THE REAL ZEROS OF THE CONFLUENT HYPERGEOMETRIC FUNCTION

By L. J. SLATER

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1. *Introduction.* This paper contains a discussion of various points which arise in the numerical evaluation of the small real zeros of the confluent hypergeometric function

$$M(a, b; x) = \sum_{n=0}^{\infty} (a)_n x^n/(b)_n n!$$

where $$(a)_n = a(a+1)(a+2)\dots(a+n-1).$$

There are two distinct problems, first the determination of those values of x for which $M(a, b; x) = 0$, given a and b, and secondly the study of the curves represented by $M(a, b; x) = 0$, for fixed values of x. These curves all lie on the surface $M(a, b; x) = 0$, of course.

A series is developed for a zero $x = c$ in terms of functions of its first approximation k. Similar expansions are given for the derivative $M'(c)$, for the zero c' of the derivative and for $M(c')$ the turning value of the function. These results are an extension of analogous expansions for the Airy integrals given by Miller (3). The four results are suitable for numerical calculation by hand methods.

Cyclic approximations are given for these four functions. These are essentially the familiar Newton method of approximation based on the expansions in series, but being simpler in algebraic structure they are more convenient for use on electronic calculators. The corresponding approximations for the cases $M(a, b; x) = 0$, b and x fixed, and $M(a, b; x) = 0$, a and x fixed, are discussed and sketches are given of the curves $M(a, b; x) = 0$ for $x = 1$ and $x = 2$. All these methods of calculation have been programmed and tested on the electronic calculator EDSAC I in the University Mathematical Laboratory, Cambridge, by the author. The paper concludes with a table of the smallest real zeros in x of $M(a, b; x)$ over the range

$$a = -4{\cdot}0\,(0{\cdot}2) - 2{\cdot}0\,(0{\cdot}1) - 0{\cdot}1, \quad b = 0{\cdot}1\,(0{\cdot}1)\,1{\cdot}2\,(0{\cdot}2)\,2{\cdot}4.$$

2. *Expansions for the zeros.* Let $y(x)$ be any solution of

$$xy'' + (b-x)y' = ay, \tag{1}$$

the differential equation satisfied by $M(a, b; x)$, and let k be a first approximation to a zero

$$c = k + h \tag{2}$$

of $y(x)$. Then a second approximation is $k - u$, where $y(k)/y'(k) = u$, by Newton's method. We note that c is a constant, independent of k and that we can express h as a function of u and k where u is itself a function of k. Then, by differentiating (2) with respect to k, we find

$$1 + \frac{\partial h}{\partial k} + \frac{\partial h}{\partial u}\frac{du}{dk} = 0.$$

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But $y/y' = u$, and so
$$\frac{du}{dk} = 1 - u^2 y''/y.$$

Also, by (1) $y''/y = a/k + (k-b)/u$. Hence
$$\frac{du}{dk} = 1 - u^2 a k^{-1} - u(k-b),$$

and so
$$0 = 1 + \frac{\partial h}{\partial k} + \frac{\partial h}{\partial u}[1 - u^2 a k^{-1} + u(b-k)]. \tag{3}$$

Suppose that $h = -u + a_2 u^2 + \ldots + a_n u^n + \ldots$, where $a_n = a_n(k)$. Then
$$\frac{\partial h}{\partial k} = \frac{\partial a_2}{\partial k} u^2 + \ldots + \frac{\partial a_n}{\partial k} u^n + \ldots,$$

and
$$\frac{\partial h}{\partial u} = -1 + 2a_2 u + \ldots + (n+1)\, a_{n+1} u^n + \ldots.$$

If we substitute these values in (3) and equate the coefficient of u^n to zero, we find
$$2a_2 + k - b = 0,$$

$$\frac{\partial a_2}{\partial k} + 3a_3 + \frac{a}{k} + 2a_2(b-k) = 0,$$

and in general
$$\frac{\partial a_n}{\partial k} + (n+1)\, a_{n+1} - (n-1)\, a_{n-1} \frac{a}{k} - n a_n (k-b) = 0. \tag{4}$$

When we solve these difference equations we have, after some reduction,
$$c = k - u + a_2 u^2 + \ldots + a_n u^n + \ldots, \tag{5}$$

where

$a_2 = \frac{1}{2}(b-k)$,

$a_3 = \frac{1}{3}(\frac{1}{2} - ak^{-1} - (b-k)^2)$,

$a_4 = -\frac{1}{4}[\frac{1}{3} ak^{-2} + (b-k)\{\frac{7}{6} - 2ak^{-1} - (b-k)^2\}]$,

$a_5 = -\frac{1}{5}[(b-k)^4 - (b-k)^2(\frac{23}{12} - 3ak^{-1}) - \frac{5}{6}(b-k)\, ak^{-2} + \frac{7}{24} + a^2 k^{-2} + \frac{1}{6} ak^{-3} - ak^{-1}]$,

and in general
$$a_{n+1} = \frac{1}{n+1}\left[n(k-b)\, a_n - \frac{\partial a_n}{\partial k} + \frac{a}{k}(n-1)\, a_{n-1}\right].$$

To find an expansion for the value of the derivative $y'(c)$ at any zero $x = c$, let us write
$$y'(c) = y'(k)\,(1+\lambda).$$

Then λ satisfies the partial differential equation
$$0 = \left(\frac{a}{k} + \frac{k-b}{u}\right)(1+\lambda) + \frac{1}{u}\frac{\partial \lambda}{\partial k} + \left(\frac{1}{u} - \frac{ua}{k} + b - k\right)\frac{\partial \lambda}{\partial u}. \tag{6}$$

Let
$$\lambda = a_1 u + a_2 u^2 + \ldots + a_n u^n + \ldots.$$

Then
$$\frac{\partial \lambda}{\partial k} = \frac{\partial a_1}{\partial k} u + \frac{\partial a_2}{\partial k} u^2 + \ldots + \frac{\partial a_n}{\partial k} u^n + \ldots,$$

and
$$\frac{\partial \lambda}{\partial u} = a_1 + 2a_2 u + 3a_3 u^2 + \ldots + (n+1)\, a_{n+1} u^n + \ldots.$$

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When these values are substituted in (6) we find

$$ka_1+k(k-b)=0,$$

$$a+k(k-b)a_1+k\frac{\partial a_1}{\partial k}+2ka_2-k(k-b)a_1=0,$$

and in general

$$a_{n+1}=\frac{1}{n+1}\left[(k-b)(n-1)a_n+\frac{a}{k}(n-2)a_{n-1}-\frac{\partial a_n}{\partial k}\right]. \tag{7}$$

Hence
$$y'(c)=y'(k)(1+a_1u+a_2u^2+\ldots+a_nu^n+\ldots), \tag{8}$$
where

$$a_1=b-k,$$
$$a_2=\tfrac{1}{2}(1-ak^{-1}),$$
$$a_3=\tfrac{1}{3}[\tfrac{1}{2}(1-ak^{-1})(k-b)-\tfrac{1}{2}ak^{-2}],$$
$$a_4=\tfrac{1}{4}[\tfrac{1}{3}(k-b)^2(1-ak^{-1})-\tfrac{1}{2}(k-b)ak^{-2}-\tfrac{1}{2}(1-ak^{-1})(\tfrac{1}{3}-ak^{-1})-\tfrac{1}{3}ak^{-3}],$$
$$a_5=\tfrac{1}{5}[\tfrac{1}{4}(k-b)^3(1-ak^{-1})-\tfrac{11}{24}(k-b)^2ak^{-2}+(k-b)\{(1-ak^{-1})(\tfrac{17}{24}ak^{-1}-\tfrac{7}{24})-\tfrac{1}{2}ak^{-2}\}$$
$$-\tfrac{7}{12}ak^{-2}(ak^{-1}-\tfrac{1}{2}+\tfrac{3}{7}k^{-2})].$$

This expansion is useful for general solutions of (1), but it is not always necessary to use it for $M'(a,b;c)$ when c is known since

$$M'(a,b;x)=\frac{a}{b}M(a+1,b+1;x), \tag{9}$$

and satisfactory programmes exist for the evaluation of the series $M(a,b;x)$ on electronic computers.

Next let us suppose c' is a turning value of $y(x)$, that is to say, c' is a zero of $y'(x)$, with a first approximation k'. Also let

$$v=y'(k')/k'^2y(k').$$

Then, proceeding in a similar way, we have

$$c'=k'+a_1v+a_2v^2+\ldots+a_nv^n+\ldots, \tag{10}$$

where

$$a_1=-k'^3/a,$$
$$a_2=k'^5[k'(k'-b)+1]/2a^2,$$
$$a_3=-k'^7[k'a+\tfrac{1}{2}(7k'^2-6k'b+5)+(k'^2-k'b-2)(k'^2-k'b+1)]/3a^3,$$
$$a_4=k'^9[(3k'^2-3k'b-6+7k')(ak'+\tfrac{5}{2}k'^2-2bk'+\tfrac{1}{2}+k'^4-2bk'^3+b^2k'^2)$$
$$+ak'+5k'^2-2bk'+4k'^4-6k'^3b+2k'^2b^2+3ak'(k'^2-bk'+1)]/12a^4,$$

and in general

$$a_{n+1}=k'^3\left[k'^2(n-1)a_{n-1}+\left(b-k'+\frac{2}{k'}\right)na_n-\frac{\partial a_n}{\partial k'}\right]\Big/(n+1)a. \tag{11}$$

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Also for the actual value of the solution at its turning point c' we find that

$$y(c') = y(k')(1 + a_1 v + a_2 v^2 + \ldots + a_n v^n + \ldots), \tag{12}$$

where

$$a_1 = 0,$$
$$a_2 = k'^5/2a,$$
$$a_3 = k'^2[k'(k'-b) + \tfrac{1}{2}]/3a^2,$$
$$a_4 = k'^9[-9k'^2 + 8bk' - \tfrac{7}{2} - \tfrac{3}{2}ak' - (3k'^2 - 3bk' - 6)(k'^2 - bk' + \tfrac{1}{2})]/12a^3$$

and in general

$$a_{n+1} = k'^3\left[k'^2(n-2)a_{n-1} - \frac{\partial a_n}{\partial k'} - n\left(k' - b - \frac{2}{k'}\right)a_n\right]\Big/(n+1)a. \tag{13}$$

Again for the solution $M(a, b; x)$ if a table of the zero c is available over a sufficiently wide range of a and b, the values of c' can be deduced directly from the table. A reasonable first approximation for the k of the solution $M(a, b; x)$ is

$$k = \pi^2(N + \tfrac{1}{2}b - \tfrac{3}{4})^2/\{2(b - 2a)\}, \tag{14}$$

where x_N is the Nth real positive zero from the plane $x = 0$. This approximation, due to Tricomi, is effective as a starting value even for $N = 1$.

The accuracy of the above approximation in series is limited and varies according to the values of a and b, and the accuracy of the initial approximation k. Two calculations starting from different initial values of k should be made, so that the number of significant figures can be determined.

3. *Nesting processes.* The expansion (5) is useful in the evaluation of the zeros in x of $M(a, b; x)$ with the help of an ordinary calculating machine, but if high accuracy is required, and an electronic calculator is available, it is simpler to draw up a programme based on the nesting processes which will now be discussed. For the solution $M(a, b; x)$ let us calculate an initial estimate of the zero, k, by (14), and using Newton's approximation, let

$$x_1 = k - y(k)/y'(k).$$

Now by (9) $y'(k)$ is known and we have

$$x_1 = k - \frac{bM(a, b; k)}{aM(a+1, b+1; k)}.$$

Similarly

$$x_2 = x_1 - \frac{bM(a, b; x_1)}{aM(a+1, b+1; x_1)},$$

and so the cycle proceeds until

$$x_{n+1} = x_n - \frac{bM(a, b; x_n)}{aM(a+1, b+1; x_n)}, \tag{15}$$

and the latest correction made to x_n is negligible. This method gives seven-decimal accuracy after only four or five cycles with

$$N = 1, \quad a = -4{\cdot}0\,(0{\cdot}1) - 0{\cdot}1 \quad \text{and} \quad b = 0{\cdot}1\,(0{\cdot}1)\,2{\cdot}5.$$

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For $y(c)$, c' and $y(c')$, similar processes for the solution $M(a,b;x)$ which are convenient for use on electronic computers are given by

$$y'(x_{n+1}) = \frac{a}{b} M(a+1,b+1;x_n) + (b-x_n) M(a,b;x_n), \tag{16}$$

$$x'_{n+1} = x'_n - \frac{x'_n M(a+1,b+1;x'_n)}{bM(a,b;x'_n)} \tag{17}$$

and

$$y(x'_{n+1}) = M(a,b;x'_n) - \frac{x'_n a M(a+1,b+1;x'_n)}{2b^2 M(a,b;x'_n)}, \tag{18}$$

where

$$x'_0 = k' = \pi^2(N+\tfrac{1}{2}b-\tfrac{1}{4})^2/2(b-2a-1). \tag{19}$$

4. *Zeros in a*, We now seek an answer to the question: for what values of a does $M(a,b;x) = 0$, when b and x are fixed? Let us express $M(a+\epsilon,b;x)$ as a series in ϵ. We find by direct expansion that

$$M(a+\epsilon,b;x) = \sum_{t=0}^{\infty} \epsilon^t \sum_{r=t}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \Sigma_{r,t}, \tag{20}$$

where

$$\Sigma_{r,t} = \sum_{p_0=0}^{r-t} \sum_{p_1=1+p_0}^{r-t+1} \cdots \sum_{p_t=p_{t-1}+1}^{r-1} \frac{1}{(a+p_1)(a+p_2)\ldots(a+p_t)}, \tag{21}$$

that is,

$$M(a+\epsilon,b;x) = M(a,b;x) + \epsilon \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \sum_{n=1}^{r} \frac{1}{a+n-1} + O(\epsilon^2). \tag{22}$$

Hence, neglecting ϵ^2, we have

$$\epsilon = [M(a+\epsilon,b;x) - M(a,b;x)]/A(a),$$

where

$$A(a) = \sum_{r,1} = \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \sum_{n=1}^{r} \frac{1}{a+n-1} \tag{23}$$

$$= \frac{\partial}{\partial a} M(a,b;x). \tag{24}$$

Thus, given a first approximation a_0 to a zero a of $M(a,b;x)$, since we have

$$a_1 = a_0 - M(a_0,b;x) \Big/ \frac{\partial}{\partial a_0} M(a_0,b;x)$$

as our second approximation, in general the cyclic process is given by

$$a_n = a_{n-1} - M(a_{n-1},b;x)/A(a_{n-1}). \tag{25}$$

For example, if $b = 2$ and $x = 1$, a first approximation to the real zero nearest $a = 0$ is

$$a_0 = -2{\cdot}8.$$

Also by direct calculation $M(a_0,b;x) = 0{\cdot}00829$ and $A(a_0) = 0{\cdot}1771$.

Hence $a_1 = -2{\cdot}7532$. In fact $a = -2{\cdot}75358$, so that only one cycle here has given two further significant figures. The value of the derivative at the zero a is of course $A(a)$.

For the value of the zero a' of the derivative it is necessary to calculate

$$\partial^2 M(a,b;x)/\partial a^2.$$

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We have, by direct expansion of $A(a+\epsilon)$,

$$A_1(a) = \partial A(a)/\partial a$$
$$= \sum_{r=2}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\left(\frac{1}{a} + \ldots + \frac{1}{a+r-1} \right)^2 - \frac{1}{a^2} - \ldots - \frac{1}{(a+r-1)^2} \right]. \tag{26}$$

Thus, given a first approximation a_0' to a zero of the derivative $A(a)$, we have

$$a_1' = a_0' - A(a_0)/A_1(a_0),$$

and, in general, the cyclic process is given by

$$a_{n+1}' = a_n' - A(a_n')/A_1(a_n'). \tag{27}$$

The value of the function $M(a, b; x)$ at a zero a' of its derivative with respect to a can then be calculated by direct evaluation of

$$y(a') = M(a', b; x).$$

5. *The zeros in b.* Next we require to find for what values of b, $M(a, b; x) = 0$, when a and x are fixed; thus let us expand $M(a, b+\eta; x)$ in powers of η. Then

$$M(a, b+\eta; x) = M(a, b; x) + \eta B(b) + O(\eta^2), \tag{28}$$

where
$$B(b) = -\sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \sum_{n=1}^{r} \frac{1}{b+n-1}, = \frac{\partial M(a, b; x)}{\partial b}, \tag{29}$$

and the required cyclic process is given by

$$b_n = b_{n-1} - M(a, b_{n-1}; x)/B(b_{n-1}). \tag{30}$$

The value of the derivative $\partial M(a, b; x)/\partial b$ at the zero β is of course $B(\beta)$.

For the zero β' of this derivative it is necessary to calculate $\partial^2 M(a, b; x)/\partial b^2$. We have by direct expansion in series

$$B(b+\eta) = B(b) + \eta B_1(b) + O(\eta^2), \tag{31}$$

where
$$B_1(b) = \sum_{r=1}^{\infty} \frac{(a)_r x^r}{(b)_r r!} \left[\left(\frac{1}{b} + \ldots + \frac{1}{b+r-1} \right)^2 + \frac{1}{b^2} + \ldots + \frac{1}{(b+n-1)^2} \right]. \tag{32}$$

Then
$$\partial^2 M(a, b; x)/\partial b^2 = \partial B(b)/\partial b = B_1(b),$$

and so the cyclic process is given by

$$b_n' = b_{n-1}' - B(b_{n-1}')/B_1(b_{n-1}'). \tag{33}$$

The value of
$$y(\beta') = M(a, \beta'; x)$$

can then be evaluated directly from its expansion in series.

6. *The distribution of the real zeros.* The problem of the distribution of the real zeros of $M(a, b; x)$ in x has been studied by Tricomi ((5), § 3·5) and the corresponding problem for the Whittaker function has been discussed by Buchholz ((1), § 1·7). A synopsis of these results and references to other work is given by Erdelyi ((2), § 6·16). The distribution of the zeros in a and b does not seem to have received much attention. Several substantial tables of $M(a, b; x)$ have been constructed by the author, using the

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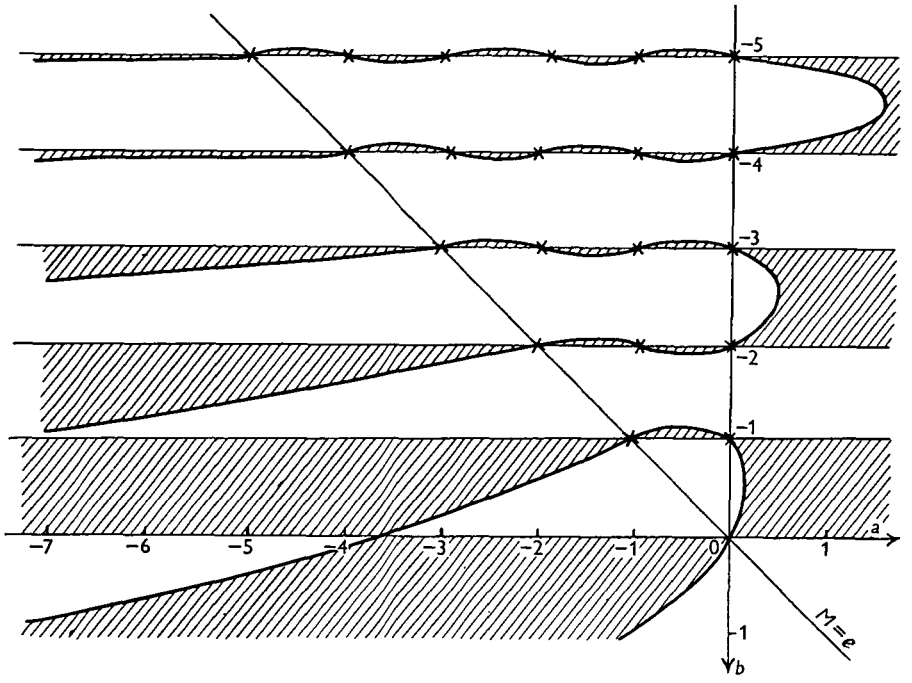


Fig. 1. $x=1$.

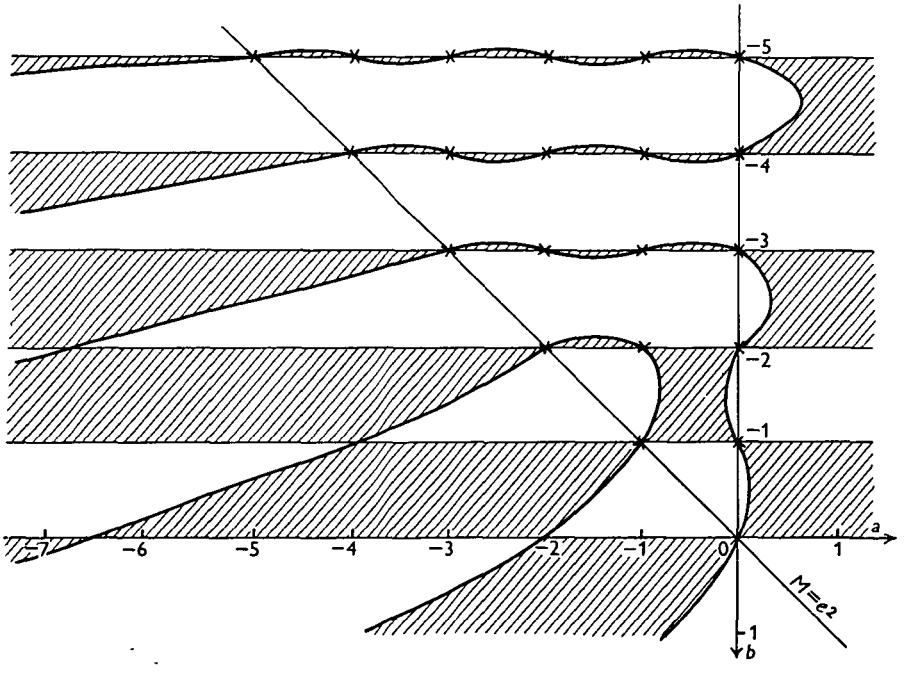


Fig. 2. $x=2$.

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electronic calculator EDSAC I. These include a table calculated by direct summation of series, over the range

$$a = -1{\cdot}0\,(0{\cdot}1)\,1{\cdot}0, \quad b = 0{\cdot}1\,(0{\cdot}1)\,1{\cdot}0, \quad x = 0{\cdot}1\,(0{\cdot}1)\,6{\cdot}5\,(0{\cdot}5)\,10{\cdot}0,$$

from which were made the first estimates of the real positive zeros in x over this range of a and b. Part of this table has been published (4). Also the author has constructed two tables, using recurrence relations, over the range

$$a = -10{\cdot}0\,(0{\cdot}1)\,2{\cdot}0, \quad b = -10{\cdot}0\,(0{\cdot}1)\,2{\cdot}0,$$

for $x = 1$ and $x = 2$. These have been used to check by interpolation the accuracy of the approximations developed in §§ 4 and 5 above for the zeros in a and b. They were also used to verify that the actual structure of the function $M(a, b; x)$ is in agreement with the theoretical results of Tricomi.

The two figures sketch the structure of the function revealed by these tabulations, the shaded areas being the regions in which the function is negative. Fig. 1 shows the curves in which the surface $M(a, b; x) = 0$ cuts the plane $x = 1$. The function $M(a, b; x)$ is not defined on the vertical planes $b = 0, -1, -2, \ldots$ and it can assume any value at the vertical lines through the points $(0, 0)$, $(0, -1)$, $(-1, -1)$, $(0, -2)$, $(-1, -2)$, $(-2, -2), \ldots$ marked with crosses. On the vertical plane $a = 0$, $M(a, b; x) = 1$ for all b, and on the vertical plane $a = b$, $M(a, b; x) = e^x$ for all a and x. Fig. 2 shows the curves in which the surface $M(a, b; x) = 0$ cuts the plane $x = 2$. It can be seen on comparing the two sketches that the function has a saddle point between $x = 1$ and $x = 2$. By direct calculation on EDSAC I this was found to be at

$$a = -0{\cdot}34235, \quad b = -1{\cdot}5708, \quad x = 1{\cdot}4564.$$

At this point $M(a, b; x)$ has a maximum in b, a minimum in a and a zero in x. Similar saddle points will occur in each strip $-N < b < -N + 1$. The second point is near $a = -1{\cdot}38$, $b = -2{\cdot}55$, $x = 2{\cdot}47$.

7. *A table of zeros in x.* It is well known that any analytic function can be expressed as an infinite product in terms of its zeros (Watson (6), § 7·5). Since the confluent hypergeometric function is analytic except when b is zero or a negative integer, we have

$$M(a, b; x) = e^{ax/b} \prod_{n=0}^{\infty} (1 - x/x_n)\, e^{x/x_n},$$

where $x_0, x_1, \ldots$ are the real and imaginary zeros of the function. This fact leads us to suppose that

$$(1 - x/x_0) \exp(ax/b + x/x_0),$$

where x_0 is the real zero nearest to the plane $x = 0$, will provide a useful first approximation to the numerical value of $M(a, b; x)$ for values of x near x_0.

The following table is of the values of x_0 for which $M(a, b; x) = 0$, over the range

$$a = -4{\cdot}0\,(0{\cdot}2) - 1{\cdot}0\,(0{\cdot}1) - 0{\cdot}1, \quad b = 0{\cdot}1\,(0{\cdot}1)\,1{\cdot}2\,(0{\cdot}2)\,2{\cdot}4.$$

It forms part of a larger table over the range

$$a = -4{\cdot}0\,(0{\cdot}1) - 0{\cdot}1, \quad b = 0{\cdot}1\,(0{\cdot}1)\,2{\cdot}5.$$

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This table was calculated on EDSAC I to eight decimal places. Hence the error is not more than two units in the seventh decimal.

No serious attempt has been made to study the next surface on which $M(a, b; x_1) = 0$. However, a single run on EDSAC I, lasting about one hour, gave the following values of the next zero x_1, and these will provide a helpful starting point for further study.

Values of x_1 at which $M(a, b; x) = 0$

$b =$	0·1	0·2	0·3	0·4	0·5
$a = -2·0$	2·14880 88	2·29544 51	2·44017 54	2·58321 60	2·72474 49
$-1·9$	2·28349 16	2·43664 54	2·58761 05	2·73663 47	2·88392 26
$-1·8$	2·43848 89	2·59883 17	2·75665 80	2·91225 38	3·06585 48
$-1·7$	2·61942 09	2·78775 08	2·95318 16	3·11604 53	3·27661 55

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Values of x_0 at which $M(a, b; x) = 0$

$b =$	0·1	0·2	0·3	0·4	0·5	0·6	a
$a = -4·0$	0·02590 78	0·05352 31	0·08270 43	0·11333 19	0·14530 35	0·17853 00	$-4·0$
$-3·8$	0·02725 38	0·05626 90	0·08689 57	0·11900 75	0·15249 63	0·18726 81	$-3·8$
$-3·6$	0·02874 73	0·05931 20	0·09153 49	0·12528 21	0·16043 92	0·19690 74	$-3·6$
$-3·4$	0·03041 41	0·06270 30	0·09669 76	0·13225 58	0·16925 64	0·20759 50	$-3·4$
$-3·2$	0·03228 60	0·06650 54	0·10247 80	0·14005 26	0·17910 09	0·21951 23	$-3·2$
$-3·0$	0·03440 34	0·07079 89	0·10899 38	0·14882 74	0·19016 35	0·23288 51	$-3·0$
$-2·8$	0·03681 82	0·07568 53	0·11639 53	0·15877 71	0·20268 59	0·24799 82	$-2·8$
$-2·6$	0·03959 76	0·08129 65	0·12487 64	0·17015 47	0·21697 82	0·26521 61	$-2·6$
$-2·4$	0·04283 09	0·08780 68	0·13469 20	0·18329 22	0·23344 53	0·28501 36	$-2·4$
$-2·2$	0·04663 93	0·09545 13	0·14618 45	0·19863 31	0·25262 62	0·30801 99	$-2·2$
$-2·0$	0·05119 11	0·10455 49	0·15982 46	0·21678 40	0·27525 51	0·33508 89	$-2·0$
$-1·8$	0·05672 79	0·11557 98	0·17627 75	0·23859 77	0·30235 84	0·36740 86	$-1·8$
$-1·6$	0·06360 79	0·12920 67	0·19651 56	0·26531 23	0·33541 75	0·40668 45	$-1·6$
$-1·4$	0·07238 79	0·14648 18	0·22201 97	0·29879 77	0·37665 43	0·45545 87	$-1·4$
$-1·2$	0·08398 14	0·16909 96	0·25516 18	0·34202 15	0·42956 49	0·51770 18	$-1·2$
$-1·0$	0·10000 00	0·20000 00	0·30000 00	0·40000 00	0·50000 00	0·60000 00	$-1·0$
$-0·9$	0·11054 47	0·22012 64	0·32894 15	0·43713 15	0·54480 16	0·65203 19	$-0·9$
$-0·8$	0·12357 83	0·24477 52	0·36411 44	0·48196 35	0·59858 98	0·71419 38	$-0·8$
$-0·7$	0·14010 11	0·27567 24	0·40779 72	0·53721 21	0·66443 91	0·78986 07	$-0·7$
$-0·6$	0·16173 42	0·31555 72	0·46354 99	0·60707 04	0·74705 02	0·88415 45	$-0·6$
$-0·5$	0·19128 98	0·36906 09	0·53728 03	0·69839 96	0·85403 26	1·00529 53	$-0·5$
$-0·4$	0·23411 73	0·44470 78	0·63961 58	0·82334 00	0·99868 55	1·16751 37	$-0·4$
$-0·3$	0·30182 31	0·56019 88	0·79200 44	1·00591 69	1·20695 84	1·39828 59	$-0·3$
$-0·2$	0·42537 31	0·75993 80	1·04632 32	1·30289 37	1·53918 36	1·76075 91	$-0·2$
$-0·1$	0·72703 16	1·20342 40	1·58016 05	1·90320 51	2·19258 90	2·45881 88	$-0·1$

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Values of x_0 at which $M(a, b; x) = 0$

	$b=$	0·7	0·8	0·9	1·0	1·1	1·2	a
$a=$	−4·0	0·21293 33	0·24844 41	0·28500 07	0·32254 76	0·36103 46	0·40041 59	−4·0
	−3·8	0·22324 09	0·26034 19	0·29850 64	0·33767 63	0·37779 91	0·41882 72	−3·8
	−3·6	0·23459 99	0·27344 05	0·31336 12	0·35430 11	0·39620 54	0·43902 45	−3·6
	−3·4	0·24718 02	0·28793 19	0·32977 86	0·37265 66	0·41650 88	0·46128 33	−3·4
	−3·2	0·26119 07	0·30405 16	0·34802 01	0·39302 97	0·43902 04	0·48593 85	−3·2
	−3·0	0·27689 08	0·32209 17	0·36840 95	0·41577 45	0·46412 43	0·51340 28	−3·0
	−2·8	0·29460 70	0·34241 91	0·39135 23	0·44133 41	0·49229 95	0·54419 03	−2·8
	−2·6	0·31475 62	0·36550 06	0·41736 38	0·47027 00	0·52415 22	0·57895 02	−2·6
	−2·4	0·33787 94	0·39194 08	0·44710 88	0·50330 54	0·56046 12	0·61851 47	−2·4
	−2·2	0·36469 17	0·42253 61	0·48146 16	0·54138 80	0·60224 49	0·66396 96	−2·2
	−2·0	0·39615 95	0·45835 92	0·52159 51	0·58578 64	0·65086 23	0·71676 03	−2·0
	−1·8	0·43362 18	0·50089 05	0·56912 30	0·63823 96	0·70817 12	0·77885 71	−1·8
	−1·6	0·47899 14	0·55223 59	0·62633 11	0·70120 23	0·77678 52	0·85302 37	−1·6
	−1·4	0·53510 39	0·61550 11	0·69657 55	0·77826 37	0·86051 15	0·94327 18	−1·4
	−1·2	0·60635 92	0·69547 76	0·78500 75	0·87490 75	0·96514 25	1·05568 22	−1·2
	−1·0	0·70000 00	0·80000 00	0·90000 00	1·00000 00	1·10000 00	1·20000 00	−1·0
	−0·9	0·75888 50	0·86541 05	0·97164 85	1·07763 19	1·18338 79	1·28893 94	−0·9
	−0·8	0·82892 89	0·94291 59	1·05625 10	1·16901 22	1·28126 39	1·39305 95	−0·8
	−0·7	0·91376 55	1·03637 62	1·15786 85	1·27838 33	1·39803 52	1·51691 93	−0·7
	−0·6	1·01887 44	1·15158 21	1·28256 70	1·41205 79	1·54023 94	1·66726 23	−0·6
	−0·5	1·15298 99	1·29771 21	1·43991 63	1·57995 68	1·71811 55	1·85462 01	−0·5
	−0·4	1·33112 03	1·49044 27	1·64618 10	1·79887 13	1·94893 39	2·09670 44	−0·4
	−0·3	1·58200 88	1·75960 56	1·93215 19	2·10045 49	2·26513 62	2·42668 47	−0·3
	−0·2	1·97114 63	2·17271 84	2·36714 89	2·55566 24	2·73918 28	2·91842 47	−0·2
	−0·1	2·70808 56	2·94434 51	3·17028 02	3·38779 57	3·59829 52	3·80284 33	−0·1

	$b=$	1·4	1·6	1·8	2·0	2·2	2·4	a
$a=$	−4·0	0·48169 84	0·56610 04	0·65337 06	0·74329 19	0·83567 52	0·93035 39	−4·0
	−3·8	0·50342 89	0·59117 55	0·68180 66	0·77509 81	0·87085 49	0·96890 59	−3·8
	−3·6	0·52723 00	0·61859 93	0·71286 33	0·80979 05	0·90918 03	1·01085 71	−3·6
	−3·4	0·55341 51	0·64872 20	0·74692 55	0·84778 70	0·95110 13	1·05668 79	−3·4
	−3·2	0·58236 51	0·68196 76	0·78445 85	0·88959 28	0·99715 97	1·10697 54	−3·2
	−3·0	0·61454 76	0·71885 56	0·82603 08	0·93582 21	1·04801 44	1·16242 06	−3·0
	−2·8	0·65054 33	0·76002 91	0·87234 42	0·98723 25	1·10447 52	1·22388 34	−2·8
	−2·6	0·69108 28	0·80629 48	0·92427 68	1·04476 90	1·16755 08	1·29243 21	−2·6
	−2·4	0·73709 93	0·85867 94	0·98294 27	1·10962 80	1·23851 41	1·36941 29	−2·4
	−2·2	0·78980 50	0·91851 24	1·04978 00	1·18334 85	1·31900 00	1·45654 96	−2·2
	−2·0	0·85080 66	0·98754 84	1·12667 99	1·26794 92	1·41114 56	1·55609 11	−2·0
	−1·8	0·92228 37	1·06815 75	1·21618 80	1·36613 76	1·51780 94	1·67103 75	−1·8
	−1·6	1·00727 62	1·16363 04	1·32182 40	1·48164 49	1·64291 89	1·80549 97	−1·6
	−1·4	1·11017 15	1·27869 20	1·44862 03	1·61978 59	1·79204 96	1·96529 63	−1·4
	−1·2	1·23757 59	1·42041 84	1·60407 79	1·78845 01	1·97345 07	2·15901 03	−1·2
	−1·0	1·40000 00	1·60000 00	1·80000 00	2·00000 00	2·20000 00	2·40000 00	−1·0
	−0·9	1·49950 39	1·70945 03	1·91887 26	2·12784 39	2·33642 18	2·54465 28	−0·9
	−0·8	1·61545 64	1·83649 04	2·05637 59	2·27527 78	2·49332 55	2·71062 30	−0·8
	−0·7	1·75268 81	1·98619 02	2·21779 49	2·44778 31	2·67637 40	2·90374 20	−0·7
	−0·6	1·91831 19	2·16598 84	2·41086 13	2·65335 95	2·89381 47	3·13248 94	−0·6
	−0·5	2·12338 42	2·38740 53	2·64751 44	2·90433 06	3·15832 83	3·40987 98	−0·5
	−0·4	2·38641 01	2·66965 12	2·94760 48	3·22113 69	3·49090 48	3·75742 03	−0·4
	−0·3	2·74187 86	3·04839 74	3·34788 42	3·64153 18	3·93023 60	4·21468 86	−0·3
	−0·2	3·26622 24	3·60240 56	3·92926 25	4·24843 13	4·56112 85	4·86828 53	−0·2
	−0·1	4·19722 81	4·57578 31	4·94174 42	5·29739 43	5·64441 02	5·98406 45	−0·1

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