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NOTES AND PROBLEMS

ANOTHER NUMERICAL METHOD OF FINDING CRITICAL VALUES FOR THE ANDREWS STABILITY TEST

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We propose a method, alternative to that of Estrella (2003, *Econometric Theory* 19, 1128–1143), of obtaining exact asymptotic p-values and critical values for the popular Andrews (1993, *Econometrica* 61, 821–856) test for structural stability. The method is based on inverting an integral equation that determines the intensity of crossing a boundary by the asymptotic process underlying the test statistic. Further integration of the crossing intensity yields a p-value. The proposed method can potentially be applied to other stability tests that employ the supremum functional.

1. INTRODUCTION

The Andrews (1993) sup-test for a structural break whose date is unknown has become by now very popular, and is even advocated as a portmanteau testing tool (e.g., Hansen, 1999). The asymptotic distribution of the test statistic is nonstandard, and researchers typically employ critical values tabulated in Andrews (1993, 2003). These are obtained using a large number of simulations for various combinations of a test level, degrees of freedom, and truncation parameter.

Estrella (2003) points out that the critical values may be obtained exactly using a result in DeLong (1981) on parabolic boundary crossing probabilities for the Bessel process. The crossing probability, however, does not have a closed form formula but instead is represented as an infinite series whose summands involve roots of a certain polynomial of infinite order. These complications lead to quite involved numerical computations. Having coped with all of them Estrella (2003) gives a more precise version of the tables in Andrews (1993, 2003) and makes comparisons with what various existing approximate methods deliver.

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We here show that the exact critical values can be alternatively obtained using a relationship between the stochastic properties of the asymptotic process underlying the test statistic, and the intensity of crossing a boundary by this process. This relationship has the form of an integral equation, and it was documented in the statistical literature, in particular in Fortet (1943) and Durbin (1971), quite a while ago. We adapt this idea to the more special setting of the Andrews test. Then we develop the numerical algorithm that inverts the integral equation and returns the crossing intensity that is then used to obtain p-values of the test. The critical values computed via our numerical method coincide with those in Estrella (2003). As the method is based on a universal statistical relationship, it potentially can be used for tabulating asymptotic distributions of other stability tests that employ the supremum functional.

2. BOUNDARY CROSSING INTENSITY

Let r and s index time on $[0, \tau]$ and consider a nonnegative continuous Markov stochastic process $Q(r)$ on $[0, \tau]$ starting from the origin, $Q(0) = 0$. Denote by $p_r(y)$ the unconditional density of $Q(r)$ and by $p_{r|s}(y|x)$ the conditional density of $Q(r)$ given that $Q(s)$ takes value x.

Consider also a boundary $\Psi(r)$, a deterministic positive function of time possessing a finite derivative on $[0, \tau]$. We are interested in counting all crossings (from below) of $\Psi(r)$ by $Q(r)$ for the first time (also called first passages in the statistical literature). Let us denote the intensity of such crossings by $\alpha(r)$. More formally,

$$\alpha(r) = \lim_{\delta \downarrow 0} \frac{\Pr\{Q(s) < \Psi(s) \text{ for all } s \in [0, r] \text{ and } Q(s) \geq \Psi(s) \text{ for some } s \in [r, r+\delta]\}}{\delta}.$$

The statistical literature, in particular Fortet (1943) and Durbin (1971), provides the following relationship between the conditional density of the process $Q(r)$ and the intensity $\alpha(r)$ of crossing the boundary $\Psi(r)$:

$$p_r(\Psi(r)) = \int_0^r p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)ds \qquad \textbf{(1)}$$

that should hold for all $r \in [0, \tau]$. The conditions that the process $Q(r)$ should meet for (1) to be valid are spelled out in Feller (1936) and Fortet (1943, conditions α through γ, p. 178). Intuitively, the meaning of the equality in (1) is as follows: the unconditional density of $Q(r)$ at the boundary $\Psi(r)$ can be alternatively obtained via the law of total probability by counting, along the boundary from 0 to r, the total measure for those trajectories that pass through $\Psi(r)$ for the first time.

For given $Q(r)$ and $\Psi(r)$ the integral equation (1) allows one to find $\alpha(r)$. Given the intensity $\alpha(r)$, the total probability of crossing the boundary on $[0, \tau]$ can be found by integration:

$$\Pr\{Q(r) \geq \Psi(r) \text{ for some } r \in [0, \tau]\} = \int_0^\tau \alpha(r)dr. \qquad \textbf{(2)}$$

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3. ADAPTATION TO ANDREWS TEST

The Andrews (1993) test for structural stability in parametric models is a class of tests where the alternative hypothesis is that of a single structural break that occurred at unknown time. The test is based on a supremum (over time) of a certain sequential (i.e., indexed by time) statistic that is a Wald, likelihood ratio, or Lagrange multiplier statistic for equality of parameters before and after a possible break. The asymptotic distribution of the test is nonstandard. In the case where the sup-statistic exceeds a critical value the null of structural stability is rejected.

The asymptotic distribution of any of the three Andrews test statistics, before the supremum functional is taken, is

$$\frac{B_p(r)' B_p(r)}{r(1-r)} \tag{3}$$

on $[0, 1]$, where $B_p(r) = W_p(r) - r W_p(1)$ is the p-dimensional Brownian bridge and $W_p(r)$ is the p-dimensional Wiener process on $[0, 1]$. The supremum of (3) is taken over $[\pi_1, \pi_2]$, where $0 < \pi_1 < \pi_2 < 1$, and the resulting value is compared to the asymptotic critical value c_α corresponding to the test level α. The critical values obtained via simulations are tabulated in Andrews (1993, Table 1, 2003, Table 1) for most often used α, several values of $\pi_0 \equiv \pi_1 = 1 - \pi_2$, and a range of p. These critical values can also be utilized for some of the tests for multiple breaks of Bai and Perron (1998).

It is clear that the decision rule of the test is equivalent to checking if the p-dimensional squared Bessel bridge process $Q(r) = B_p(r)' B_p(r)$ visits the epigraph of the boundary

$$\Psi(r) = c_\alpha r(1-r)$$

at least once on $[\pi_1, \pi_2]$. Thus, the theory outlined in the previous section can be applied to find the probability value α of the test for given c_α, π_1, π_2, and p.

The implementation of this idea is not straightforward though. First, conditional densities should be tractable. Fortunately, in the case of squared Bessel bridge they can be easily derived. Note that, because of normality of $B_p(r)$, the conditional distribution of $B_p(r)$ given $B_p(s)$, where $s < r$, is conditionally homoskedastic normal:

$$B_p(r) | B_p(s) \sim N\left(\frac{1-r}{1-s} B_p(s), \frac{(1-r)(r-s)}{1-s} I_p\right)$$

and hence

$$Q(r) | Q(s) \sim \omega_{s,r} \chi^2\left(p, \omega_{s,r} \frac{Q(s)}{(r-s)^2}\right),$$

where

$$\omega_{s,r} = \frac{(1-r)(r-s)}{1-s}$$

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and $\chi^2(p,\lambda)$ denotes a noncentral chi-squared distribution with degrees of freedom p and noncentrality parameter λ. The unconditional distribution is then

$$Q(r) \sim \omega_{0,r}\, \chi^2(p,0).$$

Therefore, the unconditional and conditional densities entering (1) are

$$p_r(y) = (2r(1-r))^{-p/2}\, \Gamma\left(\frac{p}{2}\right)^{-1} y^{p/2-1} \exp\left(-\frac{y}{2r(1-r)}\right)$$

and

$$p_{r|s}(y|x) = \frac{1}{\omega_{s,r}}\, f_{\chi^2(p,\omega_{s,r}x/(r-s)^2)}\left(\frac{y}{\omega_{s,r}}\right),$$

where $f_{\chi^2(p,\lambda)}(z)$ denotes a density evaluated at z of a random variable distributed as $\chi^2(p,\lambda)$. This density is inconveniently represented analytically as a certain infinite summation but can be numerically evaluated in statistical packages.

Second, note that the supremum is taken over an interval whose left end, in contrast to the theory, is not zero, which requires corrections of the formulas in the previous section. Take arbitrary ordinate y such that $0 \le y < \Psi(\pi_1)$ and consider only those trajectories that start from y at $r = \pi_1$. Using the continuity and Markovianity of $Q(r)$, one can formulate the version of (1) conditional on passing through the point (π_1, y) as

$$p_{r|\pi_1}(\Psi(r)|y) = \int_{\pi_1}^{r} p_{r|s}(\Psi(r)|\Psi(s))\, \alpha(s|y) ds \tag{4}$$

for all $r \in (\pi_1, \pi_2]$, where $\alpha(s|y)$ is the crossing intensity conditional on the trajectories under consideration. Take an arbitrary small $\epsilon > 0$ and integrate both sides of (4) with respect to the measure $p_{\pi_1}(y)$ on the interval from 0 to $\Psi(\pi_1) - \epsilon$ to obtain after rearranging

$$\int_0^{\Psi(\pi_1)-\epsilon} p_{\pi_1}(y)\, p_{r|\pi_1}(\Psi(r)|y)\, dy$$
$$= \int_{\pi_1}^{r} p_{r|s}(\Psi(r)|\Psi(s)) \left(\int_0^{\Psi(\pi_1)-\epsilon} \alpha(s|y) p_{\pi_1}(y)\, dy \right) ds$$

for all $r \in (\pi_1, \pi_2]$. Now, because $p_{\pi_1}(y)\, p_{r|\pi_1}(\Psi(r)|y)$ is Lebesgue integrable and bounded on $[0, \Psi(\pi_1)]$, one can take a limit as $\epsilon \downarrow 0$ to obtain

$$\int_0^{\Psi(\pi_1)} p_{\pi_1}(y)\, p_{r|\pi_1}(\Psi(r)|y)\, dy = \int_{\pi_1}^{r} p_{r|s}(\Psi(r)|\Psi(s))\, \alpha(s) ds \tag{5}$$

for all $r \in (\pi_1, \pi_2]$, because $\int_0^{\Psi(\pi_1)} \alpha(s|y) p_{\pi_1}(y)\, dy = \alpha(s)$ by the law of total probability. The total probability of visiting the epigraph of the boundary on

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$[\pi_1, \pi_2]$ can be found by integration:

$$\alpha = \Pr\{Q(r) \geq \Psi(r) \text{ for some } r \in [\pi_1, \pi_2]\} = \int_{\Psi(\pi_1)}^{+\infty} p_{\pi_1}(y)\,dy + \int_{\pi_1}^{\pi_2} \alpha(r)dr. \tag{6}$$

The first term on the right side of (6) accounts for those trajectories that are already above the boundary prior to π_1, and the second terms account for the trajectories crossing the boundary from below after π_1.

Note that the expression $p_{r|s}(\Psi(r)|\Psi(s))$ under the integral sign on the right side of (5) diverges to infinity as s gets close to r, which indicates the presence of singularities, making the problem ill posed. The integral equation (5) thus belongs to the class of so-called Volterra equations of the first kind with a weak singularity of Abel type.

4. NUMERICAL IMPLEMENTATION

In this section we outline the numerical procedures we run to calculate the p-values. Often these procedures are standard and can be found in sources familiar to economists (e.g., Judd, 1998), but in some instances additional measures are taken to increase the speed and account for specific features of the problem. The integral at the left side of (5) is not problematic. We numerically evaluate it for any r by the very efficient Clenshaw–Curtis quadrature (Clenshaw and Curtis, 1960) based on exact integration of a polynomial approximation of the integrand over intervals on a grid formed by roots of Chebychev polynomials of a certain degree. We use the degree equaling 100, which gives a wittingly high accuracy of order 10^{-12}. In what follows, we no longer discuss evaluation of the left side of (5).

Our purpose is to obtain a good numerical approximation of $\alpha(r)$ from the integral equation (5) and then integrate it using the equality (6). For a very small ε, we split the interval $[\pi_1, \pi_2]$ into a union of segment $[\pi_1, \pi_1 + \varepsilon]$ and N segments of equal size Δ that form the grid $\{r_i = \pi_1 + \varepsilon + i\,\Delta\}_{i=0}^{N}$. Also, when $r > \pi_1 + \varepsilon$ we split the integral at the right side of (5) into two: one is

$$I_\varepsilon(r) \equiv \int_{\pi_1}^{\pi_1+\varepsilon} p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)ds,$$

and the other is

$$J_\varepsilon(r) \equiv \int_{\pi_1+\varepsilon}^{r} p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)ds.$$

Note that both integrals have singularities in $p_{r|s}(\Psi(r)|\Psi(s))$ but the former one also has a singularity in $\alpha(s)$ at π_1. Accordingly, our numerical approximation of $\alpha(r)$ is obtained differently on $(\pi_1, \pi_1 + \varepsilon]$ and on $(\pi_1 + \varepsilon, \pi_2]$. On $(\pi_1, \pi_1 + \varepsilon]$, we use a series expansion for $\alpha(s)$ and $p_{r|s}(\Psi(r)|\Psi(s))$ under the

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integral sign of the right side of (5) to derive an analytical approximation for $\alpha(r)$. On $(\pi_1+\varepsilon,\pi_2]$, we approximate the true $\alpha(r)$ by a piecewise linear function with values (say) $\{\bar\alpha_i\}_{i=0}^N$ at the gridpoints $\{r_i\}_{i=0}^N$. Some details on this algorithm follow.

Using the findings from Pötzelberger and Wang (2001) adapted to the squared Bessel process, we expand $\alpha(s)$ in the vicinity of π_1 in the following way:

$$\alpha(s)=\frac{a_{-1}}{\sqrt{s-\pi_1}}+a_0+a_1\sqrt{s-\pi_1}+a_2(\sqrt{s-\pi_1})^2+a_3(\sqrt{s-\pi_1})^3 +O\left((\sqrt{s-\pi_1})^4\right),$$

where a_{-1}, a_0, a_1, a_2, and a_3 are unknown coefficients. At the same time, from its explicit form the asymptotics of $p_{r|s}(\Psi(r)|\Psi(s))$ in the vicinity of π_1 is

$$p_{r|s}(\Psi(r)|\Psi(s))=\frac{b_{-1}(r)}{\sqrt{r-s}}+b_1(r)\sqrt{r-s}+O\left((\sqrt{r-s})^3\right),$$

where $b_{-1}(r)$ and $b_1(r)$ are certain known functions of r. For any value of r sufficiently close to π_1 one can plug the two expansions without the remainder terms into the right side integral in (5) to obtain an approximation for the coefficients a_{-1}, a_0, a_1, a_2, and a_3. Specifically, we use five values for r from the set $\{\pi_1+(j/5)\,\varepsilon\}_{j=1}^5$ to obtain five linear equations for those five unknowns. This gives us an analytical approximation for $\alpha(r)$ in the vicinity of π_1. As a by-product, we compute the value $\bar\alpha_0$, an approximation for $\alpha(\pi_1+\varepsilon)$, as $\bar\alpha_0=a_{-1}/\sqrt{\varepsilon}+a_0+a_1\sqrt{\varepsilon}+a_2(\sqrt{\varepsilon})^2+a_3(\sqrt{\varepsilon})^3$. In addition, we derive an approximate analytical representation of the integral $I_\varepsilon(r)$ for $r>\pi_1+\varepsilon$ to be used later. To that end, we use the asymptotic expansion for $\alpha(s)$ in the vicinity of π_1 and the trapezoid rule to approximate $p_{r|s}(\Psi(r)|\Psi(s))$.

Now the aim is to obtain a piecewise linear approximation on $(\pi_1+\varepsilon,\pi_2]$ represented by values $\{\bar\alpha_i\}_{i=0}^N$ at gridpoints $\{r_i\}_{i=0}^N$ (note that $\bar\alpha_0$ is already available). We solve the integral equation (5) numerically at $r\in\{r_i\}_{i=1}^N$ by applying the trapezoidal product integration method of Weiss (1972) that is designed to handle our type of singularities. Denote the segments between the gridpoints by $S_i\equiv[s_l^i,s_r^i]$, where $s_l^i=\pi_1+\varepsilon+(i-1)\Delta$ and $s_r^i=\pi_1+\varepsilon+i\Delta$, $i=1,\dots,N$. For given i and $r\ge s_r^i$, we approximate $\sqrt{r-s}\cdot p_{r|s}(\Psi(r)|\Psi(s))\,\alpha(s)$ as a function of s within S_i by a straight line $g(r,s)$ connecting its endpoints in S_i, i.e., by

$$g(r,s)=\frac{s_r^i-s}{\Delta}\sqrt{r-s_l^i}\cdot p_{r|s}\left(\Psi(r)|\Psi(s_l^i)\right)\alpha(s_l^i) +\frac{s-s_l^i}{\Delta}\sqrt{r-s_r^i}\cdot p_{r|s}\left(\Psi(r)|\Psi(s_r^i)\right)\alpha(s_r^i)$$

for all $s\in S_i$. The purpose of multiplication by $\sqrt{r-s}$ is to work with a function with no singularities. So, the integral $J_\varepsilon(r)$ is analytically approximated by

$$\int_{\pi_1+\varepsilon}^r\frac{g(r,s)}{\sqrt{r-s}}ds.$$

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Now, for any $r \in \{r_i\}_{i=1}^N$, the approximated right side of (5), $I_\varepsilon(r) + J_\varepsilon(r)$, has an analytical representation as a function of $\{\bar{\alpha}_j\}_{j=0}^i$. Thus we obtain, as i increases from 1 to N, a lower triangular system of N linear equations for N unknowns $\{\bar{\alpha}_i\}_{i=1}^N$ (note that it also involves $\bar{\alpha}_0$ computed before). This system is easily solved for $\{\bar{\alpha}_i\}_{i=1}^N$ in the way it is done in Gaussian elimination.

When the approximation for $\alpha(r)$ is computed, the p-value is numerically evaluated from the following version of (6), taking into account that the unconditional distribution is χ_p^2:

$$\alpha = \left(1 - F_{\chi_p^2}(c_\alpha)\right) + \int_{\pi_1}^{\pi_1+\varepsilon} \alpha(r)dr + \int_{\pi_1+\varepsilon}^{\pi_2} \alpha(r)dr. \tag{7}$$

Here the first integral is evaluated using the explicit series representation of $\alpha(r)$ in the vicinity of π_1, whereas the second integral is approximated by applying the trapezoid rule to the collection $\{\bar{\alpha}_i\}_{i=0}^N$ computed before.

By Brunner (1974), the accuracy in approximating $\alpha(r)$ on $(\pi_1, \pi_1 + \varepsilon]$ is of order $O(\varepsilon^2)$ because of the remainders in expansions in the vicinity of π_1. As the integration interval has length ε, the approximation error for the first integral in (7) is no larger than $O(\varepsilon^3)$. By Weiss (1972), the accuracy in approximating $\alpha(r)$ on $(\pi_1 + \varepsilon, \pi_2]$ is uniformly of order $O(\varepsilon^2) + O(1/N^2)$, where the first term is an effect of an approximation error in $\bar{\alpha}_0$ and $I_\varepsilon(r)$ and the second term is an effect of approximations by trapezoidal product integration on the N point grid. As the integration interval is bounded, the approximation error for the second integral in (7) is of order $O(\varepsilon^2) + O(1/N^2)$. Summarizing, the accuracy in evaluating the p-value α is, too, of order $O(\varepsilon^2) + O(1/N^2)$.

After numerical experimentation, we set $\varepsilon = 0.001$ and $N = 500$, which proves to be sufficient to evaluate p-values with a relative error of 0.01%. A GAUSS code with this setup computes a p-value for a maximum of 17 seconds.[1] The critical values computed via our numerical method are the same as those tabulated in Estrella (2003) in all cases.

5. CONCLUDING REMARKS

Among other things, the proposed method highlights the hidden potential of some statistical results documented in the literature but underused in contexts different from those they were initially intended for. We illustrate this here for the theory of first passage probabilities represented by Fortet (1943) and Durbin (1971), which in econometrics is used only in the context of sequential testing.

There are potentially other stability tests whose critical values may be tabulated likewise. Still, for this to happen there are several prerequisites. First of all, the aggregation of the sequential statistic must be made using the supremum functional, which only allows for association of rejection with crossing a boundary. Another requirement is tractability of conditional densities for the asymptotic process, which includes Markovianity and easy computability.

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NOTE

1. The GAUSS code is available at www.nes.ru/˜sanatoly/Papers/AndrewsTest.htm.

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