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MAXIMIZING THE TIME SPENT BY A DIFFUSION PROCESS IN AN INTERVAL

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A one-dimensional diffusion process is controlled in the interval $[-d, d]$. The aim is to maximize the survival time in $[-d, d]$. Two cost criteria, which do not depend on the control variable, are considered. Particular problems are solved explicitly.

1. INTRODUCTION

Let $x(t)$ be a one-dimensional controlled diffusion process defined by the stochastic differential equation

$$dx(t) = a[x(t)]\,dt + b_1[x(t)]u(t)\,dt + b_2[x(t)]u^2(t)\,dt + \{v[x(t)]\}^{1/2}\,dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian motion and we assume that $b_2(\cdot) > 0$ and $v(\cdot) \geq 0$. Let the initial value of $x(t)$ be $x(t_0) := x \in [-d, d]$. We want to maximize the survival time of the controlled process $x(t)$ in $[-d, d]$.

The problem of maximizing the survival time of a diffusion process in an interval has been considered, in particular, by Lefebvre and Whittle [11]. The problem that they treated is a special case of what Whittle [12, p. 298] has termed "LQG homing." In the one-dimensional case, this type of problem is such that

$$dx(t) = a\,dt + bu(t)\,dt + v^{1/2}\,dW(t) \tag{2}$$

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and the optimizer wants to minimize the expected value of the cost criterion

$$J(x,t_0) := \int_{t_0}^{\tau(x,t_0)} \frac{1}{2}\, qu^2(t)\, dt + K[x(\tau),\tau], \tag{3}$$

where q is positive and

$$\tau(x,t_0) := \inf\{t \geq t_0 \colon |x(t)| = d \,|\, x(t_0) = x\}. \tag{4}$$

Furthermore, $K(\cdot,\cdot)$ is a general termination cost function. If we take $K[x(\tau),\tau] = \lambda\tau$, then the aim is to minimize (respectively, maximize) the survival time in $[-d,d]$ if the constant λ is positive (respectively, negative) while taking the quadratic control costs into account. Whittle [13, p. 222] (see also Kuhn [6]) has considered a risk-sensitive version of this problem as well.

Whittle [12] has shown that it is sometimes possible to obtain the optimal control u^* by simply considering the corresponding uncontrolled process. However, for this result to be valid, a relationship between v, b, and q must hold and ultimate absorption of the uncontrolled process into the stopping region must be certain.

In the present article, the control variable $u(t)$ enters both linearly (if $b_1 \neq 0$) and quadratically in the stochastic differential equation that defines $x(t)$. Furthermore, we consider two cost criteria: First, we look for the control u^* that maximizes the expected value of

$$C_1(x,t_0) := \int_{t_0}^{\tau(x,t_0)} \lambda e^{-\lambda t}\, dt, \tag{5}$$

where λ is a positive constant and τ is as in (4). Next, the cost criterion C_1 is replaced by

$$C_2(x,t_0) := \int_{t_0}^{\tau(x,t_0)} 1\, dt; \tag{6}$$

that is, we want to maximize the expected survival time in $[-d,d]$.

Note that both cost criteria C_1 and C_2 do not depend explicitly on the control variable $u(t)$. Moreover, there is no constraint on $u(t)$. As will be seen later, the problem is not so much to determine u^*, but rather to compute explicitly the value function $F_i(x,t_0)$ defined by

$$F_i(x,t_0) := \max_{u(t),\, t_0 \leq t \leq \tau} E[C_i(x,t_0)] \quad \text{for } i = 1,2. \tag{7}$$

The motivation for the work presented in this article is that we wanted to extend the classic LQG formulation to a case in which there are no explicit control costs on the way. The uncontrolled process that corresponds to the process in (1) has many applications. For instance, the author (Lefebvre [9]) used the process defined by

$$dx(t) = c\, dt + [4cx(t)]^{1/2}\, dW(t), \tag{8}$$

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where c is a positive constant, to model the level of water in a reservoir, and he considered the problem of maximizing or minimizing the time taken by $x(t)$ to go from $x(0) = x \in [0, d)$ to d. The value d represented a level at which there was either a high risk of inundation or of not being able to provide water for domestic consumption or to industries.

The water reservoir control problem mentioned was formulated in the classic framework, with quadratic control costs. Now, in such a problem, the costs associated with inundations or periods of drought are obviously much higher than the control costs. That is why we wanted to consider a cost criterion which is independent of the control variable $u(t)$. In order to obtain a meaningful (and interesting) problem, we found out that the quadratic term $u^2(t)$ had to enter into the plant equation.

In Section 2, the value of u^* is obtained; then, the value function is computed in three important special cases when the cost criterion C_1 is used. Next, in Section 3, the cost criterion C_2 replaces C_1 and again the value function is obtained in these three important cases. Finally, concluding remarks are made in Section 4.

2. COMPUTATION OF u^*

Because the cost criteria C_1 and C_2 are both independent of $u(t)$, to obtain the optimal control u^* we can consider the cost criterion

$$C(x, t_0) := \int_{t_0}^{\tau(x, t_0)} L(t)\, dt, \tag{9}$$

where $L(t) = \lambda e^{-\lambda t}$ or 1.

Assuming that the value function

$$F(x, t_0) := \max_{u(t),\, t_0 \le t \le \tau} E[C(x, t_0)] \tag{10}$$

exists (in $\mathbb{R}$) and is twice differentiable, we find that it satisfies the dynamic programming equation

$$0 = \max_{u(t_0)} [L + F_{t_0} + (a + b_1 u(t_0) + b_2 u^2(t_0)) F_x + \tfrac{1}{2} v F_{xx}]. \tag{11}$$

Then, we readily obtain that u^* ($:= u^*(t_0)$) is such that

$$(b_1 + 2b_2 u^*) F_x = 0. \tag{12}$$

PROPOSITION 2.1: *If $F_x \neq 0$, then the optimal control u^* is given by*

$$u^* = -\frac{b_1}{2b_2}, \tag{13}$$

where b_2 is assumed to be strictly positive.

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Remark: We cannot always assume that $F_x \neq 0$. Indeed, in some cases we can have $F(x, t_0) = \text{constant}\ \forall x \in I \subset [-d, d]$, so that $F_x = 0\ \forall x \in I$.

COROLLARY 2.1: *If* $F(x, t_0) < \infty\ \forall x \in (0, d)$, *we have*

$$u^* = \begin{cases} -b_1/2b_2 & \text{if } x > 0 \\ \text{sgn}(b_1)\infty & \text{if } x < 0. \end{cases} \tag{14}$$

PROOF: The result follows from Proposition 2.1 and the obvious fact that the origin is the "safest" place in the interval $[-d, d]$ when the objective is to maximize the survival time in $[-d, d]$. Since $u(t)$ is unconstrained, if $x(t_0) < 0$ then the optimizer can move away from the boundary $x = -d$ instantly by choosing $u^* = \text{sgn}(b_1)\infty$. Indeed, the optimally controlled process $x^*(t)$ will then move at once from its position $x(t_0)$ to 0 in an infinitely short time. Therefore, the optimal control is *not* $u^* = -b_1/2b_2$ when $x(t_0) < 0$.

Finally, if $x(t_0) > 0$ then the optimizer cannot use infinite control, since this would minimize the survival time in $[-d, d]$ rather than maximize it. It follows that the value of $F(x, t_0)$ will vary with x if $F(x, t_0) < \infty$, so that $F_x \neq 0$ if $x > 0$ and $u^* = -b_1/2b_2$, according to Proposition 2.1. ■

Remarks:

1. If the optimizer uses $u(t) = -b_1/2b_2$ when $x < 0$, then $b_1 u(t) + b_2 u^2(t) = -b_1^2/4b_2$. Therefore, the optimizer is trying to move the process $x(t)$ downward (if $b_1 \neq 0$), which is certainly not the optimal solution. The solution $u(t) = -b_1/2b_2$ would, in fact, yield a minimum here.
2. Because of the term $b_2 u^2(t)$ (with $b_2 > 0$) in the stochastic differential equation (1), we could actually take $u(t_0) = -\text{sgn}(b_1)\infty$ when $x(t_0)$ is negative and achieve the same result. It is, however, certainly more logical to choose u^*, as above.
3. If $E[C(x, t_0)] = \infty$ when the optimizer uses $u(t) \equiv 0$, we can obviously write that $u^* \equiv 0$ and the problem is of no interest. Similarly, we can eliminate the case when $F(x, t_0) = \infty$ in the interval $(0, d)$, because it is less interesting.

Consider now the case when the cost criterion C is given by C_1. Then, the dynamic programming equation (11) becomes (writing F instead of F_1 for simplicity of notation)

$$0 = \max_{u(t_0)} [\lambda e^{-\lambda t_0} + F_{t_0} + (a + b_1 u(t_0) + b_2 u^2(t_0)) F_x + \tfrac{1}{2} v F_{xx}]. \tag{15}$$

We are interested in solving this partial differential equation in the case when $x > 0$. Then, we have

$$0 = \lambda e^{-\lambda t_0} + F_{t_0} + \left(a - \frac{b_1^2}{4b_2}\right) F_x + \frac{v}{2} F_{xx}. \tag{16}$$

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Remark: Note that we cannot have $E[C_1(x, t_0)] = \infty$ for any choice of $u(t)$, so that $F(x, t_0) < \infty \ \forall x \in (0, d)$, as required.

Next, assuming that we can interchange the maximization and the mathematical expectation, we may write that

$$F(x, t_0) = E\left[\int_{t_0}^{\tau^*(x,\, t_0)} \lambda e^{-\lambda t}\, dt\right] = E\{[e^{-\lambda t_0} - e^{-\lambda \tau^*(x,\, t_0)}]\}, \tag{17}$$

where $\tau^*(x, t_0)$ is the value of $\tau(x, t_0)$ obtained by using the optimal control u^*.

Since $\tau^*(x, t_0) = t_0 + \tau_0^*(x)$, where

$$\tau_0^*(x) := \inf\{t \geq 0\colon |x^*(t)| = d \,|\, x(0) = x\}, \tag{18}$$

we have

$$F(x, t_0) = e^{-\lambda t_0} - e^{-\lambda t_0} E[e^{-\lambda \tau_0^*(x)}] := e^{-\lambda t_0}[1 - M(x; \lambda)], \tag{19}$$

where M is the moment generating function of $\tau_0^*(x)$. Substituting into the partial differential equation (16), we obtain

$$0 = \lambda e^{-\lambda t_0} + \lambda e^{-\lambda t_0}(M - 1) - \left(a - \frac{b_1^2}{4b_2}\right) e^{-\lambda t_0} M' - \frac{v}{2}\, e^{-\lambda t_0} M'' \tag{20}$$

$\Leftrightarrow$

$$\lambda M = \left(a - \frac{b_1^2}{4b_2}\right) M' + \frac{v}{2}\, M''. \tag{21}$$

In the remainder of this article, we will consider the case when the optimally controlled process $x^*(t)$, starting from $x > 0$, *can* reach zero. Since the optimizer uses infinite control when $x < 0$, $x^*(t)$ cannot cross the origin from $x > 0$. This is tantamount to saying that the origin is a *reflecting* boundary (if it is a *regular* boundary; it could also be an *exit* boundary). It follows (see Cox and Miller [3, p. 231], for instance) that the boundary conditions are

$$M(d; \lambda) = 1 \tag{22}$$

and

$$M'(0; \lambda) = 0. \tag{23}$$

Remarks:

1. It is possible that the origin be an inaccessible boundary for $x^*(t)$. For example, if $x(t)$ satisfies the stochastic differential equation

$$dx(t) = x(t)\, dt + u^2(t)\, dt + x(t)\, dW(t) \tag{24}$$

for $x > 0$, then $x^*(t)$ is such that

$$dx^*(t) = x^*(t)\, dt + x^*(t)\, dW(t), \tag{25}$$

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so that $x^*(t)$ is a geometric Brownian motion (see Karlin and Taylor [5, p. 175]). Since it can be expressed as the exponential of a Brownian motion, it cannot move from $x > 0$ to 0. In this case, the obvious optimal control, if the initial value of $x(t)$ is $x(t_0) = 0$, would be $u^* = 0$ $(= -b_1/2b_2$, actually). Indeed, the optimizer would then obtain an infinite reward.

In general, if the origin is an inaccessible boundary for $x^*(t)$, then the boundary condition $M'(0;\lambda) = 0$ does not apply. The function M is then the solution of a first passage time problem with a single boundary. This type of problem has been considered extensively in one dimension. That is why we are interested in the case when $x^*(t)$ can actually reach 0.

2. In theory, the optimal control u^* is undetermined at $x = 0$ when the origin is a regular boundary for $x^*(t)$ because

$$F_x(0, t_0) = -e^{-\lambda t_0} M'(0;\lambda) = 0, \tag{26}$$

so that the condition

$$(b_1 + 2b_2 u^*) F_x = 0 \tag{27}$$

is fulfilled for any finite u^*. However, since $x^*(t)$ cannot cross the origin, the optimizer should take $b_1 u(t) + b_2 u^2(t)$ as small as possible when $x^*(t) = 0$. It follows that $u^* = -b_1/2b_2$, as for $x > 0$. At any rate, this is only one point and has no real influence on the value of $\tau^*(x)$.

3. Since $\tau(x, t_0) = t_0 + \tau(x,0)$, we may write that

$$F(x, t_0) := \max_{u(t),\, t_0 \le t \le \tau(x,\, t_0)} E\left[\int_{t_0}^{\tau(x,\, t_0)} \lambda e^{-\lambda t}\, dt\right] \tag{28}$$

$\Leftrightarrow$

$$F(x, t_0) = \max_{u(s), 0 \le s \le \tau(x,0)} E\left[e^{-\lambda t_0} \int_0^{\tau(x,0)} \lambda e^{-\lambda s}\, ds\right] := e^{-\lambda t_0} G(x). \tag{29}$$

Now, we find that the function $G(x)$ satisfies the ordinary differential equation (see (16))

$$0 = \lambda - \lambda G + \left(a - \frac{b_1^2}{4b_2}\right) G' + \frac{v}{2} G'' \tag{30}$$

for $x > 0$ and is subject to the boundary condition

$$G(d) = 0; \tag{31}$$

that is, we can reduce the partial differential equation (16) to an ordinary differential equation. Moreover, the interchange of the maximization and the mathematical expectation is, therefore, only needed to obtain the boundary condition $G'(0) = 0$ (since $G(x) = 1 - M(x;\lambda)$).

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To conclude this section, we will solve the ordinary differential equation (21), which is a Kolmogorov backward equation, subject to the boundary conditions (22) and (23), in three important cases. These three cases are probably the most important ones for applications. Indeed, we will consider the cases when $x^*(t)$ is either a Wiener process, an Ornstein–Uhlenbeck process, or a Bessel process. A particular field in which these diffusion processes are widely used is that of mathematical finance. More precisely, the Brownian motion process and the Ornstein–Uhlenbeck process have served as models of discount bond prices, whereas the Bessel process has been used to evaluate Asian options (see Geman and Yor [4] and Chesney et al. [2]).

The Wiener process is the basic diffusion process and, as such, has been used in numerous applications. For instance, Whittle [12] used it as a rudimentary model for the flight of an aircraft.

The Ornstein–Uhlenbeck process also appears in genetics (see Karlin and Taylor [5, p. 183]), as well as in hydrology (Labib et al. [7]) and electrical engineering (Lefebvre [8]), among other fields.

Finally, Lefebvre and Adjengue [10] proposed an integrated Bessel process to model the flow of a river and considered the problem of optimally controlling this process.

I. Suppose first that $a - (b_1^2/4b_2) \equiv \mu \in \mathbb{R}$ and $v \equiv \sigma^2 > 0$. Then, $x^*(t)$ is a Wiener process. The general solution of (21) in this case is

$$M(x;\lambda) = c_1 \exp[-(\mu + \gamma x)/\sigma^2] + c_2 \exp[(\mu + \gamma x)/\sigma^2], \tag{32}$$

where c_1 and c_2 are constants and

$$\gamma := (2\lambda + \mu^2)^{1/2}. \tag{33}$$

The solution that satisfies the boundary conditions (22) and (23) is easily found to be

$$M(x;\lambda) = \frac{\cosh(\gamma x)}{\cosh(\gamma d)}. \tag{34}$$

Note that we then have

$$F_x = -e^{-\lambda t_0} M' = -e^{-\lambda t_0} \frac{\sinh(\gamma x)\gamma}{\cosh(\gamma d)} \tag{35}$$

which is negative $\forall x \in (0, d]$. Therefore, the solution $u^* = -b_1/2b_2$ really yields a maximum (because $b_2 > 0$; see (11) and (12)).

II. Suppose next that $a - (b_1^2/4b_2) = -\alpha x$, where $\alpha > 0$, and $v \equiv \sigma^2 > 0$, as above. When $\sigma^2 = 1$, the optimally controlled process $x^*(t)$ is an Ornstein–Uhlenbeck process. We must solve the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 M''(x;\lambda) - \alpha x M'(x;\lambda) = aM(x;\lambda), \tag{36}$$

subject to $M(d;\lambda) = 1$ and $M'(x;\lambda) = 0$.

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Remark: The processes $x^*(t)$ considered in I and II have a regular boundary at the origin; therefore, they can take the value $x^*(t) = 0$ and leave the origin afterward.

Now, the general solution of (36) can be written as

$$M(x;\lambda) = c_1 \Phi\left(\frac{\lambda}{2\alpha}, \frac{1}{2}, \frac{\alpha x^2}{\sigma^2}\right) + c_2 x \Phi\left(\frac{\lambda}{2\alpha} + \frac{1}{2}, \frac{3}{2}, \frac{\alpha x^2}{\sigma^2}\right), \tag{37}$$

where $\Phi(\cdot,\cdot,\cdot)$ is a confluent hypergeometric function (see Abramowitz and Stegun [1, p. 504], for instance, where Φ is denoted by M).

Using the formula

$$\frac{d}{dz}\Phi(a,b,z) = \frac{a}{b}\Phi(a+1,b+1,z) \tag{38}$$

(see Abramowitz and Stegun [1, p. 507, formula 13.4.8]), we have

$$M'(x;\lambda) = c_1 \frac{2\lambda}{\sigma^2} x \Phi\left(\frac{\lambda}{2\alpha} + 1, \frac{3}{2}, \frac{\alpha x^2}{\sigma^2}\right) + c_2 \Phi\left(\frac{\lambda}{2\alpha} + \frac{1}{2}, \frac{3}{2}, \frac{\alpha x^2}{\sigma^2}\right)$$
$$+ c_2 \frac{2x^2}{3\sigma^2}(\alpha + \lambda)\Phi\left(\frac{\lambda}{2\alpha} + \frac{3}{2}, \frac{5}{2}, \frac{\alpha x^2}{\sigma^2}\right). \tag{39}$$

Since $\Phi(a,b,0) = 1$, the boundary condition $M'(0;\lambda) = 0$ implies that we must choose the constant c_2 equal to zero. Then, the boundary condition $M(d;\lambda) = 1$ implies that

$$M(x;\lambda) = \frac{\Phi\left(\frac{\lambda}{2\alpha}, \frac{1}{2}, \frac{\alpha x^2}{\sigma^2}\right)}{\Phi\left(\frac{\lambda}{2\alpha}, \frac{1}{2}, \frac{\alpha d^2}{\sigma^2}\right)} \tag{40}$$

for $0 \le x \le d$. With the help of the formula 13.4.8 of [1], it is a simple matter to check that $M_x > 0 \ \forall x \in (0,d]$, as required.

III. We consider finally the case when $a - (b_1^2/4b_2) = (\alpha - 1)/2x$, with $\alpha \ge 0$, and $v \equiv \sigma^2 > 0$, again. When $\sigma^2 = 1$, the process $x^*(t)$ is a Bessel process (see Karlin and Taylor [5, pp. 175 and 238]). The type of boundary at the origin depends on the parameter α. If $\sigma^2 = 1$, we find (Karlin and Taylor [5, p. 238]) that the origin is

$$\begin{array}{ll} \text{An entrance boundary} & \text{if } \alpha \ge 2 \\ \text{A regular boundary} & \text{if } 0 < \alpha < 2 \\ \text{An exit boundary} & \text{if } \alpha = 0. \end{array} \tag{41}$$

Since an entrance boundary is unattainable, we restrict ourselves to the case when $\alpha < 2$. Moreover, we eliminate the value $\alpha = 0$ as well, because an exit boundary is *absorbing*; that is, it can be reached from $x > 0$. How-

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ever, if $x^*(t) = 0$ for some t, then $x^*(s) = 0\ \forall s > t$ and we will have $\tau^*(x, t_0) = \infty$. As mentioned earlier, we are interested in the case when the origin is a *reflecting* (and thus not absorbing) boundary. Finally, if $\alpha = 1$, the Bessel process is equivalent to the absolute value of the standard Brownian motion. Since we consider the optimally controlled process in the interval $[0, d]$, we may state that this problem has already been solved in the particular case I.

We can actually limit ourselves to two special values of the parameter α, namely $\alpha = \frac{3}{2}$ and $\alpha = \frac{1}{2}$, and to the value $\sigma^2 = 1$, and this without any real loss of generality.

(a) $\alpha = \frac{3}{2}$, $\sigma^2 = 1$. The function $M(x;\lambda)$ satisfies the ordinary differential equation

$$xM''(x;\lambda) + \tfrac{1}{2}M'(x;\lambda) = 2\lambda x M(x;\lambda). \tag{42}$$

Its general solution is

$$M(x;\lambda) = x^{1/4}[c_1 K_{1/4}(\sqrt{2\lambda}x) + c_2 I_{1/4}(\sqrt{2\lambda}x)], \tag{43}$$

where $I_\nu(\cdot)$ and $K_\nu(\cdot)$ are modified Bessel functions (see Abramowitz and Stegun [1, p. 374]).

We have (Abramowitz and Stegun [1, p. 376])

$$\frac{d}{dz}K_{1/4}(z) = -K_{-3/4}(z) - \frac{1/4}{z}K_{1/4}(z) \tag{44}$$

and

$$\frac{d}{dz}I_{1/4}(z) = I_{-3/4}(z) - \frac{1/4}{z}I_{1/4}(z). \tag{45}$$

It follows that

$$M'(x;\lambda) = x^{1/4}\sqrt{2\lambda}[-c_1 K_{-3/4}(\sqrt{2\lambda}x) + c_2 I_{-3/4}(\sqrt{2\lambda}x)]. \tag{46}$$

Next, using the formulas (Abramowitz and Stegun [1, p. 375]) $K_{-\nu}(\cdot) = K_\nu(\cdot)$ and

$$\begin{aligned} K_\nu(z) &\sim (\tfrac{1}{2})\Gamma(z)(z/2)^{-\nu} \quad \text{if } \mathrm{Re}(\nu) > 0 \\ I_\nu(z) &\sim (z/2)^\nu/\Gamma(z+1) \quad \text{if } \nu \neq -1, -2, \ldots, \end{aligned} \tag{47}$$

we deduce that we will have

$$\lim_{x \downarrow 0} M'(x;\lambda) = \pm\infty, \tag{48}$$

unless we choose

$$c_2 = c_1 \Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4})/2 := \delta c_1, \tag{49}$$

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in which case, the boundary condition $M'(0;\lambda) = 0$ is fulfilled. Finally, $M(d;\lambda) = 1$ implies that

$$M(x;\lambda) = \left(\frac{x}{d}\right)^{1/4} \frac{K_{1/4}(\sqrt{2\lambda}x) + \delta I_{1/4}(\sqrt{2\lambda}x)}{K_{1/4}(\sqrt{2\lambda}d) + \delta I_{1/4}(\sqrt{2\lambda}d)} \tag{50}$$

for $0 \leq x \leq d$.

(b) $\alpha = \frac{1}{2}, \sigma^2 = 1$. The Kolmogorov backward equation that we must solve is now

$$xM''(x;\lambda) - \tfrac{1}{2}M'(x;\lambda) = 2\lambda x M(x;\lambda). \tag{51}$$

We find that

$$M(x;\lambda) = x^{3/4}[c_1 K_{3/4}\sqrt{2\lambda}x) + c_2 I_{3/4}(\sqrt{2\lambda}x)] \tag{52}$$

and that

$$M'(x;\lambda) = x^{3/4}\sqrt{2\lambda}[-c_1 K_{-1/4}(\sqrt{2\lambda}x) + c_2 I_{-1/4}(\sqrt{2\lambda}x)]. \tag{53}$$

This time, if we take the limit as x decreases to zero in $M'(x;\lambda)$, we obtain that $M'(x,\lambda) = 0$ for all constants c_1 and c_2. However, if we first assume that the reflecting boundary is at $\epsilon > 0$ and then take the limit as $\epsilon \downarrow 0$, we deduce that we must have the relationship

$$c_2 = c_1\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4})/2 = \delta c_1, \tag{54}$$

as in the case when $\alpha = \frac{3}{2}$. The boundary condition $M(d;\lambda) = 0$ yields at last the formula

$$M(x;\lambda) = \left(\frac{x}{d}\right)^{3/4} \frac{K_{3/4}(\sqrt{2\lambda}x) + \delta I_{3/4}(\sqrt{2\lambda}x)}{K_{3/4}(\sqrt{2\lambda}d) + \delta I_{3/4}(\sqrt{2\lambda}d)} \tag{55}$$

for $0 \leq x \leq d$.

In the next section, the case when the cost criterion $C_1(x,t_0)$ is replaced by $C_2(x,t_0)$ will be treated.

3. MAXIMIZATION OF $E[\tau(x,t_0)]$

When the cost criterion is

$$C_2(x,t_0) = \int_{t_0}^{\tau(x,t_0)} 1\,dt = \tau(x,t_0) - t_0 = \tau(x,0) := \tau_0(x), \tag{56}$$

the problem is time-invariant and we can assume that $t_0 = 0$ without loss of generality. The dynamic programming equation that we must solve is (writing F for F_2, again, for simplicity)

$$0 = \max_{u(0)}[1 + (a + b_1 u(0) + b_2 u^2(0))F_x + \tfrac{1}{2}vF_{xx}], \tag{57}$$

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which yields the same u^* ($=u^*(0)$ here) as in Section 2. Furthermore, we have

$$F(x) = \max_{u(t),0\le t\le\tau_0} E[\tau_0(x)]. \tag{58}$$

Assuming that we can interchange the maximization and the mathematical expectation, we may write that

$$F(x) = E[\tau_0^*(x)]. \tag{59}$$

Now, the function

$$m(x) := E[\tau_0^*(x)] \tag{60}$$

satisfies the Kolmogorov backward equation

$$\frac{1}{2}\,vm''(x) + \left(a - \frac{b_1^2}{4b_2}\right)m'(x) = -1 \tag{61}$$

and is subject to the boundary condition

$$m(d) = 0. \tag{62}$$

We will consider the same particular cases as in Section 2. It follows that the origin is a reflecting boundary. We can show that the boundary condition for the function $m(x)$ at $x = 0$ is also

$$m'(0) = 0. \tag{63}$$

Remark: Theoretically, the function $m(x)$ could be infinite. That is the case, for instance, when $x^*(t)$ is a Bessel process with $\alpha = 0$ because the origin is an exit boundary and there is a positive probability of hitting 0 before d when the starting value is $x \in (0, d)$. However, in the three cases treated in Section 2, we may state that $m(x)$ is finite.

I. When $x^*(t)$ is a Wiener process, (61) takes the form

$$\tfrac{1}{2}\sigma^2 m''(x) + \mu m'(x) = -1. \tag{64}$$

Observe that this equation is a first-order linear ordinary differential equation for $n(x) := m'(x)$. We find that

$$m(x) = -\frac{x}{\mu} - c_1\frac{\sigma^2}{2\mu}\,e^{-2\mu x/\sigma^2} + c_2 \quad \text{if } \mu \neq 0, \tag{65}$$

whereas

$$m(x) = -\frac{x^2}{\sigma^2} + c_1 x + c_2 \quad \text{if } \mu = 0. \tag{66}$$

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The boundary conditions $m(d) = 0$ and $m'(0) = 0$ imply that

$$m(x) = \begin{cases} (1/\mu)\{(d-x) + (\sigma^2/2\mu)[e^{-2\mu d/\sigma^2} - e^{-2\mu x/\sigma^2}]\} & \text{if } \mu \neq 0 \\ (d^2 - x^2)/\sigma^2 & \text{if } \mu = 0. \end{cases} \tag{67}$$

Note that $F'(x)\,(= m'(x))$ is negative $\forall x \in (0, d)$ in both cases, so that we, indeed, have a maximum.

II. Let $a - (b_1^2/4b_2) = -\alpha x$ and $v \equiv \sigma^2 > 0$, so that $x^*(t)$ is a (generalized) Ornstein–Uhlenbeck process. We must solve the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 m''(x) - \alpha x m'(x) = -1. \tag{68}$$

Its general solution may be written as

$$m(x) = -(2/\sigma^2)\int_0^x e^{\alpha y^2/\sigma^2}\left[\int_0^y e^{-\alpha z^2/\sigma^2}\,dz\right]dy + c_1\int_0^x e^{\alpha y^2/\sigma^2}\,dy + c_2. \tag{69}$$

Remark: In fact, $m(x)$ can also be expressed in terms of the error function (and the complementary error function), together with the generalized hypergeometric function (as defined in the software Mathematica, for instance).

The particular solution satisfying the boundary condition $m'(0) = 0$ is such that $c_1 = 0$. It follows that

$$m(x) = (2/\sigma^2)\int_x^d e^{\alpha y^2/\sigma^2}\left[\int_0^y e^{-\alpha z^2/\sigma^2}\,dz\right]dy \tag{70}$$

for $0 \leq x \leq d$. Hence, we have

$$m'(x) = -(2/\sigma^2)e^{\alpha x^2/\sigma^2}\int_0^x e^{-\alpha z^2/\sigma^2}\,dz < 0, \tag{71}$$

as should be.

III. To conclude, we return to the Bessel process. Contrary to Section 2, we will treat the more general case when $0 < \alpha < 2$. The ordinary differential equation (61) becomes

$$\frac{1}{2}\,m''(x) - \frac{\alpha - 1}{2x}\,m'(x) = -1. \tag{72}$$

We find that

$$m(x) = -\frac{x^2}{\alpha} + c_1\,\frac{x^{2-\alpha}}{2-\alpha} + c_2. \tag{73}$$

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Finally, applying the boundary conditions $m(d) = m'(0) = 0$, we obtain that

$$m(x) = \frac{d^2 - x^2}{\alpha} \quad \text{for } 0 \le x \le d, \tag{74}$$

so that $m'(x) < 0 \ \forall x \in (0, d)$.

4. CONCLUSION

In this article, the problem of maximizing the survival time of a diffusion process $x(t)$ in an interval, as considered by Lefebvre and Whittle [11] in particular, was modified by taking a cost criterion that does not depend explicitly on the control variable $u(t)$. Moreover, $u(t)$ appears both linearly and quadratically in the stochastic differential equation that defines $x(t)$.

We have seen that although it is not very difficult to find the optimal control u^*, the problem is interesting because to obtain explicitly the value function, we must solve a first passage time probability problem with an absorbing barrier at $x = d$ and (in the special cases presented here) a *reflecting* barrier at $x = 0$. The particular processes considered are the most important ones for applications, namely the Wiener, Ornstein–Uhlenbeck, and Bessel processes.

As a sequel to this work, one could use other cost criteria, leading to different probability problems. For instance, the cost criterion

$$C_3(x, t_0) := \int_{t_0}^{\tau(x, t_0)} te^{-\lambda t}\, dt \tag{75}$$

requires the solution of a more difficult first passage problem.

Finally, one could try to generalize the problem considered in this article to the two-dimensional case, or even to the n-dimensional case. Of course, the corresponding first passage time problem would most probably be very hard to solve explicitly, except in special instances.

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