Canad. Math. Bull. 2026, pp. 1–14
http://dx.doi.org/10.4153/S0008439526101702
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

On a sector for derivative which implies the starlikeness

Mamoru Nunokawa, Krzysztof Piejko, and Janusz Sokół

Abstract. It is known that the condition $|\arg f'(z)| < \pi/2$, $|z| < 1$, is not sufficient for an analytic function $f(z) = z + a_2 z^2 + \cdots$ in $|z| < 1$, to be starlike with respect to the origin. We look for the largest $\alpha > 0$ such that the condition $|\arg f'(z)| < \alpha\pi/2$ in $|z| < 1$ is a sufficient condition for f to be a univalent, starlike, convex, or Bazilevič function.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ while $\mathcal{A} \subset \mathcal{H}$ denote the set of all functions g that are analytic in $\mathbb{D}$ and has the series representation

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Let $\mathcal{S} \subset \mathcal{A}$ be the class of normalized univalent functions. In [15], Robertson introduced the class $\mathcal{S}^*_\alpha$ of starlike functions of order α, $\alpha < 1$

$$\mathcal{S}^*_\alpha := \left\{ g \in \mathcal{A} : \operatorname{Re} \frac{zg'(z)}{g(z)} > \alpha,\ z \in \mathbb{D} \right\}.$$

In particular, when $\alpha = 0$, the class $\mathcal{S}^*_0 := \mathcal{S}^*$ is the well-known class of starlike functions which map $\mathbb{D}$ onto a domain starlike with the origin (see [3, 12]). Furthermore, the related class $\mathcal{K}_\alpha$ of convex functions of order α may be defined in the following way:

$$\begin{aligned} \mathcal{K}_\alpha &:= \{ g \in \mathcal{A} : zg'(z) \in \mathcal{S}^*_\alpha \} \\ &= \left\{ g \in \mathcal{A} : \operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > \alpha,\ z \in \mathbb{D} \right\}, \quad \alpha < 1. \end{aligned}$$

In particular, when $\alpha = 0$, the class $\mathcal{K}_0 := \mathcal{K}$ is the well-known class of convex functions which map $\mathbb{D}$ onto a convex domain. Strohhäcker in [18] proved that $\mathcal{K}_0 \subset \mathcal{S}^*_{1/2} \subset \mathcal{S}^*_0$. Wilken and Feng in [20] showed that if $g \in \mathcal{K}_\alpha$, then $g \in \mathcal{S}^*_{\delta(\alpha)}$, where

$$\delta(\alpha) = \begin{cases} 1/\log 4, & \alpha = 1/2, \\ (1-2\alpha)/(4^{1-\alpha} - 2), & \alpha \neq 1/2. \end{cases} \tag{1.2}$$

Received by the editors April 7, 2025; revised December 21, 2025; accepted December 21, 2025.
Published online on Cambridge Core January 26, 2026.
AMS subject classification: **30C45**, 30C80.
Keywords: Analytic functions, starlike, convex, Bazilevič function.

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta < 2$,

$$\mathcal{SS}^*(\beta) := \left\{ g \in \mathcal{A} : \left| \arg \frac{zg'(z)}{g(z)} \right| < \frac{\beta\pi}{2}, \ z \in \mathbb{D} \right\}, \tag{1.3}$$

which was introduced in [2, 17]. Furthermore, $\mathcal{SK}(\beta) = \{g \in \mathcal{A} : zg'(z) \in \mathcal{SS}^*(\beta)\}$ denote the class of strongly convex functions of order β. Analogously to (1.2), in the work [14], it was proved that if $\beta \in (0,1)$ and $g \in \mathcal{SK}(\alpha(\beta))$, then $g \in \mathcal{SS}(\beta)$, where

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left(\frac{\beta n(\beta) \sin(\pi(1-\beta)/2)}{m(\beta) + \beta n(\beta) \cos(\pi(1-\beta)/2)} \right), \tag{1.4}$$

and where

$$m(\beta) = (1+\beta)^{(1+\beta)/2} \quad n(\beta) = (1-\beta)^{(\beta-1)/2}.$$

It is well known that if $g \in \mathcal{H}$ satisfies $g(0) = 0$, $g'(0) \neq 0$, and

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > 0, \quad z \in \mathbb{D},$$

then g is univalent and starlike in $\mathbb{D}$. Moreover, it is known that for $g \in \mathcal{A}$, the condition

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > 0, \quad z \in \mathbb{D}$$

is sufficient for starlikeness of order $1/2$ of g in $\mathbb{D}$. The condition

$$|zg''(z)| < 2|g'(z)|, \quad z \in \mathbb{D}$$

is also sufficient for starlikeness of g in $\mathbb{D}$.

Robertson in [16] obtained that if $g \in \mathcal{A}$ with $g(z)/z \neq 0$, $z \in \mathbb{D}$ and if there exists k, $0 < k \leq 2$, such that

$$\left| \frac{zg''(z)}{g'(z)} \right| \leq k \left| \frac{zg'(z)}{g(z)} \right|, \quad z \in \mathbb{D},$$

then $g \in \mathcal{S}^*_{2/(2+k)}$. Mocanu in [10] proved that if $g \in \mathcal{A}$ with $g(z)g'(z)/z \neq 0$, $z \in \mathbb{D}$, and

$$\left| 1 + \frac{g''(z)}{g'(z)} \right| \leq \sqrt{2} \left| \frac{zg'(z)}{g(z)} + 1 \right|, \quad z \in \mathbb{D},$$

then $g \in \mathcal{S}^*_0$.

The simplest analytic criteria for univalence have limited application because they imply special properties, such as starlikenes or close-to-convexity. These conditions are sufficient for univalence but far from necessary. Another important condition sufficient for univalence of $g \in \mathcal{H}$ is due to Nehari [11] and involves the Schwarzian derivative $\{g, z\}$:

$$|\{g,z\}| = \left| \left\{ \frac{g''(z)}{g'(z)} \right\}' - \frac{1}{2} \left\{ \frac{g''(z)}{g'(z)} \right\}^2 \right| < \frac{2}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{1.5}$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

In (1.5), the constant 2 cannot be replaced by a smaller number. On the other hand, inequality (1.5) becomes a necessary condition for univalence of g if the constant 2 is replaced by 6. Instead of Nehari's condition (1.5), it is often simpler to work directly with the ratio $g''(z)/g'(z)$. Becker [1] applied Loewner's method to show that the condition

$$\left|\frac{g''(z)}{g'(z)}\right| < \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{1.6}$$

is sufficient for the univalence of $g \in \mathcal{H}$ in $\mathbb{D}$. Becker also proved that constant 1 in the nominator cannot be replaced by a greater number. The well-known Noshiro–Warshawski theorem [13, 19] says that

$$\mathrm{Re}\{e^{i\phi}g'(z)\} > 0, \quad z \in \mathbb{D}, \tag{1.7}$$

for some $\phi \in \mathbb{R}$, is a sufficient condition for the univalence of $g \in \mathcal{A}$. On the other hand, the Noshiro–Warshawski condition with $\phi = 0$ is not a sufficient condition for $g \in \mathcal{A}$ to be starlike (see [5, p. 123]). The radius of starlikess in the class $\{g \in \mathcal{A} : \mathrm{Re}g'(z) > 0,\ |z| < 1\}$ is unknown but it is at least $4\sqrt{2} - 5 = 0.65\ldots$, [6, p. 95], [8].

Several more complicated sufficient conditions for starlikeness and for convexity are collected in Chapter 5 of [9] (see also [6]). The current work revolves around determining the optimal angle opening of the sector containing the image of f' that guaranties certain geometric properties of f, i.e., f is a starlike, convex, or Bazilevič function. We utilize the notion of the dimension of a subclass $\mathcal{R}$ of the class $\mathcal{A}$ of normalized analytic functions in $\mathbb{D}$. That is, the largest number $\varepsilon \in (0, 2]$ for which the condition $|\arg f'(z)| < \varepsilon/2$, for $f \in \mathcal{A}$, implies that $f \in \mathcal{R}$. The main objective of the article is to obtain a lower bound (sharper than the trivial one 1/2) for the dimension of the class of starlike functions.

2 Sector for derivative sufficient for starlikeness

Definition 2.1 Let $\mathcal{R}$ be a subclass of $\mathcal{A}$. Set E to be the set of all $\varepsilon \in (0, 2]$ such that condition $|\arg f'(z)| < \varepsilon/2$, where $z \in \mathbb{D}$ and $f \in \mathcal{A}$, implies $f \in \mathcal{R}$. The dimension of $\mathcal{R}$ is defined as the supremum of E and is denoted by $D\mathcal{R}\{|\arg f'(z)| < \varepsilon\pi/2\}$.

For $\phi = 0$, the known Noshiro–Warshawski condition (1.7) has the form

$$|\arg f'(z)| < \pi/2, \quad z \in \mathbb{D}$$

and this is a sufficient condition for $f \in \mathcal{A}$ to be in the class $\mathcal{S}$ of normalized univalent functions. Goodman [4] proved that for each $\varepsilon > 0$, there is a function $f \in \mathcal{A}$, which satisfies

$$|\arg f'(z)| < \pi/2 + \varepsilon, \quad z \in \mathbb{D}$$

and f is not univalent in $\mathbb{D}$. Therefore, we have

$$D\mathcal{S}\{|\arg f'(z)| < \varepsilon\pi/2\} = 1.$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

If $h(z) \in \mathcal{A}$ is such that

$$h'(z) = \left\{\frac{1+z}{1-z}\right\}^{\alpha} \quad \text{or} \quad |\arg h'(z)| < \alpha\pi/2, \quad z \in \mathbb{D},$$

then

$$1 + \frac{zh''(z)}{h'(z)} = 1 + \frac{2\alpha z}{1-z^2}.$$

This shows that there is no an $\alpha > 0$ such that the condition $|\arg\{h'(z)\}| < \alpha\pi/2$ implies that $h(z)$ is a convex function. Thus, $D\mathcal{K}\{|\arg f'(z)| < \varepsilon\pi/2\}$ doesn't exist.

For two functions f and g analytic in $\mathbb{D}$, we say that f is subordinate to g, written by $f \prec g$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

It is known [7] that for a convex function $q(z) = 1 + q_1 z + \cdots$, we have

$$[f \in \mathcal{A}, \quad f'(z) \prec q(z)] \quad \Rightarrow \quad [f(z)/z \prec q(z)].$$

Therefore, the condition

$$f'(z) \prec \sqrt{\frac{1+z}{1-z}} \quad \text{or} \quad |\arg f'(z)| < \pi/4, \quad z \in \mathbb{D}$$

implies

$$\frac{f(z)}{z} \prec \sqrt{\frac{1+z}{1-z}} \quad \text{or} \quad \left|\arg \frac{f(z)}{z}\right| < \pi/4, \quad z \in \mathbb{D}$$

and so if $|\arg\{f'(z)\}| < \pi/4$, then

$$\begin{aligned}\left|\arg \frac{zf'(z)}{f(z)}\right| &\le |\arg f'(z)| + \left|\arg \frac{f(z)}{z}\right| \\ &< \pi/4 + \pi/4 = \pi/2, \quad z \in \mathbb{D}.\end{aligned}$$

Then, we can see that

$$|\arg f'(z)| < \pi/4, \quad z \in \mathbb{D}, \tag{2.1}$$

is a sufficient condition for the starlikeness of $f \in \mathcal{A}$ and we can see that $D\mathcal{S}^*\{|\arg\{f'(z)\}| < \varepsilon\pi/2\}$ satisfies

$$D\mathcal{S}^*\{|\arg f'(z)| < \varepsilon\pi/2\} \ge 1/2. \tag{2.2}$$

In this article, we want to replace in (2.2) the number 1/2 by a greater number.

Lemma 2.2 [14] *Let $P(z)$ be analytic function in $|z| < 1$ with $P(0) = 1$, $P(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that $|\arg P(z)| < \pi\gamma/2$ for $|z| < |z_0|$ and $|\arg P(z_0)| = \pi\gamma/2$ for some $\gamma \in (0, 2]$, then we have*

$$\frac{z_0 P'(z_0)}{P(z_0)} = ik\gamma \tag{2.3}$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

for some $k \geq (a + a^{-1})/2 \geq 1$ *when* $\arg P(z_0) = \pi\gamma/2$ *or* $k \leq -(a + a^{-1})/2 \leq -1$ *when* $\arg P(z_0) = -\pi\gamma/2$, *where* $\{P(z_0)\}^{1/\gamma} = \pm ia$, *and* $a > 0$.

3 Main results

Theorem 3.1 *If* $q(z) \in \mathcal{H}$, $q(0) = 0$, $q'(0) \neq 0$, *and*

$$\left|\arg \frac{q'(z)}{q'(0)}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{3.1}$$

for some $\alpha \in (0, 2]$, *and*

$$\frac{q(z)}{z} \neq 0, \quad z \in \mathbb{D},$$

then

$$\left|\arg \frac{q(z)}{zq'(0)}\right| < \frac{\pi\gamma}{2} < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

where $\gamma \in (0, 2]$ *is the smallest positive root of the equation*

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\alpha\pi}{2}. \tag{3.2}$$

Proof Because $q(0) = 0$, the function

$$s(z) = \frac{q(z)}{zq'(0)}, \quad z \in \mathbb{D},$$

is analytic in $\mathbb{D}$, $s(0) = 1$ and it follows that

$$\frac{q'(z)}{q'(0)} = s(z) + zs'(z).$$

If there exists a point $z_1 \in \mathbb{D}$, such that

$$|\arg s(z)| < \frac{\pi\gamma}{2}, \quad (|z| < |z_1|)$$

and at the point z_1 is

$$|\arg s(z_1)| = \frac{\pi\gamma}{2},$$

then from Lemma 2.2, we have

$$\frac{z_1 s'(z_1)}{s(z_1)} = ik\gamma,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad \arg p(z_1) = \frac{\pi\gamma}{2}$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

and

$$k \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \le -1 \quad \text{when} \quad \arg p(z_1) = -\frac{\pi\gamma}{2},$$

where

$$\{p(z_1)\}^{1/\gamma} = \pm ia, \quad \text{and} \quad a > 0.$$

Therefore, we have

$$\begin{aligned}
\left|\arg \frac{q'(z_1)}{q'(0)}\right| &= \left|\arg\left\{s(z_1)\left(1 + \frac{z_1 s'(z_1)}{s(z_1)}\right)\right\}\right| \\
&= \left|\arg\{s(z_1)\} + \arg\left\{1 + \frac{z_1 s'(z_1)}{s(z_1)}\right\}\right| \\
&= \left|\frac{\pi\gamma}{2} + \arg\left\{1 + \frac{z_1 s'(z_1)}{s(z_1)}\right\}\right| \\
&= \left|\frac{\pi\gamma}{2} + \arg\{1 + i\gamma k\}\right| \\
&\ge \frac{\pi\gamma}{2} + \tan^{-1}\gamma \\
&= \frac{\alpha\pi}{2}
\end{aligned}$$

because of (3.2) but this contradicts hypothesis (3.1). This shows that

$$|\arg s(z)| < \frac{\gamma\pi}{2}, \quad z \in \mathbb{D}$$

because $\arg s(0) = 0$ and the hypothesis that there exists a point $z_1\mathbb{D}$ with $|\arg s(z_1)| = \pi\gamma/2$ gives the above contradiction. ∎

Probably, the γ in (3.2) is not the smallest bound for $|\arg q(z)/z|$. The smallest bound gives the following theorem but not in explicit form.

Theorem 3.2 (See [7]) *Let the function h be analytic and convex univalent in $\mathbb{D}$ with $h(0) = a$. Let also $p(z) = a + b_n z^n + b_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{D}$. If*

$$p(z) + \frac{zp'(z)}{c} \prec h(z), \quad z \in \mathbb{D}$$

for $\mathrm{Re}\, c \ge 0$, $c \ne 0$, *then*

$$p(z) \prec q_n(z) \prec h(z), \quad z \in \mathbb{D},$$

where $q_n(z) = \frac{c}{nz^{c/n}} \int_0^z t^{c/n-1} h(t)\mathrm{d}t$. *Moreover, the function $q_n(z)$ is convex univalent and is the best dominant in the sense that if $p(z) \prec q(z)$, then $q_n(z) \prec q(z)$.*

Therefore, for $p(z) + \frac{zp'(z)}{c} = q'(z)$, $c = 1$, we may apply the following version of the above Theorem 3.2.

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

Theorem 3.3 *Let $q'(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{D}$. If*

$$q'(z) \prec h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in \mathbb{D}, \quad \left[or\ if \quad |\arg q'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}\right],$$

then

$$q(z)/z \prec q_n(z) \prec h(z), \quad z \in \mathbb{D}, \tag{3.3}$$

where $q_n(z) = \frac{1}{nz^{1/n}} \int_0^z t^{1/n-1} h(t)\mathrm{d}t$. Moreover, the function $q_n(z)$ is convex univalent and is the best dominant in the sense that if $q(z) \prec p(z)$, then $q_n(z) \prec p(z)$.

If we assume $n = 1$, then (3.3) gives

$$q(z)/z \prec q_1(z) = \frac{1}{z} \int_0^z \left(\frac{1+t}{1-t}\right)^{\alpha} \mathrm{d}t, \quad z \in \mathbb{D}.$$

From this we may find, for a given α, the smallest γ such that

$$\left|\arg\left\{\frac{1}{z} \int_0^z \left(\frac{1+t}{1-t}\right)^{\alpha} \mathrm{d}t\right\}\right| < \frac{\pi\gamma}{2}, \quad z \in \mathbb{D}. \tag{3.4}$$

This γ is probably smaller (and better) than the γ in (3.2) but to calculate γ in (3.4) is a hard problem.

Corollary 3.4 *If $q(z) \in \mathcal{H}$, $q(z) = 0$, $q(z)/z \neq 0$, and $q'(z) \neq 0$ in $\mathbb{D}$, and*

$$|\arg q'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

for some $\alpha > 0$, such that $\alpha + \gamma < 2$, where $\gamma > 0$ is the smallest positive root of the equation

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\alpha\pi}{2},$$

then

$$\left|\arg \frac{zq'(z)}{q(z)}\right| < \frac{\pi(\alpha+\gamma)}{2}, \quad z \in \mathbb{D}.$$

Proof From Theorem 3.1, we have

$$\left|\arg \frac{q(z)}{z}\right| < \frac{\pi\gamma}{2} < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

Therefore, we obtain

$$\left|\arg\frac{zq'(z)}{q(z)}\right| \le |\arg\{q'(z)\}| + \left|\arg\frac{q(z)}{z}\right| < \frac{\alpha\pi}{2} + \frac{\pi\gamma}{2} = \frac{\pi(\alpha+\gamma)}{2}, \quad z \in \mathbb{D}.$$

■

If we take $q(z) = f(z)$, $f \in \mathcal{A}$, then Corollary 3.4 provides a sufficient condition for the function f to be strongly starlike of some order (see (1.3)).

Corollary 3.5 *If $f \in \mathcal{A}$, $f(z)/z \neq 0$ in $\mathbb{D}$, and*

$$|\arg f'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

for some $\alpha > 0$, such that $\alpha + \gamma < 2$, where $\gamma > 0$ is the smallest positive root of the equation

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\alpha\pi}{2}, \tag{3.5}$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi(\alpha+\gamma)}{2}, \quad z \in \mathbb{D}$$

or f is in the class $\mathcal{SS}(\beta)$ of strongly starlike functions of order $\beta = \alpha + \gamma$ (see (1.3)).

Therefore, the dimensions of strongly starlikeness of order $\beta = \alpha + \gamma$ in the class $\mathcal{A}$ satisfies the condition

$$D\mathcal{SS}(\beta)\{|\arg f'(z)| < \varepsilon\pi/2\} \ge \alpha, \tag{3.6}$$

where γ is described in (3.5). For example, if we consider

$$\alpha = \frac{\sqrt{3}+1}{3},$$

then from (3.5), we obtain

$$\gamma = \frac{\sqrt{3}}{3}, \quad \beta = \alpha + \gamma = \frac{2\sqrt{3}+1}{3} < 2$$

and so (3.6) becomes

$$D\mathcal{SS}((2\sqrt{3}+1)/3)\{|\arg f'(z)| < \varepsilon\pi/2\} \ge \frac{\sqrt{3}+1}{3}.$$

Corollary 3.6 *If $f \in \mathcal{A}$, $f''(z) \neq 0$, and $f'(z)/z \neq 0$ in $\mathbb{D}$, and*

$$|\arg f''(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

for some $\alpha > 0$, such that $\alpha + \gamma < 2$, where γ is the smallest positive root of the equation

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\alpha\pi}{2},$$

then

$$\left|\arg \frac{zf''(z)}{f'(z)-1}\right| < \frac{\pi(\alpha+\gamma)}{2}, \quad z \in \mathbb{D}. \tag{3.7}$$

This means that $f'(z)$ is strongly starlike of order $\alpha + \gamma$ with respect to $1 = f'(0)$.

Proof If we take $q(z) = f'(z) - 1$, $f \in \mathcal{A}$, then Corollary 3.4 gives

$$\left|\arg \frac{zq'(z)}{q(z)}\right| = \left|\arg \frac{zf''(z)}{f'(z)-1}\right| < \frac{\pi(\alpha+\gamma)}{2}, \quad z \in \mathbb{D}.$$

■

If we consider

$$\alpha = \sqrt{2} - \frac{3}{4},$$

then

$$\gamma = \sqrt{2} - 1, \quad \beta = \alpha + \gamma = 2\sqrt{2} - \frac{7}{4} < 2$$

and so Corollary 3.6 becomes

$$|\arg f''(z)| < \frac{\alpha\pi}{2} \quad \Rightarrow \quad \left|\arg \frac{zf''(z)}{f'(z)-1}\right| < \frac{\pi(2\sqrt{2}-7/4)}{2}.$$

If we take $\alpha = 1$, then Theorem 3.1 becomes the following corollary.

Corollary 3.7 *If $q(z) \in \mathcal{H}$, $q(0) = 0$, and*

$$|\arg q'(z)| < \frac{\pi}{2}, \quad z \in \mathbb{D}$$

and

$$\frac{q(z)}{z} \neq 0, \quad z \in \mathbb{D},$$

then

$$\left|\arg \frac{q(z)}{z}\right| < \frac{\pi\gamma}{2} < \frac{\pi}{2}, \quad z \in \mathbb{D},$$

where $\gamma = 0.63832\ldots$ is the smallest positive root of the equation

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\pi}{2}.$$

Theorem 3.8 *If $f \in \mathcal{A}$ and*

$$|\arg f'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

where $\alpha = 0.6165513972\ldots$ *is the smallest positive root of the equation*

$$\pi(1-\alpha) + \tan^{-1}(1-\alpha) = \frac{\pi}{2},$$

then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \pi/2, \quad z \in \mathbb{D}$$

or $f \in \mathcal{S}^*$.

Proof If we take $q(z) = f(z)$, $f \in \mathcal{A}$, then from Theorem 3.1, the condition

$$|\arg f'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \tag{3.8}$$

implies

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi\gamma}{2} < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{3.9}$$

where γ is the smallest positive root of the equation

$$\frac{\pi\gamma}{2} + \tan^{-1}\gamma = \frac{\alpha\pi}{2}. \tag{3.10}$$

This gives

$$\begin{aligned}\left|\arg \frac{zf'(z)}{f(z)}\right| &\le |\arg f'(z)| + \left|\arg \frac{f(z)}{z}\right| \\ &< \frac{\alpha\pi}{2} + \frac{\pi\gamma}{2} \\ &= \pi\gamma + \tan^{-1}\gamma, \quad z \in \mathbb{D}.\end{aligned}$$

Then, we can see that if γ is such that

$$\pi\gamma + \tan^{-1}\gamma = \frac{\pi}{2}$$

(so $\gamma = 0.38344860\ldots$), and if $\frac{\alpha\pi}{2} + \frac{\pi\gamma}{2} = \frac{\pi}{2}$, which is equivalent to

$$\alpha = 1 - \gamma = 0.6165513972\ldots,$$

then the inequality

$$|\arg f'(z)| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}$$

is a sufficient condition for the starlikeness and so $D\mathcal{S}^*\{|\arg\{f'(z)\}| < \varepsilon\pi/2\} \ge 0.6165513972\ldots$. ∎

Theorem 3.8 says that the dimension of starlikeness in the class $\mathcal{A}$ satisfies

$$D\mathcal{S}^*\{|\arg f'(z)| < \varepsilon\pi/2\} \ge 0.6165513972\ldots.$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

The number 0.6165513972... cannot be extended to 1 because there exists a function $g \in \mathcal{A}$ with $|\arg\{g'(z)\}| < \pi/2$ which is not starlike (see [5, p. 123]). Therefore, the dimension of starlikeness in the class $\mathcal{A}$ satisfies the condition

$$0.6165513972\ldots \le D\mathcal{S}^*\{|\arg\{f'(z)\}| < \varepsilon\pi/2\} < 1.$$

Conjecture. The exact value of $D\mathcal{S}^*\{|\arg f'(z)| < \varepsilon\pi/2\}$ is $\sqrt{3}/2$.

Corollary 3.9 *If $g \in \mathcal{A}$, $g'(z) \neq 0$ in $\mathbb{D}$ and*

$$\left|\arg\left(zg'(z)\right)'\right| < \alpha\pi/2, \quad z \in \mathbb{D}, \tag{3.11}$$

where $\alpha = 0.6165513972\ldots$ is the smallest positive root of the equation

$$\pi(1-\alpha) + \tan^{-1}(1-\alpha) = \frac{\pi}{2},$$

then

$$\left|\arg\left\{1 + \frac{zg''(z)}{g'(z)}\right\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D},$$

or $g \in \mathcal{K}$.

Proof Let $f(z) = zg'(z)$. Applying (3.11) and Theorem 3.8, we obtain

$$\left|\arg\frac{zf'(z)}{f(z)}\right| = \left|\arg\left\{1 + \frac{zg''(z)}{g'(z)}\right\}\right| < \pi/2, \quad z \in \mathbb{D}.$$

■

Therefore, Corollary 3.9 says that the dimension of convexity in the class $\mathcal{A}$ satisfies

$$D\mathcal{K}\{|\arg(zg'(z))'| < \varepsilon\pi/2\} \ge 0.6165513972\ldots.$$

The number 0.6165513972... cannot be extended to 1 because there exists a function $f_0(z) \in \mathcal{A}$ with $|\arg f_0'(z)| < \pi/2$ which is not starlike. If we consider $g_0(z) \in \mathcal{A}$ such that $zg_0'(z) = f_0(z)$, then $g_0(z)$ is not convex but satisfies $|\arg\left(zg_0'(z)\right)'| < \pi/2$. Therefore, the dimension of convexity under condition $|\arg\left(zg'(z)\right)'| < \alpha\pi/2$ satisfies the inequalities

$$1 > D\mathcal{K}\{|\arg(zg'(z))'| < \varepsilon\pi/2\} \ge 0.6165513972\ldots.$$

Recall that $f \in \mathcal{A}$ is in the class $\mathcal{B}(\beta)$ of the Bazilevič functions of type β if there exists a function $g \in \mathcal{S}^*$ such that

$$\left|\arg\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.12}$$

Theorem 3.10 *Let $f \in \mathcal{A}$ be such that*

$$|\arg f'(z)| < \alpha_1\pi/2, \quad z \in \mathbb{D}, \tag{3.13}$$

for some $\alpha_1 \in (0, 2]$. If there exists a function $g \in \mathcal{S}^*$ such that

$$|\arg g'(z)| < \alpha_2\pi/2, \quad z \in \mathbb{D} \tag{3.14}$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

for some $\alpha_2 \in (0,2]$, then f is a Bazilevič function of type β, whenever β satisfies

$$\alpha + \gamma_1|\beta - 1| + \beta\gamma_2 < 1, \tag{3.15}$$

where γ_t, $t = 1, 2$, is the smallest positive root of the equation

$$\frac{\pi\gamma_t}{2} + \tan^{-1}\gamma_t = \frac{\alpha_t\pi}{2}.$$

Proof By (3.8)–(3.10), we have from (3.13)

$$\left|\arg\frac{f(z)}{z}\right| < \gamma_1\pi/2, \quad z \in \mathbb{D}.$$

In the same way, by (3.14), we have

$$\left|\arg\frac{g(z)}{z}\right| < \gamma_2\pi/2, \quad z \in \mathbb{D}.$$

Therefore, we have

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right| &= \left|\arg\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\beta-1}\left(\frac{z}{g(z)}\right)^{\beta}\right\}\right| \\ &\le |\arg f'(z)| + \left|(\beta-1)\arg\frac{f(z)}{z}\right| + \left|\beta\arg\frac{z}{g(z)}\right| \\ &< \frac{\alpha\pi}{2} + \frac{\gamma_1\pi}{2}|\beta-1| + \frac{\beta\gamma_2\pi}{2} \\ &= \frac{\pi}{2}(\alpha + \gamma_1|\beta-1| + \beta\gamma_2), \quad z \in \mathbb{D}.\end{aligned}$$

■

For $\beta = 1$, the class $\mathcal{B}(\beta)$ of the Bazilevič functions becomes a subclass of well-known class $\mathcal{C}$ of close-to-convex functions.

Corollary 3.11 *Let $f \in \mathcal{A}$ be such that*

$$|\arg f'(z)| < \alpha_1\pi/2, \quad z \in \mathbb{D}, \tag{3.16}$$

for some $\alpha_1 \in (0,2]$. If there exists a function $g \in \mathcal{S}^*$ such that

$$|\arg g'(z)| < \alpha_2\pi/2, \quad z \in \mathbb{D}, \tag{3.17}$$

where

$$\alpha_2\frac{\pi}{2} = (1-\alpha_1)\frac{\pi}{2} + \tan^{-1}(1-\alpha_1),$$

then f is a close-to-convex function.

Proof By (3.8)–(3.10), we have from (3.17)

$$\left|\arg\frac{g(z)}{z}\right| < (1-\alpha_1)\pi/2, \quad z \in \mathbb{D}.$$

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

Therefore, we have

$$\begin{aligned}\left|\arg\frac{zf'(z)}{g(z)}\right| &= \left|\arg\left\{f'(z)\left(\frac{z}{g(z)}\right)\right\}\right| \\ &\leq |\arg f'(z)| + \left|\arg\frac{z}{g(z)}\right| \\ &< \frac{\alpha_1\pi}{2} + \frac{(1-\alpha_1)\pi}{2} \\ &= \frac{\pi}{2}, \quad z \in \mathbb{D}.\end{aligned}$$

■

Acknowledgements The authors are thankful to the referee for pointing out some useful suggestions.

Author contributions The authors declare that they have equal contribution in the article.

Competing interests The authors declare that they have no financial interests. The authors have no competing interest to declare that are relevant to the content of this article.

Data availability statement Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Funding Not applicable.

Compliance with ethical standards The authors did not receive support from any organization for the submitted work.

Ethical Approval Not applicable.

References

[1] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*. J. Reine Angew. Math. **255**(1972), 23–43.
[2] D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*. J. London Math. Soc. **1**(1969), no. 2, 431–443.
[3] A. Emch, *E. Study, Konforme Abbildung Einfachzusammenhangender Bereiche*, B. C. Teubner, Leipzig, 1913.
[4] A. W. Goodman, *A note on the Noshiro-Warshawski theorem*. J. Analyse Math. **25**(1972), 401–408.
[5] A. W. Goodman, *Univalent functions. Vol. I*, Mariner Publishing Co., Tampa, FL, 1983.
[6] A. W. Goodman, *Univalent functions. Vol. II*, Mariner Publishing Co., Tampa, FL, 1983.
[7] D. I. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*. Proc. Amer. Math. Soc. **52**(1975), 191–195.
[8] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes, II*. Ann. Univ. Mariae Curie–Skłodowska Sect. A **14**(1960), 19–46.

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press

[9] S. S. Miller and P. T. Mocanu, *Differential subordinations*. In: E. J. Taft and Z. Nashed (eds.), Theory and applications, Series of Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker Inc., New York, NY, 2000.
[10] P. T. Mocanu, *On a theorem of Robertson*. Babeş-Bolyai Univ., Fac. Math. Res. Sem., Seminar Geom. Function Theory 5(1986), 77–82.
[11] Z. Nehari, *The Schwarzian derivative and schlicht functions*. Bull. Amer Math. Soc. 55(1949), 545–551.
[12] R. Nevanlinna, *Über die konforme Abbildund Sterngebieten*. Overs. Finska Vetensk. Soc. Forhandl. **63**(A)(1921), no. 6, 48–403.
[13] K. Noshiro, *On the theory of schlicht functions*. J. Fac. Sci. Hokkaido Univ. Jap. **2**(1934/1935), no. 1, 129–135.
[14] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*. Proc. Japan Acad. Ser. A **69**(1993), no. 7, 234–237.
[15] M. S. Robertson, *On the theory of univalent functions*. Ann. Math. **37**(1936), 374–408.
[16] M. S. Robertson, *Certain classes of starlike functions*. Michigan Math. J. **32**(1985), 135–140.
[17] J. Stankiewicz, *Quelques problèmes extrèmaux dans les classes des fonctions α-angulairement èstoilèes*. Ann. Univ. Mariae Curie–Skłodowska, Sect. A **20**(1966), 59–75.
[18] E. Strrohhäcker, *Beiträge zür Theorie der schlichter Funktionen*. Math. Z. **37**(1933), 356–380.
[19] S. Warschawski, *On the higher derivatives at the boundary in conformal mapping*. Trans. Amer. Math. Soc. **38**(1935), 10–340.
[20] D. R. Wilken and J. Feng, *A remark on convex and starlike functions*. J. London Math. Soc. **21**(1980), no. 2, 287–290.

University of Gunma, Japan
e-mail: mamorununokawa1983@gmail.com

Rzeszów University of Technology, Poland
e-mail: piejko@prz.edu.pl

University of Rzeszów, Poland
e-mail: jsokol@ur.edu.pl

https://doi.org/10.4153/S0008439526101702 Published online by Cambridge University Press