INHOMOGENEOUS LATTICES

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1. Mahler (1) has obtained a number of valuable existence theorems for homogeneous problems in the geometry of numbers. It appears that no analogous results for inhomogeneous problems have yet been published; and although none of them present great difficulty, it is convenient for applications to have them in print.

Let K be a point-set in n-dimensional Euclidean space. An inhomogeneous lattice Λ

$$x_i = \sum_{j=1}^{n} a_{ij}u_j + x_i^{(0)} \quad (i = 1, 2, \ldots, n;\ u_1, u_2, \ldots, u_n = 0, \pm 1, \ldots),$$

of determinant $d(\Lambda) = |a_{ij}|$ is called *K-admissible* if no point P of Λ is an interior point of K. The minimum determinant $\Delta(K)$ of K is defined as the lower bound of $d(\Lambda)$ over all K-admissible lattices Λ, and is conventionally infinite when there are no K-admissible lattices. In the cases which normally occur, $\Delta(K)$ is trivially finite and the problem is to prove it non-zero; by contrast, in the homogeneous problem $\Delta(K)$ is usually trivially non-zero and the problem is to prove it finite.

We shall say that the n-dimensional vector $Y = (y_1, y_2, \ldots, y_n)$ is a vector of Λ if it can be written as $P_1 - P_2$, the vector joining two points of Λ. The determinant of n vectors is defined, up to an arbitrary sign, in the obvious way. n vectors of Λ form a basis of Λ if and only if their determinant is $\pm d(\Lambda)$; and Λ is determined by knowing a basis and one point of it. We shall write

$$|Y| = (y_1^2 + y_2^2 + \ldots + y_n^2)^{\frac{1}{2}},$$

both for vectors and for points.

THEOREM 1. *There exists a constant $\gamma_n > 0$ depending only on n with the following property: every n-dimensional lattice Λ has a reduced basis, i.e. a basis $Y_1, Y_2, \ldots, Y_n$ for which*

$$|Y_1|\,|Y_2| \ldots |Y_n| \leqslant \gamma_n d(\Lambda).$$

DEFINITION 1. *An infinite sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is called bounded if there exist positive constants c_1, c_2 such that* $d(\Lambda_r) \leqslant c_1$ *for all r,*

$$|Y| \geqslant c_2 \text{ for all vectors } Y \text{ of any } \Lambda_r.$$

DEFINITION 2. *An infinite sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is said to converge to the lattice Λ if there are bases $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$ of Λ_r and $Y_1, Y_2, \ldots, Y_n$ of Λ, and points $P^{(r)}$ of Λ_r and P of Λ such that*

$$\lim |Y_i^{(r)} - Y_i| = 0 \quad (i = 1, 2, \ldots, n)$$

and

$$\lim |P^{(r)} - P| = 0.$$

These should be compared with the corresponding results in Mahler (1). Theorem 1 is the same; the Definitions and Theorem 2 below are purely formal generalizations to the non-homogeneous case.

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THEOREM 2. *Every bounded infinite sequence of lattices contains a convergent infinite subsequence.*

Let Λ_r be a general lattice of the sequence, and let $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_n^{(r)}$ be a reduced basis for Λ_r. By Theorem 1 and Definition 1 we have

$$| Y_i^{(r)} | \leqslant c_1 \gamma_n c_2^{1-n} \quad (i = 1, 2, \ldots, n).$$

Any point of space may be written uniquely as $\sum_{i=1}^{n} \xi_i Y_i^{(r)}$ for some real ξ_i, and the points of Λ_r are exactly specified by the n congruences

$$\xi_i \equiv \xi_i^{(0)} (\mathrm{mod}\, 1) \quad (i = 1, 2, \ldots, n).$$

Let $P^{(r)}$ be the point of Λ_r determined by $0 \leqslant \xi_i < 1$ $(i = 1, 2, \ldots, n)$; then we have

$$| P^{(r)} | < | Y_1^{(r)} | + | Y_2^{(r)} | + \ldots + | Y_n^{(r)} | \leqslant n c_1 \gamma_n c_2^{1-n}.$$

Thus each of the $n(n+1)$ coordinates of $P^{(r)}$ and the $Y_i^{(r)}$ is bounded independently of r. It follows from a well-known general theorem that we can choose a subsequence of the r such that the $n(n+1)$ corresponding subsequences of coordinates all converge. Let $P, Y_1, Y_2 \ldots, Y_n$ be the point and vectors whose coordinates are the limits of the corresponding subsequences of coordinates; then it is easy to see that the lattice Λ with basis $Y_1, Y_2, \ldots, Y_n$ through the point P is the limit of the corresponding subsequence of lattices Λ_r. This proves the theorem.

We note that if $\Lambda_1, \Lambda_2, \ldots$ converge to Λ then $d(\Lambda_r) \to d(\Lambda)$ and every lattice point of Λ is a limit point of lattice points of the Λ_r. Thus in particular the limit of a convergent sequence of K-admissible lattices is itself K-admissible.

2. In view of the applications we cannot confine ourselves to star bodies, and are therefore unable to make use of distance functions; this accounts for a good deal of the ensuing complications. Since the definition of *K-admissible* deals only with interior points of K, we shall assume that K is open and non-null. If Λ, P are respectively a lattice or a point, we shall use $\lambda\Lambda$, λP to denote the lattice or point obtained by λ-fold homothetic expansion about the origin.

LEMMA 1. *For any given Λ and K, the set of values of λ in $(-\infty, \infty)$ for which $\lambda\Lambda$ is not K-admissible is open and contains a neighbourhood of $\lambda = 0$.*

As in the proof of Theorem 2, there is a $\rho > 0$ depending only on Λ such that every sphere of radius ρ contains a point of Λ; and thus every sphere of radius $\rho | \lambda |$ contains a point of $\lambda\Lambda$. Since K is open, it contains a sphere of radius δ, say, and thus $\lambda\Lambda$ is not K-admissible for $| \lambda | < \delta\rho^{-1}$.

For the other part of the Lemma, let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers converging to λ such that $\lambda_r \Lambda$ is K-admissible for each r. Then $\lambda \neq 0$ and it is easy to see that $\lambda_r \Lambda$ converges to $\lambda\Lambda$ in the sense of Definition 2, so that $\lambda\Lambda$ is K-admissible. Thus the set of values of λ for which $\lambda\Lambda$ is K-admissible is closed, and this proves the Lemma.

It follows that if Λ is K-admissible, we can find a $\lambda_0 = \lambda_0(\Lambda)$ with $0 < \lambda_0 \leqslant 1$ such that $\lambda_0 \Lambda$ is K-admissible but $\lambda\Lambda$ is not K-admissible for any $0 < \lambda < \lambda_0$. We shall need this in the proof of Theorem 3.

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Now the boundary or frontier of K consists of those limit points of points of K which are not themselves points of K, and it is a closed set. It is intuitively obvious that in finding $\Delta(K)$ we may confine ourselves to those Λ which contain points in some sense near the boundary of K. We now make these concepts more precise.

DEFINITION 3. *We say that P is an exterior boundary point of K if it is a boundary point of K and there are points of K in the segment OP arbitrarily close to P.*

In effect, this means that P lies on a part of the boundary of K which faces away from O. For example, if K is $1 < xy < 2$, the exterior boundary points of K are the points of $xy = 2$. If O is a boundary point of K, it is not to be counted as an exterior boundary point.

Henceforth we write $\mathfrak{R}_0 = \mathfrak{R}_0(K)$ for the closure of the set of exterior boundary points of K, so that $\mathfrak{R}_0$ is a closed subset of the boundary of K. We write $\mathfrak{R}_\epsilon = \mathfrak{R}_\epsilon(K)$, where $\epsilon > 0$, for the set of all points which can be written in the form λP with $1 \leqslant \lambda \leqslant 1+\epsilon$, P in $\mathfrak{R}_0$; thus $\mathfrak{R}_\epsilon$ is closed.

THEOREM 3. *$\Delta(K)$ is the lower bound of $d(\Lambda)$ for those K-admissible Λ which contain at least one point of $\mathfrak{R}_\epsilon$ for every $\epsilon > 0$.*

Suppose that Λ is K-admissible, and let λ_0 have the meaning given to it above. Then $\lambda_0\Lambda$ is K-admissible and $d(\lambda_0\Lambda) = \lambda_0^n d(\Lambda) \leqslant d(\Lambda)$, so that we may confine ourselves to lattices $\lambda_0\Lambda$. We show that all such satisfy the conditions of the theorem.

The lattice $\lambda_0(1+\epsilon)^{-1}\Lambda$ is not K-admissible, and therefore contains a point of K, which we call $\lambda_0(1+\epsilon)^{-1}P$. The point $\lambda_0 P$, being a point of $\lambda_0\Lambda$, is not in K. Let μ be the least number in $\lambda_0(1+\epsilon)^{-1} < \mu \leqslant \lambda_0$ such that μP is not in K. By the properties of the Dedekind section, μ exists and μP is an exterior boundary point of K; since $1 \leqslant \lambda_0\mu^{-1} < 1+\epsilon$, it follows that $\lambda_0 P$ lies in $\mathfrak{R}_\epsilon$ and this proves the theorem.

This is as much as we can assert in general. For example, even when $\Delta(K) > 0$ it is false that there exists a K-admissible lattice of determinant $\Delta(K)$. We take K to be the region

$$|xy| < 1 + (x^2+y^2)^{-1}$$

and K' to be the region $|xy| < 1$. By a theorem of Minkowski, $\Delta(K') = 4$ and every K'-admissible lattice Λ with $d(\Lambda) = 4$ has at least one point on the boundary of K'. Since K' and its boundary lie entirely within K, every K-admissible lattice Λ must have $d(\Lambda) > 4$. On the other hand, the lattice Λ_r through $\left(r+\frac{1}{r}, \frac{1}{r}\right)$ with basis given by the vectors $\left(2+\frac{2}{r}, 0\right)$ and $\left(0, \frac{2}{r}\right)$ is K-admissible and has $d(\Lambda_r) = 4\left(1+\frac{1}{r^2}\right)$, for every non-zero integer r; thus $\Delta(K) = 4$.

Again, it is impossible in Theorem 3 to replace $\mathfrak{R}_\epsilon$ by $\mathfrak{R}_0$ or even by the whole boundary of K. Suppose that K consists of the whole plane with the exception of closed circles of radius $(m^2+n^2+1)^{-1}$ about each integral point (m, n). It is easy to see that the K-admissible lattices are precisely the sublattices of the integral lattice. Thus $\Delta(K) = 1$, but there are no K-admissible lattices which contain a point of the boundary of K.

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3. To obtain useful results, we must therefore impose further conditions on K, and for this we require the idea of an automorphism. The real homogeneous linear substitution

$$x_i' = \sum_{j=1}^{n} a_{ij}x_j \quad (i = 1, 2, \ldots, n),$$

where the determinant of the a_{ij} is non-zero, is called an *automorphism* of K if it leaves K invariant. It is easy to see that in this case it also leaves invariant the boundary of K and the regions $\mathfrak{R}_0$ and $\mathfrak{R}_\epsilon$. As in Mahler's Theorem 17, if K admits an automorph whose determinant is not ± 1, then $\Delta(K)$ is zero or infinite.

DEFINITION 4. *K is called automorphic if there is a constant c depending only on K with the following property: to every point X of the closure of K there corresponds an automorphism Ω of K with determinant ± 1 such that $|\Omega X| \leqslant c$.*

For the remainder of this paper we shall always suppose that K is automorphic, and shall denote by c the c of this definition.

LEMMA 2. *Let $\epsilon > 0$ be given. Then $\Delta(K)$ is the lower bound of $d(\Lambda)$ for those K-admissible Λ which contain a point P of $\mathfrak{R}_\epsilon$ such that $|P| \leqslant c(1+\epsilon)$.*

Suppose that Λ is a K-admissible lattice satisfying the conditions of Theorem 3. It is enough to obtain a K-admissible lattice Λ' satisfying the conditions of Lemma 2 with $d(\Lambda') = d(\Lambda)$. Now let P be a point of Λ in $\mathfrak{R}_\epsilon$. There is a point X of $\mathfrak{R}_0$ and a λ with $1 \leqslant \lambda \leqslant 1+\epsilon$ such that $P = \lambda X$. Now let Ω be an automorphism of K with determinant ± 1 such that $|\Omega X| \leqslant c$. Then ΩP lies in $\mathfrak{R}_\epsilon$ and has $|\Omega P| = \lambda |\Omega X| \leqslant c(1+\epsilon)$. Thus $\Omega\Lambda$ satisfies the conditions of the Lemma and we have $d(\Omega\Lambda) = d(\Lambda)$.

When $\Delta(K) > 0$ we say that Λ is a *critical* lattice if it is K-admissible and has $d(\Lambda) = \Delta(K)$. We can now state and prove our main result.

THEOREM 4. *Suppose that K is automorphic and satisfies the following condition: every straight line containing a point of $\mathfrak{R}_0$ contains a point of K. Then $\Delta(K) > 0$; and if $\Delta(K)$ is finite there exists a critical lattice of K containing a point of $\mathfrak{R}_0$.*

The proof of this consists of two quite independent parts; we therefore state Lemma 3 below as a half-way house. In practice it is just as useful as the theorem and has the advantage of giving an estimate for $\Delta(K)$; but it is considerably less elegant.

LEMMA 3. *Suppose that there exist $\delta > 0$, $\epsilon > 0$ with the following property: any line through any point P of $\mathfrak{R}_\epsilon$ such that $|P| \leqslant c(1+\epsilon)$ contains an interval of length δ lying in K. Then $\Delta(K) \geqslant \gamma_n^{-1}\delta^n$; and if $\Delta(K)$ is finite there exists a critical lattice of K containing a point of $\mathfrak{R}_0$.*

We know that we may confine ourselves to lattices Λ satisfying the conditions of Lemma 2. Now let Y be any lattice vector of Λ. Then $P + kY$ ($k = 0, \pm 1, \ldots$) is a point of Λ and therefore cannot lie in K. It follows from the conditions of the lemma that $|Y| \geqslant \delta$, and hence from Theorem 1 that $d(\Lambda) \geqslant \gamma_n^{-1}\delta^n$. This proves the first part of the lemma.

Now let $\epsilon_1 = \epsilon, \epsilon_2, \ldots$ be a decreasing sequence of strictly positive numbers whose limit is 0; then we can find a sequence of lattices Λ_r and a sequence of points $P^{(r)}$ such that

(i) Λ_r is K-admissible and $d(\Lambda_r) \to \Delta(K)$;

(ii) $P^{(r)}$ is a point of Λ_r in $\mathfrak{R}_{\epsilon_r}$ with $|P^{(r)}| \leqslant c(1+\epsilon_r)$.

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It follows that the Λ_r form an infinite bounded sequence, and so by Theorem 2 include a subsequence which converges to some lattice Λ. Clearly Λ is a critical lattice and contains all the points of accumulation of the $P^{(r)}$. These points of accumulation must lie in $\mathfrak{R}_0$ since they lie in every $\mathfrak{R}_{\epsilon_r}$, and this concludes the proof of the lemma.

I know of no case in which the bound for $\Delta(K)$ given by this lemma is best possible. This is because the homogeneous result we have chosen to use in obtaining it is that for the n-dimensional sphere; and though this is the natural body to use in an existence theorem, it is not the most suitable in any particular case. If we replace it by the n-dimensional cube, we can obtain best possible results for a number of bodies.

To prove Theorem 4, it remains to show that under the conditions of the theorem we can choose δ, ϵ so as to satisfy the conditions of Lemma 3. We consider the space of ordered pairs (P, Y), where P is a point and Y a unit vector in n-dimensional Euclidean space. We say that (P', Y') lies in an ϵ-neighbourhood of (P, Y) if $|P'-P|<\epsilon$ and $|Y'-Y|<\epsilon$. With the topology thus induced, the Heine-Borel Theorem holds for this space. With every pair (P, Y) there is associated the line through P parallel to Y. Now suppose that P is in $\mathfrak{R}_0$; then by the conditions of the theorem this line contains a point Q of K, and there is a sphere of radius η with centre Q lying entirely in K. We can now find a neighbourhood of (P, Y) so small that if (P', Y') is any element of it the line associated with (P', Y') has distance at most $\frac{1}{3}\eta$ from Q. Now consider the closed set of (P, Y) given by the conditions: P is in $\mathfrak{R}_0$ and $|P| \leqslant c$. By the Heine-Borel Theorem we can cover this with a finite number of the neighbourhoods defined above; in other words

There exist finite sets of points $Q_1, Q_2, \ldots, Q_N$ and of strictly positive constants $\eta_1, \eta_2, \ldots, \eta_N$ such that the sphere with centre Q_r and radius η_r lies in K for each $r = 1, 2, \ldots, N$; and to every line containing a point P of $\mathfrak{R}_0$ with $|P| \leqslant c$ there corresponds a Q_r whose distance from the line is at most $\frac{1}{3}\eta_r$.

Now if P' is a point whose distance from P is at most $\frac{1}{3}\eta_r$, then the line through P' parallel to the line through P considered above has distance from Q_r at most $\frac{2}{3}\eta_r$, and therefore intercepts on the sphere of centre Q_r and radius η_r an interval of length at least η_r. This interval, of course, lies entirely within K. It follows from these remarks that if we take $\delta = \operatorname{Min} \eta_r$, $\epsilon = \frac{1}{3}c^{-1}\delta$, the conditions of Lemma 3 will be satisfied. This completes the proof of the theorem.

We cannot improve Theorem 4 in any essential way. There is a body K satisfying the conditions of the theorem and having finite $\Delta(K)$ such that none of its critical lattices has more than one point on the boundary of K, while some of its critical lattices have no points on the boundary of K. The proof is a by-product of some results obtained by Dr E. S. Barnes and myself and will be published elsewhere.

It is essential to state Theorem 4 in terms of $\mathfrak{R}_0$ and not in terms of the whole boundary of K, in order to be able to apply it to regions such as $1 < xy < 2$; on the other hand we cannot replace $\mathfrak{R}_0$ by the set of all exterior boundary points (though we could of course do so in Lemma 3). Since any bounded region is *ipso facto* automorphic and has $\Delta(K) = 0$, we may take K to be

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} > 1 > x^{\frac{1}{2}} + y^{\frac{1}{2}}.$$

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For this region $\mathfrak{R}_0$ is given by $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$, and all points of $\mathfrak{R}_0$ are exterior boundary points except $(\pm 1, 0)$ and $(0, \pm 1)$. But these four are the only points of $\mathfrak{R}_0$ for which the conditions of the theorem are not satisfied. Similar remarks apply to the unbounded region given by

$$z^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) > 1 > z^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}), \quad z > 0.$$

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