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Commutator criteria for strong mixing

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Abstract. We present new criteria, based on commutator methods, for the strong mixing property of discrete flows $\{U^N\}_{N\in\mathbb{Z}}$ and continuous flows $\{e^{-itH}\}_{t\in\mathbb{R}}$ induced by unitary operators U and self-adjoint operators H in a Hilbert space $\mathcal{H}$. Our approach put into light a general definition for the topological degree of the maps $N \mapsto U^N$ and $t \mapsto e^{-itH}$ with values in the unitary group of $\mathcal{H}$. Among other examples, our results apply to skew products of compact Lie groups, time changes of horocycle flows and adjacency operators on graphs.

1. *Introduction*

In a recent series of papers [**5**, **22**, **24**, **23**], it has been shown that commutator methods can be used to study the absolutely continuous spectrum of various classes of ergodic dynamical systems. In the present work, we pursue this approach by showing that the property of strong mixing can also be established using commutator methods. More specifically, we present new criteria in terms of commutators for the strong mixing property of two abstract classes of dynamical systems: discrete flows $\{U^N\}_{N\in\mathbb{Z}}$ induced by a unitary operator U; and continuous flows $\{e^{-itH}\}_{t\in\mathbb{R}}$ induced by a self-adjoint operator H.

In the unitary case, our set-up is the following. Assume that U and A are, respectively, a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with U such that the commutator $[A, U]$ exists and is bounded in some precise sense. Then, the operator $[A, U]U^{-1}$ is bounded and self-adjoint, and one can define, for all $N \in \mathbb{N}_{\geq 1}$, the Birkhoff averages $(1/N)\sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n}$. The strong limit of these averages, namely the operator

$$D := \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n},$$

is (if it exists) a very interesting object. First, we explain why the operator D can be interpreted as a topological degree of the map $N \mapsto U^N$ with values in the unitary group $\mathsf{U}(\mathcal{H})$ of $\mathcal{H}$ (Remark 3.2(c)). Secondly, we prove the following result (see

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Theorem 3.1 for a precise statement): if the topological degree D exists and satisfies some weak regularity assumption with respect to A, then U is strongly mixing and has purely continuous spectrum in the subspace $\ker(D)^{\perp}$. (In some cases, we can even prove that U has purely absolutely continuous spectrum in the subspace $\ker(D)^{\perp}$, see Remark 3.2(b) and Corollary 3.4.) This result on strong mixing extends to a general Hilbertian setting comparable results in the framework of skew products of compact Lie groups (see [**22**, Remark 3.6] and references therein). Moreover, since it applies in particular to Koopman operators, it provides a surprisingly simple and general criterion to determine when an invertible measure-preserving transformation is strongly mixing and has purely continuous spectrum in some subspace. Analogues of all these results are also obtained in the case of a continuous flow $\{e^{-itH}\}_{t\in\mathbb{R}}$ induced by a self-adjoint operator H (see §4).

As an application of our results, we prove the strong mixing property or obtain purely absolutely continuous spectrum in appropriate subspaces for operators U and A satisfying the Weyl commutation relation (Example 3.5), for skew products of compact Lie groups (Example 3.6), for operators H and A satisfying the Weyl commutation relation (Example 4.5), for operators H and A satisfying the homogeneous commutation relation (Example 4.6), for adjacency operators on graphs (Example 4.7), and for time changes of horocycle flows on compact surfaces of constant negative curvature (Example 4.8). In all of these examples, we obtain new results, or re-obtain known results, with very simple and short proofs. In the case of skew products of compact Lie groups, our results extend previous results of Frączek, Iwanik, Lemańzyk, Rudolph and the author [**7**, **14**, **22**]. In the case of horocycle flows, we obtain a very simple proof that all time changes of horocycle flows of class C^1 on compact surfaces of constant negative curvature are strongly mixing. This complements classical results on horocycle flows of Kushnirenko [**16**] and Marcus [**19**]. The diversity of the examples covered in our paper and the simplicity of our proofs suggest that our results could be applied to various other models in ergodic theory and spectral theory.

Here is a brief description of the content of the paper. In §2, we recall the facts needed on commutators of operators and regularity classes associated to them. In §3, we prove our result on the strong mixing property for discrete flows $\{U^N\}_{N\in\mathbb{Z}}$ (Theorem 3.1) and give an interpretation of it in terms of the topological degree D (Remark 3.2(c)). We also present the application to skew products of compact Lie groups (Example 3.6). In §4, we prove our result on the strong mixing property for continuous flows $\{e^{-itH}\}_{t\in\mathbb{R}}$ (Theorem 4.1) and present the applications to adjacency operators on graphs (Example 4.7) and time changes of horocycle flows (Example 4.8).

To conclude, we note that it would be interesting to see if the commutator criteria of this paper admit an analogue for C^*-dynamical systems. Namely, given a triple $\{\mathcal{A}, G, \alpha\}$, with $\mathcal{A}$ a C^*-algebra, G a locally compact group and α a strongly continuous representation of G in the automorphism group of $\mathcal{A}$, can we find reasonable conditions on α in terms of an auxiliary map on $\mathcal{A}$ (maybe a derivation of $\mathcal{A}$ replacing the map $[A, \cdot]$ appearing in our set-up) guaranteeing that α is strong mixing?

2. *Commutators and regularity classes*

We recall in this section some basic facts on commutators of operators and regularity classes associated to them. We refer the reader to [**2**, Chs 5 and 6] for more details.

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Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot\,,\,\cdot \rangle$ antilinear in the first argument, denote by $\mathscr{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\|\cdot\|$ both for the norm on $\mathcal{H}$ and the norm on $\mathscr{B}(\mathcal{H})$. Let A be a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(A)$, and take $S \in \mathscr{B}(\mathcal{H})$. For any $k \in \mathbb{N}$, we say that S belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathscr{B}(\mathcal{H}) \tag{2.1}$$

is strongly of class C^k. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi,\, iSA\varphi \rangle - \langle A\varphi,\, iS\varphi \rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$. We denote by $[iS,\, A]$ the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the function (2.1) at $t = 0$. Moreover, if we set $A_\varepsilon := (i\varepsilon)^{-1}(e^{i\varepsilon A} - 1)$ for $\varepsilon \in \mathbb{R}\backslash\{0\}$, then we have that $\text{s-lim}_{\varepsilon \searrow 0}[iS,\, A_\varepsilon] = [iS,\, A]$ (see [**2**, Lemma 6.2.3(a)]).

Now, if H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C}\backslash\sigma(H)$. In particular, H is of class $C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi,\, (H - z)^{-1} A\varphi \rangle - \langle A\varphi,\, (H - z)^{-1}\varphi \rangle \in \mathbb{C}$$

extends continuously to a bounded form defined by the operator $[(H - z)^{-1},\, A] \in \mathscr{B}(\mathcal{H})$. In such a case, the set $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi,\, A\varphi \rangle - \langle A\varphi,\, H\varphi \rangle \in \mathbb{C}$$

is continuous in the topology of $\mathcal{D}(H)$ (see [**2**, Theorem 6.2.10(b)]). This form then extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a symmetric continuous operator $[H,\, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. In addition, the following relation holds in $\mathscr{B}(\mathcal{H})$ (see [**2**, Theorem 6.2.10(b)]):

$$[(H - z)^{-1},\, A] = -(H - z)^{-1}[H,\, A](H - z)^{-1}.$$

3. *Strong mixing for discrete flows*

We present in this section new criteria for the strong mixing property of a discrete flow $\{U^N\}_{N \in \mathbb{Z}}$ induced by a unitary operator U in a Hilbert space $\mathcal{H}$. We start with the main theorem of the section, then present some of its corollaries, and finally give some examples of applications.

We use the notation $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ for the unit circle in the complex plane and the notation $C_c^\infty(\mathbb{R}\backslash\{0\})$ for the set of C^∞ functions with compact support in $\mathbb{R}\backslash\{0\}$.

THEOREM 3.1. (Strong mixing for discrete flows) *Let U and A be, respectively, a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^1(A)$. Assume that the strong limit*

$$D := \underset{N \to \infty}{\text{s-lim}} \frac{1}{N} \sum_{n=0}^{N-1} U^n([A,\, U]U^{-1})U^{-n}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$. Then:

(a) $\lim_{N \to \infty} \langle \varphi,\, U^N \psi \rangle = 0$ *for each $\varphi \in \ker(D)^\perp$ and $\psi \in \mathcal{H}$;*

(b) $U|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

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Before the proof, we note that $[A, U]U^{-1}$ is bounded and self-adjoint. Thus, all the operators $(1/N)\sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n}$ are bounded and self-adjoint, and so is their strong limit D. It follows that the operators $\eta(D)$ with $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$ are well defined by functional calculus. We also note that $DU^n = U^n D$ for each $n \in \mathbb{Z}$. So, $\ker(D)^\perp$ is a reducing subspace for U, and $U|_{\ker(D)^\perp}$ is a well-defined unitary operator (see [**25**, Example 5.39(b)]). Finally, we note that the result of point (b) shares some similarities with the criterion for the absence of eigenvalues provided by the Virial theorem for unitary operators (see [**5**, Corollary 2.4 and Remark 2.8]).

Proof of Theorem 3.1. (a) The proof is divided into the points (a.1) and (a.2) below.

(a.1) First, we show that the set

$$\mathscr{D} := \{\varphi \in \mathcal{D}(A) \mid \varphi = \eta(D)\varphi \text{ for some } \eta \in C_c^\infty(\mathbb{R}\backslash\{0\})\}$$

is dense in $\ker(D)^\perp$. For this, we define the auxiliary set

$$\widetilde{\mathscr{D}} := \{\psi \in \mathcal{H} \mid \psi = \zeta(D)\psi \text{ for some } \zeta \in C_c^\infty(\mathbb{R}\backslash\{0\})\},$$

and note that $\mathscr{D}$ is dense in $\widetilde{\mathscr{D}}$. To see this, take $\psi \equiv \eta(D)\psi \in \widetilde{\mathscr{D}}$, let $\{\psi_n\} \subset \mathcal{D}(A)$ be a sequence such that $\lim_n \|\psi - \psi_n\| = 0$, and set $\varphi_n := \eta(D)\psi_n$ for each n. Then, it follows by the assumption $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ that $\{\varphi_n\} \subset \mathcal{D}(A)$. Moreover, there exists $\zeta \in C_c^\infty(\mathbb{R}\backslash\{0\})$ such that $\eta(x) = \zeta(x)\eta(x)$ for each $x \in \mathbb{R}$. Therefore, we have

$$\zeta(D)\varphi_n = \zeta(D)\eta(D)\psi_n = \eta(D)\psi_n = \varphi_n,$$

and thus $\{\varphi_n\} \subset \mathscr{D}$. Finally, we have that

$$\lim_n \|\psi - \varphi_n\| = \lim_n \|\eta(D)(\psi - \psi_n)\| \leq \text{Const.} \lim_n \|\psi - \psi_n\| = 0,$$

which concludes the proof of the density of $\mathscr{D}$ in $\widetilde{\mathscr{D}}$. But now, standard arguments imply that $\widetilde{\mathscr{D}}$ is dense in $\ker(D)^\perp$. Therefore, one obtains that $\mathscr{D}$ is dense in $\ker(D)^\perp$ by the transitivity of denseness.

(a.2) Since $U \in C^1(A)$, one has $U^N \in C^1(A)$ for each $N \in \mathbb{N}_{\geq 1}$, and

$$\begin{aligned}[A, U^N] = \sum_{n=0}^{N-1} U^{N-1-n}[A, U]U^n &= \left(\sum_{n=0}^{N-1} U^{N-1-n}([A, U]U^{-1})U^{n+1-N}\right)U^N \\ &= \left(\sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n}\right)U^N.\end{aligned}$$

Therefore, one obtains that

$$[A, U^N] = N D_N U^N \quad \text{with } D_N := \frac{1}{N}\sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n}. \tag{3.1}$$

Now, take $\varphi \equiv \eta(D)\varphi \in \mathscr{D}$ and $\psi \in \mathcal{D}(A)$. Then, the assumption $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$, for each $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$, implies that $D^{-1}\eta(D)\varphi \in \mathcal{D}(A)$, and (3.1) implies, for

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each $N \in \mathbb{N}_{\geq 1}$, that

$$\begin{aligned}
&|\langle \varphi, U^N \psi \rangle| \\
&\quad = |\langle DD^{-1}\eta(D)\varphi, U^N \psi \rangle| \\
&\quad \leq |\langle (D_N - D)D^{-1}\eta(D)\varphi, U^N \psi \rangle| + |\langle D^{-1}\eta(D)\varphi, D_N U^N \psi \rangle| \\
&\quad \leq \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| + \frac{1}{N}|\langle D^{-1}\eta(D)\varphi, [A, U^N]\psi \rangle| \\
&\quad = \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| + \frac{1}{N}|\langle AD^{-1}\eta(D)\varphi, U^N \psi \rangle \\
&\qquad - \langle D^{-1}\eta(D)\varphi, U^N A\psi \rangle| \\
&\quad \leq \|(D_N - D)D^{-1}\eta(D)\varphi\| \cdot \|\psi\| + \frac{1}{N}(\|AD^{-1}\eta(D)\varphi\| \cdot \|\psi\| \\
&\qquad + \|D^{-1}\eta(D)\varphi\| \cdot \|A\psi\|). \qquad (3.2)
\end{aligned}$$

Since $\text{s-lim}_{N\to\infty} D_N = D$, one infers that $\lim_{N\to\infty}\langle \varphi, U^N \psi \rangle = 0$, for each $\varphi \in \mathscr{D}$, and $\psi \in \mathcal{D}(A)$, and thus the claim follows by the density of $\mathscr{D}$ in $\ker(D)^\perp$ and the density of $\mathcal{D}(A)$ in $\mathcal{H}$.

(b) Suppose that, for a contradiction, there exist $z \in \mathbb{S}^1$ and $\varphi \in \ker(D)^\perp \backslash \{0\}$ such that $U\varphi = z\varphi$. Then, point (a) implies that

$$0 = \lim_{N\to\infty} |\langle \varphi, U^N \varphi \rangle| = \lim_{N\to\infty} |z|^N \cdot \|\varphi\|^2 = \|\varphi\|^2 \neq 0,$$

which is a contradiction. □

Remark 3.2.

(a) If A is bounded, then Theorem 3.1 gives no information. Indeed, one has in such a case that

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n} \right\| = \frac{1}{N}\|[A, U^N]U^{-N}\| \leq \frac{2\|A\|}{N} \quad \text{for each } N \in \mathbb{N}_{\geq 1}.$$

Therefore, the operator D exists, but it satisfies $D = 0$.

(b) If we make in Theorem 3.1 the extra assumption that

$$\sum_{N\geq 1} \left\| \left(\frac{1}{N} \sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n} - D \right) \varphi \right\|^2 < \infty \quad \text{for each } \varphi \in \mathcal{D}(A),$$

then we infer from (3.2) that $\sum_{N\geq 1} |\langle \varphi, U^N \varphi \rangle|^2 < \infty$ for each $\varphi \in \mathscr{D}$. So, each $\varphi \in \mathscr{D}$ belongs to the absolutely continuous subspace $\mathcal{H}_{\text{ac}}(U)$ of U (see the proof of [**14**, Lemma 5]), and thus $U|_{\ker(D)^\perp}$ has purely absolutely continuous spectrum since $\mathscr{D}$ is dense in $\ker(D)^\perp$ and $\mathcal{H}_{\text{ac}}(U)$ is closed in $\mathcal{H}$.

(c) The operator D can be interpreted as a topological degree of the map $N \mapsto U^N$ with values in the unitary group $\mathsf{U}(\mathcal{H})$ of $\mathcal{H}$. Indeed, one has

$$D = \text{s-lim}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n} = \text{s-lim}_{N\to\infty} \frac{1}{N}[A, U^N]U^{-N}.$$

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Therefore, if one considers the map $[A, \cdot]$ as a derivation on the set $\{U^N\}_{N\in\mathbb{Z}} \subset \mathsf{U}(\mathcal{H})$, then D can be interpreted as a renormalized winding number of the map $N \mapsto U^N$ with values in $\mathsf{U}(\mathcal{H})$ (the logarithmic derivative dz/z in the usual definition of winding number is replaced by the 'logarithmic derivative' $[A, U^N]U^{-N}$ associated to $[A, \cdot]$, see [**7**, **9**, **15**, **22**]). Accordingly, one can rephrase the result of Theorem 3.1(a) in the following more intuitive terms: if U is regular enough with respect to the auxiliary operator A, and if the topological degree D exists, then the flow $\{U^N\}_{N\in\mathbb{Z}}$ is strongly mixing in the subspace where D is non-zero. This result extends to a general Hilbertian setting comparable results in the framework of skew products of compact Lie groups (see [**3**, **7**, **8**, **11**, **14**, **22**, **26**]).

The fact that $DU^n = U^n D$ for each $n \in \mathbb{Z}$ does not imply that D is a function of U (one has to ask more for this, see [**1**, Ch. VI.75]). Still, it suggests considering the particular case where $D = g(U)$ for some function g. This is what we do in the next corollary.

We write χ_S for the characteristic function for a set S and we denote by $C^{k+\gamma}(\mathbb{S}^1; \mathbb{R})$, $k \in \mathbb{N}$, $\gamma \in [0, 1)$, the set of functions in $C^k(\mathbb{S}^1; \mathbb{R})$ with kth derivative Hölder continuous of order γ (with the usual identification of $\mathbb{S}^1$ with $[0, 1]$, see [**12**, §3.1.1 and Definition 3.2.5]).

COROLLARY 3.3. *Let U and A be, respectively, a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^1(A)$. Assume that the strong limit*

$$D := \operatorname*{s-lim}_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n([A, U]U^{-1})U^{-n}$$

exists and satisfies $D = g(U)$ for some $g \in C^{2+\gamma}(\mathbb{S}^1; \mathbb{R})$ with $\gamma \in (0, 1)$, and set $\Theta := \mathbb{S}^1 \backslash g^{-1}(\{0\})$. Then:

(a) $\lim_{N\to\infty} \langle \varphi, U^N \psi\rangle = 0$ *for each $\varphi \in \chi_\Theta(U)\mathcal{H}$ and $\psi \in \mathcal{H}$;*

(b) $U|_{\chi_\Theta(U)\mathcal{H}}$ *has purely continuous spectrum.*

Proof. We have

$$\chi_\Theta(U)\mathcal{H} = \chi_{\mathbb{R}\backslash\{0\}}(g(U))\mathcal{H} = \ker(g(U))^\perp.$$

Therefore, the claims will follow from Theorem 3.1 if we show that $\eta(g(U))\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$. So, take $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$. Then, the map $z \mapsto \eta(g(z))$ belongs to $C^{2+\gamma}(\mathbb{S}^1)$ by the usual properties of Hölder continuous functions, and thus the operator $\eta(g(U))$ can be written as a Fourier series $\eta(g(U)) = \sum_{n\in\mathbb{Z}} c_n U^n$ with coefficients $c_n \in \mathbb{C}$ satisfying $|c_n| \le \text{Const.}(1 + |n|)^{-(2+\gamma)}$ for all $n \in \mathbb{Z}$ (see [**12**, Corollary 3.2.10(b)]). Therefore, one obtains, for each $\varphi \in \mathcal{D}(A)$, the inequalities

$$\begin{aligned} |\langle A\varphi, \eta(g(U))\varphi\rangle - \langle \varphi, \eta(g(U))A\varphi\rangle| &\le \sum_{n\in\mathbb{Z}} |c_n| \cdot |\langle \varphi, [A, U^n]\varphi\rangle| \\ &\le \text{Const.}\|[A, U]\| \cdot \|\varphi\|^2 \sum_{n\in\mathbb{Z}} (1 + |n|)^{-(2+\gamma)}|n| \\ &\le \text{Const.}\|[A, U]\| \cdot \|\varphi\|^2. \end{aligned}$$

This implies that $\eta(g(U)) \in C^1(A)$, and thus that $\eta(g(U))\mathcal{D}(A) \subset \mathcal{D}(A)$. □

The next result is a direct consequence of Remark 3.2(b) and Corollary 3.3.

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COROLLARY 3.4. *Let U and A be, respectively, a unitary and a self-adjoint operator in a Hilbert space $\mathcal{H}$, with $U \in C^1(A)$. Assume that $[A, U]U^{-1} = g(U)$ for some $g \in C^{2+\gamma}(\mathbb{S}^1; \mathbb{R})$ with $\gamma \in (0, 1)$, and set $\Theta := \mathbb{S}^1 \setminus g^{-1}(\{0\})$. Then, $U|_{\chi_\Theta(U)\mathcal{H}}$ has purely absolutely continuous spectrum.*

We close the section with some examples of applications of the previous results.

Example 3.5. (Canonical commutation relation) Assume that $g(U) = g_0 \in \mathbb{R}\setminus\{0\}$ in Corollary 3.4. Then, we have $\chi_\Theta(U)\mathcal{H} = \mathcal{H}$, and thus Corollary 3.4 implies that U has purely absolutely continuous spectrum. In fact, we have in this case the canonical commutation relation $[A/g_0, U] = U$, which is equivalent to the Weyl commutation relation

$$e^{itA/g_0} U^n e^{-itA/g_0} = e^{itn} U^n, \quad t \in \mathbb{R}, n \in \mathbb{Z}.$$

Thus, representation theory implies that U has Haar spectrum with uniform multiplicity (see [**4**, Theorem 3]). This occurs for instance when A and U are, respectively, a number operator and a bilateral shift on a Hilbert space (the bilateral shift can be, for example, the Koopman operator of a K-automorphism in the orthocomplement of the constant functions, see [**5**, §3.1] and [**20**, Ch. 3.3]).

Example 3.6. (Skew products of compact Lie groups) Let X be a compact Lie group with normalized Haar measure μ_X and let $F \equiv \{F_t\}_{t\in\mathbb{R}}$ be a C^∞ measure-preserving flow on (X, μ_X). The family of operators $V_t : \mathsf{L}^2(X, \mu_X) \to \mathsf{L}^2(X, \mu_X)$ given by $V_t\varphi := \varphi \circ F_t$ defines a strongly continuous one-parameter unitary group with self-adjoint generator H in $\mathsf{L}^2(X, \mu_X)$, which is essentially self-adjoint on $C^\infty(X)$ and given by

$$H\varphi := -i\mathscr{L}_Y\varphi, \quad \varphi \in C^\infty(X),$$

with Y the divergence-free vector field associated to F and $\mathscr{L}_Y$ the corresponding Lie derivative.

Let G be a second compact Lie group with normalized Haar measure μ_G and neutral element e_G. Then, each function $\phi \in C(X; G)$ induces a cocycle $X \times \mathbb{Z} \ni (x, n) \mapsto \phi^{(n)}(x) \in G$ over F_1, given by

$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi \circ F_1)(x) \cdots (\phi \circ F_{n-1})(x) & \text{if } n \geq 1, \\ e_G & \text{if } n = 0, \\ (\phi^{(-n)} \circ F_n)(x)^{-1} & \text{if } n \leq -1. \end{cases}$$

The skew product T_ϕ associated to ϕ, defined by

$$T_\phi : (X \times G, \mu_X \otimes \mu_G) \to (X \times G, \mu_X \otimes \mu_G), \quad (x, g) \mapsto (F_1(x), g\phi(x)),$$

is an automorphism of $(X \times G, \mu_X \otimes \mu_G)$, and the Koopman operator

$$U_\phi\psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := \mathsf{L}^2(X \times G, \mu_X \otimes \mu_G),$$

is a unitary operator in $\mathcal{H}$.

Let $\widehat{G}$ be the set of all (equivalence classes of) finite-dimensional irreducible unitary representations of G. Then, each $\pi \in \widehat{G}$ is a C^∞ group homomorphism from G to the unitary group $\mathsf{U}(d_\pi)$ of degree $d_\pi := \dim(\pi) < \infty$, and the Peter–Weyl theorem implies that the set of all matrix elements $\{\pi_{jk}\}_{j,k=1}^{d_\pi}$ of all representations $\pi \in \widehat{G}$ forms an

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orthogonal basis of $\mathsf{L}^2(G, \mu_G)$. Accordingly, one has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{H}_j^{(\pi)} \quad \text{with } \mathcal{H}_j^{(\pi)} := \left\{ \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} \;\middle|\; \varphi_k \in \mathsf{L}^2(X, \mu_X),\, k = 1, \ldots, d_\pi \right\}, \tag{3.3}$$

and a calculation shows that U_ϕ is reduced by the decomposition (3.3), with restriction $U_{\pi,j} := U_\phi|_{\mathcal{H}_j^{(\pi)}}$ given by

$$U_{\pi,j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k,\ell=1}^{d_\pi} (V_1 \varphi_k)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell}, \quad \varphi_k \in \mathsf{L}^2(X, \mu_X).$$

Furthermore, the operator A, given by

$$A \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k=1}^{d_\pi} H\varphi_k \otimes \pi_{jk}, \quad \varphi_k \in C^\infty(X),$$

is essentially self-adjoint in $\mathcal{H}_j^{(\pi)}$, and calculations as in [**22**, Lemma 3.2(c) and Theorem 3.5] show the following: if, for each $k, \ell \in \{1, \ldots, d_\pi\}$, the function $\pi_{k\ell} \circ \phi \in C(X)$ satisfies $\mathscr{L}_Y(\pi_{k\ell} \circ \phi) \in \mathsf{L}^\infty(X)$, then $U_{\pi,j} \in C^1(A)$ with $M := [A, U_{\pi,j}](U_{\pi,j})^{-1}$ the matrix-valued multiplication operator in $\mathcal{H}_j^{(\pi)}$ with matrix elements

$$M_{k\ell} := -i\{\mathscr{L}_Y(\pi \circ \phi) \cdot (\pi^* \circ \phi)\}_{k\ell}.$$

Furthermore, one has, for each $N \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned} D_N &:= \frac{1}{N} \sum_{n=0}^{N-1} (U_{\pi,j})^n ([A, U_{\pi,j}](U_{\pi,j})^{-1})(U_{\pi,j})^{-n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} (\pi \circ \phi^{(n)})(M \circ F_n)(\pi^* \circ \phi^{(n)}), \end{aligned}$$

and D_N reduces to the Birkhoff sum $(1/N) \sum_{n=0}^{N-1} M \circ F_n$ if the matrix-valued function $\pi \circ \phi$ is diagonal.

We present now one case where the limit s-$\lim_{N\to\infty} D_N$ exists and Theorem 3.1 can be applied, but we stress that many other cases can be treated, since the method is general (see for instance [**22**, §§4.2 and 4.3]).

Skew products of tori. Suppose that $X \times G = \mathbb{T}^d \times \mathbb{T}^{d'}$, with $d, d' \geq 1$, and let $F \equiv \{F_t\}_{t\in\mathbb{R}}$ be the flow on $\mathbb{T}^d$, given by

$$F_t(x) := x + ty \pmod{\mathbb{Z}^d}, \quad t \in \mathbb{R},\ x \in \mathbb{T}^d,\ y \in \mathbb{R}^d,$$

with $y_1, y_2, \ldots, y_d, 1$ rationally independent. Then, each element $\chi_q \in \widehat{\mathbb{T}^{d'}}$ is a character of $\mathbb{T}^{d'}$ given by $\chi_q(z) := e^{2\pi i q\cdot z}$ for some $q \in \mathbb{Z}^{d'}$, one has $\mathcal{H}_j^{(\pi)} \equiv \mathcal{H}_1^{(\chi_q)} = \mathsf{L}^2(\mathbb{T}^d, \mu_{\mathbb{T}^d}) \otimes \{\chi_q\}$, F_1 is uniquely ergodic, and $\mathscr{L}_Y = y \cdot \nabla_x$. Suppose also that $\phi \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ satisfies $\phi = \xi + \eta$, where:

(i) $\xi : \mathbb{T}^d \to \mathbb{T}^{d'}$ is given by $\xi(x) := Bx \pmod{\mathbb{Z}^{d'}}$ for some integer matrix $B \in \mathsf{M}_{d'\times d}(\mathbb{Z})$;

(ii) $\eta \in C(\mathbb{T}^d; \mathbb{T}^{d'})$ is such that $\mathscr{L}_Y(q \cdot \eta) \in \mathsf{L}^\infty(\mathbb{T}^d)$ for each $q \in \mathbb{Z}^{d'}$.

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Then, a direct calculation (see [**22**, §4.1]) shows that the matrix-valued function D_N reduces to the scalar function

$$D_N = 2\pi y \cdot (B^{\mathsf{T}} q) + \frac{2\pi}{N} \sum_{n=0}^{N-1} \mathscr{L}_Y(q \cdot \eta) \circ F_n.$$

Therefore, Birkhoff's pointwise ergodic theorem and the Lebesgue dominated convergence theorem imply that

$$D := \operatorname*{s-lim}_{N\to\infty} D_N = 2\pi y \cdot (B^{\mathsf{T}} q) + 2\pi \int_{\mathbb{T}^d} d\mu_{\mathbb{T}^d} \mathscr{L}_Y(q \cdot \eta) = 2\pi y \cdot (B^{\mathsf{T}} q).$$

Since $y_1, y_2, \ldots, y_d, 1$ are rationally independent, one has $\ker(D) = \{0\}$ if $B^{\mathsf{T}} q \neq 0$. It follows from Theorem 3.1(a) that $\lim_{N\to\infty} \langle \varphi, (U_{\chi_q,1})^N \psi \rangle = 0$, for each $\varphi, \psi \in \mathcal{H}_1^{(\chi_q)}$, if $B^{\mathsf{T}} q \neq 0$, and thus U_ϕ is strongly mixing in the subspace $\bigoplus_{q\in\mathbb{Z}^{d'}, B^{\mathsf{T}} q \neq 0} \mathcal{H}_1^{(\chi_q)} \subset \mathcal{H}$.

In the case $d = d' = 1$, this implies that U_ϕ is strongly mixing in $\bigoplus_{q\in\mathbb{Z}\setminus\{0\}} \mathcal{H}_1^{(\chi_q)}$ if $\phi(x) = Bx + \eta(x)$, with $B \in \mathbb{Z}\setminus\{0\}$ and $\eta : \mathbb{T} \to \mathbb{T}$ absolutely continuous with $\eta' \in \mathsf{L}^\infty(\mathbb{T})$. Iwanik *et al* proved in [**14**, §3] the same result without the assumption $\eta' \in \mathsf{L}^\infty(\mathbb{T})$ by studying directly the Fourier coefficients of the operators $U_{\chi_q,1}$. In the case $d > 1$ or $d' > 1$, the result seems to be new, but we refer to the paper of Frączek [**7**] for closely related results in the case of extensions of rotations on $\mathbb{T}^d$ by cocycles $\mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$.

4. *Strong mixing for continuous flows*

We present in this section new criteria for the strong mixing property of a continuous flow $\{e^{-itH}\}_{t\in\mathbb{R}}$ induced by a self-adjoint operator H in a Hilbert space $\mathcal{H}$. We start with the main theorem of the section, then present some of its corollaries, and finally give some examples of applications.

THEOREM 4.1. (Strong mixing for continuous flows) *Let H and A be self-adjoint operators in a Hilbert space $\mathcal{H}$, with $(H - i)^{-1} \in C^1(A)$. Assume that the strong limit*

$$D := \operatorname*{s-lim}_{t\to\infty} \frac{1}{t} \int_0^t ds e^{isH} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{-isH}$$

exists, and suppose that $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$ for each $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}\setminus\{0\})$. Then:

(a) $\lim_{t\to\infty} \langle \varphi, e^{-itH} \psi \rangle = 0$ *for each $\varphi \in \mathcal{H}$ and $\psi \in \ker(D)^\perp$;*

(b) $H|_{\ker(D)^\perp}$ *has purely continuous spectrum.*

Before the proof, we note that $(H + i)^{-1}[iH, A](H - i)^{-1}$ is bounded and self-adjoint. Thus, all the operators $(1/t) \int_0^t ds e^{isH} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{-isH}$ are bounded and self-adjoint, and so is their strong limit D. It follows that the operators $\eta(D)$, with $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}\setminus\{0\})$, are well defined by functional calculus. We also note that $De^{itH} = e^{itH} D$ for each $t \in \mathbb{R}$. So, $D\chi_B(H) = \chi_B(H)D$ for each Borel set $B \subset \mathbb{R}$. Thus, $\ker(D)^\perp$ is a reducing subspace for H, and $H|_{\ker(D)^\perp}$ is a well-defined self-adjoint operator (see [**25**, Theorem 7.28]). Finally, we note that the result of point (b) shares some similarities with the criterion for the absence of eigenvalues provided by the Virial theorem for self-adjoint operators (see [**2**, Corollary 7.2.11]).

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Proof of Theorem 4.1. (a) Since $(H-i)^{-1} \in C^1(A)$, we have $(H-i)^{-1}\mathcal{D}(A) \subset \mathcal{D}(A)$, and the operator

$$\widetilde{A}\varphi := (H+i)^{-1}A(H-i)^{-1}\varphi, \quad \varphi \in \mathcal{D}(A),$$

is essentially self-adjoint (see [**2**, Lemma 7.2.15]). Take $\varphi \in \mathcal{D}(A)$ and $t > 0$, and define, for $\varepsilon \in \mathbb{R}\backslash\{0\}$, the operator $A_\varepsilon := (i\varepsilon)^{-1}(e^{i\varepsilon A} - 1)$. Then, we have the equalities

$$\begin{aligned}
&\langle \widetilde{A}\varphi, e^{-itH}\varphi\rangle - \langle \varphi, e^{-itH}\widetilde{A}\varphi\rangle \\
&\quad = \lim_{\varepsilon\searrow 0}(\langle \varphi, (H+i)^{-1}A_\varepsilon(H-i)^{-1}e^{-itH}\varphi\rangle - \langle \varphi, e^{-itH}(H+i)^{-1}A_\varepsilon(H-i)^{-1}\varphi\rangle) \\
&\quad = \lim_{\varepsilon\searrow 0}\int_0^t ds\frac{d}{ds}\langle \varphi, e^{i(s-t)H}(H+i)^{-1}A_\varepsilon(H-i)^{-1}e^{-isH}\varphi\rangle \\
&\quad = \lim_{\varepsilon\searrow 0}\int_0^t ds\langle \varphi, e^{i(s-t)H}(H+i)^{-1}[iH, A_\varepsilon](H-i)^{-1}e^{-isH}\varphi\rangle. \qquad (4.1)
\end{aligned}$$

Now, since $(H-i)^{-1} \in C^1(H)$, we have

$$\text{s-}\lim_{\varepsilon\searrow 0}(H+i)^{-1}[iH, A_\varepsilon](H-i)^{-1} = (H+i)^{-1}[iH, A](H-i)^{-1}.$$

Therefore, we can interchange the limit and the integral in (4.1) and obtain

$$\begin{aligned}
&\langle \widetilde{A}\varphi, e^{-itH}\varphi\rangle - \langle \varphi, e^{-itH}\widetilde{A}\varphi\rangle \\
&\quad = \int_0^t ds\langle \varphi, e^{i(s-t)H}(H+i)^{-1}[iH, A](H-i)^{-1}e^{-isH}\varphi\rangle \\
&\quad = \langle \varphi, te^{-itH}D_t\varphi\rangle,
\end{aligned}$$

with $D_t := (1/t)\int_0^t ds e^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}e^{-isH}$. Since $\mathcal{D}(A)$ is a core for $\widetilde{A}$, this implies that $e^{-itH} \in C^1(\widetilde{A})$, with

$$[\widetilde{A}, e^{-itH}] = te^{-itH}D_t. \qquad (4.2)$$

Now, let

$$\mathscr{D} := \{\psi \in \mathcal{D}(A) \mid \psi = \eta(D)\psi \text{ for some } \eta \in C_c^\infty(\mathbb{R}\backslash\{0\})\},$$

and take $\varphi \in \mathcal{D}(A)$ and $\psi \equiv \eta(D)\psi \in \mathscr{D}$. Then, the assumption $\eta(D)\mathcal{D}(A) \subset \mathcal{D}(A)$, for each $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$, implies that $D^{-1}\eta(D)\psi \in \mathcal{D}(A)$, and (4.2) implies, for each $t > 0$, that

$$\begin{aligned}
&|\langle \varphi, e^{-itH}\psi\rangle| \\
&\quad = |\langle \varphi, e^{-itH}DD^{-1}\eta(D)\psi\rangle| \\
&\quad \le |\langle \varphi, e^{-itH}(D_t - D)D^{-1}\eta(D)\psi\rangle| + |\langle \varphi, e^{-itH}D_tD^{-1}\eta(D)\psi\rangle| \\
&\quad \le \|\varphi\| \cdot \|(D_t - D)D^{-1}\eta(D)\psi\| + \frac{1}{t}|\langle \varphi, [\widetilde{A}, e^{-itH}]D^{-1}\eta(D)\psi\rangle| \\
&\quad = \|\varphi\| \cdot \|(D_t - D)D^{-1}\eta(D)\psi\| + \frac{1}{t}|\langle \widetilde{A}\varphi, e^{-itH}D^{-1}\eta(D)\psi\rangle \\
&\qquad - \langle \varphi, e^{-itH}\widetilde{A}D^{-1}\eta(D)\psi\rangle| \\
&\quad \le \|\varphi\| \cdot \|(D_t - D)D^{-1}\eta(D)\psi\| + \frac{1}{t}(\|\widetilde{A}\varphi\| \cdot \|D^{-1}\eta(D)\psi\| \\
&\qquad + \|\varphi\| \cdot \|\widetilde{A}D^{-1}\eta(D)\psi\|). \qquad (4.3)
\end{aligned}$$

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Since s-$\lim_{t\to\infty} D_t = D$, one infers that $\lim_{t\to\infty}\langle\varphi, e^{-itH}\psi\rangle = 0$, for each $\varphi \in \mathcal{D}(A)$ and $\psi \in \mathscr{D}$, and thus the claim follows by the density of $\mathcal{D}(A)$ in $\mathcal{H}$ and the density of $\mathscr{D}$ in $\ker(D)^\perp$ (the density of $\mathscr{D}$ in $\ker(D)^\perp$ can be shown as in the point (a.1) of the proof of Theorem 3.1).

(b) Suppose, for a contradiction, that there exist $\lambda \in \mathbb{R}$ and $\psi \in \ker(D)^\perp\backslash\{0\}$ such that $H\psi = \lambda\psi$. Then, point (a) implies that

$$0 = \lim_{t\to\infty} |\langle\psi, e^{-itH}\psi\rangle| = \lim_{t\to\infty} |e^{-it\lambda}| \cdot \|\psi\|^2 = \|\psi\|^2 \neq 0,$$

which is a contradiction. □

Remark 4.2. (a) If we make in Theorem 4.1 the extra assumption that

$$\left\| \left(\frac{1}{t} \int_0^t ds e^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}e^{-isH} - D \right)\psi \right\| \in \mathsf{L}^2([1,\infty), dt) \quad \text{for each } \psi \in \mathcal{D}(A),$$

then we infer from (4.3) that $|\langle\psi, e^{-itH}\psi\rangle| \in \mathsf{L}^2([1,\infty), dt)$ for each $\psi \in \mathscr{D}$. So, each $\psi \in \mathscr{D}$ belongs to the absolutely continuous subspace $\mathcal{H}_{\mathrm{ac}}(H)$ of H (see the proof of [**21**, Theorem XIII.23]), and thus $H|_{\ker(D)^\perp}$ has purely absolutely continuous spectrum, since $\mathscr{D}$ is dense in $\ker(D)^\perp$ and $\mathcal{H}_{\mathrm{ac}}(H)$ is closed in $\mathcal{H}$.

(b) A Cayley transform shows that considering Birkhoff averages of the operator $(H+i)^{-1}[iH, A](H-i)^{-1}$ in the self-adjoint case is consistent with considering Birkhoff averages of the operator $[A, U]U^{-1}$ in the unitary case. Indeed, if U and A are as in Theorem 3.1 and 1 does not belong to the point spectrum of U, then the range $\mathsf{Ran}(1-U)$ of $1-U$ is dense in $\mathcal{H}$, and the Cayley transform of U, given by

$$H\varphi := i(1+U)(1-U)^{-1}\varphi, \quad \varphi \in \mathcal{D}(H) := \mathsf{Ran}(1-U),$$

is a self-adjoint operator [**25**, Theorem 8.4(b)]. Moreover, one has the identities

$$U^{-1} = (H+i)(H-i)^{-1} \quad \text{and} \quad (H+i)^{-1} = \frac{1}{2i}(1-U),$$

and thus H is of class $C^1(A)$ with

$$(H+i)^{-1}[iH, A](H-i)^{-1} = i[A, (H+i)^{-1}](H+i)(H-i)^{-1} = -\tfrac{1}{2}[A, U]U^{-1}.$$

In analogy to the unitary case, we consider the particular case where $D = g(H)$ for some function g.

COROLLARY 4.3. *Let H and A be self-adjoint operators in a Hilbert space $\mathcal{H}$, with $(H-i)^{-1} \in C^1(A)$. Assume that the strong limit*

$$D := \text{s-}\lim_{t\to\infty} \frac{1}{t} \int_0^t ds e^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}e^{-isH}$$

exists and satisfies $D = g(H)$ for some $g \in C^2(\mathbb{R}; \mathbb{R}) \cap \mathsf{L}^\infty(\mathbb{R})$, and set $\Lambda := \mathbb{R}\backslash g^{-1}(\{0\})$. Then:

(a) $\lim_{t\to\infty}\langle\varphi, e^{-itH}\psi\rangle = 0$ *for each $\varphi \in \mathcal{H}$ and $\psi \in \chi_\Lambda(H)\mathcal{H}$;*

(b) $H|_{\chi_\Lambda(H)\mathcal{H}}$ *has purely continuous spectrum.*

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Proof. We have

$$\chi_\Lambda(H)\mathcal{H} = \chi_{\mathbb{R}\setminus\{0\}}(g(H))\mathcal{H} = \ker(g(H))^\perp,$$

and we have $\eta(g(H))\mathcal{D}(A) \subset \mathcal{D}(A)$, for each $\eta \in C_c^\infty(\mathbb{R}\setminus\{0\})$, due to [**2**, Theorem 6.2.5]†. So, the claims follow from Theorem 4.1. □

The next result is a direct consequence of Remark 4.2(a) and Corollary 4.3.

COROLLARY 4.4. *Let H and A be self-adjoint operators in a Hilbert space $\mathcal{H}$, with $(H-i)^{-1} \in C^1(A)$. Assume that $(H+i)^{-1}[iH, A](H-i)^{-1} = g(H)$, for some $g \in C^2(\mathbb{R}; \mathbb{R}) \cap L^\infty(\mathbb{R})$, and set $\Lambda := \mathbb{R}\setminus g^{-1}(\{0\})$. Then, $H|_{\chi_\Lambda(H)\mathcal{H}}$ has purely absolutely continuous spectrum.*

We close the section with some examples of applications of the previous results.

Example 4.5. (Canonical commutation relation) Assume that $g(H) = (H+i)^{-1} g_0 (H-i)^{-1}$ with $g_0 \in \mathbb{R}\setminus\{0\}$ in Corollary 4.4. Then, we have $\chi_\Lambda(H)\mathcal{H} = \mathcal{H}$, and thus Corollary 4.4 implies that H has purely absolutely continuous spectrum. In fact, we have in this case the canonical commutation relation $[iH, A/g_0] = 1$, which is equivalent to the Weyl commutation relation

$$e^{-itA/g_0} e^{isH} e^{itA/g_0} = e^{ist} e^{isH}, \quad s, t \in \mathbb{R}.$$

Thus, the Stone–von Neumann theorem implies that H has Lebesgue spectrum with uniform multiplicity.

Example 4.6. (Homogeneous commutation relation) Assume that $g(H) = (H+i)^{-1} g_0 H(H-i)^{-1}$ with $g_0 \in \mathbb{R}\setminus\{0\}$ in Corollary 4.4. Then, we have $\chi_\Lambda(H)\mathcal{H} = \ker(H)^\perp$, and thus Corollary 4.4 implies that $H|_{\ker(H)^\perp}$ has purely absolutely continuous spectrum. In fact, we have in this case the homogeneous commutation relation $[iH, A/g_0] = H$, which is equivalent to the commutation relation

$$e^{-itA/g_0} e^{isH} e^{itA/g_0} = e^{ise^t H}, \quad s, t \in \mathbb{R},$$

and it is known that this implies that $H|_{\ker(H)^\perp}$ has Lebesgue spectrum with uniform multiplicity if there exists a unitary operator W in $\mathcal{H}$ such that $We^{-itA/g_0}W^{-1} = e^{itA/g_0}$ for each $t \in \mathbb{R}$ (see [**4**, §4]). This occurs, for instance, when A and H are, respectively, the generator of dilations and the Laplacian in $\mathbb{R}^d$ (see [**2**, §1.2]), or when A and H are, respectively, the generator of the geodesic flow and the generator of the horocycle flow on a compact surface of constant negative curvature (see [**4**, §4]).

Example 4.7. (Adjacency operators on admissible graphs) Let $(X, \sim)$ be a graph, with symmetric relation $\sim$, of finite degree and with no multiple edges or loops, and let $\mathcal{H} := \ell^2(X)$ be the corresponding Hilbert space. Then, the adjacency operator on $(X, \sim)$, defined by

$$(H\varphi)(x) := \sum_{y\sim x} \varphi(y), \quad \varphi \in \mathcal{H}, x \in X,$$

† As in Corollary 3.3, one can give a more direct proof of the inclusion $\eta(g(H))\mathcal{D}(A) \subset \mathcal{D}(A)$ using Fourier series if one assumes that $g \in C^{2+\gamma}(\mathbb{R}; \mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\gamma \in (0, 1)$.

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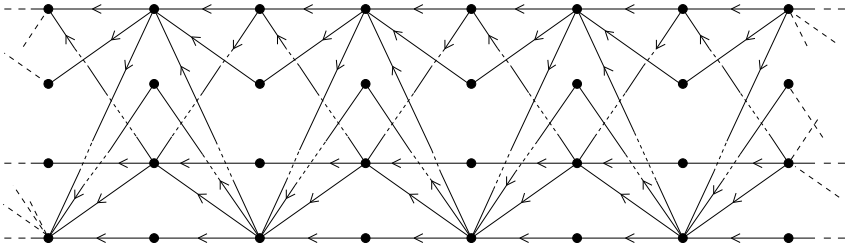


FIGURE 1. Example of an admissible graph.

is a bounded self-adjoint operator in $\mathcal{H}$. A directed graph $(X, <)$ subjacent to $(X, \sim)$ is the graph $(X, \sim)$ together with a relation $<$ on X such that, for each $x, y \in X$, $x \sim y$ is equivalent to $x < y$ or $y < x$, and one cannot have both $y < x$ and $x < y$. For each $x \in X$, we define $N^-(x) := \{y \in X \mid x < y\}$ and $N^+(x) := \{y \in X \mid y < x\}$, and note that $N(x) := N^-(x) \cup N^+(x)$ is the set of neighbours of x. When using drawings, we draw an arrow $x \leftarrow y$ if $x < y$. The following definition is taken from [**17**]. A directed graph $(X, <)$ is called admissible if:

(i) each closed path in X has the same number of positively and negatively oriented edges;

(ii) for each $x, y \in X$, one has $\#\{N^-(x) \cap N^-(y)\} = \#\{N^+(x) \cap N^+(y)\}$.

See Figure 1 for an example of an admissible graph.

Commutator calculations show that if H is the adjacency operator on an admissible graph $(X, <)$, then there exist a self-adjoint operator A and a bounded self-adjoint operator K in $\mathcal{H}$ such that $H \in C^1(A)$ with $[iH, A] = K^2$, $[K, H] = 0$, and $K^2 \in C^1(A)$ (see [**17**]). Therefore, one has

$$D := \operatorname*{s-lim}_{t\to\infty} \frac{1}{t} \int_0^t ds e^{isH}(H+i)^{-1}[iH, A](H-i)^{-1}e^{-isH} = K^2(H^2+1)^{-1},$$

and

$$\eta(D)\mathcal{D}(A) = \eta(K^2(H^2+1)^{-1})\mathcal{D}(A) \subset \mathcal{D}(A),$$

for each $\eta \in C_c^\infty(\mathbb{R}\backslash\{0\})$, due to [**2**, Proposition 5.1.5 and Theorem 6.2.5]. Since

$$\ker(D) = \ker(K^2(H^2+1)^{-1}) = \ker(K^2) = \ker(K),$$

we thus conclude by Theorem 4.1(b) and Remark 4.2(a) that $H|_{\ker(K)^\perp}$ has purely absolutely continuous spectrum.

This result was first proved in [**17**, Theorem 1.1]. Even so, it is of interest because the proof here is much simpler and shorter than in [**17**]. Similar simplifications can also be obtained for the spectral analysis of convolution operators on locally compact groups as presented in [**18**]. Details are left to the reader.

Example 4.8. (Time changes of horocycle flows) Let Σ be a compact Riemann surface of genus ≥ 2 and let $M := T^1\Sigma$ be the unit tangent bundle of Σ. The compact 3-manifold M carries a probability measure μ_Ω, induced by a canonical volume form Ω, which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle

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flow $F_1 \equiv \{F_{1,t}\}_{t\in\mathbb{R}}$; and the geodesic flow $F_2 \equiv \{F_{2,t}\}_{t\in\mathbb{R}}$. Each flow admits a self-adjoint generator H_j in $\mathsf{L}^2(M, \mu_\Omega)$, which is essentially self-adjoint on $C^\infty(M)$ and given by

$$H_j\varphi := -i\mathscr{L}_{X_j}\varphi, \quad \varphi \in C^\infty(M),$$

with X_j the divergence-free vector field associated to F_j and $\mathscr{L}_{X_j}$ the corresponding Lie derivative. Moreover, H_1 is of class $C^1(H_2)$ with $[iH_1, H_2] = H_1$.

Now, consider a C^1 vector field proportional to X_1, namely, a vector field fX_1 with $f \in C^1(M; (0, \infty))$. The vector field fX_1 admits a complete flow $\widetilde{F}_1 \equiv \{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ with generator $H := fH_1$ essentially self-adjoint on $C^1(M) \subset \mathcal{H} := \mathsf{L}^2(M, f^{-1}\mu_\Omega)$. Furthermore, [**24**, Lemmas 3.1 and 3.2] imply the following: the operator $A := f^{1/2}H_2 f^{-1/2}$ is essentially self-adjoint on $C^1(M) \subset \mathcal{H}$, and $(H - i)^{-1} \in C^1(A)$, with

$$(H + i)^{-1}[iH, A](H - i)^{-1} = (H + i)^{-1}(H\xi + \xi H)(H - i)^{-1} \quad \text{and}$$
$$\xi := \tfrac{1}{2} - \tfrac{1}{2}f^{-1}\mathscr{L}_{X_2}(f).$$

It follows, for $t > 0$, that

$$\begin{aligned} D_t &:= \frac{1}{t}\int_0^t ds e^{isH}(H + i)^{-1}[iH, A](H - i)^{-1}e^{-isH} \\ &= (H + i)^{-1}(H\xi_t + \xi_t H)(H - i)^{-1}, \end{aligned}$$

with

$$\xi_t := \frac{1}{t}\int_0^t ds e^{isH}\xi e^{-isH} = \frac{1}{t}\int_0^t ds(\xi \circ \widetilde{F}_{1,-s}).$$

Since the flow F_1 is uniquely ergodic with respect to the measure μ_Ω [**10**, p. 114], the time changed flow $\widetilde{F}_1$ is uniquely ergodic with respect to the measure $\widetilde{\mu}_\Omega := f^{-1}\mu_\Omega / \int_M f^{-1}\, d\mu_\Omega$ [**13**, Proposition 3]. Therefore,

$$\lim_{t\to\infty} \xi_t = \frac{1}{2} - \frac{1}{2}\int_M d\widetilde{\mu}_\Omega f^{-1}\mathscr{L}_{X_2}(f) = \frac{1}{2} + \frac{1}{2\int_M f^{-1}\, d\mu_\Omega}\int_M d\mu_\Omega \mathscr{L}_{X_2}(f^{-1}) = \frac{1}{2}$$

uniformly on M, and

$$D := \lim_{t\to\infty} D_t = (H + i)^{-1}(H \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot H)(H - i)^{-1} = H(H^2 + 1)^{-1} \quad \text{in } \mathscr{B}(\mathcal{H}).$$

So, Corollary 4.3 applies with $g(H) = H(H^2 + 1)^{-1}$ and $\Lambda = \mathbb{R}\backslash\{0\}$, and thus $\lim_{t\to\infty}\langle\varphi, e^{-itH}\psi\rangle = 0$ for each $\varphi \in \mathcal{H}$ and $\psi \in \ker(H)^\perp$. This shows that the time changed horocycle flow $\widetilde{F}_1$ associated to the vector field fX_1 is strongly mixing.

Thus, we have obtained a very simple proof that all time changes of horocycle flows of class C^1 on compact surfaces of constant negative curvature are strongly mixing. We refer to the work of Kushnirenko [**16**] for the same result under a smallness assumption on $\mathscr{L}_{X_2}(f)$, and to the work of Marcus [**19**] for the mixing properties of a general class of parametrizations of horocycle flows on compact surfaces of negative curvature. We also refer to the works of Forni and Ulcigrai [**6**] and of the author [**24**, **23**] for recent results on the absolutely continuous spectrum of time changes of horocycle flows in the case of constant curvature.

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