Ergod. Th. & Dynam. Sys. (2001), **21**, 1321–1332
Printed in the United Kingdom © *2001 Cambridge University Press*

Rank-one power weakly mixing non-singular transformations

TERRENCE ADAMS†, NATHANIEL FRIEDMAN‡ and CESAR E. SILVA§

† *Ohio State University, Columbus, OH 43210, USA*
‡ *State University of New York at Albany, Albany, NY 12222, USA*
§ *Williams College, Williamstown, MA 01267, USA*
(e-mail: csilva@williams.edu)

(*Received* 21 *June* 1999 *and accepted in revised form* 26 *October* 2000)

Abstract. We show that Chacon's non-singular type III_λ transformation T_λ, $0 < \lambda \leq 1$, is power weakly mixing, i.e. for all sequences of non-zero integers $\{k_1, \ldots, k_r\}$, $T_\lambda^{k_1} \times \cdots \times T_\lambda^{k_r}$ is ergodic. We then show that in infinite measure, this condition is not implied by infinite ergodic index (having all finite Cartesian products ergodic), and that infinite ergodic index does not imply 2-recurrence.

1. *Introduction*

In [**KP**], Kakutani and Parry defined an infinite measure-preserving transformation T to have *infinite ergodic index* if for all $k > 0$, the k-fold Cartesian products $T \times \cdots \times T$ are ergodic. They also showed that it may happen that $T \times T$ is ergodic but $T \times T \times T$ is not ergodic, so $T \times T$ ergodic does not imply T of infinite ergodic index. Recently, in [**DGMS**], the authors defined a transformation T to be *power weakly mixing* if for all finite sequences of non-zero integers $\{k_1, \ldots, k_r\}$, $T^{k_1} \times \cdots \times T^{k_r}$ is ergodic; they also constructed examples of rank-one infinite measure-preserving transformations that are power weakly mixing. In this paper we show that infinite ergodic index does not imply power weak mixing, and that infinite ergodic index does not imply 2-recurrence. (In fact, in our example all finite products $T \times \cdots \times T$ are ergodic but $T \times T^2$ is not conservative, hence not ergodic.) We start by showing in §2 that the non-singular type III_λ, $0 < \lambda \leq 1$, Chacon transformations of [**JS1**] are power weakly mixing. Our example of an infinite ergodic index transformation that is not power weakly mixing is presented in §3. However, §3 may be read essentially independently of §2, it only uses a standard approximation lemma from measure theory (Lemma 2.6) and the notation established in the paragraph previous to that lemma.

The non-singular Chacon transformations we study here were shown to be prime and to have trivial centralizer in [**JS1**] using non-singular coding techniques; using these same techniques the authors in [**JS1**] also had a proof (unpublished) that their Cartesian square

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

is ergodic. Transformations similar to these were shown to have two-fold non-singular minimal self-joinings in [**RS**], a property which implies prime, trivial centralizer and ergodic Cartesian square. However, it is not clear if two-fold non-singular minimal self-joinings as defined in [**RS**] implies power weak mixing. Furthermore, the notion of n-fold non-singular minimal self-joinings, for $n > 2$, is yet to be studied in detail, as questions regarding the right extension of the notion and the properties of a rational joining of more than two transformations remain open. (It was shown in [**RS**] that, in the non-singular category, to study a property such as minimal self-joinings one must restrict the non-singular joinings to a sub-class such as the rational joinings.) The methods we use here to show power weak mixing are extensions of [**AFS**], where other examples of rank-one infinite measure-preserving and non-singular transformations are constructed.

In [**JS2**], all factors of two-fold Cartesian products, and the centralizer of arbitrary Cartesian products, of these transformations are classified, where use is made of the property of power weak mixing shown in this paper.

As remarked by the referee, infinite measure-preserving null recurrent Markov shifts that are power weakly mixing have been available for some time. In fact, if $p = (p_0, p_1, \dots)$ is a Kaluza sequence and S_p is its corresponding Markov shift (cf. [**R**]) then it can be shown that S_p has infinite ergodic index if and only if

$$\sum_{n=0}^{\infty} p_n^d = \infty$$

for all $d \geq 1$. In this case S_p is power weakly mixing. (Such a sequence p_n can be obtained by choosing, for example, $p_n = 1/\ln(n + e)$, $n \geq 0$.) However, these examples are different to our rank-one constructions, as for Markov shifts of this type infinite ergodic index implies power weak mixing.

2. *Type III_λ rank-one power weakly mixing*

We define our transformations $T = T_\lambda$ inductively by the process of 'cutting and stacking' [**F, JS1**]. Let $0 < \lambda < 1$, $C_0 = \{[0, 1)\}$, $h_{-1} = 0$ and $h_0 = 1$. Assuming that column C_n of height h_n has been constructed, to obtain column C_{n+1} cut each interval (or *level*) of C_n in the proportions $\lambda/(1 + 2\lambda)$, $1/(1 + 2\lambda)$, and $\lambda/(1 + 2\lambda)$ to produce three subcolumns of C_n, numbered from left to right as $C_{n,i}$, $i = 0, 1, 2$. Then place a *spacer* (a new level) above the top of $C_{n,1}$, of the same length as the top level of $C_{n,1}$, and stack the three sub-columns from left to right, i.e. the top level of $C_{n,0}$ is sent by an affine map to the bottom level of $C_{n,1}$, the top level of $C_{n,1}$ is sent to the spacer, and the spacer at the top of $C_{n,1}$ is sent by an affine map to the bottom of $C_{n,2}$. This defines a new column C_{n+1} of height $h_{n+1} = 3h_n + 1$; levels in C_{n+1} are numbered from 1 to h_{n+1}, bottom to top. On each level the transformation is defined by the (unique orientation preserving) affine map that takes that level onto the one on top; the contraction or expansion on the level is the value of the Radon–Nikodym derivative (or Jacobian) on that level. For example, the Radon–Nikodym derivative on the first three levels of C_1 has values $1/\lambda$, 1, and λ. This defines a transformation $T = T_\lambda$ on a finite interval X of $\mathbb{R}$; we then normalize the measure to be a probability measure μ. It is clear that the resulting transformation T is conservative ergodic and that the Radon–Nikodym derivatives are powers of λ.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Given a column C_m, columns $C_{m,i}$, $i = 0, 1, 2$, are said to be *copies* of C_m in C_{m+1}. In this way one defines, for any $n > m$, copies of C_m in C_n, and for any interval I in C_m, copies of I in C_n.

Definition 2.1. We say levels I and J in C_n are $|\ell|$ *apart* if J is the $i + \ell$ level in C_n when I is the i level.

To show ergodicity of products of powers of T we show that for any two measurable sets of positive measure, there is a forward iterate of one that intersects the other in positive measure. We first obtain the necessary results for products of levels in Lemmas 2.2, 2.3 and 2.4, and then use some approximation results to extend this to arbitrary measurable sets in the product space.

LEMMA 2.2. *Suppose that k is a positive integer with ternary expansion*

$$k = \sum_{j \in R_k} a_j 3^j$$

where R_k is a subset of the non-negative integers and $a_j \in \{1, 2\}$. Let $t_k = |R_k| + \sum_{j \in R_k} a_j h_{j-1}$. If n is a positive integer and I is a level in C_n such that $T^{-t_k} I$ is in C_n, then

$$\mu(T^{kh_n} I \cap T^{-t_k} I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|} \mu(T^{-t_k} I),$$
$$\mu(T^{kh_n} I \cap T^{-t_k+1} I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|+1} \mu(T^{-t_k+1} I).$$

Proof. Note that if p is a positive integer and I is a level in C_p such that $T^{-1} I$ is in C_p, then both $T^{h_p} I$ and $T^{2h_p} I$ contain the rightmost third of $T^{-1} I$. Thus, if $q \geq p$, then

$$\mu(T^{h_q} I \cap T^{-1} I) \geq \left(\frac{\lambda}{1+2\lambda}\right) \mu(T^{-1} I),$$
$$\mu(T^{2h_q} I \cap T^{-1} I) \geq \left(\frac{\lambda}{1+2\lambda}\right) \mu(T^{-1} I)$$

and

$$\mu(T^{h_q} I \cap I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^2 \mu(I),$$
$$\mu(T^{2h_q} I \cap I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^2 \mu(I),$$

and furthermore both $T^{h_q} I \cap T^{-1} I \cap C_{q,2}$ and $T^{2h_q} I \cap T^{-1} I \cap C_{q,2}$ are each a union of levels in C_{q+1}. Now, we use the ternary expansion of k to express kh_n as a sum of heights.

First, note that h_j is formed by the relation

$$h_{j+1} = 3h_j + 1.$$

From this relation, it follows that for $m \geq 1$ and $j \geq 1$,

$$3^j h_m = h_{m+j} - h_{j-1}.$$

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Thus,

$$kh_n = \sum_{j\in R_k} a_j 3^j h_n \tag{1}$$

$$= \sum_{j\in R_k} a_j [h_{n+j} - h_{j-1}] \tag{2}$$

$$= \sum_{j\in R_k} a_j h_{n+j} - \sum_{j\in R_k} a_j h_{j-1}. \tag{3}$$

Let $R = \max(R_k)$ and

$$\sigma = \sum_{j\in R_k\setminus\{R\}} a_j h_{n+j}.$$

It may be checked, using induction, that $T^\sigma I \cap T^{-|R_k|+1} I$ contains a union of levels in C_{n+R} whose measure is at least

$$\left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|-1} \mu(T^{-|R_k|+1} I),$$

and $T^\sigma I \cap T^{-|R_k|+1} I$ is a union of levels in C_{n+R}. Hence,

$$\mu(T^{\sigma+a_r h_{n+R}} I \cap T^{-|R_k|} I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|} \mu(T^{-|R_k|} I)$$

and

$$\mu(T^{\sigma+a_r h_{n+R}} I \cap T^{-|R_k|+1} I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|+1} \mu(T^{-|R_k|+1} I).$$

Therefore, by translating $-\sum_{j\in R_k} a_j h_{j-1}$, we obtain Lemma 2.2. □

LEMMA 2.3. *Suppose that k is a positive integer with ternary expansion*

$$k = \sum_{j\in R_k} a_j 3^j$$

where R_k is a subset of the non-negative integers, $R = \max(R_k)$ and $a_j \in \{1, 2\}$. Let $t_k = |R_k| + \sum_{j\in R_k} a_j h_{j-1}$. Let ℓ be a positive integer and suppose that $n_1 < n_2 < \cdots < n_\ell$ are positive integers such that $n_{i+1} - n_i \geq R + 1$ for $1 \leq i \leq \ell - 1$. Let $H = \sum_{i=1}^{\ell} h_{n_i}$. If I is a level in C_{n_1} such that $T^{-\ell t_k} I$ is in C_{n_1}, then

$$\mu(T^{kH} I \cap T^{-\ell t_k + i} I) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{\ell(|R_k|+1)} \mu(T^{-\ell t_k+i} I)$$

for $0 \leq i \leq \ell$.

Proof. We obtain this lemma by invoking Lemma 2.2 ℓ times. First, by Lemma 2.2, since $n_2 - n_1 \geq R + 1$ both $T^{kh_{n_1}} I \cap T^{-t_k} I$ and $T^{kh_{n_1}} I \cap T^{-t_k+1} I$ contain a union of levels whose measure is at least

$$\left(\frac{\lambda}{1+2\lambda}\right)^{|R_k|+1} \mu(I).$$

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Thus, by applying Lemma 2.2 on each of the levels from C_{n_2} contained in $T^{kh_{n_1}}I \cap T^{-t_k}I$ and $T^{kh_{n_1}}I \cap T^{-t_k+1}I$, we get that all three of the sets $T^{k(h_{n_1}+h_{n_2})}I \cap T^{-2t_k}I$, $T^{k(h_{n_1}+h_{n_2})}I \cap T^{-2t_k+1}I$ and $T^{k(h_{n_1}+h_{n_2})}I \cap T^{-2t_k+2}I$ have measure at least

$$\left(\frac{\lambda}{1+2\lambda}\right)^{2(|R_k|+1)}\mu(I).$$

Finally, if we apply Lemma 2.2 a total of ℓ times, we obtain Lemma 2.3. □

LEMMA 2.4. *Let m_0 and $k_1, \ldots, k_r$ be positive integers. Let I_0 and J_0 be levels in C_{m_0} with I_0 above J_0. Let $k = \max\{k_1, \ldots, k_r\}$ and $m = m_0 + 2k$. Let I be the uppermost copy of I_0 in C_m. Then there exists a copy J of J_0 in C_m, an integer $s = s(m, k_1, \ldots, k_r)$ and $H_m = H(m, k_1, \ldots, k_r)$ such that*

$$\mu(T^{k_i H}I \cap J) \geq \left(\frac{\lambda}{1+2\lambda}\right)^s \mu(J)$$

for $i = 1, \ldots, r$. Moreover, if I' and J' are any sublevel of I and J in the same copy of C_m in C_n then there exists a positive integer $H_n = H(n, k_1, \ldots, k_r)$ such that for $i = 1, \ldots, r$,

$$\mu(T^{k_i H_n}I' \cap J') \geq \left(\frac{\lambda}{1+2\lambda}\right)^s \mu(J').$$

Proof. Let t_{k_i} and R_{k_i} be as in Lemma 2.3; put $t = \max\{t_{k_1}, \ldots, t_{k_r}\}$, and $\ell = h_{m_0}$. Then $t \leq 2k$. Then for any level L in the top C_{m_0}-copy in C_m, $T^{-\ell t_{k_i}}L \in C_m$ for $i = 1, \ldots, r$.

Put $R = \max\left(\bigcup_{i=1}^r R_{k_i}\right)$ and $\rho = \max\{|R_{k_i}| : i = 1, \ldots, r\}$.

Let $m_1 = m$ and choose $m_2 < \cdots < m_\ell$ so that $m_{j+1} - m_j \geq R+1$ for $j = 1, \ldots, \ell-1$. Let

$$H_m = \sum_{j=1}^{\ell} h_{m_j}$$

as in Lemma 2.3.

Let I be the uppermost sublevel of I_0 in C_m. Since $\ell t_{k_i} \leq h_{m_0}t \leq h_m$, for each $i = 1, \ldots, r$, $T^{-\ell t_{k_i}+j}I$, as $j = 0, \ldots, \ell$, covers $\ell+1$ consecutive levels in C_m, and so at least one of them must intersect a copy of J_0 in C_m, call it J. Therefore, by Lemma 2.3,

$$\mu(T^{k_i H_m}I \cap J) \geq \left(\frac{\lambda}{1+2\lambda}\right)^{\ell(|R_{k_i}|+1)} \mu(J)$$

for all $i = 1, \ldots, r$.

So let

$$s = \ell(\rho+1).$$

Let $n_1 = n$ and choose $n_2 < \cdots < n_\ell$ so that $n_{j+1} - n_j \geq R+1$ for $j = 1, \ldots, \ell-1$. Let

$$H_n = \sum_{j=1}^{\ell} h_{n_j}$$

as in Lemma 2.3.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Let D be any copy of C_m in C_n and let I' be the sublevel of I in D. Since $\ell t_{k_i} \leq h_{m_0} t \leq h_m$, for each $i = 1, \ldots, r$, $T^{-\ell t_{k_i}+j} I'$, as $j = 0, \ldots, \ell$, covers $\ell + 1$ consecutive levels in D, and so at least one of them must intersect a copy of J in D, call it J'. Therefore, by Lemma 2.3,

$$\mu(T^{k_i H_n} I' \cap J') \geq \left(\frac{\lambda}{1+2\lambda}\right)^s \mu(J')$$

for all $i = 1, \ldots, r$. □

LEMMA 2.5. (1) *For all $n \geq 0$,*

$$\lambda^2 \leq \frac{d\mu \circ T^{\pm h_n}}{d\mu}(x) \leq \lambda^{-2} \text{ a.e.}$$

(2) *Let ℓ be a positive integer and let $H = \sum_{i=1}^{\ell} h_{n_i}$ be as in Lemma 2.4. Then for any integers $n \geq 0$, k, k' with $0 < |k'| \leq k$,*

$$\lambda^{2k\ell} \leq \frac{d\mu \circ T^{k'H}}{d\mu}(x) \leq \lambda^{-2k\ell} \text{ a.e.}$$

Proof. Part (1) is in Proposition 3.1 in [**JS1**]. Part (2) follows from the cocycle relation and part (1) since H is a sum of column heights. □

Lemma 2.6 below is the non-singular version of a standard approximation lemma; its proof may be obtained by standard measure theory arguments using a martingale convergence theorem (a more elementary proof follows from the proof of a similar lemma in [**DGMS**]). To apply this lemma to the non-singular case we use Lemma 2.7 below (from [**M-Z**]), whose proof is immediate from the construction and is omitted.

Let r be a positive integer and ν be product measure on $\prod_{i=1}^{r} X$. A *rectangle from C_m* is a set of the form $I = I_1 \times \cdots \times I_r$, where I_i are all levels of column C_m. If

$$\nu(A \cap I) > (1 - \epsilon)\nu(A)$$

we say that I is $(1 - \epsilon)$-*full* in A. It is clear that there are 3^{n-m} copies of C_m in C_n. Let $U(m, n) = \{1, \ldots, 3^{n-m}\}$ index the copies of C_m in C_n, and let $U(m, n)^r = U(m, n) \times \cdots \times U(m, n)$ (r times).

LEMMA 2.6. (Double approximation) *Let $L = L_1 \times \cdots \times L_r$ be a rectangle from C_m, let ν be product measure, and let $0 < \epsilon < 1$. If A is any subset of L such that L is $(1-\epsilon)$-full in A, then for any δ, $0 < \delta < 1$, and any t, $0 < t < (1 - \epsilon)$, there exists an integer N such that for all $n \geq N$, there is a set $U'' \subset U(m, n)^r$ such that, for all $u = (u_1, \ldots, u_r) \in U''$, $L_{(u)}$ is $(1 - \delta)$-full of A and each $L_{(u)}$ is of the form $L_{(u)} = L_1^{u_1} \times \cdots \times L_r^{u_r}$ where $L_i^{u_i}$ is a sublevel of L_i in the u_i copy of C_m in C_n, and furthermore, if $L'' = \bigcup_{u \in U''} I_{(u)}$ then*

$$\nu(L'' \triangle L) < (1 - t)\nu(I).$$

LEMMA 2.7. *Let I and J be two levels in a column C_m. If I' and J' are any two sublevels of I and J, respectively, in a later column C_n such that I' and J' are in the same copy of C_m in C_n, then $\mu(I) = \lambda^k \mu(J)$ implies $\mu(I') = \lambda^k \mu(J')$.*

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

THEOREM 2.8. *The type III_λ transformation T_λ, $0 < \lambda < 1$, is power weakly mixing.*

Proof. Let $k_1, \ldots, k_r$ be non-zero integers and let $k = \max\{|k_i| : 0 < i \leq k\}$. We will show that $S = T_\lambda^{k_1} \times \cdots \times T_\lambda^{k_r}$ is ergodic.

Let A and B be subsets of the product space $\Pi_{i=1}^r X$ with $\nu(A) > 0$ and $\nu(B) > 0$. Choose $I_0 = I_{0,1} \times \cdots \times I_{0,r}$ and $J_0 = J_{0,1} \times \cdots \times J_{0,r}$ so that

$$\frac{\nu(A \cap I_0)}{\nu(I_0)} > 1 - \frac{1}{4}(\lambda^{2kr}), \quad \frac{\nu(B \cap J_0)}{\nu(J_0)} > 1 - \frac{1}{4}(\lambda^{2kr}),$$

and for all $i = 1, \ldots, r$, $I_{0,i}$ and $J_{0,i}$ are all in some column C_m, with $m > k$, $I_{0,i}$ is above $J_{0,i}$ if k_i is positive, and $I_{0,i}$ is below $J_{0,i}$ if k_i is negative (this latter part can be done by choosing a finer approximation in an earlier column if necessary). Let $\ell = h_{m_0}$, for each $i = 1, \ldots, r$ and R_{k_i} be as in Lemma 2.3. For each i, let I_i and J_i be the sublevels of $I_{0,i}$ and $J_{0,i}$ as in Lemma 2.4 (with the appropriate interpretation when k_i is negative), and $s = s(m, k_1, \ldots, k_r)$. Let $I = I_1 \times \cdots \times I_r$, $J = J_1 \times \cdots \times J_r$ and A and B denote $A \cap I$ and $B \cap J$, respectively. By the choice of I_0 and J_0 above, $\nu(A \cap I)/\nu(I) > \frac{3}{4}$ and $\nu(B \cap J)/\nu(J) > \frac{3}{4}$. Let α_i be such that $\mu(I_i) = \alpha_i \mu(J_i)$, $\alpha = \Pi_{i=1}^r \alpha_i$ and $\alpha_* = \min\{\alpha_i, 1\}$. Then $\nu(I) = \alpha\nu(J)$. Let

$$\beta = \alpha\lambda^{(-2k\ell)r}.$$

Choose

$$0 < \delta < (\lambda^{2k\ell})^r \left(\frac{\lambda}{1+2\lambda}\right)^{sr} (\alpha_*)^r.$$

By Lemma 2.5(2),

$$\lambda^{(2k\ell)r} \leq \frac{d\nu \circ S^H}{d\nu} \leq \lambda^{(-2k\ell)r} \text{ a.e.}$$

Apply Lemma 2.6 to I and J in C_m twice to obtain sets of indices U'' and V'' and rectangles $I_{(u)}$, and $J_{(v)}$ so that for all $u \in U''$, $v \in V''$, $I_{(u)} = I_1^{u_1} \times \cdots \times I_r^{u_r}$ is $(1 - \delta/3\beta)$-full of A and $J_{(v)} = J_1^{v_1} \times \cdots \times J_r^{v_r}$ is $(1 - \delta/3)$-full of B, $I_1^{u_1}, \ldots, I_r^{u_r}$ and $J_1^{v_1}, \ldots, J_r^{v_r}$ are in the same column C_n, with $n > m$, and for each i, $I_i^{u_i}$ and $J_i^{v_i}$ are in the u_i and v_i C_m-copy in C_n, respectively; also if $I'' = \bigcup_{u \in U''} I_{(u)}$ and $J'' = \bigcup_{v \in V''} J_{(v)}$ then

$$\nu(I'' \triangle I) < \left(\tfrac{1}{2}\right)\nu(I)$$

and

$$\nu(J'' \triangle J) < \left(\tfrac{1}{2}\right)\nu(J).$$

Since the union of the intervals corresponding to the indices in U'' covers more than half of I, Lemma 2.7 implies that

$$\nu\left(\left(\bigcup_{u \in U''} J_{(u)}\right) \triangle J\right) < \left(\tfrac{1}{2}\right)\nu(J).$$

Thus there must exist at least one index w in both U'' and V''; so we must have some $I_{(w)}$ and $J_{(w)}$, which are more than $(1 - \delta/3\beta)$- and $(1 - \delta/3)$-full of A and B, respectively. Also, $\nu(I_{(w)}) = \alpha\nu(J_{(w)})$.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Write $I_{(w)} = I_1^{w_1} \times \cdots \times I_r^{w_r}$ and $J_{(w)} = J_1^{w_1} \times \cdots \times J_r^{w_r}$. As $I_i^{w_i}$ and $J_i^{w_i}$ are in the same copy of C_m, by Lemma 2.4, if $k_i > 0$,

$$\mu(T^{k_i H} I_i^{w_i} \cap J_i^{w_i}) > \left(\frac{\lambda}{1+2\lambda}\right)^s \mu(J_i^{w_i});$$

and if $k_i < 0$,

$$\begin{aligned}
\mu(T^{k_i H} I_i^{w_i} \cap J_i^{w_i}) &= \int_{I_i^{w_i} \cap T^{-k_i H} J_i^{w_i}} \frac{d\mu \circ T^{k_i H}}{d\mu} d\mu \\
&\geq \lambda^{2k\ell} \mu(I_i^{w_i} \cap T^{-k_i H} J_i^{w_i}) \\
&\geq \left(\lambda^{2k\ell}\right)\left(\frac{\lambda}{1+2\lambda}\right)^s \mu(I_i^{w_i}) \\
&= \left(\lambda^{2k\ell}\right)\left(\frac{\lambda}{1+2\lambda}\right)^s \alpha_i \mu(J_i^{w_i}) \\
&\geq \left(\lambda^{2k\ell}\right)\left(\frac{\lambda}{1+2\lambda}\right)^s \alpha_* \mu(J_i^{w_i}).
\end{aligned}$$

Thus

$$\nu[S^H I_{(w)} \cap J_{(w)}] \geq \left(\lambda^{2k\ell}\right)^r \left(\frac{\lambda}{1+2\lambda}\right)^{sr} (\alpha_*)^r \nu(J_{(w)}) > \delta \nu(J_{(w)}).$$

Now

$$\begin{aligned}
\nu(S^H(I_{(w)} \setminus A)) &= \int_{I_{(w)} \setminus A} \frac{d\nu \circ S^H}{d\nu} d\nu \leq \lambda^{(-2k\ell)r} \nu(I_{(w)} \setminus A) \\
&\leq \lambda^{(-2k\ell)r} \frac{\delta}{3\beta} \nu(I_{(w)}) = \frac{\delta}{3\alpha} \nu(I_{(w)}) = \frac{\delta}{3} \nu(J_{(w)}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nu(S^H A \cap B) &\geq \nu(S^H I_{(w)} \cap J_{(w)}) - \nu([S^H I_{(w)} \cap J_{(w)}] \setminus [S^H A \cap B]) \\
&\geq \nu(S^H I_{(w)} \cap J_{(w)}) - \nu[S^H(I_{(w)} \setminus A)] - \nu(J_{(w)} \setminus B) \\
&\geq \delta \nu(J_{(w)}) - \frac{\delta}{3} \nu(J_{(w)}) - \frac{\delta}{3} \nu(J_{(w)}) > 0.
\end{aligned}$$

□

Remark 1. By choosing $\log \lambda_1$ and $\log \lambda_2$ irrationally related in alternate times n in the construction of T we obtain a type III_1 transformation with the same properties. Recently, after this research was completed, Hamachi and the third-named author have constructed type III_0 measures on Chacon's map for which they prove their Cartesian square is ergodic; however, these measures do not satisfy the analogue of Lemma 2.5 and their methods are different.

Remark 2. The argument we have given applies to other families of transformations. For example, the argument gives ergodicity for Cartesian products of powers of T_{λ_i} for different values of λ_i, $0 < \lambda_i < 1$.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

3. *Infinite ergodic index not power weakly mixing*

In this section we construct a transformation that has infinite ergodic index but is not power weakly mixing. We also show that our example is not 2-recurrent. We note that the first-named author has studied other constructions where the ergodicity and conservativity of two-fold products are analyzed [**A**].

We recall that a transformation T is said to be d-recurrent if, for all sets of positive measure A, there exists $n > 0$ such that $\mu(A \cap T^n A \cap \cdots \cap T^{nd} A) > 0$. As is well known, Furstenberg [**Fu**] has shown that every finite measure-preserving transformation is multiply recurrent. In [**EHH**], Eigen *et al.* showed that there exist infinite measure-preserving transformations that are d-recurrent but not $(d+1)$-recurrent, for every $d > 1$; however their examples are infinite measure-preserving odometers and hence do not have infinite ergodic index. More recently, Aaronson and Nakada [**AN**] have shown a theorem that in particular implies that an infinite Markov shift with infinite ergodic index is d-recurrent for all d. They also construct an infinite odometer with some additional properties that is not 2-recurrent.

We now construct a family of transformations. Let a_n, b_n, c_n, and d_n be sequences of non-negative integers. Let C_n be a column of height H_n. Form C_{n+1} by cutting C_n into four subcolumns of equal width and placing a_n, b_n, c_n, and d_n spacers on the first, second, third, and fourth subcolumns, respectively; then stack the subcolumns from left to right to form C_{n+1}, of height H_{n+1}. If we begin with C_1 as an interval, then the previous algorithm defines a Lebesgue measure-preserving transformation on a subset of $\mathbb{R}$. Define $p_n = H_n + a_n$, $\ell_n = H_n + b_n$, $q_n = H_n + c_n$, $m_n = H_n + d_n$, and $h_{n+1} = p_n + \ell_n + q_n + H_n$ for $n \in \mathbb{N}$. Denote the transformation by $T = T_v$ where $v = \{(p_n, \ell_n, q_n, m_n) : n \in \mathbb{N}\}$. Consider only transformations $T = T_v$ where

$$q_n \geq p_n, \quad \lim_{n\to\infty} \frac{p_n}{h_n} = \infty$$

and ℓ_n and m_n are chosen so that

$$\ell_n > n(p_n + q_n + 2h_n) \quad \text{and} \quad m_n > n(p_n + \ell_n + q_n + H_n).$$

Let us describe informally how ℓ_n and m_n are chosen, $n > 3$. m_{n-1} is chosen large enough so that once the bottom h_n intervals of C_n have entered the top d_{n-1} levels of C_n under T^2, those levels remain in the d_{n-1}-spacers until the bottom h_n levels enter the d_{n-1}-spacers under T. ℓ_n is chosen large enough so that the bottom h_n levels of the third subcolumn of C_n have moved into the m_n-spacers under T before the bottom h_n levels of the second subcolumn have exited the ℓ_n-spacers under T^2.

We first show that T has infinite ergodic index; for this we assume $c_n = a_n + 1$ for all $n \geq 1$.

The proof of the following lemma is as in Lemma 2.1 in [**AFS**].

LEMMA 3.1. *Let $c_n = a_n + 1$ for all $n \geq 1$. If I and J are any two levels in C_n that are at most ℓ apart, with I above J, then there exists and integer $H = H(n, \ell)$ such that*

$$\mu(T^H I \cap J) \geq \left(\tfrac{1}{4}\right)^{\ell} \mu(J).$$

THEOREM 3.2. *Let $c_n = a_n + 1$ for all $n \geq 1$. Then T has infinite ergodic index.*

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

Proof. We outline the main ideas since the proof is similar to that of Theorem 2.8 (and Theorem 2.2 of [**AFS**]). Let A and B be subsets of the product space of positive measure. Choose $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$ such that $\nu(A \cap I)/\nu(I) > \frac{3}{4}$, $\nu(B \cap J)/\nu(J) > \frac{3}{4}$, and for all $i = 1, \ldots, r$, I_i and J_i are all in the same column C_m, and I_i is above J_i. Let A and B denote $A \cap I$ and $B \cap J$, respectively. Let $\ell > 0$ be such that I_i and J_i are at most ℓ apart, for all i. Choose

$$0 < \delta < \left(\tfrac{1}{4}\right)^{\ell r}.$$

Let $S = T \times \cdots \times T$ (r times). Apply the double approximation lemma twice with $\epsilon = 1/4$, $t = 1/2$, and $\delta/3$ above, to obtain sets of indices U'' and V'' and rectangles I_u and J_v so that for all $u \in U''$, $v \in V''$, $I_u = I'_1 \times \cdots \times I'_r$ is $(1 - \delta/3)$-full of A and $J_v = J'_1 \times \cdots \times J'_r$ is $(1 - \delta/3)$-full of B, I'_i, J'_i are in the same column C_n, and I'_i and J'_i are in the same C_m-copy in C_n, for $i = 1, \ldots, r$. Since all levels in a column have the same length we can conclude that U'' and V'' contain more than a half of the indices in U and V, respectively.

Thus there must exist at least one index $w \in U'' \cap V''$ so that I_w and J_w are both more than $(1 - \delta/3)$-full of A and B, respectively.

By Lemma 3.1,

$$\nu[S^H I_w \cap J_w] > \left(\tfrac{1}{4}\right)^{\ell r} \nu(J_w).$$

Then, as in Theorem 2.8, one obtains that

$$\nu(S^H A \cap B) \geq \delta\nu(J_w) - \frac{\delta}{3}\nu(J_w) - \frac{\delta}{3}\nu(J_w) > 0. \qquad \square$$

THEOREM 3.3. *If the set*

$$\{n \in \mathbb{N} : |q_n - 2p_n| \leq 3h_n\}$$

is finite, then $T \times T^2$ is not conservative, hence T is not power weakly mixing. Furthermore, T is not 2-recurrent.

Proof. Choose N such that $N > 3$ and $|q_n - 2p_n| > 3h_n$ for $n \geq N$. Let A be the bottom level of C_{N+1}. Define

$$\Lambda = \{i > 0 : \mu \times \mu((T \times T^2)^i (A \times A) \cap (A \times A)) > 0\}$$

and

$$\Lambda_n = \Lambda \cap (0, \tfrac{1}{2}H_n).$$

We will prove inductively on n, that $\Lambda_n = \emptyset$. This is true for $1 \leq n \leq N$ since $\mu(T^i A \cap A) = 0$ for $0 < i < H_N$.

Suppose that our assertion holds for Λ_n. We wish to verify that $\Lambda_{n+1} = \emptyset$. Let D denote the union of the bottom h_n levels in C_n. The set A will be contained in D. Let D_j be the part of D in the jth subcolumn of C_n, $j = 1, \ldots, 4$. Below we give a table showing the integers i between 0 and $\frac{1}{2}H_{n+1}$ such that $T^i D_j \cap D_k \neq \emptyset$. The first number in each cell is the left-hand point of the interval and the second point is the right-hand point of the interval of intersection.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

i	D_1	D_2	D_3	D_4
$T^i D_1$	0	$p_n - h_n$	$p_n + \ell_n - h_n$	$p_n + \ell_n + q_n - h_n$
	h_n	$p_n + h_n$	$p_n + \ell_n + h_n$	$p_n + \ell_n + q_n + h_n$
$T^i D_2$		0	$\ell_n - h_n$	$\ell_n + q_n - h_n$
		h_n	$\ell_n + h_n$	$\ell_n + q_n + h_n$
$T^i D_3$			0	$q_n - h_n$
			h_n	$q_n + h_n$
$T^i D_4$				0
				h_n

We want to show that there is no simultaneous intersection for $i = j$ and $i = 2j$ among the 10 intervals listed. The intervals give the sets of intersection for T. To obtain the sets of intersection for T^2 divide the endpoints of the intervals by 2.

By the choice of m_{n-1} we have that

$$\frac{m_{n-1}}{2} > \frac{m_{n-1}}{(n-1)} > h_n.$$

Also, $\frac{1}{2}H_n > \frac{1}{2}m_{n-1} > h_n$. By induction it follows that $\Lambda \cap (0, h_n] = \emptyset$ and so there cannot be simultaneous intersection on this interval with any other interval in the table.

Now ℓ_n was chosen so that

$$p_n + q_n + h_n < \frac{\ell_n - h_n}{n} < \frac{\ell_n - h_n}{2}.$$

Thus, ℓ_n was chosen large enough so that we only need to compare the intervals with ℓ_n as a summand in the endpoints to other intervals with ℓ_n.

However, ℓ_n and n are sufficiently large so that

$$\begin{aligned}\frac{p_n + \ell_n + q_n + h_n}{2} &< \frac{(n-1)(p_n + \ell_n + q_n + h_n)}{n} \\ &< p_n + \ell_n + q_n + h_n - \frac{1}{n}\ell_n \\ &< \ell_n - h_n.\end{aligned}$$

Thus, T^2 will have pushed D_1 beyond D_4 before T pushes D_2 into D_3. This takes care of the worst case of intersection between intervals with ℓ_n as a summand.

Finally, we consider the case of simultaneous intersection on the interval in the first row, second column and on the interval in the third row, fourth column. The assumption on N implies that

$$\left|\frac{q_n}{2} - p_n\right| > \frac{h_n}{2} + h_n.$$

Thus there is no possibility of intersection of D_1 with D_2 under T, and intersection of D_3 with D_4 under T^2 since the distance between the centers of each of the intervals

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press

is greater than the sum of their radii. The remaining possibility cannot occur since $q_n - h_n > (p_n + h_n/2)$ as

$$\begin{aligned} q_n &> H_n > m_{n-1} \\ &> (n-1)(p_{n-1} + q_{n-1} + H_{n-1} + \ell_{n-1}) \\ &= (n-1)h_n \\ &\geq 3h_n \quad (\text{for } n \geq 4), \end{aligned}$$

by construction.

Finally, as $\mu \times \mu[(T \times T^2)^n(A \times A) \cap (A \times A)] = 0$ for all $n \neq 0$, it easily follows that for all $n \neq 0$, $\mu(A \cap T^n A \cap T^{2n} A) = 0$, hence T is not 2-recurrent. This completes the proof of the theorem. □

Finally we have the following corollary.

COROLLARY 3.4. *Infinite ergodic index does not imply power weak mixing. Furthermore, infinite ergodic index does not imply multiple recurrence.*

Proof. Choose $a_n = 3h_n$ and $c_n = a_n + 1$. Then $q_n = p_n + 1$ and $2p_n - q_n = 2H_n + 6h_n - H_n - 3h_n - 1 \geq 3h_n$. Now apply Theorems 3.2 and 3.3. □

Acknowledgement. We would like to thank the referee for several remarks that improved the exposition.

REFERENCES

[AN] J. Aaronson and H. Nakada. Multiple recurrence of Markov shifts and other infinite measure preserving transformations. *Israel J. Math.* **117** (2000), 285–310.

[A] T. Adams. Ergodicity of Cartesian products of infinite transformations. *Preprint*.

[AFS] T. Adams, N. Friedman and C. E. Silva. Rank-one weak mixing for non-singular transformations. *Israel J. Math.* **102** (1997), 269–281.

[DGMS] S. L. Day, B. R. Grivna, E. P. McCartney and C. E. Silva. Power weakly mixing infinite transformations. *New York J. Math.* **5** (1999), 17–24. http://nyjm.albany.edu:8000/j/1999/5-2.html

[EHH] S. Eigen, A. Hajian, and K. Halverson. Multiple recurrence and infinite measure preserving odometers. *Israel J. Math.* **108** (1998), 37–44.

[F] N. A. Friedman. *Introduction to Ergodic Theory*. Van Nostrand, 1970.

[Fu] H. Furstenberg. Recurrence in Ergodic Theory and Combinatorial Number Theory. Princeton University Press, Princeton, NJ, 1981.

[JS1] A. del Junco and C. E. Silva. Prime type III_λ automorphisms: an instance of coding techniques applied to non-singular maps. *Fractals and Dynamics (Okayama/Kyoto, 1992)*. Ed. Y. Takahashi. Plenum, New York, 1995, pp. 101–115.

[JS2] A. del Junco and C. E. Silva. Factors of Cartesian products of non-singular Chacon transformations, to appear.

[KP] S. Kakutani and W. Parry. Infinite measure preserving transformations with 'mixing'. *Bull. Amer. Math. Soc.* **69** (1963), 752–756.

[M-Z] E. J. Muehlegger, A. S. Raich, C. E. Silva, M. P. Touloumtzis, B. Narasimhan and W. Zhao. Infinite ergodic index Z^d actions in infinite measure. *Colloq. Math.* **82** (1999), 167–190.

[R] S. M. Rudolfer. Some metric invariants for Markov shifts. *Z. Wahrsch. verw. Geb.* **15** (1970), 202–207.

[RS] D. Rudolph and C. E. Silva. Minimal self-joinings for non-singular transformations. *Ergod. Th. & Dynam. Sys.* **9** (1989), 759–800.

https://doi.org/10.1017/S0143385701001626 Published online by Cambridge University Press