$f^*(Y)_z \cong Y_{f(z)}.$$

If Z is also étale over X, then so is $f^*(Y) = Z \times_X Y$, with open sections consisting of $T \times_X S$ for open sections $T \subseteq Z$ and $S \subseteq Y$. We also write

$$Y^n_X := Y \times_X \cdots \times_X Y$$

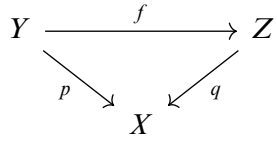
for the n-fold fiber power. When $n = 0$, by convention $Y^0_X := X$. Sometimes, it is convenient to use the following alternative (canonically isomorphic to the above) definition of Y^n_X that explicitly records the basepoint in X, hence extends "uniformly" to the case $n = 0$:

$$Y^n_X := \{(x, y_0, \dots, y_{n-1}) \in X \times Y^n \mid x = p(y_0) = \cdots = p(y_{n-1})\}. \quad (4.4)$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

REMARK 4.5. It is easily seen that an equivalent characterization of local homeomorphism $p : Y \to X$ is that both p and the diagonal embedding $Y \hookrightarrow Y \times_X Y$ are open maps. The latter can be thought of as saying that the equality relation is open in each fiber, uniformly across all fibers.

REMARK 4.6. If $p : Y \to X$ and $q : Z \to X$ are two étale spaces over the same base, and $f : Y \to Z$ is a continuous map **over** X, i.e., making the triangle



commute, then f is automatically open, since for an open p-section $S \subseteq Y$, we have

$$f(S) = \bigcup_T \big(T \cap q^{-1}(p(S \cap f^{-1}(T)))\big),$$

where T varies over open q-sections in Z. It follows that each such S is an open f-section, whence f is in fact étale.

REMARK 4.7. If $p : Y \to X$ is a second-countable étale space, then p (is not only open but) maps $\mathbf{\Sigma}^0_\alpha$ sets onto $\mathbf{\Sigma}^0_\alpha$ sets, since this is clearly true for subsets of each open section $S \subseteq Y$.

It follows that in the preceding remark, assuming Y is second-countable, f also maps $\mathbf{\Sigma}^0_\alpha$ sets onto $\mathbf{\Sigma}^0_\alpha$ sets.

DEFINITION 4.8. Let $\mathcal{L}$ be a countable language, X be a topological space. An **étale $\mathcal{L}$-structure** $\mathcal{M}$ over X is defined by "internalizing the usual definition of first-order structure into the category of étale spaces over X." Concretely, $\mathcal{M}$ consists of the following data:

- an underlying étale space $p : M \to X$ (in place of an underlying set);
- for each n-ary function symbol $f \in \mathcal{L}$, a continuous map $f^{\mathcal{M}} : M^n_X \to M$ over X;
- for each n-ary relation symbol $R \in \mathcal{L}$, an open set $R^{\mathcal{M}} \subseteq M^n_X$.

By an abuse of notation, we will often refer to these data as "the étale structure $p : \mathcal{M} \to X$."

Given such $\mathcal{M}$, we may inductively interpret each term $\tau(x_0, \dots, x_{n-1})$ in the obvious manner to get a continuous map $\tau^{\mathcal{M}} : M^n_X \to M$ over X. We may then interpret each Σ_α $\mathcal{L}_{\omega_1\omega}$ formula $\phi(x_0, \dots, x_{n-1})$ to get a subset $\phi^{\mathcal{M}} \subseteq M^n_X$, which is $\mathbf{\Sigma}^0_\alpha$ assuming that M is second-countable, by an easy induction using the three preceding remarks in the $=, \exists$ cases.

If ϕ is a sentence, then $\phi^{\mathcal{M}} \subseteq M^0_X = X$ (recall Definition 4.3); if $\phi^{\mathcal{M}} = X$, then we call $\mathcal{M}$ an **étale model of** ϕ, denoted $\mathcal{M} \models \phi$.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

REMARK 4.9. For an étale structure $p : \mathcal{M} \to X$, for each $x \in X$, we may restrict everything to the fiber over x to get an ordinary structure $\mathcal{M}_x$. Thus, we may think of $\mathcal{M}$ as a "continuous bundle of discrete structures, indexed over X," or as a "continuous map" (as explained in the introduction):

$$\begin{aligned} \mathcal{M} : X &\longrightarrow \{\text{all } \mathcal{L}\text{-structures}\} \\ x &\longmapsto \mathcal{M}_x. \end{aligned}$$

Here the right-hand side is meant in a vague, informal sense; we do not literally mean a continuous map to a topological space whose elements are $\mathcal{L}$-structures. The whole point is that $\mathcal{M}$ provides a way to regard X itself as a "space of structures." (One may make the above map precise using the *classifying topos of $\mathcal{L}$-structures*; see [30, Chapter D3].)

For each n-ary formula ϕ, the fibers $\phi_x^{\mathcal{M}}$ of $\phi^{\mathcal{M}} \subseteq M_X^n$ are the ordinary interpretations $\phi^{\mathcal{M}_x}$. In particular, for a sentence ϕ, to say that $\mathcal{M} \models \phi$ just means $\mathcal{M}_x \models \phi$ for each x.

EXAMPLE 4.10. For a countable propositional language $\mathcal{L}$, regarded as a 0-sorted first-order language as in Example 3.5, an étale $\mathcal{L}$-structure $p : \mathcal{M} \to X$ consists simply of an open set $P^{\mathcal{M}} \subseteq X$ for each 0-ary relation symbol $P \in \mathcal{L}$, hence literally corresponds to a continuous map $X \to \mathbb{S}^{\mathcal{L}}$ (whose Pth coordinate is the indicator function of $P^{\mathcal{M}}$).

REMARK 4.11. If $p : \mathcal{M} \to X$ is an étale structure, and $f : Z \to X$ is a continuous map, then we have a **pullback structure** $f^*(\mathcal{M}) \to Z$, given by pulling back the underlying étale space M and the interpretations of all the symbols (Definition 4.3). Clearly, for $z \in Z$ and a formula ϕ,

$$\begin{aligned} f^*(\mathcal{M})_z &\cong \mathcal{M}_{f(z)}, \\ \phi^{f^*(\mathcal{M})} &= f^*(\phi^{\mathcal{M}}). \end{aligned}$$

In particular, if $\mathcal{M}$ satisfies some theory, then so does $f^*(\mathcal{M})$.

DEFINITION 4.12. Let $p : \mathcal{M} \to X$ be a second-countable étale structure, and $\mathcal{F}$ be a countable fragment of $\mathcal{L}_{\omega_1\omega}$ as in Definition 3.3, with Morleyization $\mathcal{T}'$. We may then Morleyize each fiber $\mathcal{M}_x$ over $\mathcal{F}$ as in Definition 3.3; however, the resulting bundle may no longer be an étale structure in the original topologies on M, X. Instead, we refine the topologies as follows.

Let X' be X with the finer topology generated by all subbasic open sets of the form

$$p(U \cap \phi^{\mathcal{M}}) = \left\{x \in X \;\middle|\; \exists \vec{a} \in U_x \, (\phi^{\mathcal{M}_x}(\vec{a}))\right\},$$

where $U \subseteq M_X^n$ is a (basic) open set and $\phi \in \mathcal{F}$ is an n-ary formula. This topology indeed refines that of X, since we may take $n = 0$ and $\phi = \top$. Since for m-ary ϕ and n-ary ψ,

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$p(U \cap \phi^{\mathcal{M}}) \cap p(V \cap \psi^{\mathcal{M}}) = p\left((U \times_X V) \cap (\phi(x_0, \dots, x_{m-1}) \wedge \psi(x_m, \dots, x_{m+n-1}))^{\mathcal{M}}\right),$$

a basis for this topology consists of all sets $p(U \cap \phi^{\mathcal{M}})$ as above but now with ϕ "basic over $\mathcal{F}$," i.e., a finite conjunction of formulas in $\mathcal{F}$.

Now let $p' : \mathcal{M}' \to X'$ be the pullback of $\mathcal{M}$ along the identity $X' \to X$. Thus, for each $n \in \mathbb{N}$, an open basis for $M'^n_{X'}$ consists of

$$U \cap p^{-1}(p(V \cap \phi^{\mathcal{M}}))$$

for $U \subseteq M^n_X$ (basic) open, $V \subseteq M^m_X$ (basic) open for some other $m \in \mathbb{N}$, and ϕ a finite conjunction of m-ary formulas in $\mathcal{F}$. For $U = V \subseteq M^n_X$ an open section, the above set becomes simply $U \cap \phi^{\mathcal{M}}$; taking a union over all U yields that $\phi^{\mathcal{M}} \subseteq M'^n_{X'}$ is open for each n-ary $\phi \in \mathcal{F}$, so that we may canonically expand $\mathcal{M}'$ to an étale model of $\mathcal{T}'$ over X', which we call the **Morleyization of the étale structure $\mathcal{M}$** (over $\mathcal{F}$).

LEMMA 4.13. *If X above is quasi-Polish, then so is X'.*

PROOF. Adjoining each $p(U \cap \phi^{\mathcal{M}})$ (in the notation above) to the topology of X yields a quasi-Polish topology, by an induction on ϕ, using that fragments are closed under subformulas (Definition 3.3), and using (2.7)(a) and (b) to handle $\neg, \bigvee, \exists$. ⊣

REMARK 4.14. From the perspective of Remark 4.9, the "space of all models of $\mathcal{T}'$" can be seen as the "space of all $\mathcal{L}$-structures, with a finer topology (generated by $\mathcal{F}$)." The Morleyized base space X' above is then the result of "pulling back this finer topology along $\mathcal{M} : X \to \{\text{all } \mathcal{L}\text{-structures}\}$":

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle \mathcal{M}'}\downarrow & & \downarrow{\scriptstyle \mathcal{M}} \\ \{\text{all } \mathcal{T}'\text{-models}\} & \longrightarrow & \{\text{all } \mathcal{L}\text{-structures}\}. \end{array}$$

DEFINITION 4.15. Let $p : \mathcal{M} \to X$ be an étale structure. Its **isomorphism groupoid** $\mathrm{Iso}_X(\mathcal{M})$ is the set of all triples (x, y, g) where $x, y \in X$ and $g : \mathcal{M}_x \cong \mathcal{M}_y$ is an isomorphism. We equip it with the topology generated by the first and second projections to X as well as the subbasic open sets

$$[\![U \mapsto V]\!] := \left\{(x, y, g) \in \mathrm{Iso}_X(\mathcal{M}) \;\middle|\; g(U_x) \cap V_y \neq \varnothing\right\}$$

for open $U, V \subseteq M$. Equivalently, $\mathrm{Iso}_X(\mathcal{M})$ has an open basis consisting of the sets $[\![U \mapsto V]\!]$, defined in the same way, for all open $U, V \subseteq M^n_X$ where $n \in \mathbb{N}$.

Since the expression $[\![U \mapsto V]\!]$ is easily seen to preserve unions in each of the variables U, V, it is enough to consider U, V in any basis, e.g., a basis of

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

open sections. This reveals the topology on $\mathrm{Iso}_X(\mathcal{M})$ to be a generalization of the usual pointwise convergence topology on the automorphism group of a single structure, to which $\mathrm{Iso}_X(\mathcal{M})$ reduces when $X = 1$.

As its name suggests, the isomorphism groupoid $\mathrm{Iso}_X(\mathcal{M})$ is (the space of morphisms of) a **topological groupoid** with space of objects X. The domain and codomain of a morphism are

$$\mathrm{dom}(x, y, g) := x, \qquad \mathrm{cod}(x, y, g) := y.$$

These maps $\mathrm{dom}, \mathrm{cod} : \mathrm{Iso}_X(\mathcal{M}) \to X$ and the evident composition, identity, and inverse operations are easily seen to be continuous. (For the abstract notion of topological groupoid, see Definition 11.1.)

We have a natural action of $\mathrm{Iso}_X(\mathcal{M})$ on the underlying étale space $M \to X$:

$$\begin{aligned} \mathrm{Iso}_X(\mathcal{M}) \times_X M &\longrightarrow M \\ ((x, y, g), a) &\longmapsto g(a). \end{aligned}$$

In other words, each morphism $(x, y, g) : x \to y \in \mathrm{Iso}_X(\mathcal{M})$ acts as the bijection $g : M_x \to M_y$. This action is easily seen to be jointly continuous. Similarly, $\mathrm{Iso}_X(\mathcal{M})$ acts continuously on the fiber powers M_X^n. (For the abstract notion of groupoid action, again see Definition 11.1.)

We will often abuse notation and refer to $(x, y, g) \in \mathrm{Iso}_X(\mathcal{M})$ simply as $g : \mathcal{M}_x \cong \mathcal{M}_y$.

LEMMA 4.16. *If $\mathcal{L}$ is a countable language and $p : \mathcal{M} \to X$ is a second-countable étale $\mathcal{L}$-structure over a quasi-Polish space X, then $\mathrm{Iso}_X(\mathcal{M})$ is quasi-Polish.*

PROOF. First, consider the case where $\mathcal{L} = \varnothing$, so that $\mathcal{M}$ is just a second-countable étale space and $\mathrm{Iso}_X(\mathcal{M}) = \{\text{all bijections between fibers}\}$. The definition of the topology on $\mathrm{Iso}_X(\mathcal{M})$ amounts to embedding it in the fiberwise lower powerspace (Definition 2.13) of $p \times p : M \times M \to X \times X$ via

$$\begin{aligned} \mathrm{Iso}_X(\mathcal{M}) &\longrightarrow \mathcal{F}_{X\times X}(M \times M) \\ (x, y, g) &\longmapsto (x, y, \mathrm{graph}(g)). \end{aligned}$$

So we need only check that the image of this embedding is $\mathbf{\Pi}^0_2$. Fix a countable basis $\mathcal{S}$ of open sections in M. For $(x, y, F) \in \mathcal{F}_{X\times X}(M \times M)$, $F \subseteq M_x \times M_y$ is the graph of a function iff

$$\forall S \in \mathcal{S}\,\big(x \in p(S) \implies \exists T \in \mathcal{S}\,(F \cap (S \times T) \neq \varnothing)\big),$$

$$\forall S, T_1, T_2 \in \mathcal{S}\left(\begin{array}{l} F \cap (S \times T_1), F \cap (S \times T_2) \neq \varnothing \\ \quad \implies \exists \mathcal{S} \ni T_3 \subseteq T_1 \cap T_2\,(F \cap (S \times T_3) \neq \varnothing) \end{array}\right)$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

(which say that for each $a \in M_x$, there is at least, resp., at most, one $b \in M_y$ such that $(a,b) \in F$); these conditions are $\mathbf{\Pi}^0_2$ in the topology on $\mathcal{F}_{X\times X}(M \times M)$. Similarly, to say that F^{-1} is the graph of a function is a $\mathbf{\Pi}^0_2$ condition.

Now if $R \in \mathcal{L}$ is an n-ary relation symbol, then to say that $g : M_x \cong M_y$ preserves R means

$$\forall \vec{S}, \vec{T} \in \mathcal{S}^n \begin{pmatrix} x \in p((S_0 \times_X \cdots \times_X S_{n-1}) \cap R^{\mathcal{M}}) \ \&\ g \in \bigcap_i [\![S_i \mapsto T_i]\!] \\ \implies y \in p((T_0 \times_X \cdots \times_X T_{n-1}) \cap R^{\mathcal{M}}) \end{pmatrix}.$$

Similarly for g^{-1} to preserve R, and for preservation of functions. ⊣

Remark 4.17. If X and M are both zero-dimensional Polish, then so is $\mathrm{Iso}_X(\mathcal{M})$. Indeed, we may find countable bases for each M^n_X consisting of clopen sections $S \subseteq M^n_X$ such that $p(S) \subseteq X$ is also clopen. If $S, T \subseteq M^n_X$ are two such sections, then the basic open $[\![S \mapsto T]\!] \subseteq \mathrm{Iso}_X(\mathcal{M})$ has open complement $(\mathrm{dom}^{-1}(p(S)) \cap \mathrm{cod}^{-1}(p(T)) \Rightarrow [\![S \mapsto \neg T]\!]) \subseteq \mathrm{Iso}_X(\mathcal{M})$.

Remark 4.18. We have the following evident generalization of $\mathrm{Iso}_X(\mathcal{M})$: if $p : \mathcal{M} \to X$ and $q : \mathcal{N} \to Y$ are two étale structures over two base spaces, then we have a space $\mathrm{Iso}_{X,Y}(\mathcal{M},\mathcal{N})$ of isomorphisms between a fiber of $\mathcal{M}$ and a fiber of $\mathcal{N}$. If $\mathcal{M}, \mathcal{N}$ are both second-countable over quasi-Polish X, Y, then $\mathrm{Iso}_{X,Y}(\mathcal{M},\mathcal{N})$ is quasi-Polish (zero-dimensional if X, Y, M, N are).

§5. Étale structures with Σ_1 saturations. We now come to the crucial condition on an étale $\mathcal{L}$-structure, that implies that it is a "nice" parametrization of countable $\mathcal{L}$-structures obeying the usual model-theoretic results.

Definition 5.1. Let $\mathcal{M}$ be a second-countable étale $\mathcal{L}$-structure over a space X. We say that $\mathcal{M}$ **has $\mathbf{\Sigma_1}$-definable saturations of open sets**, or more succinctly **has $\mathbf{\Sigma_1}$ saturations**, if for every $n \in \mathbb{N}$ and (basic) open set $U \subseteq M^n_X$, the saturation $\mathrm{Iso}_X(\mathcal{M}) \cdot U$ under the natural action of $\mathrm{Iso}_X(\mathcal{M})$ (recall Definition 4.15) is equal to $\phi^{\mathcal{M}}$ for some Σ_1 formula $\phi(x_0, \ldots, x_{n-1})$.

Remark 5.2. From the perspective of Remark 4.9, this means that

$$\mathcal{M} : X \longrightarrow \{\text{all } \mathcal{L}\text{-structures}\}$$

is a "continuous open map onto its image." (This is literally true for a propositional language $\mathcal{L}$ via Example 4.10, and is made precise for general first-order $\mathcal{L}$ by the notion of an *open geometric surjection* between toposes; see [30, C3.1], [40, IX.6].)

Remark 5.3. Definition 5.1 may also be regarded as saying that $\mathcal{M}$ is "uniformly Σ_1-atomic": when $X = 1$, it means that every $\mathcal{L}_{\omega_1\omega}$-type realized in $\mathcal{M}$ is axiomatized by a Σ_1 formula.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

THEOREM 5.4. *If $\mathcal{L}$ is a countable language and $\mathcal{M}$ is a second-countable étale $\mathcal{L}$-structure with Σ_1 saturations over a quasi-Polish space X, then the class of countable $\mathcal{L}$-structures isomorphic to a fiber of $\mathcal{M}$ is axiomatizable by a countable Π_2 theory.*

This result can be regarded as a common generalization of the well-known special cases alluded to by the two preceding remarks: that a continuous map between (quasi-)Polish spaces which is open onto its image has $\mathbf{\Pi}^0_2$ image (the combination of (2.12) and (2.11)), and that an atomic model has a Π_2 Scott sentence (relative to a fragment; see [25, Theorem 11.5.7]). The following proof can be seen as combining these two special cases, namely, the proofs of (2.12), (2.11) in [13, Theorems 10.1 and 4.1] and the usual back-and-forth construction of Scott sentences in, e.g., [48, Section 3.3].

PROOF. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a countable open basis for the fiber power M_X^n. We use the fact that every quasi-Polish space has a countable posite representation; see [13, Section 8] for details. Briefly, each $\vec{a} \in M_X^n$ is determined by its neighborhood filter basis $\mathcal{N}_{\vec{a}} \subseteq \mathcal{U}_n$; and this map $\vec{a} \mapsto \mathcal{N}_{\vec{a}}$ yields an embedding $M_X^n \hookrightarrow \mathbb{S}^{\mathcal{U}_n}$, which thus has $\mathbf{\Pi}^0_2$ image (2.11). Using that $\mathcal{U}_n$ is a basis, one can manipulate a $\mathbf{\Pi}^0_2$ definition of this image into the following form: for a countable binary relation $\rhd_n$ consisting of certain pairs $(\mathcal{V}, U)$ of open covers $\mathcal{V} \subseteq \mathcal{U}_n$ and $U \in \mathcal{U}_n$ with $\bigcup \mathcal{V} = U$, such that

$$\mathcal{V} \rhd_n U \supseteq U' \in \mathcal{U}_n \implies \exists \mathcal{V}' \rhd_N U' \, \forall V' \in \mathcal{V}' \, \exists V \in \mathcal{V} \, (V \supseteq V')$$

("every cover of U refines to one of $U' \subseteq U$"), we have that $\mathcal{A} \subseteq \mathcal{U}_n$ is a neighborhood filter of some $\vec{a} \in M_X^n$ iff it is a $\rhd_n$-prime filter, meaning a filter satisfying the further $\mathbf{\Pi}^0_2$ condition

$$\forall \mathcal{V} \rhd_n U \, \big(U \in \mathcal{A} \implies \exists V \in \mathcal{V} \, (V \in \mathcal{A}) \big). \tag{5.5}$$

It then follows that the $\mathcal{C} \subseteq \mathcal{U}_n$ which are merely upward-closed and satisfy this condition correspond via (2.14) to arbitrary closed sets $F \in \mathcal{F}(M_X^n)$; see [13, proof of Theorem 9.2].

For each $U \in \mathcal{U}_n$, let $\phi_U(x_0, \dots, x_{n-1})$ be a Σ_1 formula defining $\mathrm{Iso}_X(\mathcal{M}) \cdot U \subseteq M_X^n$. We claim that the following Π_2 sentences axiomatize the countable $\mathcal{L}$-structures isomorphic to a fiber of $\mathcal{M}$. For each n, let $\pi_n : M_X^{n+1} \to M_X^n$ denote the projection onto the first n coordinates.

$$\forall \vec{x} \, \big(\phi_U(\vec{x}) \to \phi_V(\vec{x}) \big) \qquad \text{for } U, V \in \mathcal{U}_n, U \subseteq V, \tag{5.6}$$

$$\forall \vec{x} \, \Big(\phi_U(\vec{x}) \to \bigvee_{V \in \mathcal{V}} \phi_V(\vec{x}) \Big) \qquad \text{for } \mathcal{V} \rhd_n U, \tag{5.7}$$

$$\forall \vec{x} \, \big(\phi_U(\vec{x}) \to \psi(\vec{x}) \big) \qquad \text{for basic } n\text{-ary } \psi \text{ and } \mathcal{U}_n \ni U \subseteq \psi^{\mathcal{M}}, \tag{5.8}$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\forall \vec{x}\left(\phi_U(\vec{x}) \wedge \psi(\vec{x}) \to \bigvee_{\mathcal{U}_n \ni V \subseteq U \cap \psi^{\mathcal{M}}} \phi_V(\vec{x})\right) \qquad \text{for basic } n\text{-ary } \psi, \quad (5.9)$$

$$\forall \vec{x}\left(\exists y\, \phi_U(\vec{x}, y) \leftrightarrow \bigvee_{\mathcal{U}_n \ni V \subseteq \pi_n(U)} \phi_V(\vec{x})\right) \qquad \text{for } U \in \mathcal{U}_{n+1}, \quad (5.10)$$

$$\forall \vec{x}, y\left(\phi_U(\vec{x}) \to \bigvee_{\mathcal{U}_{n+1} \ni V \subseteq \pi_n^{-1}(U)} \phi_V(\vec{x}, y)\right) \qquad \text{for } U \in \mathcal{U}_n, \quad (5.11)$$

$$\top \to \bigvee_{U \in \mathcal{U}_0} \phi_U. \quad (5.12)$$

It is straightforward to check that each $\mathcal{M}_x$ indeed satisfies these axioms.

Now let $\mathcal{N}$ be another countable $\mathcal{L}$-structure satisfying these axioms. For each $\vec{b} \in N^n$,

$$\mathcal{C}_{\vec{b}} := \{U \in \mathcal{U}_n \mid \phi_U^{\mathcal{N}}(\vec{b})\}$$

is upward-closed and satisfies (5.5) by (5.6) and (5.7), thus corresponds to a closed $F_{\vec{b}} \subseteq M_X^n$ such that

$$F_{\vec{b}} \cap U \neq \varnothing \iff U \in \mathcal{C}_{\vec{b}} \iff \phi_U^{\mathcal{N}}(\vec{b}) \qquad \forall U \in \mathcal{U}_n.$$

Using this equivalence, for $\mathcal{N}$ to satisfy the other five axioms above means the following.

- (5.8) means $F_{\vec{b}} \cap \psi^{\mathcal{M}} \neq \varnothing \implies \psi^{\mathcal{N}}(\vec{b})$ for each basic formula ψ, which means that each $\vec{a} \in F_{\vec{b}}$ yields a well-defined partial homomorphism $\mathcal{M}_x \rightharpoonup \mathcal{N}$ given by $a_i \mapsto b_i$. (Well-definedness follows by taking $\psi(\vec{x})$ to be an equality $x_i = x_j$, so that $a_i = a_j \implies b_i = b_j$.)
- (5.9) means $F_{\vec{b}} \cap U \neq \varnothing$ & $\psi^{\mathcal{N}}(\vec{b}) \implies F_{\vec{b}} \cap U \cap \psi^{\mathcal{M}} \neq \varnothing$, i.e., the set of $\vec{a} \in F_{\vec{b}}$ for which the above partial homomorphism $a_i \mapsto b_i$ is a partial isomorphism (onto its image) is dense G_δ.
- (5.10) means $(\bigcup_{d \in N} F_{\vec{b},d}) \cap U \neq \varnothing \iff F_{\vec{b}} \cap \pi_n(U) \neq \varnothing$, i.e., $\overline{\bigcup_{d \in N} F_{\vec{b},d}} = \pi_n^{-1}(F_{\vec{b}})$.
- In particular, $\pi_n : M_X^{n+1} \to M_X^n$ restricts to a map $F_{\vec{b},d} \to F_{\vec{b}}$; (5.11) says it has dense image.
- (5.12) means $F_\varnothing \neq \varnothing$.

Consider now the forest of all $\vec{a} \in F_{\vec{b}}$, each of which determines a partial homomorphism as above, as well as all infinite branches through this forest, which we arrange into a space as follows:

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$Y_n := \bigsqcup_{\vec{b}\in N^n} F_{\vec{b}} = \left\{(\vec{b},\vec{a}) \in N^n \times M_X^n \;\middle|\; \vec{a} \in F_{\vec{b}}\right\} \subseteq N^n \times M_X^n,$$

$$Y_\omega := \varprojlim_{n\in\mathbb{N}} Y_n = \left\{(\vec{b},\vec{a}) \in N^{\mathbb{N}} \times M_X^{\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N}\,((\vec{b}|n,\vec{a}|n) \in Y_n)\right\},$$

$$Y := \bigsqcup_{n\le\omega} Y_n.$$

For $m \le n \le \omega$, let $\pi_{nm} : Y_n \to Y_m$ denote the projection. Put a topology on Y with an open basis consisting of, for each $m \in \mathbb{N}$ and (basic) open $U \subseteq Y_m$ (where Y_m has the disjoint union topology),

$$\Uparrow U := \bigsqcup_{\omega\ge n\ge m} \pi_{nm}^{-1}(U).$$

This topology is quasi-Polish; see [16, Lemma A.2], where it was called the *lax colimit* of the levels Y_n. Since each $\pi_n : F_{\vec{b},d} \to F_{\vec{b}}$ has dense image by (5.11), the closure in Y of each $F_{\vec{b}} \subseteq Y_n \subseteq Y$ is all $F_{\vec{b}|m}$ for initial segments $\vec{b}|m$ of $\vec{b}$. Thus each $\Uparrow Y_n \subseteq Y$ is dense, and so $Y_\omega = \bigcap_n \Uparrow Y_n$ is dense G_δ. And by (5.12), $Y_0 = F_\varnothing \neq \varnothing$, i.e., the forest has at least one root.

We now verify that the set of all $(\vec{b},\vec{a}) \in Y$ for which the partial homomorphism $a_i \mapsto b_i$ is an isomorphism $\mathcal{M}_x \cong \mathcal{N}$ is comeager, which will complete the proof.

- The set of $(\vec{b},\vec{a}) \in Y$ for which the partial homomorphism $a_i \mapsto b_i$ is a *partial* isomorphism is dense $\mathbf{\Pi}^0_2$, since it is the intersection, over all $n \in \mathbb{N}$, of the union of the lower levels $Y_0, \dots, Y_{n-1}$ (which is closed) and $\Uparrow$ of the disjoint union over all $\vec{b} \in N^n$ of those $\vec{a} \in F_{\vec{b}}$ for which $a_i \mapsto b_i$ is a partial isomorphism; and this latter $\Uparrow$ is dense G_δ in $\Uparrow Y_n$ by (5.9).
- The set of $(\vec{b},\vec{a}) \in Y$ for which $\vec{b}$ enumerates N is clearly G_δ, and is dense because for a nonempty basic open $\Uparrow(F_{\vec{b}} \cap U) \subseteq Y$ where $U \in \mathcal{U}_n$, also $F_{\vec{b},d} \cap \pi_n^{-1}(U) \neq \varnothing$ by (5.11).
- Finally, the set of $(\vec{b},\vec{a}) \in Y$ for which $\vec{a} \in M_X^n$ enumerates the fiber M_x in which it lies is dense $\mathbf{\Pi}^0_2$: we will show that for each open section $S \in \mathcal{U}_1$, the set of $(\vec{b},\vec{a})$ such that if $\vec{a}$ lies in a fiber over $p(S)$, then $\vec{a}$ includes a point in S (which is a $\mathbf{\Pi}^0_2$ set), is dense, which suffices because each point in M is in some open section. For a nonempty basic open $\Uparrow(F_{\vec{b}} \cap U) \subseteq Y$ where $U \in \mathcal{U}_n$, pick any $\vec{a} \in F_{\vec{b}} \cap U$, say lying in the fiber M_x. If $x \in p(S)$, then $U \times_X S \subseteq M_X^{n+1}$ is an open set such that $F_{\vec{b}} \cap \pi_n(U \times_X S) \neq \varnothing$, since it contains $\vec{a}$. Thus by the $\Longleftarrow$ direction of (5.10), for some $d \in N$, we have $F_{\vec{b},d} \cap (U \times_X S) \neq \varnothing$; an element $(\vec{a}', c)$ of this intersection then belongs to $\Uparrow(F_{\vec{b}} \cap U)$ (since $\vec{a}' \in U$) and includes $c \in S$. ⊣

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

DEFINITION 5.13. If $\mathcal{M} \to X$ is a second-countable étale structure, and a theory $\mathcal{T}$ axiomatizes the countable structures isomorphic to some $\mathcal{M}_x$, we say that ***X* parametrizes models of $\mathcal{T}$ via $\mathcal{M}$**.

Thus, Theorem 5.4 says that if X is quasi-Polish and $\mathcal{M}$ has Σ_1 saturations, then X parametrizes models of a Π_2 theory. Conversely, every Π_2 theory has a quasi-Polish parametrization with Σ_1 saturations; see Example 6.9.

LEMMA 5.14. *If $p : \mathcal{M} \to X$ is a second-countable étale structure with Σ_1 saturations, and $f : Z \to X$ is a continuous open map, then the pullback structure $f^*(\mathcal{M}) \to Z$ (Remark 4.11) also has Σ_1 saturations.*

Via Remark 5.2, this says that "the composite of open maps $\mathcal{M} \circ f$ is still open".

PROOF. Let $\pi_2 : f^*(M) \to M$ be the pullback projection (see the diagram in Definition 4.3), which is open because it is a pullback of f. It is easily seen that for $U \subseteq f^*(M)^n_Z$, we have

$$\mathrm{Iso}_Z(f^*(\mathcal{M})) \cdot U = \pi_2^{-1}(\mathrm{Iso}_X(\mathcal{M}) \cdot \pi_2(U)).$$

Thus if ϕ defines $\mathrm{Iso}_X(\mathcal{M}) \cdot \pi_2(U) \subseteq M^n_X$, then ϕ also defines $\mathrm{Iso}_Z(f^*(\mathcal{M})) \cdot U$. ⊣

REMARK 5.15. If $p : \mathcal{M} \to X$ is a second-countable étale structure with Σ_1 saturations, and $Z = \{x \in X \mid \mathcal{M}_x \models \mathcal{T}\}$ for a theory $\mathcal{T}$, then the restriction $\mathcal{M}|Z$, i.e., pullback of $\mathcal{M}$ along the inclusion $Z \hookrightarrow X$, clearly also has Σ_1 saturations, as witnessed by the same Σ_1 formulas.

(This is not an instance of the preceding lemma, since $Z \subseteq X$ need not be open.)

Results such as 5.14 and 5.15 clearly work just as well if "Σ_1 saturations" is replaced with "Σ_α saturations" throughout, for some $\alpha < \omega_1$. The following calculation provides a rigorous way of systematically making such generalizations.

LEMMA 5.16. *Let $p : \mathcal{M} \to X$ be a second-countable étale structure, and $\mathcal{F}$ be a countable fragment of $\mathcal{L}_{\omega_1\omega}$. Suppose that saturations of open sets in $\mathcal{M}$ are definable by formulas which are Σ_1 over $\mathcal{F}$. Then the Morleyized étale structure $p' : \mathcal{M}' \to X'$ (Definition 4.12) has Σ_1 saturations.*

PROOF. From Definition 4.12, $M'^n_{X'}$ has an open basis of sets of the form

$$\begin{aligned} U \cap p^{-1}(p(V \cap \phi^{\mathcal{M}})) &= \pi_1(U \times_X (V \cap \phi^{\mathcal{M}})) \\ &= \pi_1((U \times_X V) \cap \phi(x_n, \dots, x_{n+m-1})^{\mathcal{M}}) \end{aligned}$$

where $U \subseteq M^n$ is open, $V \subseteq M^m$ is open for some other $m \in \mathbb{N}$, ϕ is a finite conjunction of m-ary formulas in $\mathcal{F}$, $\pi_1 : M^n \times_X M^m \to M^n$ is the first

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

coordinate projection, and $\phi(x_n, \dots, x_{n+m-1})^{\mathcal{M}} = M_X^n \times_X \phi^{\mathcal{M}}$; since the last is isomorphism-invariant, we get

$$\begin{aligned}&\mathrm{Iso}_{X'}(\mathcal{M}') \cdot (U \cap p^{-1}(p(V \cap \phi^{\mathcal{M}})))\\ &\quad = \pi_1\big((\mathrm{Iso}_X(\mathcal{M}) \cdot (U \times_X V)) \cap \phi(x_n, \dots, x_{n+m-1})^{\mathcal{M}}\big)\\ &\quad = \big(\exists x_n, \dots, x_{n+m-1}\, \psi(x_0, \dots, x_{n+m-1}) \wedge \phi(x_n, \dots, x_{n+m-1})\big)^{\mathcal{M}}\end{aligned}$$

where ψ is a formula defining $\mathrm{Iso}_X(\mathcal{M}) \cdot (U \times_X V) \subseteq M_X^{n+m}$ which is Σ_1 over $\mathcal{F}$, hence may be replaced in $\mathcal{M}'$ by an equivalent Σ_1 formula in the Morleyized language $\mathcal{L}'$. ⊣

COROLLARY 5.17. *If $p : \mathcal{M} \to X$ is a second-countable étale structure over quasi-Polish X with Σ_α saturations, then the class of structures isomorphic to one of its fibers is $\Pi_{\alpha+1}$-axiomatizable.*

This is the Σ_α version of Theorem 5.4, and generalizes (U1) $\Longrightarrow$ (U2) of Montalbán's characterization of Scott rank [48] (see Theorem 7.4 below for other parts of the characterization).

PROOF. Let $\mathcal{F}$ be the countable fragment generated by Σ_α formulas defining the saturations of basic open sets in $\mathcal{M}$. By Lemma 5.16, the Morleyized $p' : \mathcal{M}' \to X'$ has Σ_1 saturations. Thus by Theorem 5.4, its fibers are Π_2-axiomatizable in the Morleyized language $\mathcal{L}'$, hence $\Pi_{\alpha+1}$-axiomatizable in the original language $\mathcal{L}$. ⊣

LEMMA 5.18. *If $p : \mathcal{M} \to X$ is a second-countable étale structure with Σ_1 saturations, then $\mathrm{Iso}_X(\mathcal{M})$ is an **open topological groupoid**, i.e., $\mathrm{dom}, \mathrm{cod} : \mathrm{Iso}_X(\mathcal{M}) \rightrightarrows X$ are open.*

PROOF. For open $U, V \subseteq M_X^n$, we have $\mathrm{cod}([\![U \mapsto V]\!]) = p((\mathrm{Iso}_X(\mathcal{M}) \cdot U) \cap V)$; similarly for dom. ⊣

REMARK 5.19. More generally, if $p : \mathcal{M} \to X$ and $q : \mathcal{N} \to Y$ are two étale structures, and $\mathcal{M}$ has Σ_1 saturations, then $\mathrm{cod} : \mathrm{Iso}_{X,Y}(\mathcal{M}, \mathcal{N}) \to Y$ (recall Remark 4.18) is open.

§6. Examples of parametrizations. We begin by recasting the standard zero-dimensional Polish parametrization of countably infinite models in terms of an étale structure; see [33, Definition 16.5], [25, Section 3.6].

EXAMPLE 6.1 (space of structures on $\mathbb{N}$). Let $\mathcal{L}$ be a countable language,

$$X := \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \times \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n}$$

be the zero-dimensional Polish space of $\mathcal{L}$-structures on $\mathbb{N}$, with the product topology, and

$$M := X \times \mathbb{N}$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

equipped with the first projection $p : M \to X$; this is clearly étale. On each fiber $M_x = \{x\} \times \mathbb{N} \cong \mathbb{N}$ of M, we may put the structure $\mathcal{M}_x := (M_x, x)$; that is,

$$P^{\mathcal{M}}(x, a_0, \dots, a_{n-1}) := x(P)(a_0, \dots, a_{n-1})$$

for each symbol (function or relation) $P \in \mathcal{L}$, using here the "based" representation (4.4) of M_X^n. This defines an étale $\mathcal{L}$-structure $\mathcal{M}$ over X. The isomorphism groupoid is

$$\mathrm{Iso}_X(\mathcal{M}) = \{(x, y, g) \in X \times X \times S_\infty \mid g \cdot x = y\}$$

where S_∞ is the infinite symmetric group of all bijections $\mathbb{N} \cong \mathbb{N}$, acting on X via pushforward of structure (the **logic action**). Up to topological groupoid isomorphism, we may forget about the second coordinate y above, yielding $\mathrm{Iso}_X(\mathcal{M}) \cong X \times S_\infty$ (the action groupoid of the logic action).

This $\mathcal{M}$ does not have Σ_1 saturations, however, but only Σ_2 saturations. The complement of the diagonal in M_X^2 is open and $\mathrm{Iso}_X(\mathcal{M})$-invariant, but not Σ_1-definable, since $\neq$ is not Σ_1; similarly for negations of any relation symbols in $\mathcal{L}$. In fact, this is the only issue: by definition of the product topology on X, a basic open set in $M_X^n \cong X \times \mathbb{N}^n$ is of the form

$$\{(x, a_0, \dots, a_{n-1}) \mid \phi^x(a_n, \dots, a_{n+m-1})\}$$

for some fixed $\vec{a} \in \mathbb{N}^{n+m}$ and finite conjunction $\phi(y_n, \dots, y_{n+m-1})$ of atomic $\mathcal{L}$-formulas and their negations; the $\mathrm{Iso}_X(\mathcal{M})$-saturation of this set is easily seen to be defined by the Σ_2 formula

$$\psi(y_0, \dots, y_{n-1}) := \exists y_n, \dots, y_{n+m-1} \left(\phi(y_n, \dots, y_{n+m-1}) \wedge \bigwedge_{a_i = a_j} (y_i = y_j) \wedge \bigwedge_{a_i \neq a_j} (y_i \neq y_j)\right).$$

(This is the base case of the proof of the classical Lopez-Escobar theorem; see [33, Proposition 16.9].)

We may thus Morleyize all negated atomic formulas, as in Example 3.4, to obtain an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ and a Π_2 $\mathcal{L}'$-theory $\mathcal{T}'$. Let $\mathcal{M}'$ be the unique expansion of $\mathcal{M}$ to an étale model of $\mathcal{T}'$, by interpreting the newly added relation symbols as the complements of the atomic formulas interpreted in $\mathcal{M}$ (which are clopen). Then $\mathcal{M}'$ has Σ_1 saturations, namely, given by the Morleyized formulas ψ' corresponding to the above ψ (as in Definition 3.3). A Π_2 axiomatization of the fibers of $\mathcal{M}'$ is given by $\mathcal{T}'$ together with the "infinite models" axioms for each $n \in \mathbb{N}$

$$\exists x_0, \dots, x_{n-1} \bigwedge_{i \neq j} (x_i \neq x_j). \tag{6.2}$$

If we started with a Π_2 $\mathcal{L}$-theory $\mathcal{T}$ which already has only infinite models, then we may restrict this $\mathcal{M}'$ to the models of $\mathcal{T}$ (as in Remark 5.15) to

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

obtain a parametrization of the models of the Morleyized $\mathcal{T}$; if furthermore $\mathcal{T}$ already implied that every negated atomic formula is equivalent to a Σ_1 formula, then there is no need to Morleyize.

The "infinite models" restriction in the above example may be lifted, by enlarging the parametrizing space X to include finite models. There are several ways to do so, however, that affect the saturation complexity of the resulting $\mathcal{M}$, i.e., the amount of Morleyization needed to obtain Σ_1 saturations. The simplest approach is the following:

EXAMPLE 6.3 (space of structures on $\leq \mathbb{N}$ with Δ_1 sizes). For each $N \leq \mathbb{N}$ (i.e., $N \in \mathbb{N} \cup \{\mathbb{N}\}$), define an étale structure $\mathcal{M}_N \to X_N$ over the space X_N of $\mathcal{L}$-structures on N, exactly as above. We may then simply take the disjoint unions $X := \bigsqcup_{N \leq \mathbb{N}} X_N$ and $\mathcal{M} := \bigsqcup_{N \leq \mathbb{N}} \mathcal{M}_N$. The groupoid $\mathrm{Iso}_X(\mathcal{M})$ consists of the disjoint union of the logic actions of each symmetric group S_N on X_N.

However, each $X_N \subseteq X = M_X^0$ is clopen invariant, while "the model has size N" is Σ_3-definable for finite N (by a conjunction of a sentence (6.2) and a negation of such a sentence) and Π_3-definable for infinite N (by the infinite conjunction of all (6.2)); thus this $\mathcal{M}$ only has Σ_4 saturations. Even if we first Morleyize $\neq$, i.e., use the traditional Σ_1 that includes $\neq$, we still only get Σ_3 saturations. (It is easy to see that Morleyizing the above sentences (6.2), their subformulas, their negations, and their conjunction, thereby rendering each $X_N \subseteq X$ Σ_1-definable, is sufficient to yield Σ_1 saturations, since each $\mathcal{M}_N$ individually has Σ_1 saturations by the same argument as in the preceding example.)

The worst feature of this approach is that infinite models are clopen in X but only Π_3-definable. In order to reduce this gap, we need a coarser topology on the disjoint union $X = \bigsqcup_N X_N$ that allows finite models to converge to infinite ones. In the next few examples, we assume for simplicity that $\mathcal{L}$ is relational; there is little loss of generality in doing so, since a Π_2 theory may be used to impose that a relation is the graph of a function.

EXAMPLE 6.4 (space of structures on $\leq \mathbb{N}$ with Δ_1 finite sizes). Let $\overline{\mathbb{N}} := \{0, 1, 2, \ldots, \mathbb{N}\}$, with the topology of the one-point compactification of $\mathbb{N}$, i.e., $0, 1, 2, \ldots$ are isolated points converging to $\mathbb{N}$. We may realize the disjoint union X in the preceding example as

$$\begin{aligned} X &:= \{(N, x) \mid N \in \overline{\mathbb{N}} \ \&\ x \in X_N\} \quad \text{where } X_N \text{ is as above} \\ &= \left\{ (N, x(R))_{R \in \mathcal{L}} \in \overline{\mathbb{N}} \times \prod_{R \in \mathcal{L}} 2^{\mathbb{N}^n} \;\middle|\; \forall\, n\text{-ary } R \in \mathcal{L}\, (x(R) \subseteq N^n \subseteq \mathbb{N}^n) \right\}, \end{aligned}$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

regarded now as a closed subspace of the compact zero-dimensional space $\overline{\mathbb{N}} \times \prod_{R\in\mathcal{L}} 2^{\mathbb{N}^n}$, and

$$M := \{(N, x, a) \in X \times \mathbb{N} \mid a \in N\} \subseteq \overline{\mathbb{N}} \times \prod_{R\in\mathcal{L}} 2^{\mathbb{N}^n} \times \mathbb{N},$$
$$R^{\mathcal{M}}(N, x, a_0, \dots, a_{n-1}) := x(R)(a_0, \dots, a_{n-1}) \quad \text{for } R \in \mathcal{L}.$$

A basic open set in $\overline{\mathbb{N}}$ is some finite Boolean combination of the sets $\{n, n+1, \dots, \mathbb{N}\}$ which are axiomatized by the sentences (6.2). Thus, a basic open set in M_X^n is

$$\left\{ (N, x, a_0, \dots, a_{n-1}) \;\middle|\; \begin{array}{l} (N, x) \in X \;\&\; a_0, \dots, a_{n-1} \in N \\ \;\&\; \phi^{(N,x)}(a_n, \dots, a_{n+m-1}) \wedge \psi^N \end{array} \right\} \tag{6.5}$$

for some $\vec{a} \in \mathbb{N}^{n+m}$, finite conjunction $\phi(y_n, \dots, y_{n+m-1})$ of atomic $\mathcal{L}$-formulas and their negations, and finite conjunction ψ of the sentences (6.2) and their negations. Note that in addition to the condition "$a_0, \dots, a_{n-1} \in N$", for each non-negated atomic formula in ϕ, each of its arguments among $a_n, \dots, a_{n+m-1}$ is also implicitly required to be in N. (Once we remember these implicit conditions, we may assume ϕ contains no equalities, which have constant truth value given $\vec{a}$.) All of these conditions of the form "$a_i \in N$" together mean $N \geq \max_i(a_i + 1)$, which can itself be expressed by a single sentence θ of the form (6.2). Thus the saturation of the above set is defined by

$$\exists y_n, \dots, y_{n+m-1} \left(\phi(y_n, \dots, y_{n+m-1}) \wedge \bigwedge_{a_i = a_j} (y_i = y_j) \wedge \bigwedge_{a_i \neq a_j} (y_i \neq y_j) \right) \wedge \psi \wedge \theta \tag{6.6}$$

which is Σ_3; it suffices to Morleyize negations of atomic formulas and of (6.2). As before, if we are only interested in parametrizing models of a Π_2 theory modulo which negations of atomic formulas and "the model has size $\leq n$" are already Σ_1-definable, then there is no need to Morleyize anything.

We may further reduce the need for Morleyization, provided we are willing to coarsen the topology to be quasi-Polish.

EXAMPLE 6.7 (space of structures on $\leq \mathbb{N}$ with Σ_1 lower bounds on size). Modify the preceding example by putting the *Scott* topology on $\overline{\mathbb{N}}$, with open sets $\{n, n+1, \dots, \mathbb{N}\}$. Then the ψ in (6.6) above will only be a conjunction of the sentences (6.2), no longer their negations, and so the final formula defining the saturation will be Σ_2; it suffices to Morleyize negations of atomic formulas.

If we also replace 2 with $\mathbb{S}$ in the definition of X, then the ϕ in (6.6) will become basic (and equality-free); and so we only need to Morleyize $\neq$. In other words, we obtain a parametrization of any Π_2 theory modulo which

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$\neq$ is Σ_1-definable. (This parametrization was used in [14]; its restriction to the countably infinite models was used in [5].)

REMARK 6.8. In fact, the quasi-Polishness is not essential here: by (2.12), we may find a continuous open surjection $Z \twoheadrightarrow X$ from a zero-dimensional Polish Z (e.g., $\sum : 2^{\mathbb{N}} \twoheadrightarrow \overline{\mathbb{N}}$), and then pull back $\mathcal{M} \to X$ from above to an étale structure over Z, with underlying étale space $\subseteq Z \times \mathbb{N}$ which is clearly also zero-dimensional. Thus *every* Π_2 *theory modulo which* $\neq$ *is* Σ_1*-definable has a zero-dimensional Polish parametrization via a zero-dimensional étale structure with* Σ_1 *saturations*.

The final step of removing the need to Morleyize $\neq$ requires a more drastic change. We need to render the diagonal in M_X^2 open but not closed, which clearly cannot be achieved by merely coarsening the topology on the base X while keeping the fibers subsets of some fixed set such as $\mathbb{N}$.

EXAMPLE 6.9 (space of partially enumerated structures). Let $\mathcal{L}$ be a countable relational language, and let $\mathcal{L}' := \{\sim\} \sqcup \mathcal{L}$ where $\sim$ is a binary relation symbol. Construct as in the second part of Example 6.7 a parametrization $\mathcal{M}' \to X'$ of all $\mathcal{L}'$-structures which has Σ_1 saturations modulo $\neq$. Let $X \subseteq X'$ be the structures in which $\sim$ is an equivalence relation and all relations in $\mathcal{L}$ are $\sim$-invariant. Then $\sim^{\mathcal{M}'}$, which is open in $(M')^2_{X'}$, restricted to the fibers over X is an open fiberwise equivalence relation on the étale space $M'|X \to X$, i.e., only equating pairs of elements in the same fiber. It follows that the quotient $M := (M'|X)/(\sim^{\mathcal{M}'}|X) \to X$ is also an étale space, and that the quotient map $M'|X \twoheadrightarrow M$ is open. Let $\mathcal{M}$ be the étale structure on M descended from $\mathcal{M}'|X$.

Explicitly, the space X consists of tuples $(N, \sim, x)$, consisting of $N \in \overline{\mathbb{N}}$ (with the Scott topology), an equivalence relation $\sim$ on N, and a $\sim$-invariant $\mathcal{L}$-structure x on N (both with the Sierpiński topology), so that x descends to the quotient $N/\sim$; this descended structure is then the fiber $\mathcal{M}_{(N,\sim,x)}$ of $\mathcal{M} \to X$. It is clear that $\mathcal{M}$ parametrizes all countable $\mathcal{L}$-structures up to isomorphism, since for any countable $\mathcal{L}$-structure, we may enumerate it with an initial segment $N \leq \mathbb{N}$ and then lift the structure to N.

We now check that $\mathcal{M}$ has Σ_1 saturations. A basic open set in M_X^n is the image under the quotient map of one in $(M')^n_{X'}$ as in (6.5) restricted to X, where (in the notation of that example) $\vec{a}$ is fixed, ϕ is a basic equality-free formula and ψ only asserts that "the model has size $\geq \cdots$" as in Example 6.7, since we used the Scott topology for X'. We claim that the saturation of such an image in M_X^n is defined by (6.6) with all occurrences of $\sim$ in ϕ replaced with $=$ and all $\neq$ replaced with $\top$. Suppose $([b_0], \dots, [b_{n-1}]) \in (N'/\sim')^n = M^n_{(N',\sim',x')}$ satisfies this modified version of (6.6), as witnessed by $([b_n], \dots, [b_{n+m-1}]) \in (N'/\sim')^m = M^m_{(N',\sim',x')}$; we must find $N, \sim, x$ such that $(N, \sim, x, a_0, \dots, a_{n-1})$ is in (6.5) and there is an

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

isomorphism of the quotient structures $(N, x)/{\sim} \cong (N', x')/{\sim'}$ mapping $[a_i] \mapsto [b_i]$ for $i < n$. Let $N \subseteq \mathbb{N}$ be an initial segment such that

(i) if $N' = \varnothing$, then $N = \varnothing$;
(ii) each $a_0, \dots, a_{n+m-1} \in N$ (this is consistent with (i) since $[b_0], \dots, [b_{n+m-1}] \in N'/{\sim'}$);
(iii) N is at least as big as required by ψ (consistent with (i) since $N'/{\sim'} \models \psi$);
(iv) $N \setminus \{a_0, \dots, a_{n+m-1}\}$ is at least as big as $(N'/{\sim'}) \setminus \{[b_0], \dots, [b_{n+m-1}]\}$.

By virtue of the $\bigwedge_{a_i=a_j}(y_i = y_j)$ in (6.6), $a_i \mapsto [b_i]$ is a well-defined function $\{a_0, \dots, a_{n+m-1}\} \to \{[b_0], \dots, [b_{n+m-1}]\} \subseteq N'/{\sim'}$; extend it to a surjection $h : N \twoheadrightarrow N'/{\sim'}$, which is easily seen to be possible using the above conditions. Now define $c \sim d \ :\Longleftrightarrow\ h(c) = h(d)$, and lift the rest of the quotient structure on $(N', x')/{\sim'}$ to the structure x on N. Then h descends to an isomorphism $N/{\sim} \cong N'/{\sim'}$ mapping $[a_i] \mapsto [b_i]$, whence $(N, \sim, x, a_0, \dots, a_{n-1})$ is in (6.5).

As usual, by further restricting this construction to the models of a Π_2 theory $\mathcal{T}$, we obtain that *every* Π_2 *theory has a quasi-Polish parametrization with* Σ_1 *saturations*. As in Remark 6.8, we may further pull back to a zero-dimensional Polish $Z \twoheadrightarrow X$; however, unlike there, we can no longer assume that the underlying étale space M is also zero-dimensional.

REMARK 6.10. Minor variations of Examples 6.3, 6.4, 6.7 and 6.9 are to allow arbitrary subsets $N \subseteq \mathbb{N}$, not just initial segments. (Such a variation of Example 6.7 was essentially used in [4].)

REMARK 6.11 (space of totally enumerated structures). A further variation of Example 6.9 is to again require $N = \mathbb{N}$, as in Example 6.1 (while still using the Sierpiński space for relations including the equivalence relation $\sim$). In the above verification that $\mathcal{M}$ has Σ_1 saturations, the possibility of $N \neq \mathbb{N}$ only comes into play in (i), and is clearly not needed if also $N' = \mathbb{N}$. The minor downside is that now $\mathcal{M}$ only parametrizes nonempty models. (This was the original parametrization used by Joyal–Tierney [31, VII Section 3 Theorem 1].)

Finally, we consider a parametrization of a somewhat different flavor, which only differs from that in Remark 6.11 for languages *with* function symbols. This parametrization or a variant thereof is widely used for studying classification of algebraic structures such as groups; see, e.g., [54].

EXAMPLE 6.12 (space of marked structures). Let $\mathcal{L}$ be an arbitrary countable language, possibly with function symbols. Let T be the set of $\mathcal{L}$-terms over countably many variables $a_0, a_1, \dots$. Let X' be the space of enumerated $\mathcal{L}$-structures as in Remark 6.11, where function symbols in $\mathcal{L}$ are encoded via their graphs, and where the role of the index set $\mathbb{N}$ is replaced

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

by T; let $\mathcal{M}' \to X'$ be the corresponding étale structure. Thus a point in X' consists essentially of an equivalence relation $\sim$ on T and an $\mathcal{L}$-structure on $T/\!\sim$. Let $X \subseteq X'$ be the $\mathbf{\Pi}^0_2$ subspace where $\sim$ is a congruence with respect to the usual syntactic action of function symbols on terms, and where the quotient structure on $T/\!\sim$ interprets each function symbol via this syntactic action. In other words,

$$\begin{aligned} X &\cong \Big\{(\sim, x) \;\Big|\; x \in \prod_{n\text{-ary rel } R \in \mathcal{L}} \mathbb{S}^{T^n} \;\;\&\;\; \sim \in \mathbb{S}^{T^2} \text{ is a congruence on } (T, x)\Big\} \\ &\cong \{(\sim, \widetilde{x}) \mid \sim \text{ is a congruence on } T \;\;\&\;\; \widetilde{x} \text{ interprets relation symbols on } T/\!\sim\} \\ &\cong \{\mathcal{L}\text{-structures generated by } a_0, a_1, \dots\}. \end{aligned}$$

Let $\mathcal{M} := \mathcal{M}'|X$, so that $\mathcal{M}_{(\sim,x)} = (T/\!\sim, \widetilde{x})$ is said generated structure.

Note that $X \subseteq X'$ is not $\mathrm{Iso}_{X'}(\mathcal{M}')$-invariant; thus we cannot immediately conclude that $\mathcal{M}$ has Σ_1 saturations by Remark 5.15. Nonetheless, this is the case, as we now check. Chasing through Remark 6.11, Example 6.9, and ultimately (6.5), a basic open set in M^n_X is

$$\Big\{(\sim, x, [t_0], \dots, [t_{n-1}]) \;\Big|\; (\sim, x) \in X \;\;\&\;\; \phi^x(t_n, \dots, t_{n+m-1})\Big\} \tag{6.13}$$

for some fixed terms $t_0, \dots, t_{n+m-1} \in T$ and basic $(\mathcal{L} \cup \{\sim\})$-formula ϕ not mentioning $=$; by absorbing function applications into $t_n, \dots, t_{n+m-1}$, we may assume ϕ only mentions relation symbols in $\mathcal{L}$ and $\sim$. Let $k \in \mathbb{N}$ be sufficiently large so that $t_0, \dots, t_{n+m-1}$ only mention the variables $a_0, \dots, a_{k-1}$. Then the $\mathrm{Iso}_X(\mathcal{M})$-saturation of (6.13) is defined by

$$\exists a_0, \dots, a_{k-1} \big((x_0 = t_0) \wedge \cdots \wedge (x_{n-1} = t_{n-1}) \wedge \phi'(t_n, \dots, t_{n+m-1})\big) \tag{6.14}$$

where ϕ' is ϕ with all occurrences of $\sim$ replaced with $=$. Indeed, if $\mathcal{N}$ is a countable nonempty (which the fibers of $\mathcal{M}$ all are) $\mathcal{L}$-structure which satisfies this formula under an assignment $\vec{x} \mapsto \vec{b} \in N^n$, as witnessed by some $h : \{a_0, \dots, a_{k-1}\} \to N$, we may extend h to the remaining variables $a_k, a_{k+1}, \dots$ to yield an enumeration of N, and then to all terms to obtain a surjective homomorphism $h : T \twoheadrightarrow N$ with respect to the function symbols in $\mathcal{L}$, and finally pull back the relations as well as $=$ in $\mathcal{N}$ along h to obtain $(\sim, x) \in X$ such that $h : T \twoheadrightarrow N$ descends to an isomorphism $\mathcal{M}_{(\sim,x)} \cong \mathcal{N}$ and $(\sim, x, [t_0], \dots, [t_{n-1}])$ is in (6.13) where $t_0, \dots, t_{n-1}$ are any h-preimages of $\vec{b}$.

REMARK 6.15. For finite $N \in \mathbb{N}$, one could analogously construct a space X_N as above starting from only N variables $a_0, \dots, a_{N-1}$, parametrizing N-generated structures. However, saturations would only be Σ_3, since in the defining formulas (6.14) (where without loss we can take $k := N$), we need to add a clause $\forall y \bigvee_{t \in T}(y = t)$ saying that $a_0, \dots, a_{N-1}$ are generators.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

§7. Omitting types and Scott rank. The classical Baire category theorem states that in sufficiently nice topological spaces, a countable intersection of dense open (or even $\mathbf{\Pi}^0_2$) sets is still dense. The omitting types theorem in model theory, and its infinitary variants, state that a countable conjunction of "dense Σ_1 (or even Π_2) formulas" is still "dense." It is well-known folklore that these two results are closely related, and in fact omitting types can also be reduced to Baire category via some standard coding tricks; see, e.g., [23] and the references therein. As an application of the "continuous open map" perspective of Remark 5.2, we give here an easy and transparent version of such a reduction.

THEOREM 7.1 (omitting types). *Let $\mathcal{T}$ be a satisfiable countable Π_2 theory.*

(a) *Let $\phi_i(\vec{x}_i)$ be countably many Π_2 formulas of various arities $n_i \in \mathbb{N}$, such that for each Σ_1 $\theta(\vec{x}_i)$ satisfiable in some model of $\mathcal{T}$, $\phi_i \wedge \theta$ is also satisfiable in some model of $\mathcal{T}$. Then $\mathcal{T} \cup \{\forall \vec{x}_i\, \phi_i(\vec{x}_i)\}_i$ is satisfiable.*

(b) *Let $\psi_i(\vec{x}_i)$ be countably many Σ_2 formulas of various arities $n_i \in \mathbb{N}$, such that for each Σ_1 $\theta(\vec{x}_i)$, if $\mathcal{T} \models \forall \vec{x}_i\, (\theta(\vec{x}_i) \to \psi_i(\vec{x}_i))$, then $\mathcal{T} \models \forall \vec{x}_i\, \neg\theta(\vec{x}_i)$. Then $\mathcal{T} \cup \{\forall \vec{x}_i\, \neg\psi_i(\vec{x}_i)\}_i$ is satisfiable.*

PROOF. The statements are dual; we prove (a). By Example 6.9, there is a quasi-Polish parametrization $p : \mathcal{M} \to X$ of $\mathcal{T}$ with Σ_1 saturations, i.e., a "continuous open map $\mathcal{M} : X \twoheadrightarrow \{\text{models of } \mathcal{T}\}$." Then "the $\mathcal{M}$-preimage of each dense Π_2 ϕ_i is dense $\mathbf{\Pi}^0_2$", i.e., each $\phi_i^{\mathcal{M}} \subseteq M_X^{n_i}$ is dense $\mathbf{\Pi}^0_2$, since for each open $\varnothing \neq U \subseteq M_X^{n_i}$, letting θ be Σ_1 with $\theta^{\mathcal{M}} = \mathrm{Iso}_X(\mathcal{M}) \cdot U \supseteq U$, θ is satisfiable in a model of $\mathcal{T}$, whence so is $\phi \wedge \theta$, whence $\varnothing \neq \phi^{\mathcal{M}} \wedge \theta^{\mathcal{M}} = \mathrm{Iso}_X(\mathcal{M}) \cdot (\phi^{\mathcal{M}} \cap U)$ since $\phi^{\mathcal{M}}$ is $\mathrm{Iso}_X(\mathcal{M})$-invariant. Hence, each $\phi_i^{\mathcal{M}} \subseteq M_X^{n_i}$ is comeager. It follows that each $(\forall \vec{x}_i\, \phi_i(\vec{x}_i))^{\mathcal{M}} \subseteq X$ is comeager, since dually, the p-image of each meager set in $M_X^{n_i}$ is easily seen to be meager in X, by considering a countable basis of open sections in $M_X^{n_i}$. Thus $\bigcap_i (\forall \vec{x}_i\, \phi_i(\vec{x}_i))^{\mathcal{M}}$ is also comeager, hence dense by Baire category; picking any x in this set, we have $\mathcal{M}_x \models \mathcal{T} \cup \{\forall \vec{x}_i\, \phi_i(\vec{x}_i)\}_i$. ⊣

EXAMPLE 7.2. Taking $\mathcal{T}$ to be a propositional theory, as in Example 3.5 (and the arities n_i above to all be 0), we recover the Baire category theorem for quasi-Polish spaces as a special case.

REMARK 7.3. Via Morleyization, Theorem 7.1 generalizes to an omitting types theorem for an arbitrary countable fragment $\mathcal{F}$, which generalizes the classical omitting types for $\mathcal{L}_{\omega_1\omega}$ (see, e.g., [42, Theorem 4.9]) to our more general fragments (Definition 3.3) not necessarily closed under $\neg, \wedge, \vee, \forall, \exists$.

In particular, Theorem 7.1 is sufficiently general to imply Montalbán's omitting types theorem for Π_α [48, Lemma 3.2], using which we may generalize most parts of Montalbán's characterization of Scott rank [48, Theorem 1.1] to our weaker notions of $\Sigma_\alpha, \Pi_\alpha$.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

THEOREM 7.4 (characterization of Scott rank). *Let $p : \mathcal{M} \to X$ be a second-countable étale structure with Σ_1 saturations over a quasi-Polish space X, and let $x \in X$. The following are equivalent*:

(i) *Every automorphism orbit of $\mathcal{M}_x$ is Σ_α-definable without parameters.*
(ii) *$\mathcal{M}_x$ has a $\Pi_{\alpha+1}$ Scott sentence ϕ.*
(iii) *$\mathrm{Iso}_X(\mathcal{M}) \cdot x \subseteq X$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
(iv) *For every Π_α formula $\psi(\vec{z})$ with $\psi^{\mathcal{M}_x} \neq \varnothing$, there is a Σ_α formula $\theta(\vec{z})$ with $\varnothing \neq \theta^{\mathcal{M}_x} \subseteq \psi^{\mathcal{M}_x}$.*

To recover [48, Theorem 1.1], Morleyize negated atomic formulas as in Example 3.4 and take the standard parametrization from Example 6.1. The proof is the same as in [48] but with results used therein replaced by their generalizations from this article, namely, Corollary 5.17 for (i) $\implies$ (ii), the Lopez-Escobar Theorem 10.2 for (ii) $\iff$ (iii), and Theorem 7.1 for (ii) $\implies$ (iv).

REMARK 7.5. The omitting types theorem has an analog in topos theory, called the *$\neg\neg$-subtopos*, which is the "theory of models of $\mathcal{T}$ omitting all non-isolated Π_1 types"; see [30, Example A4.5.9].

§8. Imaginary sorts and the Joyal–Tierney theorem.

DEFINITION 8.1. For an étale structure $p : \mathcal{M} \to X$, an **étale $\mathrm{Iso}_X(\mathcal{M})$-space over X** is an étale space $q : A \to X$ equipped with a (jointly) continuous action of the topological groupoid $\mathrm{Iso}_X(\mathcal{M})$, where each $g : \mathcal{M}_x \cong \mathcal{M}_y \in \mathrm{Iso}_X(\mathcal{M})$ acts via a bijection $A_x \cong A_y$. (See Definition 11.1 for the formal definition of an abstract groupoid action.)

The prototypical étale $\mathrm{Iso}_X(\mathcal{M})$-space is M (recall Definition 4.15). We may build other étale $\mathrm{Iso}_X(\mathcal{M})$-spaces via the following basic operations:

- Each fiber power M^n_X is also an étale $\mathrm{Iso}_X(\mathcal{M})$-space. More generally, a fiber product of finitely many étale $\mathrm{Iso}_X(\mathcal{M})$-spaces is again such a space, under the diagonal action.
- If $A \to X$ is an étale $\mathrm{Iso}_X(\mathcal{M})$-space, and $U \subseteq A$ is an open set invariant under the action of $\mathrm{Iso}_X(\mathcal{M})$, then U is itself an étale $\mathrm{Iso}_X(\mathcal{M})$-space. If $A = M^n_X$, and $\mathcal{M}$ has Σ_1 saturations, then such U are precisely the Σ_1-definable n-ary relations $\phi^{\mathcal{M}} \subseteq M^n_X$.
- A (possibly infinite) disjoint union of étale $\mathrm{Iso}_X(\mathcal{M})$-spaces is again such.
- If $p : A \to X$ is an étale $\mathrm{Iso}_X(\mathcal{M})$-space, and $\sim \subseteq A \times_X A$ is an $\mathrm{Iso}_X(\mathcal{M})$-invariant open (in $A \times_X A$) equivalence relation, then p and the action descend to the quotient space $A/\!\sim$ which again becomes an étale $\mathrm{Iso}_X(\mathcal{M})$-space.

We now introduce syntactic names for these last two operations, generalizing formulas.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

DEFINITION 8.2. For a countable $\mathcal{L}_{\omega_1\omega}$ theory $\mathcal{T}$, a Σ_1 **imaginary sort over** $\mathcal{T}$ is an expression

$$\Phi = \Big(\bigsqcup_i \phi_i\Big)/\Big(\bigsqcup_{i,j} \varepsilon_{ij}\Big)$$

where i, j run over the same countable index set, each ϕ_i is a Σ_1 formula in some number of free variables $n_i \in \mathbb{N}$, each ε_{ij} is a Σ_1 formula in $n_i + n_j$ variables, and $\mathcal{T}$ proves the sentences

$$\begin{aligned}
\forall \vec{x}, \vec{y}\, \big(\varepsilon_{ij}(\vec{x}, \vec{y}) \to \phi_i(\vec{x}) \wedge \phi_j(\vec{y})\big),\\
\forall \vec{x}\, \big(\phi_i(\vec{x}) \to \varepsilon_{ii}(\vec{x}, \vec{x})\big),\\
\forall \vec{x}, \vec{y}\, \big(\varepsilon_{ij}(\vec{x}, \vec{y}) \to \varepsilon_{ji}(\vec{y}, \vec{x})\big),\\
\forall \vec{x}, \vec{y}, \vec{z}\, \big(\varepsilon_{ij}(\vec{x}, \vec{y}) \wedge \varepsilon_{jk}(\vec{y}, \vec{z}) \to \varepsilon_{ik}(\vec{x}, \vec{z})\big).
\end{aligned}$$

Equivalently by the completeness theorem, in each $\mathcal{M} \models \mathcal{T}$,

$$\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}} \subseteq \bigsqcup_{i,j} M^{n_i+n_j} \cong \Big(\bigsqcup_i M^{n_i}\Big)^2 \text{ is an equivalence relation on } \bigsqcup_i \phi_i^{\mathcal{M}} \subseteq \bigsqcup_i M^{n_i}.$$

We then define the **interpretation of Ψ in $\mathcal{M}$** to be the quotient set

$$\Phi^{\mathcal{M}} := \Big(\bigsqcup_i \phi_i^{\mathcal{M}}\Big)/\Big(\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}}\Big).$$

Similarly, if $p : \mathcal{M} \to X$ is an étale model of $\mathcal{T}$, then we have an étale $\mathrm{Iso}_X(\mathcal{M})$-space $\Phi^{\mathcal{M}} \to X$ defined in the same way, as the quotient of $\bigsqcup_i \phi_i^{\mathcal{M}} \to X$ by the open equivalence relation $\bigsqcup_{i,j} \varepsilon_{ij}^{\mathcal{M}}$.

REMARK 8.3. Thus, a Σ_1 imaginary is a syntactic name for a *quotient* of a *countable disjoint union* of Σ_1*-definable subsets* of *finite products* M^n of copies of the underlying set or étale space M of an (étale) model $\mathcal{M}$. We may always distribute these four types of operations over each other to put them into this order; thus, imaginaries are themselves closed under these operations.

- If $\Phi = (\bigsqcup_i \phi_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ and $\Psi = (\bigsqcup_k \psi_k)/(\bigsqcup_{k,l} \eta_{kl})$ are imaginaries, we may define their **product sort** $\Phi \times \Psi := (\bigsqcup_{i,k} \phi_i \wedge \psi_k)/(\bigsqcup_{i,j,k,l} \varepsilon_{ij} \wedge \eta_{kl})$. There is also the **singleton sort** (nullary product) $1 := \top/\top$ where here $\top$ is regarded as having no free variables.
- If $\Phi = (\bigsqcup_i \phi_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ is a Σ_1 imaginary sort over $\mathcal{T}$, a Σ_1**-definable subsort** $\Psi \subseteq \Phi$ is given by countably many Σ_1 formulas ψ_i such that "$\bigsqcup_i \psi_i \subseteq \bigsqcup_i \phi_i$ is $\mathcal{T}$-provably $(\bigsqcup_{i,j} \varepsilon_{ij})$-invariant," which can be expressed by certain Π_2 sentences much as in Definition 8.2. Such Ψ can also be thought of as a Σ_1 imaginary in its own right, namely, $\Psi = (\bigsqcup_i \psi_i)/(\bigsqcup_{i,j} \varepsilon_{ij} \wedge \psi_i \wedge \psi_j)$, such that $\Psi^{\mathcal{M}} \subseteq \Phi^{\mathcal{M}}$ for every $\mathcal{M} \models \mathcal{T}$.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

- A **countable disjoint union** of imaginaries $\bigsqcup_i \Phi_i$ is given by simply merging the formulas.
- Finally, if $\Phi = (\bigsqcup_i \phi_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ is an imaginary, and $H \subseteq \Phi \times \Phi$ is a Σ_1-definable subsort which is $\mathcal{T}$-provably an equivalence relation, then we may define the **quotient sort** Φ/H by replacing the ε_{ij}'s with the formulas η_{ij} defining H.

THEOREM 8.4 (Joyal–Tierney [31]). *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$ theory, $p : \mathcal{M} \to X$ a second-countable étale structure with Σ_1 saturations parametrizing models of $\mathcal{T}$. Then every second-countable étale $\mathrm{Iso}_X(\mathcal{M})$-space $q : A \to X$ is isomorphic to $\Phi^{\mathcal{M}}$ for some Σ_1 imaginary Φ over $\mathcal{T}$.*

The following proof is a streamlined version of the proofs in [4, Section 1.4] and [14, Theorem 8.1] for two specific parametrizations (namely, those in Remark 6.10 and Example 6.7).

PROOF. Let $a \in A_x$, and let $a \in S \subseteq A$ be an open section. Since the identity $1_{\mathcal{M}_x} \in \mathrm{Iso}_X(\mathcal{M})$ fixes a, by continuity of the action, there are open neighborhoods $a \in S' \subseteq S$ and $1_{\mathcal{M}_x} \in [\![T \mapsto T']\!] \subseteq \mathrm{Iso}_X(\mathcal{M})$, where we may assume $T, T' \subseteq M_X^n$ are open sections, such that $[\![T \mapsto T']\!] \cdot S' \subseteq S$. Since $1_{\mathcal{M}_x} \in [\![T \mapsto T']\!]$, we have $x \in p(T \cap T')$. By replacing S with $S' \cap q^{-1}(p(T \cap T'))$ and T, T' with $T \cap T' \cap p^{-1}(q(S'))$, we get $p(T) = q(S)$ and $[\![T \mapsto T]\!] \cdot S \subseteq S$.

So we have shown that A has a basis of open sections $S \subseteq A$ for which there exists an open section $T \subseteq M_X^n$ for some n such that $p(T) = q(S)$ and $[\![T \mapsto T]\!] \cdot S \subseteq S$. For such S, T, consider

$$h_{S,T} : \mathrm{Iso}_X(\mathcal{M}) \cdot T \longrightarrow A$$
$$(g : \mathcal{M}_y \cong \mathcal{M}_z) \cdot T_y \longmapsto g \cdot S_y.$$

(Here we are abusing notation by writing T_y for the unique element of T_y, etc.) We claim that this is a well-defined, continuous $\mathrm{Iso}_X(\mathcal{M})$-equivariant map, whose image contains S. Well-definedness is because if $T_y \neq \varnothing$ then $S_y \neq \varnothing$ as $p(T) \subseteq q(S)$, and if $g \cdot T_y = g' \cdot T_{y'}$, where $g : \mathcal{M}_y \cong \mathcal{M}_z$ and $g' : \mathcal{M}_{y'} \cong \mathcal{M}_z$ and both $y, y' \in p(T)$, then $g'^{-1} \circ g : \mathcal{M}_y \cong \mathcal{M}_{y'} \in [\![T \mapsto T]\!]$ and so $g \cdot S_y = g' \cdot S_{y'}$. Equivariance is clear; and $S \subseteq \mathrm{im}(h_{S,T})$ by taking $g = 1$, using $p(T) = q(S)$. For continuity: suppose

$$h_{S,T}(g \cdot T_y) = g \cdot S_y \in S',$$

where $S' \subseteq A$ is another open section for which there is an open section $T' \subseteq M_X^{n'}$ such that $p(T') = q(S') \ni z$ and $[\![T' \mapsto T']\!] \cdot S' \subseteq S'$. Let $g^{-1} \cdot T'_z \in T'' \subseteq M_X^{n'}$ be another open section, and let

$$U := q(S \cap ([\![T' \mapsto T'']\!] \cdot S')) \subseteq X.$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

Then $S_y = g^{-1} \cdot S'_z \in [\![T' \mapsto T'']\!] \cdot S'$ whence $y \in U$ whence $g \cdot T_y \in [\![T'' \mapsto T']\!] \cdot (T \cap p^{-1}(U))$; and

$$\begin{aligned} h_{S,T}([\![T'' \mapsto T']\!] \cdot (T \cap p^{-1}(U))) &= [\![T'' \mapsto T']\!] \cdot (S \cap q^{-1}(U)) \\ &= [\![T'' \mapsto T']\!] \cdot (S \cap ([\![T' \mapsto T'']\!] \cdot S')) \quad \text{since } q|S \text{ is injective} \\ &\subseteq [\![T' \mapsto T']\!] \cdot S' \subseteq S'. \end{aligned}$$

Now since A is second-countable, hence Lindelöf, it has a countable cover by open sections $S_i \subseteq A$ for which there exist corresponding $T_i \subseteq M_X^{n_i}$ as above. For each i, let ϕ_i be a Σ_1 formula defining $\mathrm{Iso}_X(\mathcal{M}) \cdot T_i \subseteq M_X^{n_i}$. We then have a continuous equivariant surjection

$$h := \bigsqcup_i h_{S_i,T_i} : \bigsqcup_i \phi_i^{\mathcal{M}} \longrightarrow\!\!\!\!\!\rightarrow A.$$

Its congruence kernel $\sim = \{(\vec{a}, \vec{b}) \in (\bigsqcup_i \phi_i^{\mathcal{M}})^2_X \mid h(\vec{a}) = h(\vec{b})\} \subseteq (\bigsqcup_i M_X^{n_i})^2_X \cong \bigsqcup_{i,j} M_X^{n_i+n_j}$ is an invariant open set, thus there are Σ_1 formulas ε_{ij} defining $\sim$ restricted to each $M_X^{n_i+n_j}$. Then $\Phi := (\bigsqcup_i \phi_i)/(\bigsqcup_{i,j} \varepsilon_{ij})$ is a Σ_1 imaginary over $\mathcal{T}$, because the interpretation of $\bigsqcup_{i,j} \varepsilon_{ij}$ in each countable model of $\mathcal{T}$, which is isomorphic to some fiber $\mathcal{M}_x$, is the equivalence relation $\sim_x$ on $\bigsqcup_i \phi_i^{\mathcal{M}_x}$. And h descends to an isomorphism of étale $\mathrm{Iso}_X(\mathcal{M})$-spaces $\Phi^{\mathcal{M}} = (\bigsqcup_i \phi_i^{\mathcal{M}})/\sim \cong A$. ⊣

Example 8.5. If $X = 1$ and $\mathcal{M}$ is a countable structure with Σ_1-definable automorphism orbits, as in Remark 5.3, then Theorem 8.4 says that every continuous action of $\mathrm{Aut}(\mathcal{M})$ on a countable discrete space A is named by some Σ_1 imaginary of $\mathcal{M}$. To see this directly: for every $a \in A$, the stabilizer $\mathrm{Aut}(\mathcal{M})_a \subseteq \mathrm{Aut}(\mathcal{M})$ is a clopen subgroup, hence contains some basic clopen subgroup $\mathrm{Aut}(\mathcal{M}, \vec{b})$, the automorphism group of $\mathcal{M}$ expanded with finitely many fixed constants $\vec{b} \in M^n$; then the orbit $\mathrm{Aut}(\mathcal{M}) \cdot a$ is a definable quotient of the Σ_1-definable orbit $\mathrm{Aut}(\mathcal{M}) \cdot \vec{b}$. The above proof can be seen as the natural generalization of this to étale structures.

Remark 8.6. Theorem 8.4 says that the map

$$\begin{aligned} \{\Sigma_1 \text{ imaginaries over } \mathcal{T}\} &\xrightarrow{\sim} \{\text{second-countable étale } \mathrm{Iso}_X(\mathcal{M})\text{-spaces}\} \\ \Phi &\longmapsto \Phi^{\mathcal{M}} \end{aligned}$$

is surjective up to isomorphism. The full Joyal–Tierney theorem says that it is in fact an equivalence of categories; Theorem 8.4 above is merely the key ingredient. Namely, the category on the left has:

- Σ_1 imaginary sorts Φ as objects;
- subobjects of Φ are Σ_1-definable subsorts $\Psi \subseteq \Phi$ (Remark 8.3), which (modulo $\mathcal{T}$-provable equivalence) correspond bijectively to open invariant subsets of $\Phi^{\mathcal{M}}$ by Σ_1 saturations;

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

- morphisms $\Theta : \Phi \to \Psi$ are **Σ_1-definable functions**, i.e., Σ_1-definable subsorts $\Theta \subseteq \Phi \times \Psi$ which are $\mathcal{T}$-provably the graph of a function. We usually identify definable functions modulo $\mathcal{T}$-provable equivalence.

For a countable Π_2 theory $\mathcal{T}$, this category of Σ_1 imaginaries is called the **syntactic σ-pretopos** (also known as **classifying σ-pretopos**) of the theory $\mathcal{T}$, and can be thought of as a canonical algebraic representation of the syntax of $\mathcal{T}$; see [30, Section D1.4], [14, Section 10]. Now the above equivalence boils down to the standard category-theoretic fact [30, Lemma D3.5.6] that a functor preserving finite categorical limits is an equivalence iff it is surjective on objects up to isomorphism and bijective on subobjects (in the usual categorical parlance, *essentially surjective*, *conservative*, and *full on subobjects*).

§9. Interpretations and functors. Taking a more global view, the equivalence of categories in Remark 8.6 says that the map

$$\begin{aligned} \{\text{countable } \Pi_2 \text{ theories}\} &\hookrightarrow \{\text{(quasi-)Polish groupoids}\} \\ \mathcal{T} &\longmapsto \mathrm{Iso}(\text{some parametrization of } \mathcal{T}) \end{aligned} \tag{9.1}$$

is "injective": we may recover a theory from the groupoid of its parametrization. To make this statement precise, we need to discuss the appropriate structure on the class of all theories.

DEFINITION 9.2. Let $\mathcal{T}_1, \mathcal{T}_2$ be countable Π_2 theories in two respective countable languages $\mathcal{L}_1, \mathcal{L}_2$. A Σ_1 **interpretation** $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$ is a "model of $\mathcal{T}_1$ within the category of Σ_1 imaginaries over $\mathcal{T}_2$."

- First, one defines an interpretation $\mathcal{F} : \mathcal{L}_1 \to \mathcal{T}_2$ to be an "$\mathcal{L}_1$-structure in $\mathcal{T}_2$-imaginaries," consisting of an underlying Σ_1 imaginary F over $\mathcal{T}_2$, an interpretation of each n-ary $R \in \mathcal{L}_1$ as a Σ_1-definable subsort $R^{\mathcal{F}} \subseteq F^n$, and an interpretation of each n-ary $f \in \mathcal{L}_1$ as a Σ_1-definable function $f^{\mathcal{F}} : F^n \to F$.
- Next, one inductively defines, in the usual way, interpretations of $\mathcal{L}_1$-terms and formulas as definable functions and subsorts over $\mathcal{T}_2$.
- One then says that $\mathcal{F}$ satisfies an $\mathcal{L}_1$-sentence $\phi \in \mathcal{T}_1$ if its interpretation $\phi^{\mathcal{F}}$, which will be a definable subsort of the singleton sort 1 over $\mathcal{T}_2$ (recall Remark 8.3), is the entirety of 1.
- Finally, one can extend the interpretation of $\mathcal{L}_1$-formulas to imaginaries Φ over $\mathcal{T}_1$, which become interpreted as imaginaries $\Phi^{\mathcal{F}}$ over $\mathcal{T}_2$.

The categorical viewpoint of Remark 8.6 is particularly useful when dealing with interpretations: a Σ_1 interpretation $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$ is simply a functor $\Phi \mapsto \Phi^{\mathcal{F}}$ from the syntactic σ-pretopos of $\mathcal{T}_1$ to the syntactic σ-pretopos of $\mathcal{T}_2$, preserving finite limits and countable colimits. For instance, this makes it easy to define the *composition* of interpretations $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$. See [14, Section 10].

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

A **Σ_1-definable isomorphism** $h : \mathcal{F} \cong \mathcal{G} : \mathcal{T}_1 \to \mathcal{T}_2$ between two interpretations is defined in the same way as a usual isomorphism, but replacing the underlying function with a Σ_1-definable function over $\mathcal{T}_2$. Equivalently, it is a natural isomorphism between functors.

An interpretation

$$\mathcal{F} : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

can be thought of as a syntactic specification for a map

$$\begin{aligned} \{\text{models of } \mathcal{T}_1\} &\longleftarrow \{\text{models of } \mathcal{T}_2\} : \mathcal{F}^* \\ \mathcal{F}^{\mathcal{M}} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned} \tag{9.3}$$

Namely, given a (countable/étale) $\mathcal{M} \models \mathcal{T}_2$, the model $\mathcal{F}^*(\mathcal{M}) = \mathcal{F}^{\mathcal{M}} \models \mathcal{T}_1$ has underlying set/étale space $F^{\mathcal{M}}$ and each symbol $P \in \mathcal{L}_1$ interpreted as $(P^{\mathcal{F}})^{\mathcal{M}}$. Similarly, a Σ_1-definable isomorphism $h : \mathcal{F} \cong \mathcal{G}$ between interpretations yields, for each $\mathcal{M} \models \mathcal{T}_2$, an $\mathcal{L}_1$-isomorphism $h^{\mathcal{M}} : \mathcal{F}^{\mathcal{M}} \cong \mathcal{G}^{\mathcal{M}}$.

For detailed background on interpretations and imaginaries (albeit in the context of finitary first-order logic), see [28, Chapter 5]. We first give a familiar finitary example:

EXAMPLE 9.4. The construction from each integral domain $\mathcal{R}$ of its field of fractions is specified by an interpretation $\mathcal{F} : (\text{theory of fields}) \to (\text{theory of integral domains})$.

- The underlying imaginary is $F = \phi/\varepsilon$ where ϕ, ε are the formulas in the language of rings:

  $$\phi(x, y) := (y \neq 0), \qquad \varepsilon(x_1, y_1, x_2, y_2) := (x_1 y_2 = x_2 y_1).$$

  Thus given an integral domain $\mathcal{R}$, the field of fractions $\mathcal{F}^{\mathcal{R}}$ has underlying set

  $$F^{\mathcal{R}} = \{(a, b) \in R^2 \mid b \neq 0\}/\sim \quad \text{where} \quad (a_1, b_1) \sim (a_2, b_2) \ :\Longleftrightarrow\ a_1 b_2 = a_2 b_1.$$

- The operation $+$ of fields is interpreted as the definable function $+^{\mathcal{F}} = \psi/\varepsilon^3$ where

  $$\psi(x_1, y_1, x_2, y_2, x_3, y_3) := \varepsilon(x_1 y_2 + x_2 y_1, y_1 y_2, x_3, y_3).$$

  In other words, for an integral domain $\mathcal{R}$, this defines the graph of $+$ on $\mathcal{F}^{\mathcal{R}}$:

  $$[(a_1, b_1)] + [(a_2, b_2)] = [(a_3, b_3)] \iff (a_1 b_2 + a_2 b_1, b_1 b_2) \sim (a_3, b_3).$$

Similarly for the other operations. (This is not a Σ_1 interpretation, unless we include a unary relation symbol for "nonzero" in the language of integral domains.)

Next, we give an example that uses the availability of countable disjoint unions.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

EXAMPLE 9.5. We have an interpretation from the theory of groups to the theory of sets, that specifies the construction from a set X of the free group over X. Its underlying imaginary is a countable disjoint union $\bigsqcup_{\vec{s}\in\{\pm 1\}^{<\omega}} \phi_{\vec{s}}$ where for each $\vec{s} = (s_0, \dots, s_{n-1})$, the formula $\phi_{\vec{s}}(x_0, \dots, x_{n-1}) := \bigwedge_{s_i \neq s_{i+1}} (x_i \neq x_{i+1})$ says "$x_0^{s_0} \cdots x_{n-1}^{s_{n-1}}$ is a reduced word in the free group."

EXAMPLE 9.6. The notion of Σ_1 interpretation subsumes that of étale structure, which is essentially the same thing as an interpretation into a propositional theory. Indeed, recall from Example 3.5 that a quasi-Polish space X is the space of models of a countable Π_2 propositional theory $\mathcal{T}_0$. A second-countable étale space $p : A \to X$ then corresponds to a Σ_1 imaginary over $\mathcal{T}_0$ (this is the trivial case of the Joyal–Tierney Theorem 8.4, for the trivial étale structure $X \to X$). Thus, a Σ_1 interpretation $\mathcal{M}$ from another countable Π_2 theory $\mathcal{T}$ into $\mathcal{T}_0$ is a model of $\mathcal{T}$ in $\mathcal{T}_0$-imaginaries, i.e., étale spaces over X, i.e., an étale model of $\mathcal{T}$ over X. The induced map $\mathcal{M}^*$ as in (9.3), from models $x \in X$ of $\mathcal{T}_0$ to models of $\mathcal{T}$, is just the "continuous map $x \mapsto \mathcal{M}_x$" from Remark 4.9.

REMARK 9.7. Thus, a natural structure to put on the class of all countable Π_2 theories, on the left-hand side of (9.1), is that of a **2-category**, whose objects are countable Π_2 theories $\mathcal{T}$, morphisms are Σ_1 interpretations $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$, and 2-cells are Σ_1-definable isomorphisms[4] between interpretations $h : \mathcal{F} \cong \mathcal{G} : \mathcal{T}_1 \to \mathcal{T}_2$. We may visualize this 2-category as follows:

$$\mathcal{T}_1 \;\overset{\mathcal{F}}{\underset{\mathcal{G}}{\Downarrow h}}\; \mathcal{T}_2 \xrightarrow{\mathcal{H}} \mathcal{T}_3$$

For general background on 2-categories, see [30, Section B1.1], [9, I Chapter 7].

However, there remain annoying technicalities in defining a map (technically, a *2-functor*) from this 2-category of theories to groupoids, as in (9.1). This is due to non-canonical coding choices. To define such a 2-functor on objects, we have to pick, for each countable Π_2 theory $\mathcal{T}$, some particular quasi-Polish parametrization $\mathcal{M} \to X$ of it, and then take its isomorphism groupoid $\mathrm{Iso}_X(\mathcal{M})$. Worse yet, to define the 2-functor on *morphisms*, we have to pick, for an interpretation between theories $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$, some map $f : X_2 \to X_1$ between the respective chosen parametrizing spaces

[4]It makes sense to consider more generally non-invertible 2-cells which are definable homomorphisms; however, in order to reflect these on the semantic (right) side of (9.1), one should then replace $\mathrm{Iso}_X(\mathcal{M})$ with the *homomorphism category*, a topological category, which is a more involved notion than a topological groupoid. See Remark 11.10.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

which realizes the operation on models specified by $\mathcal{F}$, i.e., so that $(\mathcal{M}_1)_{f(x)} \cong \mathcal{F}^{(\mathcal{M}_2)_x}$.

It is possible to make all of these coding choices in some explicit *ad hoc* manner, and this was done in [26] and [14]. Here, we take the opportunity to illustrate a more abstract approach, via a standard trick in higher-dimensional category theory: instead of defining the 2-functor (9.1) directly, we first "cover" its domain with a bigger 2-category that is equivalent, but contains isomorphic copies that include all possible coding choices beforehand.

DEFINITION 9.8. A **quasi-Polish parametrized Π_2 theory** will mean a tuple $(\mathcal{L}, \mathcal{T}, X, \mathcal{M}, p)$, consisting of a countable Π_2 theory $\mathcal{T}$ in some countable language $\mathcal{L}$, together with some quasi-Polish parametrization X of models of $\mathcal{T}$ via an étale model $p : \mathcal{M} \to X$ with Σ_1 saturations. We will often abbreviate the tuple $(\mathcal{L}, \mathcal{T}, X, \mathcal{M}, p)$ to $(\mathcal{T}, X, \mathcal{M})$.

A **parametrized Σ_1 interpretation** $(\mathcal{F}, f) : (\mathcal{L}_1, \mathcal{T}_1, X_1, \mathcal{M}_1, p_1) \to (\mathcal{L}_2, \mathcal{T}_2, X_2, \mathcal{M}_2, p_2)$ consists of a Σ_1 interpretation $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$, a continuous map $f : X_2 \to X_1$, and a (specified, but left nameless) isomorphism of étale structures (over X_2) $f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2}$:

$$\begin{array}{ccccc} \mathcal{M}_1 & \longleftarrow & f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2} & \xleftarrow{\mathcal{F}^*} & \mathcal{M}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow & \swarrow{\scriptstyle p_2} & \\ X_1 & \xleftarrow{f} & X_2 & & \end{array}$$

Regarding $\mathcal{M}_1, \mathcal{M}_2$ as "continuous maps to countable models" as in Remark 4.9, the picture becomes

$$\begin{array}{ccc} X_1 & \xleftarrow{f} & X_2 \\ {\scriptstyle \mathcal{M}_1}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_2} \\ \{\text{models of } \mathcal{T}_1\} & \xleftarrow{\mathcal{F}^*} & \{\text{models of } \mathcal{T}_2\} \end{array} \qquad \mathcal{T}_1 \xrightarrow{\mathcal{F}} \mathcal{T}_2$$

Two consecutive parametrized interpretations may be composed in the obvious manner. A **Σ_1-definable isomorphism** between parametrized interpretations $h : (\mathcal{F}, f) \cong (\mathcal{G}, g)$ is simply one between the underlying intepretations $h : \mathcal{F} \cong \mathcal{G}$.

We thus get a 2-category of quasi-Polish parametrized Π_2 theories, parametrized Σ_1 interpretations, and Σ_1-definable isomorphisms, which admits canonical maps to both sides of (9.1):

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\begin{array}{ccc} & \{\text{quasi-Polish parametrized } \Pi_2 \text{ theories}\} & \\ \swarrow & & \searrow \\ \{\text{countable } \Pi_2 \text{ theories}\} & & \{\text{quasi-Polish groupoids}\} \end{array} \tag{9.9}$$

The left leg simply forgets about the parametrizations. The right leg, a contravariant 2-functor, takes a parametrized theory $(\mathcal{T}, X, \mathcal{M})$ to the isomorphism groupoid $\mathrm{Iso}_X(\mathcal{M})$; takes a parametrized interpretation $(\mathcal{F}, f) : (\mathcal{T}_1, X_1, \mathcal{M}_1) \to (\mathcal{T}_2, X_2, \mathcal{M}_2)$ to the continuous functor

$$\begin{aligned} (\mathcal{F}, f)^* : \mathrm{Iso}_{X_2}(\mathcal{M}_2) &\longrightarrow \mathrm{Iso}_{X_1}(\mathcal{M}_1) \\ \left(g : (\mathcal{M}_2)_x \cong (\mathcal{M}_2)_y\right) &\longmapsto \begin{pmatrix} (\mathcal{M}_1)_{f(x)} = f^*(\mathcal{M}_1)_x \cong \mathcal{F}_x^{\mathcal{M}_2} \\ \overset{g}{\cong} \mathcal{F}_y^{\mathcal{M}_2} \cong f^*(\mathcal{M}_1)_y = (\mathcal{M}_1)_{f(y)} \end{pmatrix}; \end{aligned} \tag{9.10}$$

and takes a Σ_1-definable isomorphism $h : (\mathcal{F}, f) \cong (\mathcal{G}, g)$ to the continuous natural isomorphism

$$\begin{aligned} h^* : X_2 &\longrightarrow \mathrm{Iso}_{X_1}(\mathcal{M}_1) \\ x &\longmapsto \begin{pmatrix} (\mathcal{M}_1)_{f(x)} = f^*(\mathcal{M}_1)_x \cong \mathcal{F}_x^{\mathcal{M}_2} \\ \overset{h_x}{\cong} \mathcal{G}_x^{\mathcal{M}_2} \cong g^*(\mathcal{M}_1)_x = (\mathcal{M}_1)_{g(x)} \end{pmatrix} \end{aligned} \tag{9.11}$$

between the continuous functors $(\mathcal{F}, f)^*$ and $(\mathcal{G}, g)^*$ as defined above.

THEOREM 9.12. *In the above diagram* (9.9):

(a) *The right leg is a full and faithful contravariant* 2*-functor, i.e., restricts to an equivalence between each hom-groupoid of parametrized interpretations* $(\mathcal{T}_1, X_1, \mathcal{M}_1) \to (\mathcal{T}_2, X_2, \mathcal{M}_2)$ *and the corresponding hom-groupoid of continuous functors* $\mathrm{Iso}_{X_2}(\mathcal{M}_2) \to \mathrm{Iso}_{X_1}(\mathcal{M}_1)$.

(b) *The left leg restricts to an equivalence of* 2*-categories between the full sub-*2*-category of zero-dimensional Polish parametrized* Π_2 *theories* (*meaning the base space* X *of the parametrization is zero-dimensional Polish, not necessarily the étale space* M), *and all countable* Π_2 *theories.*

Thus, we have a composite full and faithful 2*-functor, i.e., "embedding of* 2*-categories,"*

$$\{\textit{ctbl } \Pi_2 \textit{ theories}\} \simeq \{0\textit{-d Pol parametrized } \Pi_2 \textit{ theories}\} \longrightarrow \{\textit{q-Pol groupoids}\} \tag{9.13}$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

taking each countable Π_2 *theory to the isomorphism groupoid of any of its zero-dimensional Polish parametrizations with* Σ_1 *saturations.*

PROOF. (a) follows from the Joyal–Tierney theorem. Indeed, for a parametrized interpretation $(\mathcal{F}, f) : (\mathcal{T}_1, X_1, \mathcal{M}_1) \to (\mathcal{T}_2, X_2, \mathcal{M}_2)$, $\mathcal{F}$ is a model of $\mathcal{T}_1$ in the category of Σ_1 imaginaries over $\mathcal{T}_2$, which by Joyal–Tierney in the form of Remark 8.6 is equivalently a model $\mathcal{F}^{\mathcal{M}_2}$ in the category of second-countable étale $\mathrm{Iso}_{X_2}(\mathcal{M}_2)$-spaces, i.e., an étale model of $\mathcal{T}_1$ over X_2 equipped with a continuous action of $\mathrm{Iso}_{X_2}(\mathcal{M}_2)$ via isomorphisms; this action corresponds, via the isomorphism $f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2}$, to the extension of $f : X_2 \to X_1$ to a functor $\mathrm{Iso}_{X_2}(\mathcal{M}_2) \to \mathrm{Iso}_{X_1}(\mathcal{M}_1)$ via (9.10). Conversely, given such a continuous functor $f : \mathrm{Iso}_{X_2}(\mathcal{M}_2) \to \mathrm{Iso}_{X_1}(\mathcal{M}_1)$, the unique-up-to-definable-isomorphism interpretation $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$ inducing f in the above manner takes a Σ_1 imaginary over $\mathcal{T}_1$, which is equivalently by Remark 8.6 an étale $\mathrm{Iso}_{X_1}(\mathcal{M}_1)$-space $q : A \to X_1$, to the Σ_1 imaginary over $\mathcal{T}_2$ corresponding to the pullback $\mathrm{Iso}_{X_2}(\mathcal{M}_2)$-space $f^*(A)$; then $\mathcal{F}^{\mathcal{M}_2} \cong f^*(\mathcal{M}_1)$ by definition. It is straightforward to check that this correspondence is functorial via (9.11).

For (b), the left leg is essentially surjective, i.e., every countable Π_2 theory has a zero-dimensional Polish parametrization, by Example 6.9; and it is locally full and faithful, i.e., bijective on 2-cells between each fixed pair of morphisms, by definition of said 2-cells as Σ_1-definable isomorphisms. It remains only to check that it is locally (essentially) surjective, i.e., for two zero-dimensional Polish parametrized theories $(\mathcal{T}_1, X_1, \mathcal{M}_1), (\mathcal{T}_2, X_2, \mathcal{M}_2)$, every Σ_1 interpretation $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$ can be parametrized via some continuous map $f : X_2 \to X_1$ and isomorphism $f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2}$. That is, for each fiber $(\mathcal{M}_2)_x$ of $\mathcal{M}_2$, we know the model $\mathcal{F}^{(\mathcal{M}_2)_x} \models \mathcal{T}_1$ is isomorphic to some fiber $(\mathcal{M}_1)_{f(x)}$ of $\mathcal{M}_1$; we need to find the fiber $f(x)$ and the isomorphism in a continuous manner. Form the space

$$\mathrm{Iso}_{X_1,X_2}(\mathcal{M}_1, \mathcal{F}^{\mathcal{M}_2}) = \left\{(y, x, g) \;\middle|\; y \in X_1 \;\&\; x \in X_2 \;\&\; g : (\mathcal{M}_1)_y \cong \mathcal{F}^{(\mathcal{M}_2)_x}\right\}$$

as in Remark 4.18. By Remark 5.19, the second projection $\mathrm{cod} : \mathrm{Iso}_{X_1,X_2}(\mathcal{M}_1, \mathcal{F}^{\mathcal{M}_2}) \to X_2$ is open, and it is also surjective, since as noted before, each $\mathcal{F}^{(\mathcal{M}_2)_x}$ is isomorphic to some $(\mathcal{M}_1)_y$. Thus by Michael's selection theorem [44, Corollary 1.4] (see also [45], [21]; to deal with the fact that $\mathrm{Iso}_{X_1,X_2}(\mathcal{M}_1, \mathcal{F}^{\mathcal{M}_2})$ is quasi-Polish instead of Polish, use the latter paper or (2.12)), cod has a continuous section $X_2 \to \mathrm{Iso}_{X_1,X_2}(\mathcal{M}_1, \mathcal{F}^{\mathcal{M}_2})$, whose first coordinate yields f and third coordinate yields the isomorphism $f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2}$. ⊣

REMARK 9.14. If one is only interested in certain restricted kinds of theories, then it suffices in Theorem 9.12 to restrict to a full sub-2-category of zero-dimensional Polish parametrizations which are known to parametrize all such theories.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

For example, if one is only interested in theories with no finite models, and modulo which negated atomic formulas are equivalent to Σ_1 formulas (e.g., from Morleyizing to recover the traditional definition of Σ_1 as in Example 3.4), then it suffices to consider the usual Polish space of models on $\mathbb{N}$ as in Example 6.1, thereby recovering the boldface version of [26, Theorem 1.5].

§10. The Lopez-Escobar theorem and $\mathcal{L}_{\omega_1\omega}$ imaginaries. An important tool in descriptive set theory is the Baire category quantifier $\exists^*$ (and its dual $\forall^*$); see [33, Section 8.J, Exercise 22.22], [43, Section A], [18, Section 2.3–4]. Given a continuous open map $f : X \to Y$ between quasi-Polish spaces, each $A \in X$ has a "Baire-categorical image" under f:

$$\exists^*_f(A) := \{y \in Y \mid A \text{ is nonmeager in the fiber } f^{-1}(y)\}.$$

The usefulness of $\exists^*_f$ is largely because it, unlike ordinary image, preserves Borel sets; in fact, it preserves $\mathbf{\Sigma}^0_\alpha$ sets for all α.

We have an analogous result for étale structures $\mathcal{M} \to X$ with Σ_1 saturations, thought of as "maps $x \mapsto \mathcal{M}_x : X \to \{\text{all structures}\}$" as in Remark 5.2. The "fibers" of such a map should be thought of as the isomorphism orbits $\mathrm{Iso}_X(\mathcal{M}) \cdot \vec{a}$ of each $\vec{a} \in M^n_X$ (and not the fiber structures $\mathcal{M}_x$, which are the "values" of the map $x \mapsto \mathcal{M}_x$). Thus, "image" becomes "saturation."

Definition 10.1. For an étale structure $p : \mathcal{M} \to X$, open $U \subseteq \mathrm{Iso}_X(\mathcal{M})$, and any $A \subseteq M^n_X$, the **Vaught transform** is the "Baire-categorical saturation"

$$U * A := \{\vec{a} \in M^n_x \mid \exists \text{ nonmeagerly many } g \in U \cap \mathrm{cod}^{-1}(x) \text{ s.t. } (\vec{a} \in g \cdot A)\}.$$

(Here $\mathrm{cod} : \mathrm{Iso}_X(\mathcal{M}) \to X$ is the codomain map; recall Definition 4.15.)

The more common notation for $U * A$ is $A^{\triangle U^{-1}}$, including in [39] where it was first studied for groupoids; the above notation suggestive of the ordinary saturation $U \cdot A$ is from [18, Definition 4.2.1].

Theorem 10.2 (Lopez-Escobar for étale structures). *Let $\mathcal{L}$ be a countable language, $p : \mathcal{M} \to X$ be a second-countable étale $\mathcal{L}$-structure with Σ_1 saturations over a quasi-Polish X. For any $\mathbf{\Sigma}^0_\alpha$ set $A \subseteq M^n_X$, there is a Σ_α formula defining* $\mathrm{Iso}_X(\mathcal{M}) * A$.

Remark 10.3. It follows from this statement that more generally, for any basic open $[\![U \mapsto V]\!] \subseteq \mathrm{Iso}_X(\mathcal{M})$, where $U, V \subseteq M^m_X$ are open,

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

$$\begin{aligned}
&[\![U \mapsto V]\!] * A \\
&\quad = \{\vec{a} \in M_x^n \mid \{g \in [\![U \mapsto V]\!] \mid \vec{a} \in g \cdot A\} \text{ is nonmeager in } \mathrm{cod}^{-1}(x)\} \\
&\quad = \{\vec{a} \in M_x^n \mid \exists \vec{b} \in V_x \, (\{g \mid (\vec{a}, \vec{b}) \in g \cdot (A \times_X U)\} \text{ is nonmeager in } \mathrm{cod}^{-1}(x))\} \\
&\quad = \{\vec{a} \in M_x^n \mid \exists \vec{b} \in V_x \, ((\vec{a}, \vec{b}) \in \mathrm{Iso}_X(\mathcal{M}) * (A \times_X U))\}
\end{aligned}$$

is the fiberwise inverse image of V under a binary relation $\phi^{\mathcal{M}} \subseteq M_X^n \times_X M_X^m$, for some Σ_α formula ϕ with $n + m$ variables depending only on U.

Proof of Theorem 10.2. By induction on α. For $\alpha = 1$, this is just the fact that $\mathcal{M}$ has Σ_1 saturations. For a countable union of sets for which the result holds, we may take the disjunction of the formulas. By Definition 2.1 of the Borel hierarchy, it thus remains to show, assuming the result holds for all $\mathbf{\Sigma}^0_\alpha$ sets, that for any two $A, B \in \mathbf{\Sigma}^0_\alpha(M_X^n)$, the result holds for $A \setminus B \in \mathbf{\Sigma}^0_{\alpha+1}(M_X^n)$. Fix countable bases $\mathcal{U}_m$ for each M_X^m, so that $[\![U \mapsto V]\!]$ for $U, V \in \mathcal{U}_m$ form a basis for $\mathrm{Iso}_X(\mathcal{M})$. For $\vec{a} \in M_x^n$,

$$\begin{aligned}
&\vec{a} \in \mathrm{Iso}_X(\mathcal{M}) * (A \setminus B) \\
&\qquad \iff \{g \mid \vec{a} \in g(A \setminus B)\} \text{ is nonmeager in } \mathrm{cod}^{-1}(x);
\end{aligned}$$

by the property of Baire (see [33, Proposition 8.26]), this holds iff $\{g \mid \vec{a} \in g(A \setminus B)\} = \{g \mid \vec{a} \in gA\} \setminus \{g \mid \vec{a} \in gB\}$ is comeager in some nonempty basic open set $[\![U \mapsto V]\!] \cap \mathrm{cod}^{-1}(x)$, and so this is

$$\begin{aligned}
&\iff \exists m \, \exists U, V \in \mathcal{U}_m \begin{pmatrix} \{g \in [\![U \mapsto V]\!] \cap \mathrm{cod}^{-1}(x) \mid \vec{a} \in gA\} \text{ nonmeager}, \\ \{g \in [\![U \mapsto V]\!] \cap \mathrm{cod}^{-1}(x) \mid \vec{a} \in gB\} \text{ meager} \end{pmatrix} \\
&\iff \exists m \, \exists U, V \in \mathcal{U}_m \, (\vec{a} \in ([\![U \mapsto V]\!] * A) \setminus ([\![U \mapsto V]\!] * B));
\end{aligned}$$

by the induction hypothesis and preceding remark, there are Σ_α formulas ϕ_U, ψ_U such that this is

$$\begin{aligned}
&\iff \exists m \, \exists U, V \in \mathcal{U}_m \, ((\exists \vec{b} \in V_x) \, \phi_U^{\mathcal{M}}(\vec{a}, \vec{b}) \wedge \neg(\exists \vec{c} \in V_x) \, \psi_U^{\mathcal{M}}(\vec{a}, \vec{c})) \\
&\iff \exists m \, \exists U \in \mathcal{U}_m \, \exists \vec{b} \in M_x^m \, (\phi_U^{\mathcal{M}}(\vec{a}, \vec{b}) \wedge \neg \psi_U^{\mathcal{M}}(\vec{a}, \vec{b}))
\end{aligned}$$

where $\Longleftarrow$ is because $\vec{b}$ belongs to some open section $V \in \mathcal{U}_m$. So this is defined by the $\Sigma_{\alpha+1}$ formula

$$\bigvee_m \bigvee_{U \in \mathcal{U}_m} \exists y_0, \ldots, y_{m-1} \, (\phi_U(\vec{x}, \vec{y}) \wedge \neg \psi_U(\vec{x}, \vec{y})). \qquad \dashv$$

Example 10.4. Applying Theorem 10.2 to the standard Polish space of models on $\mathbb{N}$ (Example 6.1) recovers the classical Lopez-Escobar theorem [38], or rather its strengthening adapted levelwise to the Borel hierarchy by Vaught [55] (see also [33, Theorem 16.8], [25, Theorem 11.3.6]).

Applying it instead to the parametrization of Example 6.7 yields the version of the Lopez-Escobar theorem used in [14], which is the boldface

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

version of the effective "positive" (but still admitting $\neq$ as Σ_1) Lopez-Escobar theorem in [5].

We now have the Borel analogs of the material from the two preceding sections.

DEFINITION 10.5. For a second-countable étale structure $p : \mathcal{M} \to X$ over quasi-Polish X, a **(fiberwise) countable Borel** $\mathrm{Iso}_{\mathbf{X}}(\boldsymbol{\mathcal{M}})$**-space over** $\boldsymbol{X}$ is a standard Borel space A equipped with a countable-to-1 Borel map $q : A \to X$ and a Borel action of $\mathrm{Iso}_X(\mathcal{M})$.

DEFINITION 10.6. For a countable $\mathcal{L}_{\omega_1\omega}$ theory $\mathcal{T}$, a $\mathcal{L}_{\omega_1\omega}$ **imaginary sort** $\boldsymbol{\Psi} = (\bigsqcup_{\mathbf{i}} \phi_{\mathbf{i}})/(\bigsqcup_{\mathbf{i,j}} \varepsilon_{\mathbf{ij}})$ **over** $\boldsymbol{\mathcal{T}}$ is defined exactly as in Definition 8.2, except that the formulas ϕ_i, ε_{ij} may be $\mathcal{L}_{\omega_1\omega}$ instead of Σ_1. These may be interpreted in a second-countable étale $p : \mathcal{M} \to X$ over quasi-Polish X to yield a countable Borel $\mathrm{Iso}_X(\mathcal{M})$-space $\Phi^{\mathcal{M}} \to X$.

THEOREM 10.7. *Let $\mathcal{T}$ be a countable $\mathcal{L}_{\omega_1\omega}$ theory, $p : \mathcal{M} \to X$ be a second-countable étale space with Σ_1 saturations parametrizing models of $\mathcal{T}$. Then every countable Borel $\mathrm{Iso}_X(\mathcal{M})$-space $q : A \to X$ is isomorphic to $\Phi^{\mathcal{M}}$ for some $\mathcal{L}_{\omega_1\omega}$ imaginary Φ over $\mathcal{T}$.*

This was proved in [14] for a particular parametrization $\mathcal{M}$, namely, that of Example 6.7 (and implicitly in [27] for the standard parametrization of Example 6.1).

PROOF. By [18, proofs of Corollaries 4.5.13 and 4.3.9], we may topologically realize A as a second-countable étale $\mathrm{Iso}_X(\mathcal{M})$-space, after refining the topology on the space of objects X of the groupoid $\mathrm{Iso}_X(\mathcal{M})$ by adjoining countably many sets of the form $[\![U_i \mapsto V_i]\!] * B_i$ to the topology, where $U_i, V_i \subseteq M_X^{m_i}$ are open and $B_i \subseteq X$ are Borel. By Remark 10.3, each $[\![U_i \mapsto V_i]\!] * B_i = p(U_i \cap \phi_i^{\mathcal{M}})$ for some $\mathcal{L}_{\omega_1\omega}$ formula $\phi_i(x_0, \dots, x_{m_i-1})$. Morleyize the formulas ϕ_i to obtain a new theory $\mathcal{T}'$ in an expanded language $\mathcal{L}' \supseteq \mathcal{L}$, and then Morleyize the étale structure $\mathcal{M}$ via Definition 4.12, to obtain a new étale structure $p' : \mathcal{M}' \to X'$, now with Σ_1 saturations by Lemma 5.16, such that X' is X with a finer topology in which each $p(U_i \cap \phi_i^{\mathcal{M}})$ becomes open. Pulling back the topologically realized étale space A to X', we thus obtain a second-countable étale $\mathrm{Iso}_{X'}(\mathcal{M}')$-space, which by the Joyal–Tierney Theorem 8.4 is named by some Σ_1 imaginary in $\mathcal{M}'$, hence by an $\mathcal{L}_{\omega_1\omega}$ imaginary in $\mathcal{M}$. ⊣

REMARK 10.8. As in Remark 8.6, it follows that we in fact have an equivalence of categories

$$\{\mathcal{L}_{\omega_1\omega} \text{ imaginaries over } \mathcal{T}\} \xrightarrow{\sim} \{\text{countable Borel } \mathrm{Iso}_X(\mathcal{M})\text{-spaces}\}$$
$$\Phi \longmapsto \Phi^{\mathcal{M}},$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

where the subobjects on the left, namely, $\mathcal{L}_{\omega_1\omega}$-definable subsorts of $\mathcal{L}_{\omega_1\omega}$ imaginaries, correspond to $\mathrm{Iso}_X(\mathcal{M})$-invariant Borel subspaces by the Lopez-Escobar Theorem 10.2.

This recovers one of the main results of [14] (extending the boldface result of [27]). To recover the rest, we need to convert Theorem 10.7 into the 2-categorical form of Theorem 9.12.

DEFINITION 10.9. An $\mathcal{L}_{\omega_1\omega}$ **interpretation** between two $\mathcal{L}_{\omega_1\omega}$ theories is defined exactly as in Definition 9.2, except that the formulas and imaginaries used may be $\mathcal{L}_{\omega_1\omega}$ rather than Σ_1.

DEFINITION 10.10. Recall the notion of *quasi-Polish parametrized* Π_2 *theory* $(\mathcal{L}, \mathcal{T}, X, \mathcal{M}, p)$ from Definition 9.8. A **quasi-Polish parametrized $\mathcal{L}_{\omega_1\omega}$ theory** will mean such a tuple where $\mathcal{T}$ is a countable $\mathcal{L}_{\omega_1\omega}$ theory in the countable language $\mathcal{L}$, and $p : \mathcal{M} \to X$ is a second-countable étale structure over the quasi-Polish space X with $\mathcal{L}_{\omega_1\omega}$ saturations of open sets.

A **parametrized $\mathcal{L}_{\omega_1\omega}$ interpretation** $(\mathcal{F}, f) : (\mathcal{L}_1, \mathcal{T}_1, X_1, \mathcal{M}_1, p_1) \to (\mathcal{L}_2, \mathcal{T}_2, X_2, \mathcal{M}_2, p_2)$ between two such parametrized theories consists of an $\mathcal{L}_{\omega_1\omega}$ interpretation $\mathcal{F} : \mathcal{T}_1 \to \mathcal{T}_2$, a *Borel* map $f : X_2 \to X_1$, and a *Borel* isomorphism of fiberwise countable *Borel* structures $f^*(\mathcal{M}_1) \cong \mathcal{F}^{\mathcal{M}_2}$.

We have the Borel analog of the diagram (9.9):

$$\begin{array}{ccc} & \{\text{quasi-Polish parametrized } \mathcal{L}_{\omega_1\omega} \text{ theories}\} & \\ & \simeq \swarrow \qquad \searrow & \\ \{\text{countable } \mathcal{L}_{\omega_1\omega} \text{ theories}\} & & \{\text{quasi-Polish groupoids}\} \end{array} \tag{10.11}$$

in which the three 2-categories and both 2-functors are as before (see (9.10) and (9.11)), but with $\mathcal{L}_{\omega_1\omega}$ interpretations and definable isomorphisms and *Borel* functors and natural transformations.

THEOREM 10.12. *In the above diagram*:

(a) *The right leg is a full and faithful* 2*-functor.*
(b) *The left leg is an equivalence of* 2*-categories* (*on its entire domain*).

Thus, we have a composite full and faithful 2*-functor* {*ctbl* $\mathcal{L}_{\omega_1\omega}$ *theories*} $\to$ {*quasi-Polish groupoids*}.

PROOF. As in Theorem 9.12, (a) follows from Theorem 10.7 or rather Remark 10.8. For (b), essential surjectivity follows from the fact that every countable $\mathcal{L}_{\omega_1\omega}$ theory has a quasi-Polish (or even zero-dimensional Polish) parametrization with $\mathcal{L}_{\omega_1\omega}$ saturations by Morleyization and any of the examples from Section 6 that admit finite models, and local full faithfulness

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

is trivial as before. For fullness, we again follow the proof of Theorem 9.12; the added ingredient needed is that (in the notation from there) surjectivity of the Borel map $\mathrm{cod} : \mathrm{Iso}_{X_1,X_2}(\mathcal{M}_1, \mathcal{F}^{\mathcal{M}_2}) \to X_2$ is already enough to imply the existence of a *Borel* section, which follows by a straigtforward application of Kechris's large section uniformization theorem [34, Theorem 18.6*] applied to the meager ideal; see [13, Theorem 7.9]. ⊣

§11. Groupoid representations. Theorems 9.12 and 10.12 show that the operation of taking the semantics of a theory is an "embedding" from the 2-category of theories to the 2-category of groupoids (in both the continuous and Borel settings). It is natural to ask what is the image of this 2-functor, i.e., which quasi-Polish groupoids arise as the groupoid of models of a theory. The corresponding question in topos theory was answered by Moerdijk [47], who proved what can be called the topos-theoretic analog of the Yoneda lemma (which says that absent any topological structure, *every* groupoid is canonically a groupoid of isomorphisms between structures, via the left translation action on itself). In this final section, we adapt Moerdijk's result to the countable model-theoretic setting.

First, we clarify what we mean by an abstract topological groupoid (e.g., $\mathrm{Iso}_X(\mathcal{M})$) and action.

Definition 11.1. A **topological groupoid** $G \rightrightarrows X$ consists of topological spaces G, X of *morphisms* and *objects*, respectively, with continuous *domain* and *codomain* maps $\mathrm{dom}, \mathrm{cod} : G \rightrightarrows X$, as well as *identity* $1_{(-)} : X \to G$, *inverse* $(-)^{-1} : G \to G$, and *composition* $\circ$ (or simply juxtaposition) of adjacent morphisms, subject to the usual axioms of associativity, identity, and inverse.

As is common when working with topological groupoids, we identify each object $x \in X$ with the corresponding identity morphism $1_x \in G$, so that $X \subseteq G$ is a subspace.

By an **open subgroupoid** of G, we mean an open subset $U \subseteq G$ of morphisms which is closed under composition and inverse. It follows that for each $g : x \to y \in U$, the identity morphisms at both its source $1_x = g^{-1} \circ g$ and target $1_y = g \circ g^{-1}$ are in U as well, whence we may regard U as the space of morphisms of a groupoid in its own right, with objects $\{x \in X \mid 1_x \in U\} = \mathrm{dom}(U)$.

We call G a **non-Archimedean** topological groupoid if every identity morphism $1_x \in G$ has a neighborhood basis of open subgroupoids.

A **continuous action** of the groupoid G on a topological space A equipped with a continuous map $q : A \to X$, regarded as a topological bundle over X, is a map

$$\cdot : G \times_X A = \{(g, a) \in G \times A \mid \mathrm{dom}(g) = q(a)\} \longrightarrow A,$$

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

jointly continuous in both variables, taking $g : x \to y \in G$ and $a \in A_x$ to $g \cdot a \in A_y$, and obeying the usual associativity and identity axioms. We also call (A, q) equipped with the action a **G-space**, and an **étale G-space** if A is an étale space over X.

DEFINITION 11.2. Now suppose $G \rightrightarrows X$ is an *open* topological groupoid, i.e., dom, cod are open maps (see Lemma 5.18). For an open subgroupoid $U \subseteq G$ as above, let (by an abuse of notation)

$$G/U = \mathrm{dom}^{-1}(U)/U = \left\{ gU \;\middle|\; g \in G \;\&\; \mathrm{dom}(g) = 1_{\mathrm{dom}(g)} \in U \right\}$$

denote the **space of left cosets** of U. This is naturally an étale G-space over X via $\mathrm{cod} : G/U \to X$ and the left multiplication action of G. An open section $S/U \subseteq G/U$ may be identified with its lift in G, which is an open right-U-invariant subset $S \subseteq \mathrm{dom}^{-1}(U)$ such that $S^{-1}S \subseteq U$.

For two open subgroupoids $U, V \subseteq G$ and an open right-V-invariant $S \subseteq \mathrm{dom}^{-1}(V)$, we have

$$U \subseteq SS^{-1} \tag{11.3}$$
$$\iff \text{the right multiplication map } (-)S : G/U \to G/V \text{ is well-defined.}$$

Note also that such a right multiplication map, when defined, is clearly left-G-equivariant.

DEFINITION 11.4. Let $G \rightrightarrows X$ be an open non-Archimedean topological groupoid. Fix a family $\mathcal{U}$ of open subgroupoids forming a neighborhood basis for each identity morphism, and for each $U \in \mathcal{U}$, an open cover $\mathcal{S}_U$ of $\mathrm{dom}^{-1}(U)$ by open right-U-invariant sets S such that $S^{-1}S \subseteq U$ (corresponding to a cover of G/U by open sections S/U). Note that when G is second-countable, $\mathcal{U}$ and each $\mathcal{S}_U$ may be chosen to be countable, whence each G/U is a second-countable étale space over X.

The **canonical G-structure** $\mathcal{M} = \mathcal{M}_{G\mathcal{U},(\mathcal{S}_U)_U}$ (determined by the $\mathcal{U}, \mathcal{S}_U$) is the multi-sorted étale structure over X with a sort G/U for each $U \in \mathcal{U}$, and a unary function $(-)S : G/U \to G/V$ for each $U, V \in \mathcal{U}$ and $S \in \mathcal{S}_V$ obeying (11.3). The canonical left translation action of G on each G/U is an action via isomorphisms between fibers of $\mathcal{M}_G$, yielding a canonical functor

$$\iota : G \longrightarrow \mathrm{Iso}_X(\mathcal{M})$$

which is the identity on objects and sending each $g : x \to y \in G$ to its action $\mathcal{M}_x \cong \mathcal{M}_y$.

THEOREM 11.5 (Moerdijk [47]). *For any open non-Archimedean T_0 topological groupoid G with canonical structure $\mathcal{M} = \mathcal{M}_{G\mathcal{U},(\mathcal{S}_U)_U}$ as above, the canonical functor $\iota : G \to \mathrm{Iso}_X(\mathcal{M})$ is a topological embedding with* cod*-fiberwise dense image.*

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

PROOF. First, we convert to an alternate description of $\mathrm{Iso}_X(\mathcal{M})$. An isomorphism $f : \mathcal{M}_x \to \mathcal{M}_y$ in this groupoid is determined by its values $f(gU)$ for each element gU of each sort $(G/U)_x$ of $\mathcal{M}_x$. But preservation of the right multiplication maps $(-)S$ means that f is in fact determined by its values on just the identity cosets $1_x U$: indeed, for any other coset $gU \in (G/U)_x$, we may find

$$\begin{aligned} &g \in S \in \mathcal{S}_U \ \&\ 1_x \in V \in \mathcal{U} \ \&\ V \subseteq SS^{-1}, \text{ which implies} \\ &f(gU) = f(1_x VS) = f(1_x V)S. \end{aligned} \tag{11.6}$$

(This is the part most analogous to the Yoneda lemma.) Moreover, these values $f(1_x U)$ must obey

$$1_x \in U \subseteq V \in \mathcal{U} \implies f(1_x U)V = f(1_x V), \tag{11.7}$$

by finding $1_x \in S \in \mathcal{S}_U$ and $1_x \in T \in \mathcal{S}_V$ and then a neighborhood $1_x \in W \subseteq SS^{-1} \cap TT^{-1}$ in $\mathcal{U}$, so that $f(1_x U)V = f(1_x WS)V = f(1_x W)SV = f(1_x W)T = f(1_x WT) = f(1_x V)$. In other words, we have a canonical injection

$$\{\text{isomorphisms } f : \mathcal{M}_x \cong \mathcal{M}_y \in \mathrm{Iso}_X(\mathcal{M})\} \hookrightarrow \varprojlim_{1_x \in U \in \mathcal{U}} (G/U)_y \tag{11.8}$$

into the set of all families $(f(1_x U) \in (G/U)_y)_{1_x \in U \in \mathcal{U}}$ obeying the coherence condition (11.7).

Moreover, the above injections (11.8) together form a topological embedding of $\mathrm{Iso}_X(\mathcal{M})$ into the inverse limit topology on the right-hand side, in the following sense. For each $U \in \mathcal{U}$ and open right-U-invariant $S \subseteq \mathrm{dom}^{-1}(U)$, the set

$$[\![U/U \mapsto S/U]\!] = \left\{ f : \mathcal{M}_x \cong \mathcal{M}_y \;\middle|\; 1_x \in U \ \&\ f(1_x U) \subseteq S \right\} \subseteq \mathrm{Iso}_X(\mathcal{M}), \tag{11.9}$$

which is the preimage under (11.8) of the coherent families containing a fiber of the open section $S/U \subseteq G/U$, is open. We claim that these sets together with the maps $\mathrm{dom}, \mathrm{cod} : \mathrm{Iso}_X(\mathcal{M}) \rightrightarrows X$ suffice to generate the topology of $\mathrm{Iso}_X(\mathcal{M})$ (from Definition 4.15). Indeed, by (11.6), a subbasic open $[\![S/U \mapsto T/U]\!] \subseteq \mathrm{Iso}_X(G)$, where $S/U, T/U \subseteq G/U$ are basic open sections, may be written in terms of the sets (11.9) as

$$[\![S/U \mapsto T/U]\!] = \bigcup_{\substack{\mathcal{U} \ni V \subseteq SS^{-1} \\ S' \subseteq S \\ T'/V \subseteq G/V \\ T'S' \subseteq T}} [\![V/V \mapsto T'/V]\!]$$

(where $T'/V \subseteq G/V$ is an open section containing $f(1_x V)$ in (11.6)).

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

Now the ι-preimage of such a set $[\![U/U \mapsto S/U]\!]$ consists of all $g : x \to y \in G$ such that $1_x \in U$ and $gU \in S/U$, which is just the open right-U-invariant $S \subseteq G$. Since G is non-Archimedean, its open right-U-invariant sets over all $U \in \mathcal{U}$ are easily seen to form a basis; thus ι is an embedding.

Finally, the image of ι is cod-fiberwise dense: from above, $\mathrm{Iso}_X(\mathcal{M})$ has an open basis of sets

$$\bigcap_{i<n} [\![U_i/U_i \mapsto S_i/U_i]\!] \cap \mathrm{dom}^{-1}(V) \cap \mathrm{cod}^{-1}(W)$$

where $U_0, \dots, U_{n-1} \in \mathcal{U}$, $S_i/U_i \subseteq G/U_i$ are open sections, and $V, W \subseteq X$ are open. If such a set is nonempty in some $\mathrm{cod}^{-1}(y)$, then it contains some isomorphism $f : \mathcal{M}_x \to \mathcal{M}_y$ where $x = 1_x \in V \cap \bigcap_i U_i$, $y \in W$, and each $f(1_x U_i) \subseteq S_i$. Let $U \in \mathcal{U}$ with $1_x \in U \subseteq V \cap \bigcap_i U_i$, and pick $g \in f(1_x U)$. Then $g : x \to y \in G$ with each $gU_i \supseteq gU \subseteq f(1_x U) \subseteq f(1_x U_i)$, whence $gU_i = f(1_x U_i)$ since both are left cosets of U_i, whence $\iota(g) \in \bigcap_{i<n} [\![U_i/U_i \mapsto S_i/U_i]\!] \cap \mathrm{dom}^{-1}(V) \cap \mathrm{cod}^{-1}(y)$. ⊣

Remark 11.10. The inverse limits on the right-hand side of (11.8) above may be patched together, over all $x, y \in X$, into a topological category, which can be seen as a "left completion" of G analogous to the left completion of a Polish automorphism group [6]. This is studied in depth by Moerdijk [47] in the topos-theoretic context.

Theorem 11.11. *For an open non-Archimedean quasi-Polish groupoid G, with canonical second-countable étale structure $\mathcal{M} = \mathcal{M}_{G\mathcal{U},(\mathcal{S}_U)_U}$ with respect to some countable $\mathcal{U}, \mathcal{S}_U$ as above, the canonical functor $\iota : G \to \mathrm{Iso}_X(\mathcal{M})$ is a topological groupoid isomorphism.*

Proof. By Theorem 11.5, we may identify G with a cod-fiberwise dense quasi-Polish (hence cod-fiberwise dense $\mathbf{\Pi}^0_2$) subgroupoid of $\mathrm{Iso}_X(\mathcal{M})$; since $(-)^{-1}$ is a homeomorphism, $G \subseteq \mathrm{Iso}_X(\mathcal{M})$ is also dom-fiberwise dense $\mathbf{\Pi}^0_2$. By Pettis's theorem for the left translation action of the open quasi-Polish groupoid G on the bundle $\mathrm{cod} : \mathrm{Iso}_X(\mathcal{M}) \to X$ (proved the same way as Pettis's theorem for Polish groups; see [18, 4.2.8]), it follows that $G = GG = \mathrm{Iso}_X(\mathcal{M})\mathrm{Iso}_X(\mathcal{M}) = \mathrm{Iso}_X(\mathcal{M})$. ⊣

Remark 11.12. We note that the last step above of applying Pettis's theorem also has precedent in topos theory, namely, the "closed subgroupoid theorem" of Johnstone [29]; see [17]. In the terminology of [46], [47], we have shown that an open quasi-Polish groupoid is non-Archimedean iff it is *étale-complete*.

Corollary 11.13. *The 2-functor from Theorem 9.12 is an equivalence of 2-categories*

$$\{\textit{countable } \Pi_2 \textit{ theories}\} \simeq \{\textit{open non-Archimedean quasi-Polish groupoids}\}$$

(*with continuous functors and natural isomorphisms on the right*).

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

COROLLARY 11.14. *The 2-functor from Theorem 10.12 is an equivalence of 2-categories*

$$\{\textit{countable } \mathcal{L}_{\omega_1\omega} \textit{ theories}\} \simeq \{\textit{open non-Archimedean quasi-Polish groupoids}\}$$

(*with Borel functors and natural isomorphisms on the right*).

As before (see Theorem 9.12), these equivalences of 2-categories take a theory to the isomorphism groupoid of any of its parametrizations, hence are not completely canonical (depending on a choice of parametrization). Also, in the latter result, we may replace "quasi-Polish" with "zero-dimensional Polish," due to the freedom to Morleyize arbitrarily (see Remark 9.14).

REMARK 11.15. It may seem a bit strange that Theorem 10.12 and Corollary 11.14 are mostly about the Borel setting, yet still mention quasi-Polish spaces and groupoids. This is an instance of the subtle interactions between topological and Borel structure in the presence of a group(oid) action, as exemplified by results, such as Pettis's theorem and the Becker–Kechris theorem; see [18] for an extended discussion on this point.

To our knowledge, it is an open problem to give a purely Borel-theoretic characterization of the standard Borel groupoids equivalent to the isomorphism groupoid of some countable $\mathcal{L}_{\omega_1\omega}$ theory.

Acknowledgments. I would like to thank Anush Tserunyan for providing useful comments on a draft of this article, as well as the anonymous referee for several helpful comments and suggestions.

Funding. Research supported by NSF grant DMS-2224709.

REFERENCES

[1] G. AHLBRANDT and M. ZIEGLER, *Quasi-finitely axiomatizable totally categorical theories*. ***Annals of Pure and Applied Logic***, vol. 30 (1986), no. 1, pp. 63–82.

[2] M. ARTIN, A. GROTHENDIECK, and J. L. VERDIER, ***Théorie des Topos et Cohomologie Étale des Schémas***, vol. 2, Lecture Notes in Mathematics, 270, Springer-Verlag, Berlin-New York, 1971.

[3] C. J. ASH and J. KNIGHT, ***Computable Structures and the Hyperarithmetical Hierarchy***, Studies in Logic and the Foundations of Mathematics, 144, North-Holland, Amsterdam, 2000.

[4] S. AWODEY and H. FORSSELL, *First-order logical duality*. ***Annals of Pure and Applied Logic***, vol. 164 (2013), no. 3, pp. 319–348.

[5] N. BAZHENOV, E. FOKINA, D. ROSSEGGER, A. A. SOSKOVA, and S. V. VATEV, *A Lopez-Escobar Theorem for Continuous Domains*. ***The Journal of Symbolic Logic***, vol. 90 (2025), no. 2, pp. 854–871.

[6] H. BECKER, *Polish group actions: dichotomies and generalized elementary embeddings*. ***Journal of the American Mathematical Society***, vol. 11 (1998), no. 2, pp. 397–449.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

[7] H. Becker and A. S. Kechris, ***The Descriptive Set Theory of Polish Group Actions***, London Mathematical Society Lecture Note Series, 232, Cambridge University Press, Cambridge, 1996.

[8] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for Metric Structures*, ***Model Theory With Applications to Algebra and Analysis*** (Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie, editors), vol. 2, London Mathematical Society Lecture Note Series, 350, Cambridge University Press, 2008, pp. 315–427.

[9] F. Borceux, ***Handbook of Categorical Algebra***, Encyclopedia of Mathematics and its Applications, 50–52, Cambridge University Press, Cambridge, 1994.

[10] F. Borceux and J. Rosický, *Purity in algebra*. ***Algebra Universalis***, vol. 56 (2007), no. 1, pp. 17–35.

[11] C. Butz and I. Moerdijk, *Representing topoi by topological groupoids*. ***The Journal of Symbolic Logic*** vol. 130 (1998), no. 3, pp. 223–235.

[12] O. Caramello, *Atomic toposes and countable categoricity*. ***Applied Categorical Structures***, vol. 20 (2012), no. 4, pp. 379–391.

[13] R. Chen, *Notes on quasi-Polish spaces*, preprint, 2018, https://arxiv.org/abs/1809.07440.

[14] ———, *Borel functors, interpretations, and strong conceptual completeness for* $\mathcal{L}_{\omega_1\omega}$. ***Transactions of the American Mathematical Society***, vol. 372 (2019), no. 12, pp. 8955–8983.

[15] ———, *Representing Polish groupoids via metric structures*, preprint, 2019, https://arxiv.org/abs/1908.03268.

[16] ———, *Decompositions and measures on countable Borel equivalence relations*. ***Ergodic Theory and Dynamical Systems***, vol. 41 (2021), no. 12, pp. 3671–3703.

[17] ———, *On the Pettis–Johnstone theorem for localic groups*, preprint, 2021, https://arxiv.org/abs/2109.12721.

[18] ———, *Structural, point-free, non-Hausdorff topological realization of Borel groupoid actions*. ***Forum of Mathematics, Sigma***, vol. 12 (2024), Paper No. e35. 53pp.

[19] M. de Brecht, *Quasi-Polish spaces*. ***Annals of Pure and Applied Logic***, vol. 164 (2013), no. 3, pp. 356–381.

[20] M. de Brecht and T. Kawai, *On the commutativity of the powerspace constructions*. ***Methods in Computational Science***, vol. 15 (2019), no. 3, p. 13. 25pp.

[21] M. de Brecht, A. Pauly, and M. Schröder, *Overt choice*. ***Computability***, vol. 9 (2020), nos. 3–4, pp. 169–191.

[22] I. Di Liberti, *Weak saturation and weak amalgamation property*. ***The Journal of Symbolic Logic*** vol. 84 (2019), no. 3, pp. 929–936.

[23] C. J. Eagle and F. D. Tall, *Omitting types and the Baire category theorem*. preprint, 2017, https://arxiv.org/abs/1710.05889.

[24] M. P. Fourman and R. J. Grayson, *Formal spaces*, ***The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*** (A. S. Troelstra and D. van Dalen, editors), Studies in Logic and the Foundations of Mathematics, 110, North-Holland, Amsterdam, 1982, pp. 107–122,

[25] S. Gao, ***Invariant Descriptive Set Theory***, Pure and Applied Mathematics (Boca Raton), 293, CRC Press, Boca Raton, FL, 2009.

[26] M. Harrison-Trainor, A. Melnikov, R. Miller, and A. Montalbán, *Computable functors and effective interpretability*. ***The Journal of Symbolic Logic***, vol. 82 (2017), no. 1, pp. 77–97.

[27] M. Harrison-Trainor, R. Miller, and A. Montalbán, *Borel functors and infinitary interpretations*. ***The Journal of Symbolic Logic***, vol. 83 (2018), no. 4, pp. 1434–1456.

[28] W. Hodges, ***Model Theory***, Encyclopedia of Mathematics and its Applications, 42, Cambridge University Press, Cambridge, 1993.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

[29] P. T. Johnstone, *A constructive "closed subgroup theorem" for localic groups and groupoids*. ***Cahiers de Topologie et Géométrie Différentielle Catégoriques***, vol. 30 (1989), no. 1, pp. 3–23.

[30] ———, ***Sketches of an Elephant: A Topos Theory Compendium***, Oxford Logic Studies, 44, The Clarendon Press; Oxford University Press, Oxford, 2002.

[31] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck*. ***Memoirs of the American Mathematical Society***, vol. 51 (1984), no. 309, pp. vii+71pp.

[32] C. R. Karp, ***Languages with Expressions of Infinite Length***, North-Holland, Amsterdam, 1964.

[33] A. S. Kechris, ***Classical Descriptive Set Theory***, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[34] ———, *Classical descriptive set theory; corrections and updates*. 2018. https://www.pma.caltech.edu/documents/5656/CDST-corrections.pdf.

[35] ———, ***Global Aspects of Ergodic Group Actions***, American Mathematical Society, Providence, RI, 2010.

[36] W. Kubiś, *Weak Fraïssé categories*. ***Theory and Applications of Categories***, vol. 38 (2022), Paper No. 2, 27–63.

[37] M. Lieberman, J. Rosický, and S. Vasey, *Forking independence from the categorical point of view*. ***Advances in Mathematics***, vol. 346 (2019), pp. 719–772.

[38] E. G. K. Lopez-Escobar, *An interpolation theorem for denumerably long formulas*. ***Fundamenta Mathematicae***, vol. 57 (1965), pp. 253–272.

[39] M. Lupini, *Polish groupoids and functorial complexity*. ***Transactions of the American Mathematical Society***, vol. 369 (2017), no. 9, pp. 6683–6723.

[40] S. M. Lane and I. Moerdijk, ***Sheaves in Geometry and Logic: A First Introduction to Topos Theory***, Corrected reprint of the 1992 edition, Universitext, Springer-Verlag, New York, 1994.

[41] M. Makkai and G. E. Reyes, ***First Order Categorical Logic***, Lecture Notes in Mathematics, 611, Springer-Verlag, Berlin-New York, 1977.

[42] D. Marker, ***Lectures on Infinitary Model Theory***, Lecture Notes in Mathematics, 46, Cambridge University Press, Cambridge, 2016.

[43] J. Melleray and T. Tsankov, *Generic representations of abelian groups and extreme amenability*. ***Israel Journal of Mathematics***, vol. 198 (2013), no. 1, pp. 129–167.

[44] E. Michael, *Continuous selections. II*. ***Annals of Mathematics***, vol. 64 (1956), no. 2, pp. 562–580.

[45] ———, *Selected selection theorems*, ***The American Mathematical Monthly***, vol. 63 (1956), no. 4, pp. 233–238.

[46] I. Moerdijk, *The classifying topos of a continuous groupoid. I*. ***Transactions of the American Mathematical Society***, vol. 310 (1988), pp. 629–668.

[47] ———, *The classifying topos of a continuous groupoid II*. ***Cahiers de Topologie et Géométrie Différentielle Catégoriques***, vol. 31 (1990), pp. 137–168.

[48] A. Montalbán, *A robuster Scott rank*. ***Proceedings of the American Mathematical Society***, vol. 143 (2015), no. 12, pp. 5427–5436.

[49] ———, ***Computable Structure Theory: Within the Arithmetic***, Perspectives in Logic, Cambridge University Press, Cambridge Association for Symbolic Logic, Ithaca, NY, 2021.

[50] M. Prest, *Definable additive categories: Purity and model theory*, ***Memoirs of the American Mathematical Society***, vol. 210 (2011), no. 987, pp. vi+109pp.

[51] R. L. Sami, *Polish group actions and the Vaught conjecture*. ***Transactions of the American Mathematical Society***, vol. 341 (1994), no. 1, pp. 335–353.

[52] V. Selivanov, *Towards a descriptive set theory for domain-like structures*. ***Theoretical Computer Science***, vol. 365 (2006), no. 3, pp. 258–282.

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press

[53] B. R. Tennison, ***Sheaf Theory***, London Mathematical Society Lecture Note Series, 20, Cambridge University Press, New York, 1975.

[54] S. Thomas, *On the complexity of the quasi-isometry and virtual isomorphism problems for finitely generated groups*. ***Groups, Geometry, and Dynamics***, vol. 2 (2008), no. 2, pp. 281–307.

[55] R. Vaught, *Invariant sets in topology and logic*. ***Fundamenta Mathematicae***, vol. 82 (1974), no. 3, pp. 269–294.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI
USA
E-mail: ruiyuan@umich.edu

https://doi.org/10.1017/bsl.2025.10086 Published online by Cambridge University Press