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Expansive algebraic actions of discrete residually finite amenable groups and their entropy

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Abstract. We prove an entropy formula for certain expansive actions of a countable discrete residually finite group Γ by automorphisms of compact abelian groups in terms of Fuglede–Kadison determinants. This extends an earlier result proved by the first author under somewhat more restrictive conditions. The main tools for this generalization are a representation of the Γ-action by means of a 'fundamental homoclinic point' and the description of entropy in terms of the renormalized logarithmic growth rate of the set of Γ_n-fixed points, where $(\Gamma_n, n \geq 1)$ is a decreasing sequence of finite index normal subgroups of Γ with trivial intersection.

1. *Introduction*

For a countable discrete amenable group Γ and an element $f = \sum f_\gamma \gamma$ in the integral group ring $\mathbb{Z}\Gamma$, consider the quotient $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ of $\mathbb{Z}\Gamma$ by the left ideal $\mathbb{Z}\Gamma f$ generated by f. It is a discrete abelian group with a left Γ-action by multiplication. The Pontryagin dual

$$X_f = \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f} \tag{1.1}$$

is a compact abelian group with a left action of Γ by continuous group automorphisms. More explicitly, X_f is the following closed subshift of the full shift $(\mathbb{R}/\mathbb{Z})^\Gamma = \widehat{\mathbb{Z}\Gamma}$ with Γ-action given by $(\gamma \cdot x)_{\gamma'} = x_{\gamma^{-1}\gamma'}$. The elements of X_f are the sequences $(x_{\gamma'})$ in $(\mathbb{R}/\mathbb{Z})^\Gamma$ which satisfy the equations

$$\sum_{\gamma'} f_{\gamma'} x_{\gamma\gamma'} = 0 \quad \text{in } \mathbb{R}/\mathbb{Z} \text{ for all } \gamma \text{ in } \Gamma.$$

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For $\Gamma = \mathbb{Z}^d$ the theory of such dynamical systems has been extensively studied (see, e.g., [**12**]). For general Γ the investigation of such systems was initiated in [**5**].

The various definitions of entropy (topological, metric and measure theoretic with respect to Haar measure) all coincide for this action (see, e.g., [**3**, §2]). We denote this common entropy by h_f.

If f is a unit in the convolution algebra $L^1(\Gamma)$, then the action of Γ on X_f is expansive. If in addition f is positive in the von Neumann algebra $\mathcal{N}\Gamma$ and Γ has a 'log-strong Følner sequence', it was shown in [**3**] that

$$h_f = \log \det_{\mathcal{N}\Gamma} f,$$

where $\det_{\mathcal{N}\Gamma}$ is the Fuglede–Kadison determinant [**6, 10**]. Currently, the only groups known to have a log-strong Følner sequence are the virtually nilpotent groups, so this condition is unfortunate, even though it is only needed for the inequality $h_f \leq \log \det_{\mathcal{N}\Gamma} f$ in [**3**].

Recall that a countable group Γ is residually finite if there is a sequence Γ_n of normal subgroups of finite index whose intersection is trivial.

Our main result is given in the following theorem.

THEOREM 1.1. *Let Γ be a countable discrete amenable and residually finite group and f an element of $\mathbb{Z}\Gamma$. Then the action of Γ on X_f is expansive if and only if f is a unit in $L^1(\Gamma)$. In this case we have the formula*

$$h_f = \log \det_{\mathcal{N}\Gamma} f. \tag{1.2}$$

For example, Γ could be any finitely generated solvable subgroup of a matrix group over a field.

If $f = \sum f_\gamma \gamma$ is invertible in $L^1(\Gamma)$ then $\det_{\mathcal{N}\Gamma} f = \det_{\mathcal{N}\Gamma} f^*$, where $f^* = \sum f_\gamma \gamma^{-1}$, since the Fuglede–Kadison determinant of an invertible operator is equal to that of its adjoint. From (1.2) we therefore obtain that $h_f = h_{f^*}$. This question had been left open in [**3**] even for the integral Heisenberg group.

Dynamically, the proof of formula (1.2) is based on a description of the entropy for expansive Γ-actions as a renormalized logarithmic growth rate of the number of Γ_n-fixed points, where $(\Gamma_n, n \geq 1)$ is a decreasing sequence of finite index normal subgroups with trivial intersection. In the case of expansive $\mathbb{Z}^d$-actions this relation between entropy and periodic points is known and the methods can be carried over to our case. It remains to prove that $\det_{\mathcal{N}\Gamma} f$ is the limit of certain finite-dimensional determinants. This is a well-known problem in the context of L^2-invariants for residually finite groups (cf. [**10**, Ch. 13]). There one tries to approximate L^2-Betti numbers, L^2-signatures, etc. of coverings with covering group Γ by the corresponding quantities for the finite groups Γ/Γ_n. However, for the combinatorial L^2-torsion which involves Fuglede–Kadison determinants there is no general result in this direction.

Nonetheless, in our situation which is quite favourable analytically it is not difficult to prove the required approximation property of $\det_{\mathcal{N}\Gamma} f$ directly.

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2. *Group rings*

Let Γ be a countable discrete group with identity element $1 = 1_\Gamma$ and integral group ring $\mathbb{Z}\Gamma$. We denote by $L^\infty(\Gamma)$ the set of all bounded maps from Γ to $\mathbb{R}$, write every $w \in L^\infty(\Gamma)$ as (w_γ) with $w_\gamma \in \mathbb{R}$ for every $\gamma \in \Gamma$, and denote by $\|w\|_\infty = \sup_{\gamma\in\Gamma} |w_\gamma|$ the supremum norm on $L^\infty(\Gamma)$. For $p \in [1, \infty)$ we set

$$L^p(\Gamma) = \left\{ w \in L^\infty(\Gamma) : \|w\|_p := \left(\sum_{\gamma\in\Gamma} |w_\gamma|^p \right)^{1/p} < \infty \right\}.$$

For $\gamma \in \Gamma$ we define $e(\gamma) \in L^1(\Gamma) \subset L^\infty(\Gamma)$ by

$$e(\gamma)_{\gamma'} = \begin{cases} 1 & \text{if } \gamma = \gamma', \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Every h in $L^1(\Gamma)$ can be uniquely written as a convergent series

$$h = \sum_{\gamma\in\Gamma} h_\gamma e(\gamma) \tag{2.2}$$

with $h_\gamma \in \mathbb{R}$ and $\sum_{\gamma\in\Gamma} |h_\gamma| < \infty$. In this notation, multiplication in the convolution algebra $L^1(\Gamma)$ takes the form

$$\begin{aligned} h \cdot h' = \sum_{\gamma,\gamma'\in\Gamma} h_\gamma h'_{\gamma'} e(\gamma\gamma') &= \sum_{\gamma\in\Gamma} \left(\sum_{\gamma'\in\Gamma} h_{\gamma\gamma'^{-1}} h'_{\gamma'} \right) e(\gamma) \\ &= \sum_{\gamma\in\Gamma} \left(\sum_{\gamma'\in\Gamma} h_{\gamma'} h'_{\gamma'^{-1}\gamma} \right) e(\gamma). \end{aligned} \tag{2.3}$$

The involution $h \mapsto h^*$ in $L^1(\Gamma)$ is defined by

$$h^* = \sum_{\gamma\in\Gamma} h_{\gamma^{-1}} e(\gamma) = \sum_{\gamma\in\Gamma} h_\gamma e(\gamma^{-1}) \tag{2.4}$$

and satisfies the condition that $(g \cdot h)^* = h^* \cdot g^*$ for $g, h \in L^1(\Gamma)$. An element $h \in L^1(\Gamma)$ is *self-adjoint* if $h^* = h$.

Note that the integral group ring $\mathbb{Z}\Gamma$ of finite formal sums $\sum_\gamma a_\gamma \gamma$ with $a_\gamma \in \mathbb{Z}$ can be viewed as a subring of $L^1(\Gamma)$ by identifying $\sum_\gamma a_\gamma \gamma$ with $\sum_\gamma a_\gamma e(\gamma)$.

Multiplication in $L^1(\Gamma)$ can be extended further to commuting (left) actions $(h, w) \mapsto \lambda_h w$ and $(h, w) \mapsto \rho_h w$ of $L^1(\Gamma)$ on $L^\infty(\Gamma)$ by setting

$$\begin{aligned} (\lambda_h w)_\gamma &= (h \cdot w)_\gamma = \sum_{\gamma'\in\Gamma} h_{\gamma\gamma'^{-1}} w_{\gamma'} = \sum_{\gamma'\in\Gamma} h_{\gamma'} w_{\gamma'^{-1}\gamma}, \\ (\rho_h w)_\gamma &= (w \cdot h^*)_\gamma = \sum_{\gamma'\in\Gamma} w_{\gamma\gamma'^{-1}} h^*_{\gamma'} = \sum_{\gamma'\in\Gamma} w_{\gamma\gamma'} h_{\gamma'} \end{aligned} \tag{2.5}$$

for every $h \in L^1(\Gamma)$, $w \in L^\infty(\Gamma)$ and $\gamma \in \Gamma$. The *left* and *right shift actions* λ and ρ of Γ on $L^\infty(\Gamma)$ are given by

$$\lambda^\gamma w = \lambda_{e(\gamma)} w, \quad \rho^\gamma w = \rho_{e(\gamma)} w$$

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or, equivalently, by

$$(\lambda^\gamma w)_{\gamma'} = w_{\gamma^{-1}\gamma'}, \quad (\rho^\gamma w)_{\gamma'} = w_{\gamma'\gamma} \tag{2.6}$$

for every $w \in L^\infty(\Gamma)$ and $\gamma, \gamma' \in \Gamma$.

For the following lemmas we fix $h \in L^1(\Gamma)$ and define the linear operators $\rho_h, \rho_{h^*} : L^\infty(\Gamma) \longrightarrow L^\infty(\Gamma)$ by (2.4) and (2.5). For $v \in L^p(\Gamma)$ and $w \in L^q(\Gamma)$ with $(1/p) + (1/q) = 1$, $1 \le p, q \le \infty$, we set

$$\langle v, w \rangle = \sum_{\gamma \in \Gamma} v_\gamma w_\gamma. \tag{2.7}$$

For $1 < p \le \infty$ this pairing identifies $L^p(\Gamma)$ with the dual $L^q(\Gamma)'$ as Banach spaces and $L^p(\Gamma)$ acquires a weak $*$-topology. In this topology (v_n) converges to v if and only if $\lim_{n\to\infty}\langle v_n, w\rangle = \langle v, w\rangle$ for all w in $L^q(\Gamma)$. Note that

$$\begin{aligned} \langle v, \rho_h w\rangle &= \langle v, w \cdot h^*\rangle = \langle v \cdot h, w\rangle, \\ \langle v, \lambda_h w\rangle &= \langle v, h \cdot w\rangle = \langle h^* \cdot v, w\rangle \end{aligned} \tag{2.8}$$

for every $h \in L^1(\Gamma)$. Finally we put, for every $h \in L^1(\Gamma)$ and $1 \le p \le \infty$,

$$K_p(h) = \{g \in L^p(\Gamma) : \rho_h g = 0\}, \quad V_p(h) = \rho_h(L^p(\Gamma)). \tag{2.9}$$

3. *Algebraic Γ-actions*

Definition 3.1. Let Γ be a countable discrete group. An *algebraic Γ-action* is a homomorphism $\alpha : \gamma \mapsto \alpha^\gamma$ from Γ into the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

If α is an algebraic Γ-action on a compact abelian group X then α is *expansive* if there exists an open neighbourhood $\mathcal{U}$ of the identity in X with $\bigcap_{\gamma\in\Gamma} \alpha^\gamma(\mathcal{U}) = 0$.

In this section we consider algebraic Γ-actions of a very special nature. Fix a countable discrete group Γ and consider the compact abelian group $X = \mathbb{T}^\Gamma$ consisting of all maps from Γ to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ under point-wise addition. Each $x \in X$ is written as $(x_\gamma) = (x_\gamma, \gamma \in \Gamma)$, where $x_\gamma \in \mathbb{T}$ denotes the value of x at $\gamma \in \Gamma$. We identify the Pontryagin dual $\widehat{X}$ of X with the group ring $\mathbb{Z}\Gamma$ under the pairing

$$\langle f, x\rangle = e^{2\pi i \sum_{\gamma\in\Gamma} f_\gamma x_\gamma}, \tag{3.1}$$

where $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma \in \mathbb{Z}\Gamma$ and $x = (x_\gamma) \in X$.

The left and right shift actions λ and ρ of Γ on X are defined by

$$(\lambda^\gamma x)_{\gamma'} = x_{\gamma^{-1}\gamma'} \quad \text{and} \quad (\rho^\gamma x)_{\gamma'} = x_{\gamma'\gamma} \tag{3.2}$$

for every $\gamma, \gamma' \in \Gamma$ and $x \in X$. As in (2.5) we can extend these actions of Γ to commuting (left) actions λ and ρ of $\mathbb{Z}\Gamma$ on X by setting

$$\lambda_f x = \sum_{\gamma\in\Gamma} f_\gamma \lambda^\gamma x \quad \text{and} \quad \rho_f x = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma x \tag{3.3}$$

for $f \in \mathbb{Z}\Gamma$ and $x \in X$ (cf. (2.4)). For every $f \in \mathbb{Z}\Gamma$, the maps $\lambda_f, \rho_f : X \longrightarrow X$ are continuous group homomorphisms which are dual to left and right multiplication by f^* and f, respectively, on $\widehat{X} = \mathbb{Z}\Gamma$.

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For the following discussion we set

$$L^\infty(\Gamma, \mathbb{Z}) = \{w = (w_\gamma) \in L^\infty(\Gamma) : w_\gamma \in \mathbb{Z} \text{ for every } \gamma \in \Gamma\}. \tag{3.4}$$

We fix $f \in \mathbb{Z}\Gamma$, set

$$X_f = \ker(\rho_f) = \{x \in X : \rho_f x = 0\} = \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}, \tag{3.5}$$

and denote by

$$\alpha_f = \lambda|_{X_f} \tag{3.6}$$

the restriction to X_f of the Γ-action λ on X. Since $\alpha_f^\gamma \in \mathrm{Aut}(X_f)$ for every $\gamma \in \Gamma$, α_f is an algebraic Γ-action on the compact abelian group X_f.

The map $\eta\colon L^\infty(\Gamma) \longrightarrow X$, given by

$$\eta(w)_\gamma = w_\gamma (\mathrm{mod}\ 1) \tag{3.7}$$

for every $w = (w_\gamma) \in L^\infty(\Gamma)$ and $\gamma \in \Gamma$, is a continuous surjective group homomorphism with

$$\eta \circ \lambda^\gamma = \lambda^\gamma \circ \eta, \quad \eta \circ \rho^\gamma = \rho^\gamma \circ \eta \tag{3.8}$$

for every $\gamma \in \Gamma$, and the *linearization*

$$W_f = \eta^{-1}(X_f) = \rho_f^{-1}(L^\infty(\Gamma, \mathbb{Z})) = \{w \in L^\infty(\Gamma) : \rho_f w \in L^\infty(\Gamma, \mathbb{Z})\} \tag{3.9}$$

of X_f is a weak*-closed and λ-invariant subgroup with $\ker(\eta) = L^\infty(\Gamma, \mathbb{Z}) \subset W_f$.

THEOREM 3.2. *Let Γ be a countable group, $f \in \mathbb{Z}\Gamma$, and let α_f be the algebraic Γ-action on X_f defined in (3.5) and (3.6). The following conditions are equivalent:*

(1) *the action α_f is expansive;*
(2) *$K_\infty(f) = \{0\}$ (cf. (2.5) and (2.9));*
(3) *f is invertible in $L^1(\Gamma)$.*

Proof. This follows by combining [**5**, Theorem 8.1] with [**2**, Theorem 1]. For the convenience of the reader we include the argument.

Let d be the usual metric

$$d(s_1, s_2) = \min\{|\tilde{s}_1 - \tilde{s}_2| : \tilde{s}_i \in \mathbb{R} \text{ and } s_i = \tilde{s}_i \ (\mathrm{mod}\ 1) \text{ for } i = 1, 2\} \tag{3.10}$$

on $\mathbb{T}$. If there exists a non-zero element $v \in K_\infty(f)$ then $\eta(cv) \in X_f$ for every $c \in \mathbb{R}$, and by choosing $|c|$ sufficiently small we can find, for every $\varepsilon > 0$, a non-zero element $x^{(\varepsilon)} \in X_f$ with $d(x_\gamma^{(\varepsilon)}, 0) < \varepsilon$ for every $\gamma \in \Gamma$. This proves that α_f is non-expansive.

Conversely, if α_f is non-expansive, then we can find a non-zero element $x \in X_f$ with $d(x_\gamma, 0) < (3\|f\|_1)^{-1}$ for every $\gamma \in \Gamma$. We choose $\tilde{x} \in \eta^{-1}(\{x\}) \subset W_f = \eta^{-1}(X_f)$ with $|\tilde{x}_\gamma| < (3\|f\|_1)^{-1}$ for every $\gamma \in \Gamma$. By definition of X_f, $\rho_f \tilde{x} \in L^\infty(\Gamma, \mathbb{Z})$, and the smallness of the coordinates of $\tilde{x}$ implies that $\tilde{x} \in K_\infty(f)$. This proves that (1) if and only if (2).

If $K_\infty(f) = \{0\}$ then $V_1(f^*)$ is dense in $L^1(\Gamma)$ by (2.8) and the Hahn–Banach theorem. Since the group of units in $L^1(\Gamma)$ is open, $V_1(f^*)$ contains a unit. Hence there exists a $g \in L^1(\Gamma)$ with $g \cdot f = 1$. By [**7**, p. 122], f is invertible in $L^1(\Gamma)$, which proves (3). The implication (3) $\Rightarrow$ (2) is obvious. □

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4. *Homoclinic points*

Definition 4.1. Let α be an algebraic action of a countable discrete group Γ on a compact abelian group X with identity element 0. A point $x \in X$ is *α-homoclinic* (or simply *homoclinic*) if $\lim_{\gamma\to\infty} \alpha^\gamma x = 0$, i.e. if for every neighbourhood U of 0 in X there is a finite subset F of Γ with $\alpha^\gamma x \in U$ for all $\gamma \in \Gamma \setminus F$.

The set $\Delta_\alpha(X)$ of all α-homoclinic points in X is a subgroup of X, called the *homoclinic group of α*. Following [**8**] we call an α-homoclinic point $x \in X$ *fundamental* if the homoclinic group $\Delta_\alpha(X)$ is generated by the orbit $\{\alpha^\gamma x : \gamma \in \Gamma\}$ of x.

PROPOSITION 4.2. *Let Γ be a countable group and $f \in \mathbb{Z}\Gamma$ an element which is invertible in $L^1(\Gamma)$.*

If $w_f^\Delta = f^{-1} \in L^1(\Gamma)$ and $\xi = \eta \circ \rho_{w_f^\Delta} : L^\infty(\Gamma, \mathbb{Z}) \longrightarrow X_f$ (cf. (2.5) and (3.5)–(3.7)), then ξ is a surjective group homomorphism with the following properties:

(1) $\ker(\xi) = \rho_f(L^\infty(\Gamma, \mathbb{Z}))$;

(2) $\xi \circ \lambda^\gamma = \alpha_f^\gamma \circ \xi$ *for every* $\gamma \in \Gamma$;

(3) ξ *is continuous in the weak*-topology on closed, bounded subsets of* $L^\infty(\Gamma, \mathbb{Z})$.

Proof. We set $\tilde{w} = (f^*)^{-1} = (f^{-1})^*$. By definition, $\rho_f \tilde{w} = \tilde{w} \cdot f^* = e(1)$ and hence $\tilde{w} \in W_f$ and $x_f^\Delta = \eta(\tilde{w}) \in X_f$ (cf. (3.9)). Since $\tilde{w} \in L^1(\Gamma)$, $x_f^\Delta \in \Delta_{\alpha_f}(X_f)$.

The λ-invariance of W_f implies that $\rho_{w_f^\Delta} h = \lambda_h \tilde{w} \in W_f$ for every $h \in \mathbb{Z}\Gamma$. Since W_f is weak*-closed and $\rho_{w_f^\Delta}$ is weak*-continuous on bounded subsets of $L^\infty(\Gamma, \mathbb{Z})$, it follows that

$$\rho_{w_f^\Delta}(L^\infty(\Gamma, \mathbb{Z})) \subset W_f.$$

In order to prove that $\rho_{w_f^\Delta}(L^\infty(\Gamma, \mathbb{Z})) = W_f$ we fix $w \in W_f$, set $v = \rho_f w \in L^\infty(\Gamma, \mathbb{Z})$, and obtain that $w = \rho_{w_f^\Delta} v$. The group homomorphism

$$\xi = \eta \circ \rho_{w_f^\Delta} : L^\infty(\Gamma, \mathbb{Z}) \longrightarrow X_f \tag{4.1}$$

is thus surjective, and the equivariance of ξ is obvious.

If $B \subset L^\infty(\Gamma, \mathbb{Z})$ is a closed, bounded subset, then the weak*-topology coincides with the topology of coordinate-wise convergence, and ξ is obviously continuous in that topology. □

THEOREM 4.3. *Let Γ be a countable residually finite discrete group and $f \in \mathbb{Z}\Gamma$. If the algebraic Γ-action α_f on the compact abelian group X_f in (3.5) and (3.6) is expansive, and if $w_f^\Delta = f^{-1} \in L^1(\Gamma)$ is the inverse of f in $L^1(\Gamma)$ described in Theorem 3.2, then $x_f^\Delta = \eta((w_f^\Delta)^*) \in X_f$ is a fundamental homoclinic point of α_f.*

Proof. Since $\tilde{w} = (w_f^\Delta)^* \in L^1(\Gamma)$, $x_f^\Delta \in \Delta_{\alpha_f}(X_f)$. If $x \in \Delta_{\alpha_f}(X_f)$ is an arbitrary α_f-homoclinic point, then we can find a $w \in \eta^{-1}(\{x\})$ with $\lim_{\gamma\to\infty} w_\gamma = 0$ and hence with $v = \rho_f(w) \in L^1(\Gamma, \mathbb{Z}) = L^1(\Gamma) \cap L^\infty(\Gamma, \mathbb{Z})$. The point $w = \rho_{w_f^\Delta} v$ lies in the group generated by the λ-orbit $\{\lambda^\gamma \tilde{w} : \gamma \in \Gamma\}$ of $\tilde{w}$, and by applying η we see that x lies in the group generated by the α_f-orbit of x_f^Δ. This proves that x_f^Δ is fundamental. □

THEOREM 4.4. (Specification theorem) *Let Γ be a countable residually finite group and $f \in \mathbb{Z}\Gamma$ an element such that the algebraic Γ-action α_f on the compact abelian group X_f*

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in (3.5) and (3.6) is expansive. Then there exists, for every $\varepsilon > 0$, a finite subset $F_\varepsilon \subset \Gamma$ with the following property: if C_1, C_2 are subsets of Γ with $F_\varepsilon C_1 \cap F_\varepsilon C_2 = \varnothing$, then we can find, for every pair of points $x^{(1)}, x^{(2)} \in X_f$, a point $y \in X_f$ with $d(x_\gamma^{(i)}, y_\gamma) < \varepsilon$ for every $\gamma \in C_i, i = 1, 2$ (cf. (3.10)).

For the proof of Theorem 4.4 we need an elementary lemma.

LEMMA 4.5. *For every $x \in X_f$ there exists an element $v \in L^\infty(\Gamma, \mathbb{Z})$ with $\xi(v) = x$ and $\|v\|_\infty \leq \|f\|_1/2$.*

Proof. Choose $w \in W_f = \eta^{-1}(X_f) \subset L^\infty(\Gamma)$ with $\eta(w) = x$ and $-1/2 \leq w_\gamma < 1/2$ for every $\gamma \in \Gamma$. Then $v = \rho_f w \in L^\infty(\Gamma, \mathbb{Z})$, $\|v\|_\infty \leq \|f\|_1/2$ and $\xi(v) = x$ by Proposition 4.2. □

Proof of Theorem 4.4. Consider the point $w_f^\Delta = f^{-1} \in L^1(\Gamma)$ described in Theorem 3.2. We can find, for every $\varepsilon > 0$, a finite set $F_\varepsilon \subset \Gamma$ with $\sum_{\gamma \in \Gamma \setminus F_\varepsilon} |(w_f^\Delta)_\gamma| < \varepsilon / \|f\|_1$. Put

$$\tilde{w}_\gamma = \begin{cases} (w_f^\Delta)_\gamma & \text{if } \gamma \in F_\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and note that $|(\rho_{w_f^\Delta} y)_\gamma - (\rho_{\tilde{w}} y)_\gamma| < \varepsilon/2$ for every $\gamma \in \Gamma$ and every $y \in L^\infty(\Gamma, \mathbb{Z})$ with $\|y\|_\infty \leq \|f\|_1/2$.

Lemma 4.5 allows us to find points $v^{(i)} \in L^\infty(\Gamma, \mathbb{Z})$ such that $\xi(v^{(i)}) = x^{(i)}$ and $\|v^{(i)}\| \leq \|f\|_1/2$ for $i = 1, 2$. Let $v \in L^\infty(\Gamma, \mathbb{Z})$ be a point with $\|v\|_\infty \leq \|f\|_1/2$ and $v_\gamma = v_\gamma^{(i)}$ for $\gamma \in C_i F_\varepsilon^{-1}, i = 1, 2$, where $F_\varepsilon^{-1} = \{\gamma^{-1} : \gamma \in F_\varepsilon\}$. Then $d(x_\gamma^{(i)}, \eta(\rho_{\tilde{w}} v^{(i)})_\gamma) < \varepsilon/2$ for every $\gamma \in \Gamma$ and $(\rho_{\tilde{w}} v^{(i)})_\gamma = (\rho_{\tilde{w}} v)_\gamma$ for every $\gamma \in C_i, i = 1, 2$. By setting $y = \xi(v)$ we obtain that $d(x_\gamma^{(i)}, y_\gamma) < \varepsilon$ for every $\gamma \in C_i, i = 1, 2$. This proves our claim. □

THEOREM 4.6. *Let Γ be a countable residually finite group and $f \in \mathbb{Z}\Gamma$ an element such that the algebraic Γ-action α_f on the compact abelian group X_f in (3.5) and (3.6) is expansive. Then the homoclinic group $\Delta_{\alpha_f}(X_f)$ is countable and dense in X_f.*

Proof. The countability of $\Delta_{\alpha_f}(X_f)$ is proved exactly as in [**8**, Lemma 3.2]. In order to verify that $\Delta_{\alpha_f}(X_f)$ is dense in X_f we consider the continuous surjective group homomorphism $\xi : L^\infty(\Gamma, \mathbb{Z}) \longrightarrow X_f$ in Proposition 4.2. We set $B = \{v \in L^\infty(\Gamma, \mathbb{Z}) : \|v\|_\infty \leq \|f\|_1/2\}$ and note that $\xi(B) = X_f$ by Lemma 4.5, and that the restriction of ξ to B is continuous in the weak*-topology. Since $B \cap L^1(\Gamma, \mathbb{Z})$ is countable and weak*-dense in B, the countable set $\xi(B \cap L^1(\Gamma, \mathbb{Z})) \subset \Delta_{\alpha_f}(X_f)$ is dense in X_f. □

We end this section with a lemma which implies that the point $w_f^\Delta \in L^1(\Gamma)$ appearing in the Theorems 4.3 and 4.4 decays rapidly.

PROPOSITION 4.7. *Let Γ be a countable discrete group and $f \in \mathbb{Z}\Gamma$. If α_f is expansive there exists, for every $L > 0$ and $\varepsilon > 0$, a finite subset $F(L, \varepsilon) \subset \Gamma$ with the following property: if $w \in W_f$ satisfies the condition that $\|w\|_\infty \leq L$ and $(\rho_f w)_\gamma = 0$ for every $\gamma \in F(L, \varepsilon)$, then $|w_1| < \varepsilon$. If the group Γ is residually finite, finitely generated and has polynomial growth, then the point w_f^Δ has exponential decay in the word metric on Γ.*

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Proof. We argue by contradiction and assume that there exist an $\varepsilon > 0$, an increasing sequence (F_n) of finite subsets in Γ with $\bigcup_{n\geq 1} F_n = \Gamma$ and a sequence $(w^{(n)}, n \geq 1)$ in W_f such that, for every $n \geq 1$, $\|w^{(n)}\|_\infty \leq L$, $|w_1^{(n)}| > \varepsilon$ and $\rho_f w_\gamma^{(n)} = 0$ for every $\gamma \in F_n$. If $w^{(0)}$ is the limit of a weak*-convergent subsequence of $(w^{(n)})$, then $|w_1^{(0)}| \geq \varepsilon$ and $w^{(0)} \in \ker(\rho_f)$, contrary to our hypothesis that α_f is expansive and $\ker(\rho_f)$ is therefore equal to zero.

Now suppose that Γ is residually finite, finitely generated and has polynomial growth. We choose and fix a finite symmetric set of generators F of Γ and write δ_F for the word metric on Γ. For every $\gamma \in \Gamma$ we denote by $\ell_F(\gamma)$ the length of the shortest expression of $\gamma \in \Gamma$ in terms of elements of F, where $\ell_F(1) = 0$. Since Γ has polynomial growth there exist constants c, $M \geq 1$ such that the set

$$B_F(n) = \{\gamma \in \Gamma : \ell_F(\gamma) \leq n\} \tag{4.2}$$

has cardinality at most cM^n for every $n \geq 0$.

The point $\tilde{w} = (w_f^\Delta)^* \in L^1(\Gamma)$ satisfies the fact that $\rho_f \tilde{\omega} = e(1)$. We set $L = \|\tilde{w}\|_\infty$ and use the first part of this proposition to find an $R \geq 1$ such that $|w_1| < L/2$ for every $w \in W_f$ with $\|w\|_\infty \leq L$ and $\rho_f w \equiv 0$ on $B_F(R)$. Our choice of R guarantees that $|\tilde{w}_\gamma| < L/2$ for every $\gamma \in B_F(2R) \setminus B_F(R)$ and, by induction, that $|\tilde{w}_\gamma| < L/2^k$ for every $k \geq 1$ and every $\gamma \in B_F(kR) \setminus B_F((k-1)R)$.

This proves exponential decay of the coordinates of $\tilde{w}$ (and hence of w_f^Δ) in the word metric on Γ. □

5. *Entropy and periodic points*

Let Γ be a countable discrete group and $K \subset \Gamma$ a finite set. A finite set $Q \subset \Gamma$ is *left* (K, ε)-*invariant* if

$$\sum_{\gamma \in K} |\gamma Q \triangle Q|/|Q| < \varepsilon,$$

and *right* (K, ε)-*invariant* if

$$\sum_{\gamma \in K} |Q\gamma \triangle Q|/|Q| < \varepsilon.$$

If Q satisfies both these conditions it is (K, ε)-*invariant*.

A sequence $(Q_n, n \geq 1)$ of finite subsets of Γ is a *left Følner sequence* if there exists, for every finite subset $K \subset \Gamma$ and every $\varepsilon > 0$, an $N \geq 1$ such that Q_n is left (K, ε)-invariant for every $n \geq N$. The definitions of *right* and *two-sided Følner sequences* are analogous. The group Γ is *amenable* if it has a left Følner sequence. If Γ is amenable it also has right and two-sided Følner sequences.

Let Γ be a countable residually finite discrete group. If $(\Gamma_n, n \geq 1)$ is a sequence of finite index normal subgroups in Γ we say that

$$\lim_{n\to\infty} \Gamma_n = \{1\} \tag{5.1}$$

if we can find, for every finite set $K \subset \Gamma$, an $N \geq 1$ with $\Gamma_n \cap (K^{-1}K) = \{1\}$ for every $n \geq N$. Clearly, such sequences exist.

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THEOREM 5.1. *Let Γ be a countable residually finite amenable group, $f \in \mathbb{Z}\Gamma$, and let α_f be the Γ-action on X_f defined in (3.5) and (3.6). If $X_f \neq 0$ and α_f is expansive then $h(\alpha_f) > 0$.*

Proof. Let $(\Gamma_n, n \geq 1)$ be a decreasing sequence of finite index normal subgroups of Γ with $\bigcap_{n\geq 1} \Gamma_n = \{1\}$.

For notational simplicity set $\tilde{w} = (w_f^{\Delta})^*$ (cf. Theorem 4.3). The homoclinic group $\Delta_{\alpha_f}(X_f)$ is dense in X_f by Theorem 4.6 and, since $X_f \neq 0$ by assumption, the fundamental homoclinic point $\tilde{x} = x_f^{\Delta} = \eta(\tilde{w})$ is non-zero. Hence there exist a $\gamma_0 \in \Gamma$ and a finite subset $F \subset \Gamma$ with $\tilde{w}_{\gamma_0} \notin \mathbb{Z}$ and $\sum_{\gamma\in\Gamma\setminus F} |\tilde{w}_\gamma| < d(\tilde{x}_{\gamma_0}, 0)/2$, where $d(\cdot,\cdot)$ is the metric on $\mathbb{T}$ defined in (3.10). We choose n sufficiently large so that the sets γF, $\gamma \in \Gamma_n$, are all disjoint. For every $\omega \in \Omega = \{0,1\}^{\Gamma_n}$ we define a point $x^{(\omega)} \in X_f$ by setting $w^{(\omega)} = \sum_{\gamma\in\Gamma_n} \omega_\gamma \lambda^\gamma \tilde{w}$ and $x^{(\omega)} = \eta(w^{(\omega)}) = \sum_{\gamma\in\Gamma_n} \omega_\gamma \lambda^\gamma \tilde{x}$.

We fix $\gamma \in \Gamma_n$ for the moment and consider two points $\omega, \omega' \in \Omega$ with $\omega_\gamma \neq \omega'_\gamma$. Then $d(x^{(\omega)}_{\gamma\gamma_0}, x^{(\omega')}_{\gamma\gamma_0}) > d(\tilde{x}_{\gamma_0}, 0)/2$.

If $(Q_m, m \geq 1)$ is a left Følner sequence in Γ then

$$\lim_{m\to\infty} \frac{|Q_m \cap \Gamma_n\gamma_0|}{|Q_m|} = |\Gamma/\Gamma_n|^{-1}.$$

We conclude that the set $\{x^{(\omega)} : \omega \in \Omega\}$ contains, for every $m \geq 1$, a $(Q_m, d(\tilde{x}_{\gamma_0}, 0)/2)$-separated set of cardinality $2^{|Q_m\cap\Gamma_n\gamma_0|}$, and hence that $h(\alpha_f) \geq |\Gamma/\Gamma_n|^{-1} \cdot \log 2 > 0$. □

In order to find out more about the actual value of $h(\alpha_f)$ we take a look at the periodic points of α_f. Fix a subgroup $\Gamma' \subset \Gamma$ and denote by

$$\mathrm{Fix}_{\Gamma'}(X_f) = \{x \in X_f : \alpha_f^\gamma x = x \text{ for every } \gamma \in \Gamma'\} \tag{5.2}$$

the subgroup of Γ'-*invariant points* in X_f. Clearly, $\mathrm{Fix}_{\Gamma'}(X_f)$ is Γ'-invariant, and $\mathrm{Fix}_{\Gamma'}(X_f)$ is Γ-invariant if and only if Γ' is a normal subgroup of Γ. Next we set

$$\begin{aligned} L^\infty(\Gamma)^{\Gamma'} &= \{w \in L^\infty(\Gamma) : \lambda^\gamma w = w \text{ for every } \gamma \in \Gamma'\}, \\ W_f^{\Gamma'} &= W_f \cap L^\infty(\Gamma)^{\Gamma'}, \\ L^\infty(\Gamma,\mathbb{Z})^{\Gamma'} &= L^\infty(\Gamma,\mathbb{Z}) \cap L^\infty(\Gamma)^{\Gamma'}. \end{aligned} \tag{5.3}$$

We write $\xi : L^\infty(\Gamma,\mathbb{Z}) \longrightarrow X_f$ for the group homomorphism described in Proposition 4.2.

PROPOSITION 5.2. *Let Γ be a countable residually finite group, $f \in \mathbb{Z}\Gamma$, and let α_f be the Γ-action on X_f defined in (3.5) and (3.6). For every subgroup $\Gamma' \subset \Gamma$ of finite index,*

$$\mathrm{Fix}_{\Gamma'}(X_f) = \xi(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}) \cong L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}/\rho_f(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}). \tag{5.4}$$

Proof. From the equivariance of ξ it is clear that $\xi(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}) \subset \mathrm{Fix}_{\Gamma'}(X_f)$. Conversely, if $x \in \mathrm{Fix}_{\Gamma'}(X_f)$, then we can find $w \in W_f^{\Gamma'} \subset L^\infty(\Gamma)^{\Gamma'}$ with $\eta(w) = x$ (cf. (3.7)), and the point $v = \rho_f w \in L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}$ satisfies the fact that $\xi(v) = x$. This proves that $\xi(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}) = \mathrm{Fix}_{\Gamma'}(X_f)$, and Proposition 4.2 guarantees that $\ker(\xi) \cap L^\infty(\Gamma,\mathbb{Z})^{\Gamma'} = \rho_f(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'})$. It follows that $\mathrm{Fix}_{\Gamma'}(X_f) = \xi(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}) \cong L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}/\rho_f(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'})$, as claimed. □

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COROLLARY 5.3. *If* $|\Gamma'\backslash\Gamma| < \infty$ *then* $|\mathrm{Fix}_{\Gamma'}(X_f)| = |\det(\rho_f|_{L^\infty(\Gamma)^{\Gamma'}})|$.

Proof. If the coset space $\Gamma'\backslash\Gamma$ is finite then $L^\infty(\Gamma)^{\Gamma'} \cong \mathbb{R}^{\Gamma'\backslash\Gamma}$ and we denote by $\rho_f|_{L^\infty(\Gamma)^{\Gamma'}}$ the restriction to $L^\infty(\Gamma)^{\Gamma'} \cong \mathbb{Z}^{\Gamma'\backslash\Gamma}$ of the linear map $\rho_f\colon L^\infty(\Gamma) \longrightarrow L^\infty(\Gamma)$. Then $\rho_f(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}) \subset L^\infty(\Gamma,\mathbb{Z})^{\Gamma'}$ and the absolute value of the determinant $\det(\rho_f|_{L^\infty(\Gamma)^{\Gamma'}})$ is equal to $|L^\infty(X,\mathbb{Z})^{\Gamma'}/\rho_f(L^\infty(\Gamma,\mathbb{Z})^{\Gamma'})| = |X_f^{\Gamma'}|$. □

Remark 5.4. Since α_f is expansive, $\ker(\rho_f) = 0$ by Theorem 3.2, and the proof of Theorem 3.2 shows that there exists, for every pair of distinct points $x, x' \in X_f$, a $\gamma \in \Gamma$ with $d(x_\gamma, x'_\gamma) \geq (3\|f\|_1)^{-1}$ (cf. (3.10)).

If $\Gamma' \subset \Gamma$ is a subgroup with finite index, and if $Q \subset \Gamma$ is a *fundamental domain* of the right coset space $\Gamma'\backslash\Gamma$, i.e. a finite subset such that $\{\gamma Q : \gamma \in \Gamma'\}$ is a partition of Γ, then the preceding paragraph implies that $\mathrm{Fix}_{\Gamma'}(X_f)$ is $(Q, (3\|f\|_1)^{-1})$-*separated* in the sense that there exists, for any two distinct points $x, x' \in X_f^{\Gamma'}$, a $\gamma \in Q$ with $d(x_\gamma, x'_\gamma) \geq (3\|f\|_1)^{-1}$.

For the terminology used in our next proposition we refer to the beginning of this section and to Remark 5.4.

PROPOSITION 5.5. [**13**] *Let* Γ *be a countable residually finite amenable group and let* $(\Gamma_n, n \geq 1)$ *be a sequence of finite index normal subgroups with* $\lim_{n\to\infty}\Gamma_n = \{1\}$ *(cf. (5.1)). Then there exists, for every finite subset* $K \subset \Gamma$ *and every* $\bar\varepsilon > 0$, *an integer* $M = M(K,\bar\varepsilon) \geq 1$ *such that every* Γ_n *with* $n \geq M$ *has a* $(K,\bar\varepsilon)$*-invariant fundamental domain* Q_n *of the coset space* Γ/Γ_n.

Proof. This is a slight reformulation of [**13**, Theorem 1] requiring only very minor changes in the proof. We describe these changes briefly, using essentially the same notation and terminology as in [**13**].

Fix a finite set $K \subset \Gamma$ and $\bar\varepsilon > 0$ and let ε and η be sufficiently small positive numbers whose sizes will be clear by examining the course of the proof. Choose $N \geq 1$ with

$$\left(1 - \frac{\varepsilon}{2}\right)^N < \frac{\bar\varepsilon}{200}$$

and use the amenability of Γ to find an increasing sequence $(F_j, j \geq 1)$ such that:

(i) F_1 is (K,ε)-invariant and $1 \in F_1$;

(ii) for $1 \leq j < N$, $F_{j+1} \supset F_j$ and F_{j+1} is $(F_j F_j^{-1}, \eta)$-invariant.

Since $\bigcap_{n\geq 1}\Gamma_n = \{1\}$ we can find an $M \geq 1$ with $\Gamma_M \cap F_N^{-1}F_N = \{1\}$. We fix $n \geq M$ and write $\theta : \Gamma \longrightarrow \Gamma/\Gamma_n = G$ for the quotient map which is injective on F_N.

The argument in the proof of [**13**, Theorem 1] allows us to find—for appropriately chosen ε, η—finite subsets $\{\gamma_{i,j} : i = 1,\dots,m_j, j = 1,\dots,N\}$ and $\{\gamma'_{i,j} : i = 1,\dots,m'_j, j = 1,\dots,N\}$ of Γ and, for each $j = 1,\dots,N$, sets $F_{i,j} \subset F_j, i = 1,\dots,m_j$, $F'_{i,j} \subset F_j, i = 1,\dots,m'_j$, such that the following conditions are satisfied:

(1) for every i, j, $|F_{i,j}| \geq (1-\varepsilon)|F_j|$ and $|F'_{i,j}| \geq (1-\varepsilon)|F_j|$;

(2) for every i, j, the sets $F_{i,j}$ and $F'_{i,j}$ are $(K, \bar\varepsilon/2)$-invariant;

(3) the sets $\theta(F_{i,j}\gamma_{i,j})$, $i = 1,\dots,m_j$, $j = 1,\dots,N$, are disjoint;

(4) the sets $\theta(\gamma'_{i,j}F'_{i,j})$, $i = 1,\dots,m'_j$, $j = 1,\dots,N$, are disjoint;

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(5) the sets

$$E = \bigcup_{j=1}^{N} \bigcup_{i=1}^{m_j} \theta(F_{i,j}\gamma_{i,j}), \quad E' = \bigcup_{j=1}^{N} \bigcup_{i=1}^{m'_j} \theta(\gamma'_{i,j}F'_{i,j})$$

satisfy

$$|E| \geq \left(1 - \frac{\bar{\varepsilon}}{100}\right)|G|, \quad |E'| \geq \left(1 - \frac{\bar{\varepsilon}}{100}\right)|G|.$$

We set $F = E \cap E'$ and choose a set $Q \supset F$ such that $|Q| = |G|$ and $\theta(Q) = G$. Then Q is a $(K, \bar{\varepsilon})$-invariant fundamental domain of the coset space Γ/Γ_n. □

COROLLARY 5.6. *Let Γ be a countable residually finite amenable group and let $(\Gamma_n, n \geq 1)$ be a sequence of finite index normal subgroups with $\lim_{n\to\infty} \Gamma_n = \{1\}$. Then there exists a Følner sequence $(Q_n, n \geq 1)$ such that Q_n is a fundamental domain of Γ/Γ_n for every $n \geq 1$.*

THEOREM 5.7. *Let Γ be a countable residually finite amenable group and let $(\Gamma_n, n \geq 1)$ be a sequence of finite index normal subgroups with $\lim_{n\to\infty} \Gamma_n = \{1\}$.*

If $f \in \mathbb{Z}\Gamma$, and if the algebraic Γ-action α_f on X_f in (3.5) and (3.6) is expansive, then

$$\begin{aligned} h(\alpha_f) &= \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| \\ &= \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |L^\infty(\Gamma, \mathbb{Z})^{\Gamma_n}/\rho_f(L^\infty(\Gamma, \mathbb{Z})^{\Gamma_n})| \\ &= \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |\det(\rho_f|_{L^\infty(\Gamma)^{\Gamma_n}})|, \end{aligned} \tag{5.5}$$

where $\mathrm{Fix}_{\Gamma_n}(X_f) \subset X_f$ is the subgroup of Γ_n-periodic points in (5.2) and $L^\infty(\Gamma)^{\Gamma_n} \subset L^\infty(\Gamma)$ is defined in (5.3) (cf. Corollary 5.3).

Proof. We choose a Følner sequence $(Q_n, n \geq 1)$ in Γ such that Q_n is a fundamental domain of Γ/Γ_n for every $n \geq 1$ (cf. Corollary 5.6). Proposition 5.2 and Corollary 5.3 show that there exists, for every $n \geq 1$, a $(Q_n, (3\|f\|_1)^{-1})$-separated set (in the sense of Remark 5.4) of cardinality

$$|\mathrm{Fix}_{\Gamma_n}(X_f)| = |L^\infty(\Gamma, \mathbb{Z})^{\Gamma_n}/\rho_f(L^\infty(\Gamma, \mathbb{Z})^{\Gamma_n})| = |\det(\rho_f|_{L^\infty(\Gamma)^{\Gamma_n}})|.$$

Since $(Q_n, n \geq 1)$ is Følner and $|Q_n| = |\Gamma/\Gamma_n|$ this implies that

$$h(\alpha_f) \geq \limsup_{n\to\infty} \frac{1}{|Q_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)|.$$

Conversely, let $\delta > 0$, $\varepsilon < \delta/3$, and let F_ε be a finite symmetric set with $\sum_{\gamma\in\Gamma\setminus F_\varepsilon} |w_\gamma^\Delta| < \varepsilon/\|f\|_1$ (cf. the proof of Theorem 4.4). The sets $P_n = Q_n \cap \bigcap_{\gamma\in F_\varepsilon} Q_n\gamma$, $n \geq 1$, form a Følner sequence with $\lim_{n\to\infty} |P_n|/|Q_n| = 1$.

We fix $n \geq 1$ for the moment and choose a maximal set $S_{n,\delta} \subset X_f$ which is (P_n, δ)-separated in the sense of Remark 5.4. For every $x \in S_{n,\delta}$ we find $w(x) \in W_f \subset L^\infty(\Gamma)$ with $\|w(x)\|_\infty \leq \|f\|_1/2$ and $\eta(w(x)) = x$ (Lemma 4.5) and write $v(x) \in L^\infty(\Gamma, \mathbb{Z})^{\Gamma_n}$ for the unique point with $v(x)_\gamma = (\rho_f w(x))_\gamma$ for every $\gamma \in Q_n$. Our choice of F_ε implies

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that the points $\{\xi(v(x)) : x \in S_{n,\delta}\} \subset \mathrm{Fix}_{\Gamma_n}(X_f)$ are $(P_n, \delta/3)$-separated and therefore distinct, and Proposition 5.2 shows that $|S_{n,\delta}| \le |\mathrm{Fix}_{\Gamma_n}(X_f)|$.

Since $(P_n, n \ge 1)$ is Følner and $\lim_{n\to\infty} |P_n|/|Q_n| = 1$ this implies that

$$h(\alpha_f) = \lim_{n\to\infty} \frac{1}{|P_n|} \log |S_{n,\delta}| \le \liminf_{n\to\infty} \frac{1}{|Q_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)|,$$

which completes the proof of (5.5). □

COROLLARY 5.8. *Let Γ be a countable residually finite amenable group, and let $f, g \in \mathbb{Z}\Gamma$ be elements such that the algebraic Γ-actions α_f and α_g on X_f and X_g in (3.5) and (3.6) are expansive. If α_{f^*} and $\alpha_{f\cdot g}$ are given as in (3.5) and (3.6) with f replaced by f^* and $f \cdot g$, respectively, then α_{f^*} and $\alpha_{f\cdot g}$ are expansive, $h(\alpha_{f^*}) = h(\alpha_f)$ and $h(\alpha_{f\cdot g}) = h(\alpha_f) + h(\alpha_g)$.*

Proof. The expansiveness of α_{f^*} and $\alpha_{f\cdot g}$ is clear from Theorem 3.2, and the entropy formulae follow from the usual properties of determinants (cf. (5.5)). □

6. *Entropy and Fuglede–Kadison determinants*

In this section, $L^p(\Gamma, \mathbb{C})$ will denote the *complex* L^p-space of Γ for $1 \le p \le \infty$ with its conjugate linear involution $w \mapsto w^*$ given by $(w^*)_\gamma = \bar{w}_{\gamma^{-1}}$, $\gamma \in \Gamma$.

The von Neumann algebra $\mathcal{N}\Gamma$ of a discrete group Γ can be defined as the algebra of left Γ-equivariant bounded operators of $L^2(\Gamma, \mathbb{C})$ to itself. Thus a bounded operator A on $L^2(\Gamma, \mathbb{C})$ belongs to $\mathcal{N}\Gamma$ if and only if we have

$$\lambda^\gamma A(v) = A(\lambda^\gamma(v))$$

for all v in $L^2(\Gamma, \mathbb{C})$ and all γ in Γ.

The homomorphism of $\mathbb{C}$-algebras with involution

$$\rho : L^1(\Gamma, \mathbb{C}) \longrightarrow \mathcal{N}\Gamma$$

mapping f to the operator ρ_f with $\rho_f(v) = v \cdot f^*$ is injective because $\rho_f(e(1)) = f^*$. In the following we will sometimes view ρ as an inclusion both of $L^1(\Gamma, \mathbb{C})$ and of $\mathbb{C}\Gamma$ into $\mathcal{N}\Gamma$ and omit it from the notation.

The von Neumann trace on $\mathcal{N}\Gamma$ is the linear form

$$\mathrm{tr}_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \longrightarrow \mathbb{C}$$

mapping A to $\mathrm{tr}_{\mathcal{N}\Gamma}(A) = \langle A(e(1)), e(1)\rangle$. The trace is faithful in the sense that $\mathrm{tr}_{\mathcal{N}\Gamma} A = 0$ for a positive operator A in $\mathcal{N}\Gamma$ implies that $A = 0$. Moreover $\mathrm{tr}_{\mathcal{N}\Gamma}$ vanishes on commutators and satisfies the estimate $|\mathrm{tr}_{\mathcal{N}\Gamma} A| \le \|A\|$. On $L^1(\Gamma, \mathbb{C})$ it is given by $\mathrm{tr}_{\mathcal{N}\Gamma}(w) = w(1)$. All this is easy to check.

The Fuglede–Kadison determinant of A in $(\mathcal{N}\Gamma)^*$ is the positive real number

$$\det_{\mathcal{N}\Gamma} A = \exp(\tfrac{1}{2}\mathrm{tr}_{\mathcal{N}\Gamma}(\log AA^*)). \tag{6.1}$$

Note here that the operator AA^* is invertible and positive so that $\log AA^*$ is defined by the functional calculus. If E_λ is a spectral resolution for AA^* we can also write

$$\det_{\mathcal{N}\Gamma} A = \exp\left(\frac{1}{2}\int_0^\infty \log\lambda \, d\,\mathrm{tr}_{\mathcal{N}\Gamma}(E_\lambda)\right). \tag{6.2}$$

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If the group Γ is finite we have $\mathcal{N}\Gamma = \mathbb{C}\Gamma$ and

$$\det_{\mathcal{N}\Gamma} A = |\det A|^{1/|\Gamma|}. \tag{6.3}$$

We refer to [**10**, Example 3.13] or [**3**, Proposition 3.1] for a discussion of the case $\Gamma = \mathbb{Z}^n$, or see below.

For positive A in $(\mathcal{N}\Gamma)^*$ we have

$$\det_{\mathcal{N}\Gamma} A = \exp \operatorname{tr}_{\mathcal{N}\Gamma}(\log A).$$

For operators $0 \leq A \leq B$ in $(\mathcal{N}\Gamma)^*$ this implies the inequality

$$\det_{\mathcal{N}\Gamma} A \leq \det_{\mathcal{N}\Gamma} B. \tag{6.4}$$

It is a non-trivial fact that $\det_{\mathcal{N}\Gamma}$ defines a *homomorphism*

$$\det_{\mathcal{N}\Gamma} : (\mathcal{N}\Gamma)^* \longrightarrow \mathbb{R}_+^*.$$

The intuitive reason for this is the Campbell–Hausdorff formula. The actual proof in [**6**] for II_1-factors and the analogous argument for group von Neumann algebras in [**10**, §3.2] is different in order to avoid convergence problems.

The following remark on the disintegration of $\det_{\mathcal{N}\Gamma}$ serves only to give some further background on this determinant. The required theory can be found in [**4**].

Let $Z(\Gamma)$ be the centre of Γ. For a character χ in $\widehat{Z(\Gamma)}$ consider the Hilbert space

$$L^2(\Gamma, \mathbb{C})_\chi = \{h \text{ in } L^2(\Gamma, \mathbb{C}) : \rho^\gamma h = \chi(\gamma)h \text{ for all } \gamma \text{ in } Z(\Gamma)\}.$$

By the spectral theorem applied to the commuting unitary operators ρ^γ with γ in $Z(\Gamma)$ we have a direct integral decomposition

$$L^2(\Gamma, \mathbb{C}) = \int_{\widehat{Z(\Gamma)}} L^2(\Gamma, \mathbb{C})_\chi \, d\mu(\chi)$$

where μ is the Haar probability measure on the compact abelian group $\widehat{Z(\Gamma)}$. Correspondingly, we obtain a disintegration

$$\mathcal{N}\Gamma = \int_{\widehat{Z(\Gamma)}} \mathcal{N}_\chi \Gamma \, d\mu(\chi)$$

where $\mathcal{N}_\chi \Gamma$ is the von Neumann algebra of bounded operators on $L^2(\Gamma, \mathbb{C})_\chi$ which commute with the left Γ-action λ.

The Fuglede–Kadison determinant $\det_{\mathcal{N}_\chi \Gamma}$ is defined similarly as above and we obtain the formula

$$\log \det_{\mathcal{N}\Gamma} A = \int_{\widehat{Z(\Gamma)}} \log \det_{\mathcal{N}_\chi \Gamma} A_\chi \, d\mu(\chi).$$

Here A_χ is the restriction of A to an invertible operator from $L^2(\Gamma, \mathbb{C})_\chi$ to itself. If Γ is commutative we find that

$$\log \det_{\mathcal{N}\Gamma} A = \int_{\hat{\Gamma}} \log |a(\chi)| \, d\mu(\chi),$$

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if A_χ acts on the one-dimensional space $L^2(\Gamma,\mathbb{C})_\chi$ by multiplication with $a(\chi)\in\mathbb{C}^*$. For f in $\mathbb{C}\Gamma$ in particular we obtain

$$\log\det_{\mathcal{N}\Gamma} f=\int_{\hat\Gamma}\log|\hat f(\chi)|\,d\mu(\chi) \tag{6.5}$$

where $\hat f$ is the Fourier transform of f.

We now begin with some preparations for the proof of the entropy formula in Theorem 1.1. Let Γ be a countable residually finite discrete group and let $(\Gamma_n, n\geq 1)$ be a sequence of finite index normal subgroups with $\lim_{n\to\infty}\Gamma_n=\{1\}$. Set $\Gamma^{(n)}=\Gamma_n\backslash\Gamma$. For f in $L^1(\Gamma,\mathbb{C})$ the bounded operator $\rho_f : L^2(\Gamma,\mathbb{C})\to L^2(\Gamma,\mathbb{C})$ given by right convolution with f^* satisfies the norm estimate

$$\|\rho_f\|\leq\|f\|_1. \tag{6.6}$$

The group Γ acts via λ on $L^\infty(\Gamma,\mathbb{C})$ and we have an isomorphism of finite-dimensional $\mathbb{C}$-vector spaces

$$L^\infty(\Gamma,\mathbb{C})^{\Gamma_n}\cong L^\infty(\Gamma^{(n)},\mathbb{C}) \tag{6.7}$$

given by viewing left Γ_n-invariant functions on Γ as functions on $\Gamma_n\backslash\Gamma$. Since ρ_f is left Γ_n-equivariant it induces an endomorphism of $L^\infty(\Gamma,\mathbb{C})^{\Gamma_n}$ and hence an endomorphism of $L^\infty(\Gamma^{(n)},\mathbb{C})=\mathbb{C}\Gamma^{(n)}=L^2(\Gamma^{(n)},\mathbb{C})$ which we denote by

$$\rho_f^{(n)} : L^2(\Gamma^{(n)},\mathbb{C})\longrightarrow L^2(\Gamma^{(n)},\mathbb{C}). \tag{6.8}$$

Consider the map 'integration along the fibres':

$$L^1(\Gamma,\mathbb{C})\longrightarrow L^1(\Gamma^{(n)},\mathbb{C}) \tag{6.9}$$

given by sending $f:\Gamma\to\mathbb{C}$ to the function $f^{(n)}:\Gamma^{(n)}\to\mathbb{C}$ defined by

$$f^{(n)}(\delta)=\sum_{\gamma\in\delta}f(\gamma),$$

for all residue classes δ in $\Gamma^{(n)}$. A short calculation shows that (6.9) is a homomorphism of $\mathbb{C}$-algebras with involution such that $\|f^{(n)}\|_1\leq\|f\|_1$. Moreover we have

$$\rho_f^{(n)}=\rho_{f^{(n)}} : L^2(\Gamma^{(n)},\mathbb{C})\longrightarrow L^2(\Gamma^{(n)},\mathbb{C}). \tag{6.10}$$

By the estimate (6.6) applied to $f^{(n)}$ and $\Gamma^{(n)}$ we have $\|\rho_{f^{(n)}}\|\leq\|f^{(n)}\|_1$. Using (6.10) we obtain the uniform estimate for all $n\geq 1$,

$$\|\rho_f^{(n)}\|\leq\|f\|_1. \tag{6.11}$$

From the definition of $\rho_f^{(n)}$ the relation $\rho_{fg}^{(n)}=\rho_f^{(n)}\rho_g^{(n)}$ follows. Hence $\rho_f^{(n)}$ is invertible if f is invertible in $L^1(\Gamma,\mathbb{C})$ and we have $(\rho_f^{(n)})^{-1}=\rho_{f^{-1}}^{(n)}$. Together with (6.11) this gives the estimate

$$\|(\rho_f^{(n)})^{-1}\|\leq\|f^{-1}\|_1\quad\text{for } f \text{ in } L^1(\Gamma,\mathbb{C})^* \text{ and all } n\geq 1. \tag{6.12}$$

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Note that by (6.3) we have

$$\log \det_{\mathcal{N}\Gamma^{(n)}} f^{(n)} = \frac{1}{|\Gamma/\Gamma_n|} \log |\det \rho_{f^{(n)}}| = \frac{1}{|\Gamma/\Gamma_n|} \log |\det(\rho_f|_{L^\infty(\Gamma,\mathbb{C})^{\Gamma_n}})|. \tag{6.13}$$

Hence Theorem 1.1 of the introduction follows from Theorem 3.2, Theorem 5.7 and the following result.

THEOREM 6.1. *Let Γ be a countable residually finite discrete group and $(\Gamma_n, n \geq 1)$ a sequence of finite index normal subgroups with $\lim_{n\to\infty} \Gamma_n = \{1\}$. For f in $L^1(\Gamma, \mathbb{C})^*$ we have*

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \det_{\mathcal{N}\Gamma^{(n)}} f^{(n)}.$$

Proof. Because of the relation $(ff^*)^{(n)} = f^{(n)} f^{(n)*}$ the assertion means

$$\mathrm{tr}_{\mathcal{N}\Gamma} \log \rho_g = \lim_{n\to\infty} \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} \log \rho_{g^{(n)}} \tag{6.14}$$

for $g = ff^*$ in $L^1(\Gamma, \mathbb{C})^*$. However, $\rho_g = \rho_f \rho_f^*$ and $\rho_{g^{(n)}} = \rho_{f^{(n)}} \rho_{f^{(n)}}^*$ are positive operators on $L^2(\Gamma, \mathbb{C})$. By the estimate (6.6) applied to g and g^{-1} and the estimates (6.11) and (6.12) applied to g instead of f it follows that the spectra $\sigma(\rho_g)$ and $\sigma(\rho_{g^{(n)}})$ lie in the closed interval $I = [\|g\|_1^{-1}, \|g\|_1]$.

Fix $\varepsilon > 0$. By the Weierstrass approximation theorem there exists a real polynomial Q such that

$$\sup_{t\in I} |\log t - Q(t)| \leq \varepsilon.$$

Since the spectra of ρ_g and $\rho_{g^{(n)}}$ lie in I it follows that we have

$$\|\log \rho_g - Q(\rho_g)\| \leq \varepsilon \quad \text{and} \quad \|\log \rho_{g^{(n)}} - Q(\rho_{g^{(n)}})\| \leq \varepsilon.$$

Using the estimate $|\mathrm{tr}_{\mathcal{N}\Gamma} A| \leq \|A\|$ we find

$$\begin{aligned}
&|\mathrm{tr}_{\mathcal{N}\Gamma} \log \rho_g - \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} \log \rho_{g^{(n)}}| \\
&\quad \leq |\mathrm{tr}_{\mathcal{N}\Gamma}(\log \rho_g - Q(\rho_g))| + |\mathrm{tr}_{\mathcal{N}\Gamma} Q(\rho_g) - \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} Q(\rho_{g^{(n)}})| \\
&\qquad + |\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}(\log \rho_{g^{(n)}} - Q(\rho_{g^{(n)}}))| \\
&\quad \leq \|\log \rho_g - Q(\rho_g)\| + |\mathrm{tr}_{\mathcal{N}\Gamma} Q(\rho_g) - \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} Q(\rho_{g^{(n)}})| + \|\log \rho_{g^{(n)}} - Q(\rho_{g^{(n)}})\| \\
&\quad \leq 2\varepsilon + |\mathrm{tr}_{\mathcal{N}\Gamma} Q(g) - \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} Q(g^{(n)})|.
\end{aligned}$$

Hence formula (6.14) is a consequence of the following Lemma 6.2. □

LEMMA 6.2. *For any f in $L^1(\Gamma, \mathbb{C})$ and any complex polynomial $Q(t)$ we have*

$$\mathrm{tr}_{\mathcal{N}\Gamma} Q(f) = \lim_{n\to\infty} \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} Q(f^{(n)}).$$

Proof. Since $Q(f)$ is in $L^1(\Gamma, \mathbb{C})$ and $Q(f)^{(n)} = Q(f^{(n)})$ it suffices to prove the assertion for $Q(t) = t$, i.e. that

$$\mathrm{tr}_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}$$

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for all f in $L^1(\Gamma,\mathbb{C})$. Writing $f=\sum_\gamma f_\gamma e(\gamma)$, we have $f^{(n)}=\sum_\gamma f_\gamma e(\bar\gamma)$ where $\bar\gamma=\Gamma_n\gamma$. Thus we obtain

$$\mathrm{tr}_{\mathcal{N}\Gamma} f=f_1 \quad\text{and}\quad \mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}=\sum_{\gamma\in\Gamma_n} f_\gamma.$$

Fix some $\varepsilon>0$. Since f is in $L^1(\Gamma,\mathbb{C})$ we have $\sum_{\gamma\in\Gamma}|f_\gamma|<\infty$. Hence there is a finite subset K of Γ with $1\in K$ such that we have $\sum_{\gamma\in\Gamma\setminus K}|f_\gamma|<\varepsilon$. Since $\lim_{n\to\infty}\Gamma_n=\{1\}$ we can find an index $N\geq 1$ such that $\Gamma_n\cap K^{-1}K=\{1\}$ for all $n\geq N$. Since $1\in K$ it follows that $\Gamma_n\cap K=\{1\}$ for all $n\geq N$ as well. For $n\geq N$ we therefore obtain the estimate

$$|\mathrm{tr}_{\mathcal{N}\Gamma} f-\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}|=\left|f_1-\sum_{\gamma\in\Gamma_n} f_\gamma\right|\leq\sum_{\gamma\in\Gamma_n\setminus\{1\}}|f_\gamma|\leq\sum_{\gamma\in\Gamma\setminus K}|f_\gamma|<\varepsilon.$$

Since $\varepsilon>0$ was arbitrary the lemma is proved. □

Remark 6.3. For f in $\mathbb{Z}\Gamma$ the element $f^{(n)}$ lies in $\mathbb{Z}\Gamma^{(n)}$. Hence $\mathrm{tr}_{\mathcal{N}\Gamma} f$ and $\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}$ are integers and it follows that we have $\mathrm{tr}_{\mathcal{N}\Gamma} f=\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}$ for all $n\gg 0$, cf. [**1**, Lemma 2.2] or [**11**, Lemma 5.5].

For an element f in $L^1(\Gamma,\mathbb{C})$ let $\sigma_f=\lim_{n\to\infty}\|f^n\|_1^{1/n}\leq\|f\|_1$ be the spectral radius of f. We have

$$\log\det_{\mathcal{N}\Gamma} f=\tfrac{1}{2}\mathrm{tr}_{\mathcal{N}\Gamma}\log\rho_{ff^*}\leq\tfrac{1}{2}\|\log\rho_{ff^*}\|\overset{(6.6)}{\leq}\tfrac{1}{2}\log\|ff^*\|_1\leq\log\|f\|_1.$$

Hence

$$\log\det_{\mathcal{N}\Gamma} f=\frac{1}{n}\log\det_{\mathcal{N}\Gamma} f^n\leq\log\|f^n\|_1^{1/n}. \tag{6.15}$$

Therefore,

$$\log\det_{\mathcal{N}\Gamma} f\leq\log\sigma_f.$$

If f is in $L^1(\Gamma,\mathbb{C})^*$ inequality (6.15) applied to f^{-1} gives

$$\log\det_{\mathcal{N}\Gamma} f\geq-\log\|f^{-n}\|_1^{1/n}$$

and in the limit

$$\log\det_{\mathcal{N}\Gamma} f\geq-\log\sigma_{f^{-1}}.$$

Together with Theorem 1.1 this gives the following corollary.

COROLLARY 6.4. *In the situation of Theorem 1.1, if $f\in\mathbb{Z}\Gamma$ is a unit in $L^1(\Gamma,\mathbb{C})$ we have the estimates*

$$\log\|f^{-1}\|_1^{-1}\leq\log(\sigma_{f^{-1}})^{-1}\leq h_f\leq\log\sigma_f\leq\log\|f\|_1.$$

Example 6.5. For $\Gamma=\mathbb{Z}^n$ the spectrum of the commutative normed $*$-algebra $L^1(\mathbb{Z}^n)$ is the real n-torus $\mathbb{T}^n\cong\mathbb{S}^n$, where $\mathbb{S}=\{z\in\mathbb{C}:|z|=1\}$ (cf. [**14**, XI 2, Example 2]). Hence an element f in $L^1(\mathbb{Z}^n)$ is invertible if and only if the Fourier series $\hat f$ does not vanish in any point of $\mathbb{S}^n$ (Wiener's theorem). By [**14**, XI 2, Theorem 1], we have

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$\sigma_f = \max_{x\in\mathbb{S}^n} |\hat{f}(x)|$ and $\sigma_{f^{-1}} = \max_{x\in\mathbb{S}^n} |\widehat{f^{-1}}(x)| = \max_{x\in\mathbb{S}^n} |\hat{f}(x)^{-1}|$. For an element f in $\mathbb{Z}[\mathbb{Z}^n]$ which is invertible in $L^1(\mathbb{Z}^n)$ the corollary therefore gives the estimates

$$\min_{x\in\mathbb{S}^n} \log|\hat{f}(x)| = \log(\sigma_{f^{-1}})^{-1} \le h_f \le \log\sigma_f = \max_{x\in\mathbb{S}^n} \log|\hat{f}(x)|.$$

Of course, this estimate follows directly from the formula [**9**]

$$h_f = \int_{\mathbb{S}^n} \log|\hat{f}(x)|\, d\mu(x),$$

which in our case is also a special case of Theorem 1.1 and (6.5).

COROLLARY 6.6. *In the situation of Theorem 1.1, consider elements f, g in $\mathbb{Z}\Gamma \cap L^1(\Gamma,\mathbb{C})^*$ which satisfy $0 \le f \le g$ in $\mathcal{N}\Gamma$. Then we have $h_f \le h_g$ and equality $h_f = h_g$ holds if and only if $f = g$.*

Proof. The first assertion follows from inequality (6.4) applied to the operators ρ_f and ρ_g on $L^2(\Gamma,\mathbb{C})$. By the monotonicity of the logarithm the operator $A = \log\rho_g - \log\rho_f$ in $\mathcal{N}\Gamma$ is positive. If the entropies are equal, i.e. if $h_f = h_g$, then it follows from Theorem 1.1 that

$$\mathrm{tr}_{\mathcal{N}\Gamma}(\log\rho_f) = \mathrm{tr}_{\mathcal{N}\Gamma}(\log\rho_g).$$

However, this means that $\mathrm{tr}_{\mathcal{N}\Gamma}A = 0$ and hence $A = 0$ since $\mathrm{tr}_{\mathcal{N}\Gamma}$ is faithful. Applying exp to the equality $\log\rho_f = \log\rho_g$ gives the second assertion. □

The following corollary is an application of our theory to the structure of $(\mathcal{N}\Gamma)^*$.

COROLLARY 6.7. *For a group Γ as in Theorem 1.1 let f be an element of $\mathbb{Z}\Gamma$ which is invertible in $L^1(\Gamma)$ but does not have a left inverse in $\mathbb{Z}\Gamma$. Then f is not contained in the commutator subgroup of $(\mathcal{N}\Gamma)^*$.*

Proof. If this were not the case we would have $\det_{\mathcal{N}\Gamma} f = 1$ and hence $h_f = 0$ by Theorem 1.1. By assumption we have $\mathbb{Z}\Gamma \ne \mathbb{Z}\Gamma f$ and hence $X_f \ne 0$. However, then Theorem 5.1 tells us that $h_f > 0$, a contradiction. □

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