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On the accumulation sets of exponential rays

JIANXUN FU† and GAOFEI ZHANG‡

† *Department of Mathematics, Nanjing University, Nanjing, 210093, PR China*
(e-mail: jianxunf@163.com)

‡ *Department of Mathematics, Qufu Normal University, Qufu, 273165, PR China*
(e-mail: zhanggf@nju.edu.cn)

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Abstract. We show that there exist non-landing exponential rays with bounded accumulation sets. By introducing folding models of certain rays, we prove that each of the corresponding accumulation sets is an indecomposable continuum containing part of the ray, an indecomposable continuum disjoint from the ray or a Jordan arc.

1. *Introduction*

Let $\lambda \neq 0$ and $E_\lambda(z) = \lambda e^z$. A point $z \in \mathbb{C}$ is called an escaping point if $E_\lambda^n(z) \to \infty$ as $n \to \infty$. It was known that all the escaping points of E_λ are organized in rays [**15**]. This result was recently extended to transcendental entire functions in the Eremenko–Lyubich class with finite order [**14**]. Such rays are called dynamic rays. According to [**15**], every escaping point of E_λ is either on a dynamic ray, or the landing point of a unique ray. As an analogue of Douady–Hubbard's theorem in polynomial dynamics (cf. [**9**, Theorem 18.10]), all periodic rays land for exponential maps with non-escaping singular value [**11**, **16**].

The non-landing rays for exponential maps were constructed in a series of works by several authors [**4**, **6**, **7**, **12**]. One of the common features of these non-landing rays is that each of the corresponding accumulation sets is unbounded and an indecomposable continuum containing some ray in the extended complex plane. Recently, Benini and Rocha constructed non-landing exponential rays through parabolic bifurcation but did not investigate the topologies of the accumulation sets [**1**]. Thus, it is natural to ask whether there exists a dynamic ray with bounded accumulation set and what kind of topology the accumulation set may have once such a ray exists. To answer it, we will prove the *further statement of the Main Theorem* (cf. §3.3) in this paper, which implies the following theorem.

MAIN THEOREM. *There exist non-landing rays for E_λ, each of whose accumulation sets is bounded and can be either an indecomposable continuum or a Jordan arc.*

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We construct these rays for E_λ with $\lambda = 2\pi i$. The construction can be adapted to all post-singularly finite exponential maps. The starting point of our proof is the folding phenomena, which has already been observed by Devaney, Jarque, Rocha and Rempe in their articles cited above: if a ray is close enough to the origin, then the preimages of the ray extend far away to negative infinity, and the further preimages will fold somewhere. The organization of our proof is as follows. First, we introduce the concept of the folding model. Then we choose the address of the ray carefully so that the folding of the ray realizes the prescribed model. Finally, we classify the folding models in a combinatorial way and prove that each of the corresponding accumulation sets is an indecomposable continuum containing part of the ray, an indecomposable continuum disjoint from the ray or a Jordan arc.

Remark 1.1. We would like to mention that the existence of the ray with bounded accumulation set was also obtained by Rempe and Worsley in their recent work [**13**]. In particular, they proved that, apart from certain necessary exceptions, the accumulation sets of such rays are arc-like.

The organization of the paper is as follows.

In §2, we first recall the basic concepts for the dynamics of E_λ: dynamical partition, itineraries, tails and rays. We then prove the hyperbolic expansion lemma (Lemma 2.2) which plays an essential role in this work.

In §3, we introduce the concept of the *folding model*. Then we describe the general idea for constructing the candidate ray from a given folding model. After this, we define the *running-back sequences* for folding models by which we can classify the corresponding accumulation sets into three types.

In §4, we prove the bounded-wiggling lemma (Lemma 4.1). We use the lemma to guarantee that within any two adjacent folding points of the ray, the wiggling is uniformly bounded.

In §5, we show that the candidate ray constructed in §3 realizes the given folding model.

In §6, we prove that the accumulation set of any candidate ray is a bounded subset of the plane.

In §7, we prove that all the corresponding accumulation sets can be classified into three types according to the running-back sequences: an indecomposable continuum containing part of the ray, an indecomposable continuum disjoint from the ray and a Jordan arc.

2. *Preliminaries*

In this section, we first recall some basic concepts and properties of the iteration of an exponential family. The reader may refer to [**5**, **10**] for a thorough background of this subject. We then prove a hyperbolic expansion lemma (cf. Lemma 2.2). To fix the idea let us set

$$\lambda = 2\pi i$$

and restrict to the map $E_\lambda : z \to \lambda e^z$ throughout the paper.

2.1. *Dynamical partition, itineraries, tails and rays.*

Note that λ is a repelling fixed point of E_λ. Following [**2**, **7**], let γ_λ be the unique ray of E_λ landing at λ and let η_λ

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be the pre-image of γ_λ which lands at the origin. Then $E_\lambda^{-1}(\eta_\lambda)$ consists of infinitely many disjoint curves, each of which extends from the left to right across the entire plane. We denote these curves by t_j, $j \in \mathbb{Z}$, so that $t_{j+1} = t_j + 2\pi i$ and zero lies in the strip bounded by t_1 and t_0. Let T_j, $j \in \mathbb{Z}$, be the open strip bounded above by t_{j+1} and below by t_j. Then, for $z \in \bigcup_{j\in\mathbb{Z}} T_j$, we can define the *itinerary* of z as the sequence of integers $\underline{s}(z) = (s_0 s_1 s_2 \cdots)$, where $s_j \in \mathbb{Z}$ and $s_j = k$ if and only if $E_\lambda^j(z) \in T_k$. Let $I(\underline{s})$ be the set consisting of all z with the itinerary $\underline{s}$.

For our purpose, we are only concerned with the itineraries of the form

$$(1, -1, 0, 1_{k_1}; 1, -1, 0, 1_{k_2}; \ldots; 1, -1, 0, 1_{k_n}; \ldots),$$

where 1_{k_i} denotes the block of 1 of length k_i for any $i \in \mathbb{N}^*$.

For $b > 0$, set

$$H_b = (T_{-1} \cup T_0 \cup T_1) \cap \{z \mid \mathrm{Re}(z) > b\}.$$

We say that the orbit of z *tends directly to* ∞ in H_b if the entire orbit of z lies in H_b and $\mathrm{Re}(E_\lambda^{n+1}(z)) > \mathrm{Re}(E_\lambda^n(z))$ for all $n \geq 0$. Let $\omega_{\underline{s}}(b)$ denote the set of points in H_b whose orbits tend directly to ∞ with itinerary $\underline{s}$.

LEMMA 2.1. (Cf. [**2**, **6**, **7**]) *For* $\underline{s} \in \{(s_0 s_1 s_2 \cdots) | |s_j| \leq 1 \text{ for each } j\}$, *there exists* $\zeta \in \mathbb{R}^+$ *such that* $\omega_{\underline{s}} = \omega_{\underline{s}}(\zeta)$ *is a continuous curve of the form* $(t, h_{\underline{s}}(t))$ *with* $\zeta \leq t < \infty$.

We call $\omega_{\underline{s}} = \omega_{\underline{s}}(\zeta)$ the *tail* of $I(\underline{s})$. For given $\underline{s} = (s_0 s_1 s_2 \cdots)$, let $\sigma(\underline{s}) = (s_1 s_2 \cdots)$. Denote by $L_{s_j} : \mathbb{C}\backslash\eta_\lambda \to T_{s_j}$, $j \in \mathbb{Z}$, the branches of the inverse of E_λ. Then the set

$$\gamma_{\underline{s}} = \bigcup_{n=0}^{\infty} L_{s_0} \circ \cdots \circ L_{s_{n-1}}(\omega_{\sigma^n(\underline{s})})$$

is called the dynamic ray with address $\underline{s}$. Note that the ray $\gamma_{\underline{s}}$ can be parameterized as a continuous curve $\gamma_{\underline{s}}(t)$, $t \in (0, \infty)$, such that $\gamma_{\underline{s}}(t) = \omega_{\underline{s}}(t)$ for $t \geq \zeta$. We say that $\gamma_{\underline{s}}$ *lands* if the accumulation set $\bigcap_{t>0} \overline{\gamma_{\underline{s}}((0, t))}$ contains only one point. Otherwise, we say that $\gamma_{\underline{s}}$ is *non-landing*.

2.2. *Hyperbolic expansion.* Let $X = \mathbb{C}\backslash\{0, \lambda\}$ and let ρ_X denote the hyperbolic density function of X. For $r > 0$ and $z \in \mathbb{C}$, let $B_r(z)$ denote the Euclidean disk with center z and radius r.

LEMMA 2.2. *For any* $r > 0$, *there exists a* $\delta = \delta(r) \in (0, 1)$ *depending only on* r *such that if* $z \notin B_r(0) \cup B_r(\lambda) \cup \{k\lambda \mid k \in \mathbb{Z}\}$, *then*

$$\rho_X(z) < \delta \cdot \rho_X(E_\lambda(z))|E_\lambda'(z)|. \tag{2.1}$$

Proof. Let $Y = \mathbb{C}\backslash E_\lambda^{-1}(\lambda)$ and let ρ_Y denote the hyperbolic density function of Y. Since $E_\lambda : Y \to X$ is a holomorphic covering map,

$$\rho_Y(z) = \rho_X(E_\lambda(z))|E_\lambda'(z)|.$$

Note that $Y \subset X$. Thus it is sufficient to prove that

$$\rho_X(z) < \tfrac{1}{2}\rho_Y(z) \tag{2.2}$$

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holds for all z with $|z|$ large enough. To see this, suppose that $|z| \gg 1$. We take a positive integer m such that $|m\lambda| \le |z| < |(m+1)\lambda|$ and let $Z = \mathbb{C}\backslash\{0, m\lambda\}$. Now (2.2) follows from

$$\rho_X(z) \preceq \frac{1}{|z| \ln |z|}$$

and

$$\rho_Y(z) > \rho_Z(z) = \rho_X\left(\frac{z}{m}\right)\frac{1}{m} \succeq \frac{1}{|z|}. \qquad \square$$

Since $L_1(T_{-1})$ is bounded away from zero, λ and $\{k\lambda \mid k \in \mathbb{Z}\}$, from Lemma 2.2 we have the following corollary.

COROLLARY 2.3. *There is a $0 < \delta < 1$ such that, for all $z \in L_1(T_{-1})$, (2.1) holds. Therefore, for all $k \ge 1$ and $z \in \mathbb{C}\backslash\{0, \lambda\}$,*

$$|g'(z)| \cdot \rho_X(g(z)) < \delta \cdot \rho_X(z),$$

where $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$.

From the proof of Lemma 2.2, it follows that, for any $0 < \delta < 1$, (2.1) holds for all z with $|z|$ large enough. Thus we obtain the following corollary.

COROLLARY 2.4. *For any $0 < \delta < 1$, there is an $N \ge 1$ such that, for all $k > N$ and $z \in X$,*

$$|g'(z)| \cdot \rho_X(g(z)) < \delta \cdot \rho_X(z),$$

where $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$.

Notation. In the following, we will always use a hyperbolic metric in $X = \mathbb{C}\backslash\{0, \lambda\}$ rather than a Euclidean metric in $\mathbb{C}$. We use $\mathrm{dist}(\cdot, \cdot)$ and $\mathrm{diam}(\cdot)$ to denote, respectively, the hyperbolic distance and the hyperbolic diameter in X. For any $R > 0$ and a set A, we denote by $U_R(A)$ the R-neighborhood of A with respect to hyperbolic metric in X.

3. *Folding models*

3.1. *Definition of folding models.*

Definition 3.1. A folding model F is the union of a sequence of directed polyline J_i satisfying the following rules.

(1) J_1 is a directed horizontal segment.

(2) For $k \ge 2$, J_k is a directed polyline starting from the endpoint of J_{k-1} such that the vertical and horizontal segments appear in J_k alternatively.

(3) For $k \ge 2$, there is a $1 \le m_k \le k-1$ such that when running through J_k in its direction, one runs back along $J_{k-1}, J_{k-2}, \ldots, J_{k-m_k}$ in order. For the purpose of this work, we require that, in this process, each J_i, $k - m_k \le i \le k-1$, is on the left hand of J_k.

The sequence $\{m_k\}_{k\ge 2}$ is called the *folding sequence* of F (see Figure 1). From the above definition, the following proposition is clear.

PROPOSITION 3.1. *An integer sequence $\{m_k\}_{k\ge 2}$ is a folding sequence for some folding model F if and only if $1 \le m_k \le k-1$ for all $k \ge 2$.*

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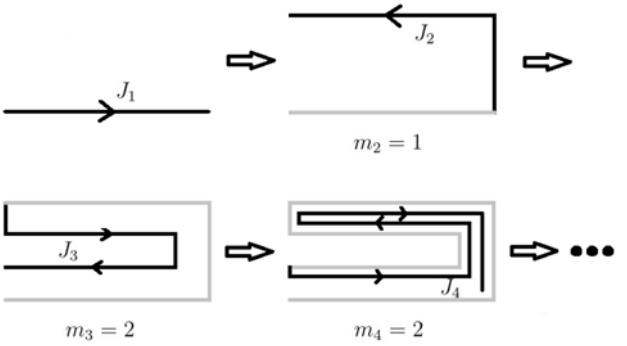


FIGURE 1. Given the folding sequence $\{m_k\}_{k\geq 2} = \{1, 2, 2, \ldots\}$, the folding model can be constructed as above. The black directed polyline represents J_k and the grey ones represent all J_i, $1 \leq i \leq k-1$.

3.2. *Realize the folding models by rays.* We first describe how the folding phenomena is produced. Let us start from a tail ω in the strip T_1. First, pulling back ω by L_1^k, we get $L_1^k(\omega)$ contained in T_1 with the endpoint of $L_1^k(\omega)$ very close to $\lambda = 2\pi i$ if k is large. Next, pulling back $L_1^k(\omega)$ by L_0, we get $L_0 \circ L_1^k(\omega) \subset T_0$ with the endpoint very close to the origin. Then, pulling back $L_0 \circ L_1^k(\omega)$ by L_{-1}, we get $L_{-1} \circ L_0 \circ L_1^k(\omega) \subset T_{-1}$, which extends to the far left. At last, pulling back $L_{-1} \circ L_0 \circ L_1^k(\omega)$ by L_1, we get

$$L_1 \circ L_{-1} \circ L_0 \circ L_1^k(\omega) \subset T_1.$$

Note that $L_1 \circ L_{-1} \circ L_0 \circ L_1^k(\omega)$ folds at some point contained in a bounded subset of the plane independent of k, with the endpoint reaching to the far right when k is large.

The itineraries of our candidate rays have the form

$$(1, -1, 0, 1_{k_1}; 1, -1, 0, 1_{k_2}; \ldots; 1, -1, 0, 1_{k_n}; \ldots).$$

Let us describe how to get the sequence $\{k_i\}$ from the folding sequence $\{m_i\}$ of a folding model F. First, take $k_1 \geq 1$. Suppose that k_i, $1 \leq i \leq n-1$, have been given. Let us define k_n as follows.

Let $h \geq 1$ be any integer. Recall that γ_λ be the unique ray of E_λ landing at λ with address $(1, 1, 1, \ldots)$ and let ω_λ be its tail. Define

$$f_n(z) = E_\lambda^{h+3}(z) \quad \text{and} \quad g_n(z) = L_1 \circ L_{-1} \circ L_0 \circ L_1^h,$$

and, for $1 \leq i \leq n-1$, define

$$f_i(z) = E_\lambda^{k_i+3}(z) \quad \text{and} \quad g_i(z) = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_i}.$$

Let γ_n^0 denote ω_λ and let z_n^0 denote its endpoint. For $0 \leq i \leq n-1$, let γ_i^0 be the tail of $g_{i+1} \circ \cdots \circ g_n(\omega)$ and let z_i^0 be its endpoint. Then for $0 \leq i \leq n$ and $i - n \leq j \leq i$, since we have defined z_i^0 and γ_i^0 for $1 \leq i \leq n$, we can define z_i^j and γ_i^j through

$$g_i(z_i^j) = z_{i-1}^{j-1} \quad \text{and} \quad g_i(\gamma_i^j) = \gamma_{i-1}^{j-1}$$

(see Figure 2). Note that z_i^j is the endpoint of γ_i^j and both of them depend on the number n.

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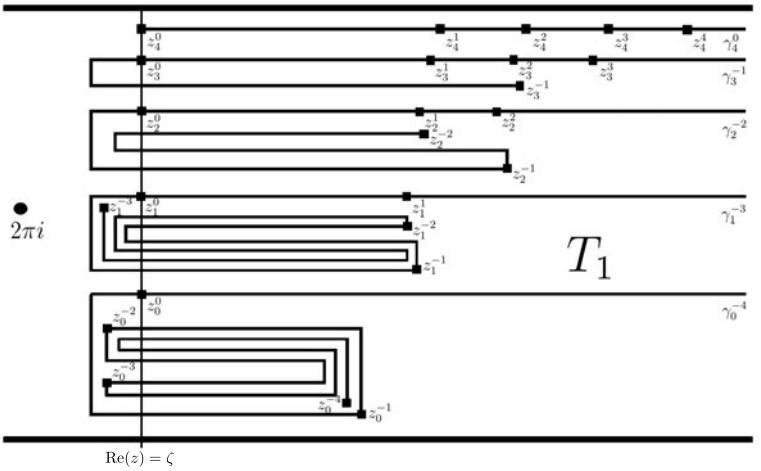


FIGURE 2. Here is the process to determine k_4 from a folding sequence $\{m_k\}_{k\geq 2} = \{1, 2, 2, \ldots\}$. In this case, $m_4 = 2$. When h is large, $\gamma_3^{-1} = g_4(\gamma_4^0)$ folds at some point with bounded hyperbolic distance from z_3^0 and extends to the far right. We define k_4 to be the least h such that $\mathrm{Re}(z_3^{-1}) \geq \mathrm{Re}(z_3^2)$. Then the arc between z_3^0 and z_3^{-1} runs back along the arc between z_3^0 and z_3^2 and, moreover, they will be folded in the same way successively by g_i for $1 \leq i \leq 3$. This fact will be guaranteed by Proposition 5.4 in §5.

To determine k_n, note that when h is large, $\gamma_{n-1}^{-1} = g_n(\gamma_n^0)$ extends γ_{n-1}^0 and folds at some point with bounded hyperbolic distance from z_{n-1}^0, with the endpoint z_{n-1}^{-1} reaching to the right as far as want provided that h is large enough. On the other hand, we shall see that $z_{n-1}^{m_n}$ belongs to a bounded set independent of h (cf. Lemma 5.1). This implies that $\mathrm{Re}(z_{n-1}^{-1}) \geq \mathrm{Re}(z_{n-1}^{m_n})$ when h is large enough. Then we define k_n to be the least h such that $\mathrm{Re}(z_{n-1}^{-1}) \geq \mathrm{Re}(z_{n-1}^{m_n})$. See Figure 2 for an explanation of how to determine k_4 from a folding sequence $\{m_k\}_{k\geq 2} = \{1, 2, 2, \ldots\}$.

For the sequence $\{k_i\}$ determined above, we will show in §5 that the ray with the address

$$(1, -1, 0, 1_{k_1}; 1, -1, 0, 1_{k_2}; \ldots; 1, -1, 0, 1_{k_n}; \ldots)$$

realizes the given folding model.

3.3. *Classification of the accumulation sets.* Let F be a folding model and let $\{m_k\}_{k\geq 2}$ be its folding sequence. Define

$$\chi_n(k) = \begin{cases} 1, & k - m_k \leq n \leq k - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

and

$$a_n = \sum_{k=n+1}^{\infty} \chi_n(k). \tag{3.2}$$

We call a_n the *running-back sequence* of F. By definition, a_n is the number of J_k $(k > n)$ which runs back along J_n.

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PROPOSITION 3.2. *A sequence* $\{a_n\}$ *with* $a_n \in \mathbb{N}^* \cup \{\infty\}$ *is a running-back sequence of some folding model* F *if and only if* $a_{n+1} \geq a_n - 1$.

Proof. Let us first prove the necessity. Since $m_{n+1} \geq 1$, J_{n+1} must run back along J_n. Thus $\chi_n(n+1) = 1$ and then

$$a_n - 1 = \chi_n(n+2) + \chi_n(n+3) + \cdots . \tag{3.3}$$

Claim. For any $i \geq 2$, if $\chi_n(n+i) = 1$, then $\chi_{n+1}(n+i) = 1$. Let us prove the claim. From (3.1) and $\chi_n(n+i) = 1$,

$$n + i - m_{n+i} \leq n \leq n + i - 1.$$

This, together with $i \geq 2$, implies that

$$n + i - m_{n+i} \leq n + 1 \leq n + i - 1.$$

From (3.1), it follows that $\chi_{n+1}(n+i) = 1$. The claim has been proved. From the claim it follows that $\chi_{n+1}(n+i) \geq \chi_n(n+i)$ for all $i \geq 2$. This, together with (3.3) and

$$a_{n+1} = \sum_{k=n+2}^{\infty} \chi_{n+1}(k) = \chi_{n+1}(n+2) + \chi_{n+1}(n+3) + \cdots ,$$

implies that $a_{n+1} \geq a_n - 1$.

Now let us prove the sufficiency. Suppose $\{a_n\}$ is given such that $a_n \in \mathbb{N}^* \cup \{\infty\}$ and $a_{n+1} \geq a_n - 1$. For $n \geq 2$, define

$$m_n = \Lambda(a_{n-1}) + \Lambda(a_{n-2} - 1) + \cdots + \Lambda(a_1 - (n-2)), \tag{3.4}$$

where $\Lambda(x) = 1$ if $x \geq 1$ and $\Lambda(x) = 0$ otherwise. Since $a_{n-1} \geq 1$, by definition $1 \leq m_n \leq n - 1$ for all $n \geq 2$. Hence, by Proposition 3.1, $\{m_n\}_{n\geq 2}$ is a folding sequence of some folding model F. For $k \geq 1$, from (3.4)

$$m_{n+k} = \Lambda(a_{n+k-1}) + \cdots + \Lambda(a_n - (k-1)) + \cdots + \Lambda(a_1 - (n+k-2)). \tag{3.5}$$

For the sequence $\{m_n\}_{n\geq 2}$ defined as above, let $\chi_n(k)$ be defined as in (3.1).

We claim that

$$a_n \geq k \Leftrightarrow \chi_n(n+k) = 1 \quad \text{for all } k \geq 1. \tag{3.6}$$

We first prove that $a_n \geq k \Leftrightarrow m_{n+k} \geq k$. Since $a_{i+1} \geq a_i - 1$ for all $i \geq 1$,

$$a_{n+k-1} \geq a_{n+k-2} - 1 \geq \cdots \geq a_n - (k-1) \geq \cdots \geq a_1 - (n+k-2).$$

From (3.5) and the definition of Λ,

$$m_{n+k} \geq k \Leftrightarrow a_n - (k-1) \geq 1 \Leftrightarrow a_n \geq k.$$

But, for $k \geq 1$, by (3.1), $\chi_n(n+k) = 1 \Leftrightarrow m_{n+k} \geq k$. This proves (3.6).

For the $\{m_n\}_{n\geq 2}$ defined above, from (3.2), the running-back sequence can be computed as

$$\sum_{k=n+1}^{\infty} \chi_n(k) = \sum_{k=1}^{\infty} \chi_n(n+k) = \sum_{1\leq k\leq a_n} 1 = a_n.$$

Note that the second equation comes from (3.6). This completes the proof of the lemma. □

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The main task of the paper is to prove the following statement.

FURTHER STATEMENT OF THE MAIN THEOREM. *Let F be a folding model with running-back sequence $\{a_n\}$. Then E_λ has a ray modeled by F such that the accumulation set is a bounded subset of the plane. Moreover:*

(1) *if $a_i = \infty$ for some i, then the accumulation set is an indecomposable continuum containing part of the ray;*
(2) *if all a_i are finite and there are infinitely many a_i such that $a_i \geq 2$, then the accumulation set is an indecomposable continuum disjoint from the ray; and*
(3) *if $a_i = 1$ for all i sufficiently large, then the accumulation set is a Jordan arc.*

4. *Bounded-wiggling lemma*

Take $r_0 > 0$ such that

$$L_1(T_{-1}) \subset \Omega = \{z \mid |z - \lambda| > r_0\} \cap \{z \mid \mathrm{Re}(z) > -1/r_0\}.$$

Let $\Gamma \subset T_1 \cap \Omega$ be an injective curve(finite or infinite). For any two points $x, y \in \Gamma$, let $\Gamma_{x,y} \subset \Gamma$ be the subarc within x and y. We denote by $\mathrm{dist}(\cdot, \cdot)$ and $\mathrm{diam}(\cdot)$, respectively, the distance and the diameter with respect to hyperbolic metric in $\mathbb{C}\backslash\{0, \lambda\}$. Then we define

$$W(\Gamma) = \sup \mathrm{diam}(\Gamma_{x,y}),$$

where the sup is taken over all $\Gamma_{x,y}$ with $x, y \in \Gamma$ and $\mathrm{Re}(x) = \mathrm{Re}(y)$. For $K > 0$, we say that the *wiggling* of Γ is bounded by K if $W(\Gamma) \leq K$.

Recall that T_{-1} is bounded by t_0 and t_{-1}. Let $\alpha = \ln(d_0/2\pi)$, where $d_0 = \mathrm{dist}(0, t_{-1})$. Let $\Gamma \subset T_1 \cap \Omega$ be an injective curve with endpoints a and b. Let $k \geq 1$ be an integer large enough and consider the map $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$. We say that the map g *folds* Γ into two pieces if $\mathrm{Re}(c) \leq \alpha$ for some $c \in g(\Gamma)$. The point c is called the *folding point* which separates $g(\Gamma)$ into two curves, each of which is still contained in $T_1 \cap \Omega$.

Remark 4.1. In the application, one can take any point c' to be the folding point provided that the diameter of the subarc between c and c' is bounded by some universal constant. Later we take this point so that it is close to the line $\mathrm{Re}(z) = \zeta$.

Let $k_1, \ldots, k_n$ be $n \geq 2$ integers and let $g_i = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_i}$, $1 \leq i \leq n$. Let us define the folding points of $g_1 \circ \cdots \circ g_n(\Gamma)$ by induction. Suppose the folding points of $g_2 \circ \cdots \circ g_n(\Gamma)$ have been defined. We say that z is a *folding point* of $g_1 \circ \cdots \circ g_n(\Gamma)$ if either:

(1) there is a folding point w of $g_2 \circ \cdots \circ g_n(\Gamma)$ such that $z = g_1(w)$; or
(2) there is a subarc Λ of $g_2 \circ \cdots \circ g_n(\Gamma)$ with the endpoints being two adjacent folding points of $g_2 \circ \cdots \circ g_n(\Gamma)$ such that z is the folding point of $g_1(\Lambda)$.

From the definition, the folding points of $g_2 \circ \cdots \circ g_n(\Gamma)$ are mapped to the folding points of $g_1 \circ \cdots \circ g_n(\Gamma)$ by g_1 and the map $g_1 \circ \cdots \circ g_n$ folds Γ into at most 2^n pieces. Hence $g_1 \circ \cdots \circ g_n(\Gamma)$ has at most $2^n - 1$ folding points.

Our goal in this section is to prove the following lemma.

LEMMA 4.1. (Bounded-wiggling) *Suppose the wiggling of Γ is bounded by K_0. Then there exists a K depending only on K_0 such that the wiggling of each of the above pieces is bounded by K.*

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It is sufficient to prove the following lemma.

LEMMA 4.2. *There exist* $0 < \delta_1 < 1$ *and* $K_1 > 0$ *such that for any injective curve* $\Gamma \subset T_1 \cap \Omega$ *and all integer* $k \geq 1$ *large enough, if* $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$ *folds* Γ *into two pieces* Θ *and* Π*, then*

$$\max\{W(\Theta), W(\Pi)) \leq \delta_1 \cdot W(\Gamma) + K_1.$$

Otherwise,

$$W(g(\Gamma)) \leq \delta_1 \cdot W(\Gamma) + K_1.$$

In fact, assuming Lemma 4.2, Lemma 4.1 follows by taking

$$K = K_0 + \frac{K_1}{1 - \delta_1},$$

which is obtained by successively applying Lemma 4.2 to g_n, g_{n-1}, ..., and g_1.

Recall that γ_λ denotes the fixed ray landing on λ. Let ω_λ denote its tail. Let us first prove Lemma 4.2 in the case that $\Gamma = \omega_\lambda$.

LEMMA 4.3. *There exists a* $K_2 > 0$ *such that, for any* $k \geq 1$*, if* $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$ *folds* ω_λ *into two pieces* Θ *and* Π*,*

$$\max\{W(\Theta), W(\Pi)) \leq K_2.$$

Otherwise,

$$W(g(\omega_\lambda)) \leq K_2.$$

Proof. We parameterize γ_λ as a continuous curve $\gamma_\lambda(t)$, $0 \leq t < \infty$, such that $\lim_{t \to 0+} \gamma_\lambda(t) = \lambda$. Let $\Gamma_\lambda(t) = L_1 \circ L_{-1} \circ L_0(\gamma_\lambda(t))$, $0 < t < \infty$. By setting

$$t_1 = \min\{t \mid \mathrm{Re}(\Gamma_\lambda(t)) = \zeta\} \quad \text{and} \quad t_2 = \max\{t \mid \mathrm{Re}(\Gamma_\lambda(t)) = \zeta\},$$

we write $\Lambda_1 = \{\Gamma_\lambda(t) \mid t_2 \leq t < \infty\}$, $\Lambda_2 = \{\Gamma_\lambda(t) \mid t_1 \leq t \leq t_2\}$ and $\Lambda_3 = \{\Gamma_\lambda(t) \mid 0 < t \leq t_1\}$. Note that $W(\Lambda_1) = 0$ by Lemma 2.1, and $\mathrm{diam}(\Lambda_2) < C_3$ with some $C_3 > 0$ depending only on ζ.

We now prove that $W(\Lambda_3)$ is finite. Let z_λ^0 be the endpoint of ω_λ. There exists an $N \geq 1$ such that, for all $n \geq N$, $z_n = L_1 \circ L_{-1} \circ L_0 \circ L_1^n(z_\lambda^0) \in \Lambda_3$. Since γ_λ spirals around λ clockwise, we get $|L_{-1} \circ L_0 \circ L_1^n(z_\lambda^0)| < |L_{-1} \circ L_0 \circ L_1^{n+1}(z_\lambda^0)|$ for $n \geq N$, and thus

$$\mathrm{Re}(z_n) < \mathrm{Re}(z_{n+1}). \tag{4.1}$$

Let A denote the subarc of ω_λ within z_λ^0 and $E_\lambda(z_\lambda^0)$ and let $\Gamma_{z_n, z_{n+1}}$ denote the subarc of Λ_3 within z_n and z_{n+1}. Since $L_1 \circ L_{-1} \circ L_0 \circ L_1^{n+1}(A) = \Gamma_{z_n, z_{n+1}}$ and $L_1 \circ L_{-1} \circ L_0 \circ L_1^{n+1}$ contracts the hyperbolic metric in $\mathbb{C} \backslash \{0, \lambda\}$,

$$\mathrm{diam}(\Gamma_{z_n, z_{n+1}}) < \mathrm{diam}(A) \quad \text{for all } n \geq 1. \tag{4.2}$$

Since $\mathrm{Re}(z_n) \to +\infty$ as $n \to +\infty$ and the vertical width of T_1 is bounded, from (4.1) and (4.2) one can easily derive that $W(\Lambda_3) < C_2$ for some $C_2 > 0$.

Suppose g folds ω_k into two pieces Θ and Π. Then each of the two pieces is contained in either $\Lambda_1 \cup \Lambda_2$ or $\Lambda_3 \cup \Lambda_2$. Otherwise, $g(\omega_k)$ is contained in $\Lambda_1 \cup \Lambda_2$. Thus, combining with the definition of Λ_1, Λ_2 and Λ_3, there is a $K_2 > 0$ such that

$$\max\{W(\Theta), W(\Pi)\} < K_2 \quad \text{or} \quad \max\{W(g(\omega_\lambda))\} < K_2. \qquad \square$$

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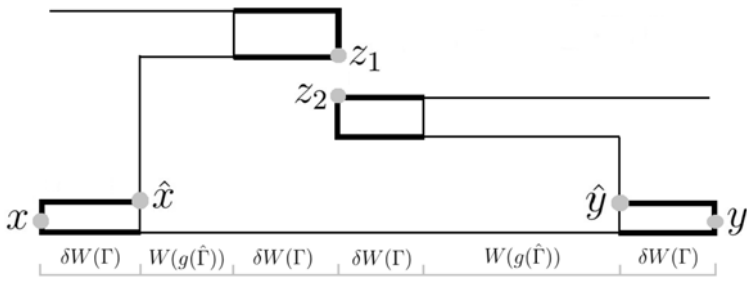


FIGURE 3. The different parts of Γ relative to $\hat{\Gamma}$ are emphasized in bold. Here $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{diam}(\Gamma_{z_1,z_2}) < 4\delta W(\Gamma) + 2W(g(\hat{\Gamma})) + C$, with $C > 0$ being some universal constant.

Suppose $\Gamma \subset T_1 \cap \Omega$ is an injective curve. By replacing some subarcs $\Gamma_{a,b} \subset \Gamma$ with $\mathrm{Re}(a) = \mathrm{Re}(b)$ by the vertical segment $[a, b]$, if necessary, we get an injective curve $\hat{\Gamma} \subset T_1 \cap \Omega$ such that $W(\hat{\Gamma}) < K_0$ for some universal $K_0 > 0$. This, and the fact that $W(\omega_\lambda) = 0$, implies the following lemma.

LEMMA 4.4. *There exists a $D > 0$ such that, for any $\Gamma \subset T_1 \cap \Omega$, one can parameterize $\hat{\Gamma}$ as $\eta(t)$, $\zeta \le t < \infty$ such that* $\mathrm{dist}(\eta(t), \gamma_\lambda(t)) < D$ *for all $\zeta \le t < \infty$.*

As a corollary of Lemmas 4.3 and 4.4, we obtain the following lemma.

LEMMA 4.5. *There exists a $K_3 > 0$ such that, for any $k \ge 1$, if $g = L_1 \circ L_{-1} \circ L_0 \circ L_1^k$ folds $\hat{\Gamma}$ into two pieces $\hat{\Theta}$ and $\hat{\Pi}$, then*

$$\max\{W(\hat{\Theta}), W(\hat{\Pi})) \le K_3.$$

Otherwise,

$$W(g(\hat{\Gamma})) \le K_3.$$

Now let us prove Lemma 4.2. First, we assume that g does not fold Γ. Without loss of generality, we may assume that g does not fold $\hat{\Gamma}$ either. In Figure 3, we illustrate the worst possible case that the wiggling may be enlarged by replacing the arcs $g([a, b])$ by $g(\Gamma_{a,b})$. Note that $\mathrm{diam}(g(\Gamma_{a,b})) < \delta W(\Gamma)$ by Corollary 2.3. It follows that

$$W(g(\Gamma)) < 4\delta W(\Gamma) + 2W(g(\hat{\Gamma})) + C, \tag{4.3}$$

where $C > 0$ is some universal constant. Here we use the fact that the diameter of a set $E \subset T_1 \cap \Omega$ can be computed as its horizontal diameter with the error bounded by some universal constant. That is, let $x \in E$ and $y \in E$ be such that $\mathrm{Re}(x) = \min\{\mathrm{Re}(z) \mid z \in E\}$ and $\mathrm{Re}(y) = \max\{\mathrm{Re}(z) \mid z \in E\}$. Then $\mathrm{diam}(E) < \mathrm{dist}(x, y) + C_0$ for some universal $C_0 > 0$. By Corollary 2.4, we can assume that $\delta < 1/4$ by taking k large enough.

Next we assume that g folds Γ into two pieces Θ and Π. There are two subcases. In the first subcase, g does not fold $\hat{\Gamma}$. Then, by the same reasoning as above, the inequality (4.3) holds with $g(\Gamma)$ replaced by Θ or Π. In the second subcase, g folds $\hat{\Gamma}$ into two pieces $\hat{\Theta}$ and $\hat{\Pi}$. Again by the same reasoning as above, we get inequality (4.3) with $g(\Gamma)$ replaced by Θ or Π and $g(\hat{\Gamma})$ replaced by $\hat{\Theta}$ or $\hat{\Pi}$.

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Let $K_3 > 0$ be the constant guaranteed by Lemma 4.5. Now Lemma 4.2 follows in all of the above cases by taking $\delta_1 = 4\delta$ and $K_1 = 2K_3 + C$. The proof of Lemma 4.1 is complete.

5. *Realization of the folding model*

Let F be a given folding model and let $\{k_i\}$ be the sequence obtained in §3. The goal of this section is to prove that the ray with the address

$$\underline{s}^* = (1, -1, 0, 1_{k_1}; 1, -1, 0, 1_{k_2}; \ldots; 1, -1, 0, 1_{k_n}; \ldots)$$

realizes the folding model.

For $n \geq 1$, recall that, in §3.2, we defined z_i^j by the tail ω_λ, where $0 \leq i \leq n$, $i - n \leq j \leq i$. If we replace ω_λ by any tail $\omega_{\underline{s}} \subset T_1$, we can define $z_i^j(\underline{s})$ in a similar way. More precisely, let $g_i = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_i}$ for all $1 \leq i \leq n$. Let $\gamma_n^0(\underline{s})$ denote $\omega_{\underline{s}}$ and let $z_n^0(\underline{s})$ denote its endpoint. For $0 \leq i \leq n-1$, let $\gamma_i^0(\underline{s})$ be the tail of $g_{i+1} \circ \cdots \circ g_n(\omega_{\underline{s}})$ and let $z_i^0(\underline{s})$ be its endpoint. Then, for $0 \leq i \leq n$ and $i - n \leq j \leq i$, since we have defined $z_i^0(\underline{s})$ and $\gamma_i^0(\underline{s})$ for $0 \leq i \leq n$, we can define $z_i^j(\underline{s})$ and $\gamma_i^j(\underline{s})$ through

$$g_i(z_i^j(\underline{s})) = z_{i-1}^{j-1}(\underline{s}) \quad \text{and} \quad g_i(\gamma_i^j(\underline{s})) = \gamma_{i-1}^{j-1}(\underline{s}).$$

Note that $z_i^j(\underline{s})$ is the endpoint of $\gamma_i^j(\underline{s})$ and both of them depend on the number n. We denote by $[z_k^i(\underline{s}), z_k^j(\underline{s})]$ the subarc of $\gamma_k(\underline{s})$ within $z_k^i(\underline{s})$ and $z_k^j(\underline{s})$ if it is well defined.

LEMMA 5.1. *There exists a $D_0 > 0$ such that, for all $\omega_{\underline{s}} \subset T_1$ and all $-1 \leq l \leq n-1$,* $\operatorname{dist}(z_{n-1}^l, z_{n-1}^l(\underline{s})) < D_0$.

Proof. First note that z_{n-1}^0, $z_{n-1}^0(\underline{s})$, z_n^0 and $z_n^0(\underline{s})$ are contained in T_1 and have the same real part ζ; there is a universal $D_1 > 0$ such that

$$\operatorname{dist}(z_{n-1}^0, z_{n-1}^0(\underline{s})) < D_1$$

and $\operatorname{dist}(z_n^0, z_n^0(\underline{s})) < D_1$. Since g_n contracts the hyperbolic metric by Corollary 2.3, it follows that

$$\operatorname{dist}(z_{n-1}^{-1}, z_{n-1}^{-1}(\underline{s})) \leq \operatorname{dist}(z_n^0, z_n^0(\underline{s})) < D_1.$$

For $1 \leq l \leq n-1$,

$$z_{n-1-l}^0 - z_{n-1-l}^0(\underline{s}) = \int_\Gamma (g_{n-l} \circ \cdots \circ g_{n-1})'(\xi)\, d\xi, \tag{5.1}$$

where Γ is the straight segment connecting z_{n-1}^l and $z_{n-1}^l(\underline{s})$. Note that, for all $\xi \in \Gamma$, the points ξ, $g_{n-1}(\xi)$, $g_{n-2} \circ g_{n-1}(\xi)$, $\ldots$, and $g_{n-l} \circ \cdots \circ g_{n-1}(\xi)$ are all contained in the half plane H_ζ. By the same reasoning as in the proof of [**8**, Lemma 3.2],

$$|\arg((g_{n-l} \circ \cdots \circ g_{n-1})'(\xi))| < \epsilon,$$

where $\epsilon > 0$ can be arbitrarily small provided that ζ is large enough. By noting that the left hand of (5.1) is an imaginary number, Γ must be almost vertical provided that $\epsilon > 0$ is small enough. Since $T_1 \cap H_\zeta$ is like a half horizontal strip with vertical width 2π, it

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follows that the Euclidean length of Γ is less than 3π. Note that z_{n-1}^l and $z_{n-1}^l(\underline{s})$ are bounded away from 0 and 2π, and hence

$$\operatorname{dist}(z_{n-1}^l, z_{n-1}^l(\underline{s})) < D_2$$

for some universal $D_2 > 0$. This completes the proof of Lemma 5.1. □

LEMMA 5.2. *There exists a $D_3 > 0$ such that, for all $n \geq 1$,*

$$\operatorname{dist}(z_{n-1}^{-1}, z_{n-1}^{m_n}) < D_3.$$

Proof. Recall that k_n is defined to be the least integer such that $\operatorname{Re}(z_{n-1}^{-1}) \geq \operatorname{Re}(z_{n-1}^{m_n})$. Let $g_n = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_n}$ and $\hat{g}_n = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_n-1}$. Since ω_λ is the tail of the fixed ray, the points z_{n-1}^j, $0 \leq j \leq n-1$, do not change, when we replace k_n by $k_n - 1$ in the definition of z_i^j. In particular, $z_{n-1}^{m_n}$ denotes the same point in the two situations. By the definition of k_n,

$$\operatorname{Re}(\hat{g}_n(z_n^0)) < \operatorname{Re}(z_{n-1}^{m_n}) < \operatorname{Re}(g_n(z_n^0)).$$

Recall that $g_n(z_n^0) = z_{n-1}^{-1}$. All three points $\hat{g}_n(z_n^0)$, $z_{n-1}^{m_n}$ and $g_n(z_n^0)$ belong to T_1 with bounded imaginary parts. From the above inequality, the lemma will follow once we have proved that $\operatorname{dist}(\hat{g}_n(z_n^0), g_n(z_n^0))$ is bounded by some constant depending only on ζ. To see this, we first note that $\hat{g}_n(z_n^0) = g_n(E_\lambda(z_n^0))$ and $\operatorname{dist}(E_\lambda(z_n^0), z_n^0) < D_4$ for some $D_4 > 0$ depending only on ζ. Since g_n contracts the hyperbolic metric by Corollary 2.3,

$$\operatorname{dist}(\hat{g}_n(z_n^0), g_n(z_n^0)) = \operatorname{dist}(g_n(E_\lambda(z_n^0), g_n(z_n^0)) \leq \operatorname{dist}(E_\lambda(z_n^0), z_n^0) < D_4.$$

This implies the lemma. □

LEMMA 5.3. *There exists a $D_5 > 0$ such that, for all tails $\omega_{\underline{s}} \subset T_1$,*

$$\operatorname{dist}(z_{n-1}^{-1}(\underline{s}), z_{n-1}^{m_n}(\underline{s})) < D_5.$$

Proof. By Lemmas 5.1 and 5.2,

$$\begin{aligned}
&\operatorname{dist}(z_{n-1}^{-1}(\underline{s}), z_{n-1}^{m_n}(\underline{s})) \\
&\quad \leq \operatorname{dist}(z_{n-1}^{-1}(\underline{s}), z_{n-1}^{-1}) + \operatorname{dist}(z_{n-1}^{-1}, z_{n-1}^{m_n}) + \operatorname{dist}(z_{n-1}^{m_n}, z_{n-1}^{m_n}(\underline{s})) \\
&\quad \leq D_5.
\end{aligned}$$

□

PROPOSITION 5.4. *There exists a $D_6 > 0$ such that, for every $n \geq 2$, there is a homeomorphism $\tilde{\phi}_{n-1}$ from $[z_{n-1}^0(\underline{s}), z_{n-1}^{m_n}(\underline{s})]$ to $[z_{n-1}^0(\underline{s}), z_{n-1}^{-1}(\underline{s})]$ such that, for any $\xi \in [z_{n-1}^0(\underline{s}), z_{n-1}^{m_n}(\underline{s})]$,*

$$\operatorname{dist}(\xi, \tilde{\phi}_{n-1}(\xi)) < D_6, \tag{5.2}$$

and, moreover,

$$\operatorname{dist}(g_i \circ \cdots \circ g_{n-1}(\xi), g_i \circ \cdots \circ g_{n-1}(\tilde{\phi}_{n-1}(\xi))) < \delta^{n-i} \cdot D_6 \tag{5.3}$$

for any $1 \leq i \leq n-1$.

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Proof. Note that both $[z_{n-1}^0(\underline{s}), z_{n-1}^{m_n}(\underline{s})]$ and $[z_{n-1}^0(\underline{s}), z_{n-1}^{-1}(\underline{s})]$ are contained in T_1, and that $\operatorname{dist}(z_{n-1}^{-1}(\underline{s}), z_{n-1}^{m_n}(\underline{s})) < D_5$ by Lemma 5.3. By Lemma 4.1, it follows that the wiggling of both $[z_{n-1}^0(\underline{s}), z_{n-1}^{m_n}(\underline{s})]$ and $[z_{n-1}^0(\underline{s}), z_{n-1}^{-1}(\underline{s})]$ is bounded by some universal $C > 0$. This implies the existence of $\tilde{\phi}_{n-1}$ so that (5.2) holds for some universal $D_6 > 0$. The inequality (5.3) then follows from (5.2) and Corollary 2.3. □

From Proposition 5.4 and the construction of $\{k_n\}$ in §3.2, we see that, for any $n \geq 1$, $[z_{n-1}^0(\underline{s}), z_{n-1}^{-1}(\underline{s})]$ runs back along $[z_{n-1}^0(\underline{s}), z_{n-1}^{m_n}(\underline{s})]$ (cf. (5.2)) and, moreover, the two arcs will be folded in the same way successively by g_i for $1 \leq i \leq n-1$ (cf. (5.3)). Thus, for any $n \geq 1$, the ray $g_1 \circ \cdots \circ g_n(\omega_\lambda)$ realizes the first few folds of the model F (compare Figures 1 and 2). But the problem is that $g_1 \circ \cdots \circ g_n(\omega_\lambda)$ represents a different ray as n increases.

To solve this problem, let us take

$$\tau^n(\underline{s}^*) = (1, -1, 0, 1_{k_{n+1}}; 1, -1, 0, 1_{k_{n+2}}; 1, -1, 0, 1_{k_{n+3}}; \ldots), \quad n \geq 1.$$

At the same time, we provide that the number n in $\tau^n(\underline{s}^*)$ is the number which occurs in the definition of $\gamma_i^j(\tau^n(\underline{s}^*))$. Then, for all $0 \leq i \leq n-1$ and $i-(n-1) \leq j \leq i$,

$$\gamma_i^j(\tau^n(\underline{s}^*)) = \gamma_i^j(\tau^{n-1}(\underline{s}^*)) \quad \text{and} \quad z_i^j(\tau^n(\underline{s}^*)) = z_i^j(\tau^{n-1}(\underline{s}^*)). \tag{5.4}$$

This implies that, for any $0 \leq k \leq n-1$,

$$\gamma_0^{-k}(\tau^{n-1}(\underline{s}^*)) = \gamma_0^{-k}(\tau^n(\underline{s}^*)) \quad \text{and} \quad z_0^{-k}(\tau^n(\underline{s}^*)) = z_0^{-k}(\tau^{n-1}(\underline{s}^*)). \tag{5.5}$$

Since $\gamma_0^{-(k+1)}(\tau^n(\underline{s}^*))$ extends $\gamma_0^{-k}(\tau^n(\underline{s}^*))$, from the first equation of (5.5), it follows that $\gamma_0^{-(k+1)}(\tau^n(\underline{s}^*))$ extends $\gamma_0^{-k}(\tau^{n-1}(\underline{s}^*))$. In particular, by taking $k = n-1$, it follows that $\gamma_0^{-n}(\tau^n(\underline{s}^*))$ extends $\gamma_0^{-(n-1)}(\tau^{n-1}(\underline{s}^*))$. On the other hand, from Proposition 5.4 and the construction of $\{k_n\}$ in §3.2, $\gamma_0^{-n}(\tau^n(\underline{s}^*))$ folds in the same way as $g_1 \circ \cdots \circ g_n(\omega_\lambda)$, which realizes the first few folds of the model F. In fact, if we set $I_k(\underline{s}^*) = [z_0^{-k+1}(\tau^n(\underline{s}^*)), z_0^{-k}(\tau^n(\underline{s}^*))] \subset \gamma_0^{-n}(\tau^n(\underline{s}^*))$, $1 \leq k \leq n$, then $I_k(\underline{s}^*)$ corresponds to the polyline J_k of the folding model F. Since $\gamma_0^{-n}(\tau^n(\underline{s}^*))$ extends $\gamma_0^{-(n-1)}(\tau^{n-1}(\underline{s}^*))$ for all $n \geq 1$, it follows that

$$\gamma_0(\underline{s}^*) = \bigcup_{n \geq 1} \gamma_0^{-n}(\tau^n(\underline{s}^*)) \tag{5.6}$$

realizes the model F.

By the construction, it follows that $\gamma_0(\underline{s}^*)$ is the ray with the address $\underline{s}^*$. Let $\gamma_0 = \gamma_0(\underline{s}^*)$. For $n \geq 1$, let $\gamma_n = f_n \circ \cdots \circ f_1(\gamma_0)$ and let γ_n^0 be the tail of γ_n. Recall that $g_l = L_1 \circ L_{-1} \circ L_0 \circ L_1^{k_l}$, $1 \leq l \leq n$. For any two integers $j \leq i$, by taking n large enough, $0 \leq i \leq n$, $i - n \leq j \leq i$. Thus we can define $\gamma_i^j(\tau^n(\underline{s}^*))$ and $z_i^j(\tau^n(\underline{s}^*))$ (note that $\tau^n(\underline{s}^*)$ is the address of γ_n^0). By (5.4), this definition does not depend on n. From now on, let us reuse the notation γ_i^j and z_i^j to denote $\gamma_i^j(\tau^n(\underline{s}^*))$ and $z_i^j(\tau^n(\underline{s}^*))$, respectively. Note that γ_0^0 is the tail of γ_0 and $I_k = [z_0^{-k+1}, z_0^{-k}] \subset \gamma_0$, $k \geq 1$.

For later use, we need to understand how γ_n^0 is folded by $g_{i+1} \circ \cdots \circ g_n$ for any $0 \leq i \leq n$. To this end, let us define inductively the set of *folding points* of

$$\gamma_i^{i-n} = g_{i+1} \circ \cdots \circ g_n(\gamma_n^0),$$

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which is denoted by Z_i. Let $n \geq 1$ be fixed and set $Z_n = \{z_n^0\}$. For any $0 \leq i \leq n-1$, suppose that Z_{i+1} has been defined. Let us define Z_i as follows.

Let $W_0 = \{z_i^0\}$. Suppose that W_j has been defined for $j \geq 0$. Define W_{j+1} to be the set consisting of all those points,

$$z,$$
$$g_{i+1} \circ \tilde{\phi}_{i+1} \circ f_{i+1}(z) \quad \text{if it is well defined,}$$
$$\cdots,$$
$$g_{i+1} \circ \cdots \circ g_{n-1} \circ \tilde{\phi}_{n-1} \circ f_{n-1} \circ \cdots \circ f_{i+1}(z) \quad \text{if it is well defined,}$$

where $z \in W_j$ and $\tilde{\phi}_l$, $i+1 \leq l \leq n-1$, are the maps defined in Proposition 5.4. Define

$$Z_i = g_{i+1}(Z_{i+1}) \cup W_{n-i}.$$

Roughly speaking, after the folding by g_{i+1}, the folding points of $\gamma_{i+1}^{(i+1)-n}$ are mapped to the folding points of γ_i^{i-n}; besides these, the map g_{i+1} produces new folding points all of which are contained in the set W_{n-i}. To get a clear understanding of the definition, the reader may refer to Figure 2 and find all the folding points of γ_1^{-3}. From the definition, one may easily derive the following lemma.

LEMMA 5.5. *Let $0 \leq i < k < n$ and $z \in [z_k^0, z_k^{m_{k+1}}]$. Then $g_{i+1} \circ \cdots \circ g_k(z) \in Z_i$ if and only if $g_{i+1} \circ \cdots \circ g_k(\tilde{\phi}_k(z)) \in Z_i$. In the case $0 \leq i = k < n$ and $z \in [z_k^0, z_k^{m_{k+1}}]$, then $z \in Z_i$ if and only if $\tilde{\phi}_k(z) \in Z_i$.*

6. *The accumulation set is bounded*

Let $\underline{s}^*$ and $\gamma_0 = \gamma_0(\underline{s}^*)$ be, respectively, the address and the ray obtained in §5. In this section, we will prove that the accumulation set of γ_0 is bounded. The other assertions of the *further statement of the Main Theorem* (cf. §3.3) will be proved in §7.

For any $R > 0$ and a set A, we denote by $U_R(A)$ the R-neighborhood of A with respect to hyperbolic metric in $\mathbb{C}\backslash\{0, \lambda\}$. Then we prove the following lemma.

LEMMA 6.1. *There exists a $D_7 > 0$ such that $\gamma_0 \backslash \gamma_0^0$ is contained in $U_{D_7}(I_1)$.*

Proof. Let $\tilde{d}_H(\cdot)$ denote the semi-Hausdorff distance between compact subsets of the hyperbolic metric space X. That is, for any compact subsets A and B of X,

$$\tilde{d}_H(A, B) = \sup_{x \in A} \inf_{y \in B} \operatorname{dist}(x, y).$$

Note that I_n runs back along I_{n-1}, I_{n-2}, ... and I_{n-m_n}. Let $\Pi = [z_{n-1}^0, z_{n-1}^{m_n}]$ and $\Theta = [z_{n-1}^0, z_{n-1}^{-1}]$. Then

$$g_1 \circ \cdots \circ g_{n-1}(\Pi) = I_{n-1} \cup \cdots \cup I_{n-m_n} \quad \text{and} \quad g_1 \circ \cdots \circ g_{n-1}(\Theta) = I_n.$$

By Proposition 5.4, $\tilde{d}_H(\Theta, \Pi) < D_6$. This, together with Corollary 2.3, implies that

$$\begin{aligned} &\tilde{d}_H(I_n, I_1 \cup \cdots \cup I_{n-1}) \\ &\quad \leq \tilde{d}_H(I_n, I_{n-m_n} \cup \cdots \cup I_{n-1}) \\ &\quad \leq \delta^{n-1} \tilde{d}_H(\Theta, \Pi) \\ &\quad < \delta^{n-1} \cdot D_6, \end{aligned}$$

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where $0 < \delta < 1$ is the constant in Corollary 2.3. It follows that

$$\begin{aligned}
&\tilde{d}_H(I_n, I_1) \\
&\quad \le \tilde{d}_H(I_n, I_1 \cup \cdots \cup I_{n-1}) + \tilde{d}_H(I_{n-1}, I_1 \cup \cdots \cup I_{n-2}) + \cdots + \tilde{d}_H(I_2, I_1) \\
&\quad \le (\delta^{n-1} + \delta^{n-2} + \cdots + 1) D_6 \\
&\quad \le \frac{D_6}{1-\delta}
\end{aligned}$$

holds for all $n \ge 1$. This implies the lemma. □

Since the accumulation set of γ_0 is equal to that of $\gamma_0 \backslash \gamma_0^0$, by Lemma 6.1 we obtain the following proposition.

PROPOSITION 6.2. *The accumulation set of γ_0 is bounded in the plane.*

For $k \ge 1$, recall that a_k is the number of J_i $(i > k)$ which runs back along J_k. Suppose $a_k \ne \infty$. Then there exists some $l_0 \ge k+2$ such that J_l does not run back along J_k for all $l \ge l_0$. In particular, $a_i \ne \infty$ for all $1 \le i \le k-1$. The following result will be used in the proof of Proposition 7.11.

LEMMA 6.3. *Suppose that $a_k \ne \infty$ for some $k \ge 1$ and let l_0 be the integer as above. Then there exists a $D_8 > 0$ such that $\gamma_l \backslash \gamma_l^{l-k} \subset U_{D_8}([z_l^0, z_l^{l-k}])$ for all $l \ge l_0$.*

Proof. For any $n \ge l+2$, let $\Pi = [z_{n-1}^0, z_{n-1}^{n-1-k}]$ and $\Theta = [z_{n-1}^0, z_{n-1}^{-1}]$. Since J_l does not run back along J_i, $1 \le i \le k$, it follows that $n-1-k \ge m_n$ and hence $[z_{n-1}^0, z_{n-1}^{n-1-k}] \supseteq [z_{n-1}^0, z_{n-1}^{m_n}]$. By Proposition 5.4, $\tilde{d}_H(\Theta, \Pi) < D_6$. By considering $g_{l+1} \circ \cdots \circ g_{n-1}(\Pi)$ and $g_{l+1} \circ \cdots \circ g_{n-1}(\Theta)$, Lemma 6.3 follows from a similar argument as in the proof of Lemma 6.1. □

7. *The classification of accumulation sets*

Our goal in this section is to prove the remaining part of *further statement of the Main Theorem* (cf. §3.3). That is, we will classify the accumulation sets of the rays by the running-back sequences. Before this, let us first prove the following two lemmas.

LEMMA 7.1. *Let $\{a_n\}_{n\ge1}$ and $\{m_n\}_{n\ge2}$ be, respectively, the running-back sequence and the folding sequence of a folding model F. Then there are infinitely many a_i with $a_i \ge 2$ if and only if there are infinitely many m_k with $m_k \ge 2$.*

Proof. Note that $m_2 = 1$. Suppose that $m_k \ge 2$ for some $k \ge 3$. Then both J_{k-1} and J_k trace back along J_{k-2}. So $a_{k-2} \ge 2$. This proves the necessity. Suppose that $a_i \ge 2$ for some $i \ge 1$. If $m_l = 1$ for all $l > i$, then none of J_l, $l \ge i+1$, trace back along J_i. It follows that $a_i = 1$. This is a contradiction. So we must have an $l \ge i$ such that $m_l \ge 2$. This proves the sufficiency. □

By similar reasoning to the above we can get the following lemma.

LEMMA 7.2. *Let $\{a_n\}_{n\ge1}$ and $\{m_n\}_{n\ge2}$ be, respectively, the running-back sequence and the folding sequence of a folding model F. Then $a_i = 1$ for all i sufficiently large if and only if $m_j = 1$ for all j sufficiently large.*

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7.1. *Indecomposable continua cases.* In this subsection, we assume that there are infinitely many a_k with $a_k \geq 2$. According to Lemma 7.1, this is equivalent to there being infinitely many m_n in $\{m_n\}_{n\geq 2}$ such that $m_n \geq 2$. Without loss of generality, we assume that there is a sequence of odd (or even) integers, n_k, such that $m_{n_k} \geq 2$. Under this assumption, we will first prove that the accumulation set of γ_0 is an indecomposable continuum and then prove that the accumulation set contains part of the ray if $a_i = \infty$ for some i and is disjoint from the ray if all a_i are finite.

For any $k \geq 1$, set

$$\tilde{f}_k = f_k \circ \cdots \circ f_1,$$
$$\tilde{g}_k = g_1 \circ \cdots \circ g_k$$

and

$$\phi_{k-1} = \tilde{g}_{k-1} \circ \tilde{\phi}_{k-1} \circ \tilde{f}_{k-1},$$

where $\tilde{\phi}_{k-1}$ is the map guaranteed by Proposition 5.4. According to [**3**], a ray is a continuous and injective curve defined on $[0, \infty)$.

LEMMA 7.3. (Curry's lemma [**3**]) *Let $\eta : [0, \infty) \to \mathbb{C}$ be a ray such that η accumulates everywhere upon itself and, moreover, the closure of η does not separates the plane. Then the accumulation set of η is an indecomposable continuum.*

Let us construct such a ray for γ_0. To this end, we first parameterize $I_{n_1} = [z_0^{-n_1+1}, z_0^{-n_1}]$ as an injective curve $\omega_{n_1}(t)$, $0 \leq t \leq 1$, with $\omega_{n_1}(0) = z_0^{-n_1+1}$ and $\omega_{n_1}(1) = z_0^{-n_1}$. Then we define inductively a sequence of curves $\omega_k(t)$ by setting

$$\omega_k(t) = \phi_{k-1} \circ \omega_{k-1}(t), \quad 0 \leq t \leq 1, k > n_1.$$

Remark 7.1. Note that the map ϕ_{k-1} maps I_{k-1} homeomorphically to a subarc of I_k with order reversed. In particular, ω_k maps $[0, 1]$ to a subarc of $I_k = [z_0^{-(k-1)}, z_0^{-k}]$ for all $k \geq n_1$. Since n_1 is odd, it follows that, for all odd $k \geq n_1$, as t changes from 0 to 1, ω_t changes from some point near $z_0^{-(k-1)}$ to some point near z_0^{-k}. That is, when k is odd, the points $z_0^{-(k-1)}$, $\omega_k(0)$, $\omega_k(1)$ and z_0^{-k} appear in γ_0 in order (see Figure 4).

LEMMA 7.4. *ω_k converge uniformly on* $[0, 1]$.

Proof. Note that, for every $t \in [0, 1]$,

$$\omega_{k-1}(t) = \tilde{g}_{k-1}(\tilde{f}_{k-1} \circ \omega_{k-1}(t))$$

and

$$\omega_k(t) = \tilde{g}_{k-1}(\tilde{\phi}_{k-1} \circ \tilde{f}_{k-1} \circ \omega_{k-1}(t)).$$

Hence

$$\begin{aligned} &\operatorname{dist}(\omega_{k-1}(t), \omega_k(t)) \\ &\quad < \delta^{k-1} \cdot \operatorname{dist}(\tilde{f}_{k-1} \circ \omega_{k-1}(t), \tilde{\phi}_{k-1}(\tilde{f}_{k-1} \circ \omega_{k-1}(t))) \\ &\quad < \delta^{k-1} \cdot D_6 \end{aligned}$$

by Proposition 5.4. This proves the lemma. □

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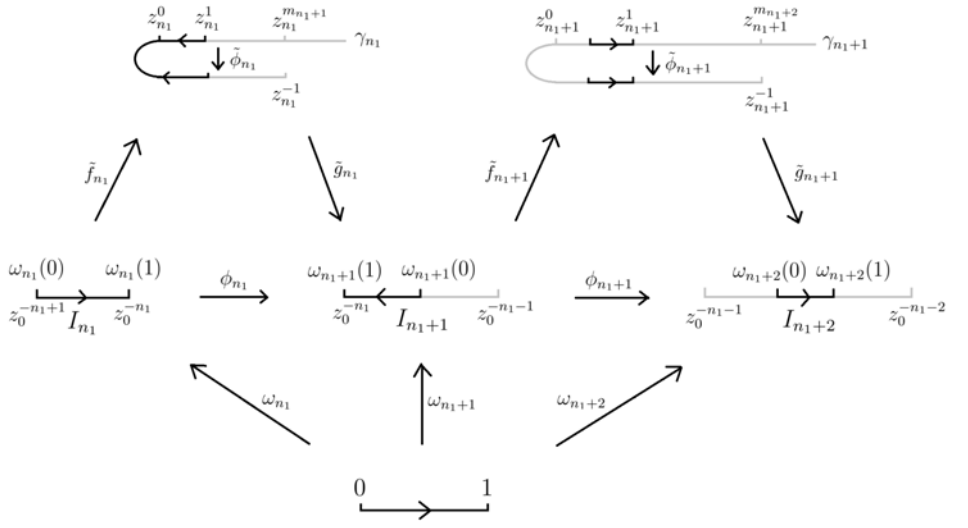


FIGURE 4. This is the sketch map of the definition of ω_{n_1+1} and ω_{n_1+2}.

For $0 \le t \le 1$, define $\eta(t)$ to be the limit of $\omega_k(t)$, $0 \le t \le 1$. To extend η to be a curve on $[0, 2]$, let us extend ω_k from $[0, 1]$ to $[0, 2]$ for all $k \ge n_2$. By Remark 7.1, the points $z_0^{-(n_2-1)}$, $\omega_{n_2}(0)$, $\omega_{n_2}(1)$ and $z_0^{-n_2}$ appear in γ_0 in order. Let us first parameterize $[\omega_{n_2}(1), z_0^{-n_2}]$ as a continuous curve $\omega_{n_2}(t)$, $1 \le t \le 2$, such that $\omega_{n_2}(2) = z_0^{-n_2}$. This extends ω_{n_2} as a continuous curve defined on $[0, 2]$. For $1 \le t \le 2$ and $k > n_2$, we also define

$$\omega_k(t) = \phi_{k-1} \circ \omega_{k-1}(t).$$

This extends all ω_k to continuous curves on $[0, 2]$. The same argument as in the proof of Lemma 7.4 implies that ω_k converges uniformly on $[0, 2]$. We define the limit to be $\eta(t)$, $0 \le t \le 2$.

In this way, for each $k \ge 1$ and $n_k \le l < n_{k+1}$, we can define a continuous curve $\omega_l(t)$, $0 \le t \le k$, such that, for any given $k_0 \ge 1$, $\{\omega_l\}_{l \ge n_{k_0}}$ converges uniformly on $[0, k_0]$. Hence we get a limit curve $\eta(t)$, $0 \le t < \infty$. By making n_k larger, if necessary, we can get the following lemma.

LEMMA 7.5. *For every $\epsilon > 0$, there is an $k \ge 1$ such that, for all $l \ge n_k$,*

$$\operatorname{dist}(\omega_l(t), \eta(t)) < \epsilon$$

for all $t \in [0, k]$.

LEMMA 7.6. *By reparameterization, if necessary, η is one-to-one.*

Proof. Suppose it were not true. Then there exist $0 \le t_1 < t_2 < \infty$ and $d > 0$ such that $\eta(t_1) = \eta(t_2)$ and

$$\operatorname{diam}(\{\eta(t) \mid t_1 \le t \le t_2\}) > d.$$

By Lemma 4.1 (bounded-wiggling), we can assume that, for all $n \ge 1$ and $0 \le i \le n$, $W(\Gamma) < K_0$, where $\Gamma \subset \gamma_i^{i-n}$ is any subarc between two adjacent folding points of γ_i^{i-n}.

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Let $D > K_0$ be an arbitrarily large number and take $l \geq 1$ such that

$$d > \delta^l \cdot D,$$

where $0 < \delta < 1$ is the constant in Corollary 2.3. Take $k \geq 1$ such that $t_1, t_2 \in [0, k]$ and let $m = \max\{l, n_k\}$. Then

$$\tilde{f}_m \circ \omega_n(t), \quad n \geq n_k,$$

converges uniformly to $\tilde{f}_m \circ \eta(t)$ on $[0, k]$.

Suppose that $n < m$. Note that $\{\omega_{n+1}(t) \mid t_1 \leq t \leq t_2\} \subset I_{n+1}$ and $\tilde{f}_{n+1}(I_{n+1}) = [z^0_{n+1}, z^1_{n+1}]$ is contained in the tail of γ_{n+1}. Since $m > n$, it follows that $\{\tilde{f}_m \circ \omega_{n+1}(t) \mid t_1 \leq t \leq t_2\} \subset \tilde{f}_m(I_{n+1})$ is contained in the tail of γ_m and thus contains no folding points of γ_m in its interior. In particular, for $t_1 < t < t_2$, $\tilde{f}_m \circ \omega_m(t)$ is not a folding point for γ_m.

Suppose that $n = m$. We write

$$\begin{aligned}
&\tilde{f}_m \circ \omega_{m+1}(t) \\
&\quad = \tilde{f}_m \circ \tilde{g}_m \circ \tilde{\phi}_m \circ \tilde{f}_m \circ \omega_m(t) \\
&\quad = \tilde{\phi}_m \circ \tilde{f}_m \circ \omega_m(t), \quad t_1 \leq t \leq t_2.
\end{aligned}$$

Since we have proved that $\tilde{f}_m \circ \omega_m(t)$ is not a folding point for γ_m, by regarding $\tilde{f}_m \circ \omega_m(t)$ as z in the second assertion of Lemma 5.5, $\tilde{f}_m \circ \omega_{m+1}(t)$ is not a folding point for γ_m.

Suppose that $n > m$. We write

$$\begin{aligned}
&\tilde{f}_m \circ \omega_{n+1}(t) \\
&\quad = \tilde{f}_m \circ \tilde{g}_n \circ \tilde{\phi}_n \circ \tilde{f}_n \circ \omega_n(t) \\
&\quad = (g_{m+1} \circ \cdots \circ g_n) \circ \tilde{\phi}_n \circ (f_n \circ \cdots \circ f_{m+1}) \circ \tilde{f}_m \circ \omega_n(t), \quad t_1 \leq t \leq t_2.
\end{aligned}$$

By regarding $(f_n \circ \cdots \circ f_{m+1}) \circ \tilde{f}_m \circ \omega_n(t)$ as z in the first assertion of Lemma 5.5, $\tilde{f}_m \circ \omega_n(t)$ is a folding point for γ_m if and only if $\tilde{f}_m \circ \omega_{n+1}(t)$ is a folding point for γ_m. Since we have proved that $\tilde{f}_m \circ \omega_{m+1}(t)$ is not a folding point for γ_m, by induction,

$$\{\tilde{f}_m \circ \omega_n(t) \mid t_1 \leq t \leq t_2\}$$

contains no folding points in its interior for all $n \geq n_k$.

On the other hand, by Corollary 2.3,

$$\operatorname{diam}(\{\tilde{f}_m \circ \eta(t) \mid t_1 \leq t \leq t_2\}) > \operatorname{diam}(\{\eta(t) \mid t_1 \leq t \leq t_2\})/\delta^l > d/\delta^l > D.$$

Since $\tilde{f}_m \circ \omega_n(t)$ converge uniformly to $\tilde{f}_m \circ \eta(t)$ on $[t_1, t_2]$ and $\eta(t_1) = \eta(t_2)$, by taking n large enough,

$$\operatorname{diam}(\{\tilde{f}_m \circ \omega_n(t) \mid t_1 \leq t \leq t_2\}) > D$$

and

$$\operatorname{dist}(\tilde{f}_m \circ \omega_n(t_1), \tilde{f}_m \circ \omega_n(t_2))$$

is as small as we want. Since $\{\tilde{f}_m \circ \omega_n(t) \mid t_1 \leq t \leq t_2\}$ is contained in T_1, by taking D large enough, $W(\{\tilde{f}_m \circ \omega_n(t) \mid t_1 \leq t \leq t_2\}) > K_0$. Since $\{\tilde{f}_m \circ \omega_n(t) \mid t_1 \leq t \leq t_2\}$ contains no folding points of γ_m in its interior, this is a contradiction with the choice of K_0. The lemma has been proved. □

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LEMMA 7.7. *For any $0 \le t_0 < \infty$, there is a sequence $t_k \to \infty$ such that $\eta(t_k) \to \eta(t_0)$. That is, η accumulates everywhere upon itself.*

Proof. Take an arbitrary $t_0 \in [0, \infty)$. It is sufficient to prove that, for every $\epsilon > 0$ and every $T > 0$, there is a $t \ge T$ such that $\operatorname{dist}(\eta(t), \eta(t_0)) < \epsilon$. Take $k \ge T$ large enough such that $t_0 \in [0, k]$. By Lemma 7.5, we can assume that

$$\operatorname{dist}(\omega_{n_{k+1}-2}(t_0), \eta(t_0)) < \epsilon/3 \quad \text{and} \quad \operatorname{dist}(\omega_{n_{k+1}}(t), \eta(t)) < \epsilon/3 \quad \text{for all } t \in [0, k+1].$$

Thus, it is sufficient to prove that

$$\operatorname{dist}(\omega_{n_{k+1}}(t), \omega_{n_{k+1}-2}(t_0)) < \epsilon/3,$$

for some $t \in [k, k+1]$.

To this end, note that by the definition of $\omega_{n_{k+1}}$ and Remark 7.1,

$$\begin{aligned} \tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}}([0,k])) &= \phi_{n_{k+1}-1} \circ \tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}-1}([0,k])) \\ &\subset \phi_{n_{k+1}-1} \circ \tilde{f}_{n_{k+1}-1}(I_{n_{k+1}-1}). \end{aligned} \tag{7.1}$$

Since $m_{n_{k+1}} \ge 2$,

$$\tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}}([0,k+1])) = \tilde{f}_{n_{k+1}-1}(I_{n_{k+1}}) \supset \phi_{n_{k+1}-1} \circ \tilde{f}_{n_{k+1}-1}(I_{n_{k+1}-2}). \tag{7.2}$$

Then, from the fact that $\omega_{n_{k+1}-2}(t_0) \in I_{n_{k+1}-2}$ and from (7.1)–(7.2), we can get a $t \in [k, k+1]$ such that

$$\tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}}(t)) = \phi_{n_{k+1}-1} \circ \tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}-2}(t_0)).$$

It follows that $\operatorname{dist}(\tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}}(t)), \tilde{f}_{n_{k+1}-1}(\omega_{n_{k+1}-2}(t_0))) < D_6$ by Proposition 5.4. Thus, by Corollary 2.3,

$$\operatorname{dist}(\omega_{n_{k+1}}(t), \omega_{n_{k+1}-2}(t_0)) < D_6 \cdot \delta^{n_{k+1}-1} < \epsilon/3$$

provided that k is large enough. It follows that $|\eta(t) - \eta(t_0)| < \epsilon$. The lemma has been proved. □

LEMMA 7.8. *The closure of η does not separate the plane.*

Proof. The argument is same as that in the proof of [**6**, Theorem 5.1]. Suppose this were not true. Let U be one of the bounded components of the complement of the closure of η. Then, for every $n \ge 0$, $E_\lambda^n(U)$ is contained in one of the strips T_1, T_0 and T_{-1}. Thus E_λ is normal in U. It follows that U is contained in the Fatou set of E_λ. This contradicts the fact that the Fatou set of E_λ is empty. The lemma has been proved. □

The ray η we have constructed satisfies the conditions of Lemma 7.3. Hence, we get the following corollary.

COROLLARY 7.9. *The closure of η is an indecomposable continuum.*

PROPOSITION 7.10. *Suppose there are infinitely many a_k with $a_k \ge 2$. Then the accumulation set of γ is the closure of η. Thus the accumulation set of γ is an indecomposable continuum.*

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Proof. Note that, for each $t \in [0, \infty)$, $\eta(t) = \lim_{k\to\infty} \omega_k(t)$ and $\omega_k(t) \in I_k$ for all k large enough. This implies that η and thus its closure are contained in the accumulation set of γ. For the other direction, take a point z in the accumulation set of the ray and let $\epsilon > 0$ be an arbitrarily small number. Then, for all $N \geq 1$, there is an $n > N$ and a point $w \in I_n$ such that $\operatorname{dist}(z, w) < \epsilon/4$. Note that w is not necessarily parameterized. We take n_k such that $n_k - 2 > n$. For $k \geq 1$, recall that

$$\phi_k = g_1 \circ \cdots \circ g_k \circ \tilde{\phi}_k \circ f_k \circ \cdots \circ f_1$$

denotes the homeomorphism from I_k and a subarc of I_{k+1}. Then

$$x = \phi_{n_k-3} \circ \cdots \circ \phi_n(w) \in I_{n_k-2}.$$

Since $m_{n_k} \geq 2$, there is a $t \in [k, k+1]$ such that $\operatorname{dist}(\omega_{n_k}(t), x) < \epsilon/4$. If we take N and thus n_k large enough, $\operatorname{dist}(\omega_{n_k}(t), \eta(t)) < \epsilon/4$ and $\operatorname{dist}(x, w) < \epsilon/4$. This implies that $\operatorname{dist}(\eta(t), z) < \epsilon$. □

Recall that, in the beginning of this section, we assumed that there are infinitely many m_n in $\{m_n\}_{n\geq 2}$ such that $m_n \geq 2$. Now we have two subcases.

In the first case, there is some a_i such that $a_i = \infty$. Let Λ denote the accumulation set of γ_0.

PROPOSITION 7.11. *If there exists an $n \geq 1$ such that $a_n = \infty$, then Λ is an indecomposable continuum containing part of the ray γ_0. Moreover, the accumulation set on γ_0, $\Lambda \cap \gamma_0$, is just equal to $\bigcup_{k\geq n} I_k$ if the above n is the least one.*

Proof. Since $a_{k+1} \geq a_k - 1$ for all $k \geq 1$ and $a_n = \infty$, we must have $a_k = \infty$ for all $k \geq n$. This implies that Λ is an indecomposable continuum by Proposition 7.10. If we suppose that $n \geq 1$ is the least integer such that $a_n = \infty$, let us prove that $\Lambda \cap \gamma_0 = \bigcup_{k\geq n} I_k$.

To prove $\Lambda \cap \gamma_0 \supset \bigcup_{k\geq n} I_k$, it suffices to prove that $\Lambda \supset \bigcup_{k\geq n} I_k$. That is, given any $k \geq n$ and $z \in I_k$, we must prove that, for any $\epsilon > 0$ and $N \geq 1$, there is a $j > N$ and a $w \in I_j$ satisfying $\operatorname{dist}(z, w) < \epsilon$. In fact, since $a_k = \infty$, there are infinitely many $j > N$ such that I_j runs back along I_k. Let $\tilde{z}_{j-1} = \tilde{f}_{j-1}(z)$, $\tilde{w}_{j-1} = \tilde{\phi}_{j-1}(\tilde{z}_{j-1})$ and $w = \tilde{g}_{j-1}(\tilde{w}_{j-1})$. Note that $w \in [z_0^{-j}, z_0^{-j+1}] = I_j$. Then

$$\operatorname{dist}(z, w) \leq \delta^{j-1} \cdot \operatorname{dist}(\tilde{z}_{j-1}, \tilde{w}_{j-1}) < \delta^{j-1} \cdot D_6$$

by Corollary 2.3 and Proposition 5.4. Since there are infinitely many $j \geq N$, we can guarantee that $\operatorname{dist}(z, w) < \epsilon$ by taking j large enough. This proves that $\Lambda \cap \gamma_0 \supset \bigcup_{k\geq n} I_k$.

To prove that $\Lambda \cap \gamma_0 \subset \bigcup_{k\geq n} I_k$, it suffices to prove that $\Lambda \cap \gamma_0^{-(n-1)} \backslash \{z_0^{-(n-1)}\} = \emptyset$. Recall that we denote by $U_R(A)$ the R-neighborhood of a set A with respect to the hyperbolic metric in $\mathbb{C}\backslash\{0, \lambda\}$. Now we consider the ray $\gamma_l = \tilde{f}_l(\gamma_0)$ by taking l large enough. By Lemma 6.3, $\gamma_l \backslash \gamma_l^{l-(n-1)}$ is contained in $U_{D_8}([z_l^0, z_l^{l-(n-1)}])$. This implies that

$$\tilde{f}_l(\Lambda) \subset U_{D_8}([z_l^0, z_l^{l-(n-1)}]). \tag{7.3}$$

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Since $[z_l^0, z_l^{l-(n-1)}]$ and $\gamma_l^{l-(n-1)}$ have a common endpoint $z_l^{l-(n-1)}$ and are contained in the tail γ_l^0 which is almost straight, there is some constant $D_9 > 0$, depending only on D_8, such that

$$U_{D_8}([z_l^0, z_l^{l-(n-1)}]) \cap \gamma_l^{l-(n-1)} \subset U_{D_9}(z_l^{l-(n-1)}). \tag{7.4}$$

From (7.3) and (7.4),

$$\Lambda \cap \gamma_0^{-(n-1)} = \tilde{g}_l(\tilde{f}_l(\Lambda) \cap \gamma_l^{l-(n-1)}) \subset U_\epsilon(z_0^{-(n-1)})$$

with $\epsilon = D_9 \cdot \delta^l$ and $0 < \delta < 1$ being the constant in Corollary 2.3. Since ϵ can be arbitrarily small provided that l is large enough,

$$\Lambda \cap \gamma_0^{-(n-1)} = z_0^{-(n-1)}.$$

This completes the proof of the Proposition 7.11. □

In the second case, all a_i are finite integers and, moreover, there are infinitely many a_i with $a_i \geq 2$.

PROPOSITION 7.12. *If all a_i are finite integers and there are infinitely many a_i such that $a_i \geq 2$, then Λ is an indecomposable continuum disjoint from the ray γ_0.*

Proof. It suffices to prove that, for any $n \geq 1$, $\Lambda \cap \gamma_0^{-(n-1)} \setminus \{z_0^{-(n-1)}) = \emptyset$. The argument is completely the same as used in the proof of Proposition 7.11. □

7.2. *Jordan arcs.* In this subsection, we only need to prove the following fact.

PROPOSITION 7.13. *If there is some $n \geq 1$ such that $a_k = 1$ for all $k \geq n$, then Λ is a Jordan arc.*

Proof. Since $a_k = 1$ for all $k \geq n$, we can assume that $m_j = 1$ for all $j \geq l$ by Lemma 7.2. Let us parameterize I_l as an injective curve $w_l(t)$, $0 \leq t \leq 1$. Suppose we have defined w_{k-1} for $k \geq l + 1$. As in §7.1,we define

$$\omega_k(t) = g_1 \circ \cdots g_{k-1} \circ \tilde{\phi}_{k-1} \circ f_{k-1} \circ \cdots f_1 \circ \omega_{k-1}(t).$$

By the same argument as in the proof of Lemma 7.4, it follows that w_k converges uniformly to a continuous curve $\eta(t)$, $0 \leq t \leq 1$. By the same argument as in the proof of Lemma 7.6, η is injective up to reparameterization if necessary. This implies that $\Lambda = \{\eta(t) \mid 0 \leq t \leq 1\}$ is a Jordan arc. The proof of Proposition 7.13 is complete. □

This completes the proof of the *further statement of the Main Theorem* (cf. §3.3).

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REFERENCES

[**1**] A. M. Benini and M. M. Rocha. Satellite parabolic perturbations in exponential dynamics. *Preprint*, 2012.
[**2**] C. Bodelón, R. L. Devaney, L. Goldberg, M. Hayes, J. Hubbard and G. Roberts. Hairs for the complex exponential family. *Internat. J. Bifur. Chaos Appl. Sci. Engrg* **9** (1999), 1517–1534.

https://doi.org/10.1017/etds.2017.33 Published online by Cambridge University Press

[**3**] S. Curry. One-dimensional nonseparating plane continua with disjoint ε-dense subcontinua. *Topology Appl.* **39** (1991), 145–151.
[**4**] R. L. Devaney. Knaster-like continua and complex dynamics. *Ergod. Th. & Dynam. Sys.* **13** (1993), 627–634.
[**5**] R. L. Devaney. Complex exponential dynamics. *Handbook of Dynamical Systems.* Vol. 3. Eds. H. Broer, B. Hasselblatt and F. Takens. Elsevier, North-Holland, 2010, pp. 125–224.
[**6**] R. L. Devaney and X. Jarque. Indecomposable continua in exponential dynamics. *Conform. Geom. Dyn.* **6** (2002), 1–12 (electronic).
[**7**] R. L. Devaney, X. Jarque and M. M. Rocha. Indecomposable continua and Misiurewicz points in exponential dynamics. *Internat. J. Bifur. Chaos Appl. Sci. Engrg* **15**(10) (2005), 3281–3293.
[**8**] R. L. Devaney and M. Krych. Dynamics of $\exp(z)$. *Ergod. Th. & Dynam. Sys.* **4**(1) (1984), 35–52.
[**9**] J. Milnor. *Dynamics in One Complex Variable*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.
[**10**] L. Rempe. Dynamics of exponential maps. *Doctoral Thesis*, Christian-Albrechts-Universität Kiel, 2003.
[**11**] L. Rempe. A landing theorem for periodic rays of exponential maps. *Proc. Amer. Math. Soc.* **134** (2006), 2639–2648.
[**12**] L. Rempe. On non-landing dynamic rays of exponential maps. *Ann. Acad. Sci. Fenn.* **32** (2007), 353–369.
[**13**] L. Rempe and S. Worsley. *Preprint*, 2015.
[**14**] G. Rottenfusser, J. Ruckert, L. Rempe and D. Schleicher. Dynamic rays of bounded-type entire functions. *Ann. of Math. (2)* **173** (2010), 77–125.
[**15**] D. Schleicher and J. Zimmer. Escaping points of exponential maps. *J. Lond. Math. Soc. (2)* **67** (2003), 380–400.
[**16**] D. Schleicher and J. Zimmer. Periodic points and dynamic rays of exponential maps. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 327–354.

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