18 Elastic Scattering Theory

The quantum theory of scattering is a complex subject and is covered in more detail in a course on quantum field theory. In a first approximation, a collision or scattering is a process in which two particles, initially very far apart (and therefore not interacting), approach each other, interact (exchanging energy and momentum), and finally move apart. In this chapter we will introduce the main concepts and derive the most important formulas for elastic scattering, in a nonrelativistic framework. This will allow us to appreciate how the information obtained from scattering experiments allows us to explore the intricacies of quantum particles that are otherwise inaccessible.

We have been able to use the Schrödinger equation to describe atomic and molecular systems because we know the theory of electromagnetic interactions in great detail. However, if we want to use this equation to describe nuclear systems or, more generally, interactions between elementary particles – assuming that it is still valid for such levels of organization of matter – the first difficulty is that these interactions are not necessarily electromagnetic. There are four known fundamental interactions: the electromagnetic and the gravitational, already known in classical physics, plus the so-called strong (or nuclear) and weak interactions. At present, the Standard Model of particle interactions includes the strong, weak and electromagnetic interactions, but not the gravitational ones. Research aimed at integrating all four interactions into a general theory of fields and matter is currently very active. The fact that we are talking about a model rather than a theory is a recognition that there is still plenty of room (and need) for fundamental research.

18.1 Collision Phenomena

Due to the microscopic nature of the basic quantum corpuscles, the only method available to investigate their structure and interactions – to determine their dependence on the various parameters and variables such as the type of particles involved, energies, charges, orbital momenta, spins, relative distances, and so on – is to produce collisions between them and then analyze the outcomes under controlled conditions. The example of Rutherford's historic experiment, which led to the establishment of the atomic planetary model, shows that scattering experiments, even against unknown targets, can be used not only to find out the nature and characteristics of the interaction, but also to determine the structure of the particles involved in the scattering process.

These observations justify a careful study of Schrödinger's theory for the case of dispersion, that is, when the spectrum is continuous, which usually corresponds to

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positive total energy of the system. From a physical point of view, a simple general situation is as follows. A beam of particles with controlled and known momentum and energy is directed against a target of (possibly) known structure (a very thin metallic plate deposited on a suitable substrate, a gas, a crystal, etc.), which is at rest in the laboratory.[1] As a result of the collisions occurring in the target, the bombarding particles (which are independent of each other, assuming a low density of the beam) are scattered and leave the target in all possible directions and states.

Elastic processes – the only ones we will study – occur when the particles are scattered with the same energy as they hit; inelastic processes occur when the target is excited or ionized, or some particles are transformed into others, and so on. Whatever the process, the particles emerging from the interaction are received by suitable detectors placed very far from the interaction region (in what we will call the asymptotic region, where the interaction is null or negligible), which record each particle that hits them (in the form of traces on photographic film, or in a cloud or bubble chamber, or as light emission in a scintillation crystal, etc.). In addition to the obvious information about the average number of particles that hit the detector per second per unit area, it is often possible to deduce some of the properties of the recorded particle (nature, mass, charge, momentum, spin, energy, etc.). These are the data provided by the experimenter, and they constitute the material for further theoretical analysis of the problem.

18.1.1 Laboratory and center-of-mass frames

The first thing to note is that the description of the experiment is naturally carried out in a coordinate system linked to the laboratory, in which the target is (usually) at rest (this is the so-called laboratory L frame), while the theoretical description is more simply done in the reference frame in which the center of mass of the entire physical system is at rest (CM frame), since any uniform motion of the CM is irrelevant. When the target particle is very massive with respect to the projectile (e.g., atomic nuclei bombarded by electrons), both reference systems are almost coincident, since the CM remains very close to the target at all times, practically blending in with it. However, as is often the case, when the target and projectile masses are comparable, the two descriptions differ substantially. For example, from the kinematics of the collision between two particles it follows that the scattering angles θ_L in the L frame and θ_{CM} in the CM frame, as shown in Fig. 18.1, are related by (problem P18.1)

$$\tan\theta_{1_L} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + m_1/m_2}. \tag{18.1}$$

This formula shows that the scattering angles measured in both systems are practically the same if $m_1/m_2 \ll 1$, but if this condition is not fulfilled, they can be very different. For example, the backscatter in the CM system ($\theta_{CM} = 180^o$) corresponds to a deviation of $\theta_L = 90^o$ of the incident particle in the laboratory frame when $m_1 = m_2$. On the other hand, part of the energy available in the L system is associated with the

[1] There are important exceptions to this rule. In collision rings, for example, two beams are accelerated in opposite directions and eventually brought to a head-on collision. This device greatly increases the energy available for the reaction by avoiding the energy required to accelerate the center of mass.

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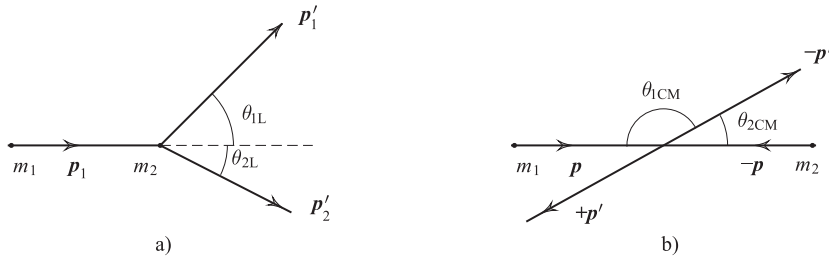


Figure 18.1 Diagram of an elastic collision between particles 1 and 2, seen from (a) the laboratory frame, (b) the CM frame.

CM motion and is not used for the scattering. Thus, the collision energy is less than that used to accelerate the particles according to the formula

$$E_{CM} = \left(1 - \frac{m_1}{M}\right) E_L, \quad M = m_1 + m_2. \tag{18.2}$$

We see that the loss can be significant for comparable masses. A very important result is that which relates the number of particles $n(\theta_{CM})$ that hit per unit area per unit time on a detector whose aperture $d\Omega_{CM}$ and angular position are defined in the CM system, to the corresponding quantity defined in the L system, $n(\theta_L)$ (see problem P18.2),

$$n(\theta_L) = \frac{\left[1 + 2\frac{m_1}{m_2}\cos\theta_{CM} + \left(\frac{m_1}{m_2}\right)^2\right]^{3/2}}{1 + \frac{m_1}{m_2}\cos\theta_{CM}} n(\theta_{CM}). \tag{18.3}$$

In the following we will work directly in the CM system; for simplicity we will omit the CM indices, which are to be understood.

18.1.2 Scattering amplitude and scattering cross section

In elastic collisions, the wave functions are eigenfunctions of the Hamiltonian corresponding to the energy E determined by the bombardment conditions. From the radial Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right) u(r) = Eu(r) \tag{18.4}$$

we see that if the product $V(r)u(r)$ vanishes at infinity, then the asymptotic behavior of the radial function u at infinity is $u \sim e^{ikr}$, with $k = \sqrt{2mE}/\hbar$; this asymptotic behavior is obtained if we restrict ourselves to potentials such that $rV(r) \to 0$ for $r \to \infty$, which are called short-range potentials. In the following, V will represent a short-range potential. This excludes the Coulomb potential from the analysis, although we will have the opportunity to include it in our results later on.

When the interaction is due to a short-range potential, we can consider that it takes place entirely within a sphere of finite radius R, so that for distances greater than R, the motion is free. In particular, both the incident and the target particles can then be considered as free. In this case, the kinetic energy of the relative motion of the

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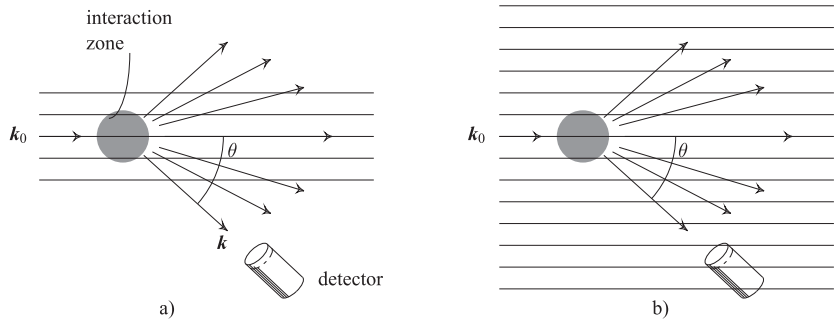


Figure 18.2 A typical scattering experiment seen from the laboratory frame. The target produces a short-range potential of radius R. (a) In the physical situation, the particle beam has a finite width. (b) The simplest theoretical description assumes that the incident beam corresponds to a plane monochromatic wave filling the whole space. The nonscattered particles should not be counted, as they are only part of the (idealized) description.

free particles is the total energy of the system. Since the incident particles move along parallel trajectories before entering the interaction region, as shown in Fig. 18.2(a), the orbital momentum about the origin, which is placed at the center of the target (where we assume the interaction to originate) varies from particle to particle, so the wave function will be a superposition of all possible orbital momentum states. Here we restrict ourselves to the case of spinless particles.

Based on the previous discussion, the wave function describing the ensemble of particles scattered in the asymptotic region ($r \to \infty$) can be written as the product of the asymptotic radial function $R(r) = u(r)/r$ with an angular wave function, which is not an angular momentum eigenstate and which we will denote by $f(\theta, \varphi)$. We therefore write

$$\psi_{\text{disp}} = f(\theta, \varphi)\frac{u(r)}{r} = f(\theta, \varphi)\frac{e^{ikr}}{r} \quad (r \to \infty), \tag{18.5}$$

where $f(\theta, \varphi)$ is the *scattering amplitude* and

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad E > 0. \tag{18.6}$$

Since the detector is always placed very far away from the target to ensure that the scattering process is completed, we use this wave function to calculate the number of particles that are counted, to avoid having to describe the scattering process in more detail. The complete wave function has an additional component corresponding to the particles not scattered by the target, as can be seen in Fig. 18.2(b). Since these form a packet of free particles ψ_{inc}, we can write the total wave function away from the target as

$$\psi = \psi_{\text{inc}} + f(\theta, \varphi)\frac{e^{ikr}}{r}. \tag{18.7}$$

It is possible to considerably simplify the theoretical description by replacing ψ_{inc} with a plane wave whose energy is equal to the mean energy of the packet, as shown in Fig. 18.2(b). Using the conventional normalization of one incident particle per unit

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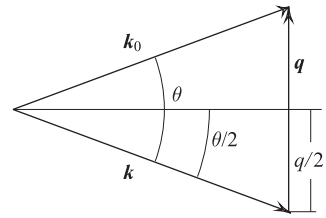


Figure 18.3 The vector q represents the momentum transferred from the particle to the target.

volume ($\rho_{\text{inc}} = 1$), we write ψ as follows, with the z-axis along the direction of incidence,[2]

$$\psi = e^{ikz} + f(\theta, \varphi)\frac{e^{ikr}}{r}. \tag{18.8}$$

The parameter k in the first term is the magnitude of the momentum of the incident particles $\boldsymbol{k}_0 = \widehat{\mathbf{a}}_z k$; the k in the scattered component (spherical wave) is the magnitude of the momentum $\boldsymbol{k}$ of the scattered particles, which, for very large distances from the target, is essentially oriented along $\widehat{\mathbf{a}}_r = \boldsymbol{r}/r$. Since the dispersion is elastic, $|\boldsymbol{k}_0| = |\boldsymbol{k}| = k$, but the two vectors have different directions, so that there is a momentum transfer $\boldsymbol{q}$ given by

$$\boldsymbol{k}_0 = \boldsymbol{k} + \boldsymbol{q}. \tag{18.9}$$

The quantity q is easily determined from Fig. 18.3, where θ is the scattering angle in the CM frame; it is clear that

$$q = 2k \sin\frac{\theta}{2}. \tag{18.10}$$

This is the momentum with which the target will move if it is initially at rest.

To count the number of particles hitting the detector we use (18.8) without the plane-wave contribution (counters are never placed exactly in front of the target). Take a small area dA; if the detector is placed at a large distance r from the target and its aperture is $d\Omega$ ester-radians, then $dA = r^2 d\Omega$; see Fig. 18.4. In the time dt all particles within the solid angle $d\Omega$ that are at a distance not greater than $dr = v\,dt$, where v is the average velocity of the particle flow, $v = \hbar k/m$, will enter the detector.

Since, on the other hand, the density of scattered particles is $|\psi_{\text{sc}}|^2$, the number of particles hitting the detector during dt is

$$dn = \rho dV = |\psi_{\text{sc}}|^2 dV = |\psi_{\text{sc}}|^2 r^2 v d\Omega dt. \tag{18.11}$$

The average number of particles registered is ηdn, where η is the counting efficiency of the detector; in the following we will use $\eta = 1$, which is often far from reality. Obviously, the count depends not only on the time during which the particles are recorded, but also on how many have been sent to the target during that time. In order to make the final result independent of these circumstantial elements, it is customary to report

[2] To study the scattering of photons by atoms, it is necessary to build correct asymptotic states, taking into account that the radiation and the atom are never disconnected; see Cohen–Tannoudji, Dupont–Roc, and Grynberg (1992).

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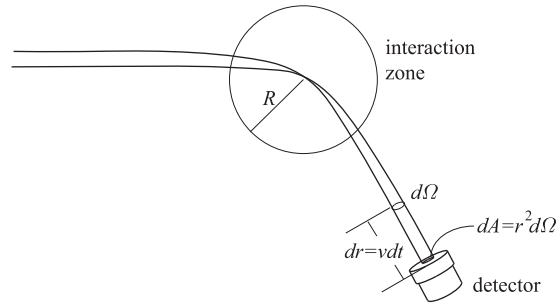


Figure 18.4 All the particles inside the cone with the base $r^2 d\omega$ and the height $dr = vdt$ will hit the detector within the time interval dt.

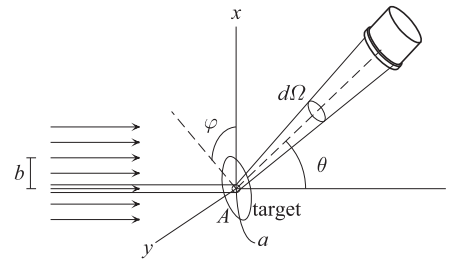


Figure 18.5 The fraction of particles incident on the target area A are scattered toward the detector. Alternatively, the particles incident on the effective target area $a = d\sigma$ will hit the detector. b is the impact parameter.

the value of dn properly normalized. Since the number of particles hitting the target per unit area and per unit time is $j_{\text{inc}} = \rho_{\text{inc}} v_{\text{inc}} = \hbar k/m$ (remember that the normalization $\rho_{\text{inc}} = 1$ was adopted), it is customary to report the quantity

$$d\sigma \equiv \frac{dn/dt}{j_{\text{inc}}}, \tag{18.12}$$

which is the number of particles detected per unit time per unit incident flow. The quantity $d\sigma$ is called differential *scattering cross section* (or *effective cross section*), because it has area dimensions,

$$\left[\frac{|\psi|^2 dV}{\rho_{\text{inc}} v dt}\right] = \left[\frac{1}{\rho_{\text{inc}} v dt}\right] = L^3 L^{-1} T T^{-1} = L^2. \tag{18.13}$$

Substituting dn and $j_{\text{inc}} = \hbar k/m$ in Eq. (18.12) and using Eq. (18.5), we obtain the important formula for the differential cross section,

$$\frac{d\sigma}{d\Omega} = |f(\theta, \varphi)|^2. \tag{18.14}$$

To give this effective scattering area a geometric sense, we turn to Fig. 18.5. In this figure, A represents the area of the target, while a represents the effective scattering area in the direction (θ, φ), that is, all particles falling on a have been deflected toward the detector with aperture $d\Omega$. On the other hand, if N is the number of particles hitting the target in one second, the incident flux is

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$$j_{\text{inc}} = \frac{N}{A}. \tag{18.15}$$

Now let n be the number of particles that hit the detector in one second. From the definition of a, it follows that $n/M = a/A$, that is,

$$a = \frac{n}{N}A = \frac{n}{N/A} = \frac{n}{j_{\text{inc}}}. \tag{18.16}$$

But the ratio n/j_{inc} is exactly the scattering cross section given by (18.12), therefore a is the same as the effective cross section,

$$a = d\sigma. \tag{18.17}$$

18.2 The Born Approximation

For Eq. (18.14) to be useful it is necessary to determine the scattering amplitude, which is equivalent to solving the Schrödinger equation, at least for the asymptotic region. This can be done in different ways, depending on the specific conditions. Here we will study the Born approximation, which is applicable when the collision energy is high, or when the scattering potential is low, or when both conditions are met. To do this, we write the Schrödinger equation for the problem

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi = \frac{\hbar^2 k^2}{2m}\psi \tag{18.18}$$

in the equivalent form

$$(\nabla^2 + k^2)\psi = \frac{2m}{\hbar^2}V\psi. \tag{18.19}$$

This can be transformed into an integral equation by the simple expedient of considering the right side as a source $\rho(r)$, so that

$$\frac{2m}{\hbar^2}V\psi = -4\pi\rho; \tag{18.20}$$

Equation (18.19) becomes thus a Helmholtz equation with a source,

$$(\nabla^2 + k^2)\psi = -4\pi\rho. \tag{18.21}$$

Comparing with Exercise E7.3, we see that the solution of this equation can be written in the form

$$\psi_{\text{sc}}(\boldsymbol{r}) = \int G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\,d^3r' = -\frac{m}{2\pi\hbar^2}\int G(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi(\boldsymbol{r}')\,d^3r', \tag{18.22}$$

where $G(\boldsymbol{r},\boldsymbol{r}')$ is the Green function of the operator $\nabla^2 + k^2$, that is, a solution of (18.21), which can be written as a linear combination of the functions G_+ and G_-, see Eqs. (7.87) and (7.88). In the present case, the asymptotic scattering solution is an *outgoing* spherical wave e^{ikr}/r (see (18.5)), of the form of G_+. So we take $G = G_+ = e^{ikR}/R$, with $R = |\boldsymbol{r} - \boldsymbol{r'}|$, and get

$$\psi_{\text{sc}}(\boldsymbol{r}) = -\frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi(\boldsymbol{r}')d^3r'. \tag{18.23}$$

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This is the required integral equation, which replaces the Schrödinger differential equation plus the boundary condition (outgoing wave at infinity). To this expression we must add an arbitrary solution of the homogeneous equation $(\nabla^2 + k^2)\psi = 0$. To obtain (18.8) we choose the solution e^{ikz}, so that

$$\psi(\boldsymbol{r}) = e^{ikz} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi(\boldsymbol{r}')d^3r'. \tag{18.24}$$

It is important to note that the ψ appearing in the integrand of (18.24) is the complete one.

The potential $V(\boldsymbol{r}')$ in (18.23) and (18.24) is nonzero only in the region where $r' \leq R \ll r$. Therefore, the denominator can be approximated as $|\boldsymbol{r}-\boldsymbol{r}'| \simeq r$, while in the exponential we have to go one order further in the approximation and write

$$k|\boldsymbol{r}-\boldsymbol{r}'| = kr - \boldsymbol{k}\cdot\boldsymbol{r}', \quad |\boldsymbol{r}-\boldsymbol{r}'| \doteq r - \boldsymbol{r}'\cdot\hat{\mathbf{a}}_r, \tag{18.25}$$

where $\hat{\mathbf{a}}_r$ is a unit vector along the direction of $\boldsymbol{r}$. With these approximations, the wave function in the asymptotic region becomes

$$\psi(\mathbf{r}) = e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int V(\boldsymbol{r}')\psi(\boldsymbol{r}')e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}d^3r'. \tag{18.26}$$

Comparing with Eq. (18.8), we obtain the following exact, but formal, expression for the scattering amplitude:

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r})\psi(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}d^3r. \tag{18.27}$$

The original problem has thus been transformed into one of solving the integral equation (18.26) for $\psi(r)$. Since it is not normally possible to do this in closed form, it is customary to use an iteration method assuming that ψ_{sc} is small relative to ψ_{inc}.[3] This will be the case if (a) the particles are very energetic and cross the interaction region very quickly, so that they are slightly deflected from their initial trajectory by the scattering centers, or (b) the scattering potential is weak, or finally (c) both occur. Then the ψ in the integrand can be approximated by $\psi(\boldsymbol{r}) \simeq e^{ikz} = e^{i\boldsymbol{k_0}\cdot\boldsymbol{r}}$, giving the asymptotic wave function

$$\psi(\boldsymbol{r}) = e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int V(\boldsymbol{r}')e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}d^3r', \tag{18.28}$$

where $\boldsymbol{q} = \boldsymbol{k}_0 - \boldsymbol{k}$ is the transferred momentum, see Eq. (18.9). This gives for the scattering amplitude in the first Born approximation

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r})e^{i\boldsymbol{q}\cdot\boldsymbol{r}}d^3r. \tag{18.29}$$

We see that, up to a numerical factor, the scattering amplitude is given in this approximation by the Fourier transform of the scattering potential. This is a very useful result and we will have the opportunity to apply it to some important problems. For now, let us use it to draw some general conclusions.

[3] Think of light passing through a window pane; most of it passes through the glass and only a small percentage (usually less than 5%) is reflected, that is, scattered.

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18.2.1 Scattering by a periodic potential

We first consider the case of a periodic potential, which can be written in the form

$$V(\boldsymbol{r}) = \sum A_{\mathbf{n}} \exp 2\pi i \left[\frac{x}{a}n_1 + \frac{y}{b}n_2 + \frac{z}{c}n_3\right] = \sum A_{\mathbf{n}} \exp i\,\boldsymbol{g}_{\mathbf{n}} \cdot \boldsymbol{r}. \tag{18.30}$$

This can be, for example, the potential generated by a crystal lattice with parameters (a, b, c), similar to the Kronig and Penney model discussed in Section 6.3 or any other periodic structure of atomic dimensions; $\boldsymbol{g}_{\mathbf{n}}$ is the corresponding reciprocal lattice, $\boldsymbol{g}_{\mathbf{n}} = 2\pi\left(\frac{n_1}{a}\mathbf{i} + \frac{n_2}{b}\mathbf{j} + \frac{n_3}{c}\mathbf{k}\right)$. Applying Eq. (18.29), we get

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\sum A_{\mathbf{n}} \int e^{i(\boldsymbol{q}+\boldsymbol{g}_{\mathbf{n}})\cdot\mathbf{r}'} d^3r' = -\frac{4\pi^2 m}{\hbar^2}\sum A_{\mathbf{n}}\delta\left(\boldsymbol{q}+\boldsymbol{g}_{\mathbf{n}}\right), \tag{18.31}$$

which indicates that the momentum transferred during the interaction is quantized and can only take the discrete values

$$-\hbar\boldsymbol{q}_{\mathbf{n}} = \hbar\left(\boldsymbol{k}_{\mathbf{n}} - \boldsymbol{k}_0\right) = \hbar\mathbf{g}_n = 2\pi\hbar\left(\frac{n_1}{a}\mathbf{i} + \frac{n_2}{b}\mathbf{j} + \frac{n_3}{c}\mathbf{k}\right). \tag{18.32}$$

We conclude that when electrons (quantum corpuscles in general) are scattered by a periodic structure, the momentum transferred in the direction of the periodicity is quantized. This result was first formulated by W. Duane (1872–1935) in 1923, shortly before the advent of QM, as a principle derived from the old quantization rules. A. Landé and other authors even considered it to be the basic principle of quantization at that time, now it can be seen as a consequence of Schrödinger's theory.

It should be clear that what matters in the preceding derivation is the existence of a periodicity capable of influencing the behavior of matter, and not the nature of that periodicity. This means that different sources of periodicity can lead to a similar quantization of the exchanged momentum. An interesting example is the phenomenon known as the Kapitza[4]–Dirac effect, which consists of the diffraction of a molecular beam by a stationary electromagnetic wave, that is, the inverse phenomenon of the diffraction of light by a grating.

18.2.2 Scattering by a weak potential

Returning to the general case, let us now consider a problem in which the scattering potential is so weak that Eq. (18.29) holds even at very low energies, when $kR \ll 1$. It follows from Eq. (18.10) that in this case $qR \ll 1$ will also hold, and therefore the exponent of the integral in (18.29) will remain small throughout the integration region where $V(r')$ is different from zero; neglecting it, we can write

$$f(\theta,\varphi) \simeq f(0), \tag{18.33}$$

[4] Pyotr Leonidovich Kapitsa (1894–1984) was an outstanding Russian–Soviet physicist. His series of experiments to study liquid helium culminated in the discovery of superfluidity in 1937. In 1978 Kapitsa was awarded the Nobel Prize in Physics for his fundamental inventions and discoveries in the field of low-temperature physics.

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where

$$f(0) = -\frac{m}{2\pi\hbar^2}\int V(\boldsymbol{r})d^3r = -\frac{m}{2\pi\hbar^2}\overline{V}R^3 \tag{18.34}$$

is the forward scattering amplitude, in all cases (even at high energies); $\overline{V}$ is an average potential such that $\int V(\boldsymbol{r})d^3r \equiv \overline{V}R^3$, and R is the range of the potential.

According to (18.33), at low energies the scattering produced by a short-range potential is isotropic. The *total cross section*, defined as the integral over all directions of the differential cross section – that is, as the total number of particles scattered in all directions for each particle incident on a unit area of the target – and given by

$$\sigma \equiv \int_\Omega \frac{d\sigma}{d\Omega}d\Omega = \int |f(\theta,\varphi)|^2 d\Omega, \tag{18.35}$$

is thus independent of the energy for low energies,

$$\sigma = \int |f(\theta,\varphi)|^2 d\Omega \simeq 4\pi |f(0)|^2 = \frac{1}{\pi}\left(\frac{m\overline{V}R^3}{\hbar^2}\right)^2. \tag{18.36}$$

The case of high energies corresponds to $kR \gg 1$; except within a narrow cone where the angle θ is small, which we exclude, also $qR \gg 1$ and the exponential of the integral in Eq. (18.29) oscillates violently, vanishing the integral. The effective contribution to the integral occurs within the cone for which $qR = 2kR\sin(\theta/2) \leq 1$ holds. Since $kR \gg 1$, this condition is only satisfied for angles θ for which $\sin\theta \leq 1/kR \ll 1$, so the differential section has a sharp peak around the direction of incidence of width $1/kR$, and vanishes for larger scattering angles. We conclude that the scattering produced by a short-range potential is highly anisotropic at high bombardment energies, as shown in Fig. 18.6. In this case we can approximate as follows:

$$\begin{aligned}\sigma &\simeq 2\pi\int_0^{1/kR} |f(\theta)|^2 \sin\theta d\theta \simeq 2\pi |f(0)|^2 \int_0^{1/kR} \theta d\theta \\ &= \frac{\pi |f(0)|^2}{k^2R^2} = \frac{1}{4\pi}\left(\frac{m\overline{V}R^2}{\hbar^2 k}\right)^2,\end{aligned} \tag{18.37}$$

that is,

$$\sigma = \frac{1}{8\pi}\left(\frac{\overline{V}R^2}{\hbar}\right)^2 \frac{m}{E}. \tag{18.38}$$

We see that at high energies the scattering cross section decreases with the energy. This is because the interaction time decreases as the bombardment energy increases.

The region of validity of the Born approximation can be evaluated by noting that the approximation is satisfactory only in the case where the total section σ is small with respect to the geometric area presented by the target,

$$\sigma \ll \pi R^2, \tag{18.39}$$

since otherwise ψ_{sc} cannot be neglected in (18.26) and (18.27), This condition can alternatively be expressed as a condition on $\overline{V}$ as follows. For low energies we substitute σ given by Eq. (18.36) into (18.39) to obtain

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$$|\overline{V}| \ll \frac{\pi\hbar^2}{mR^2}, \tag{18.40}$$

while at high energies Eq. (18.38) requires that

$$|\overline{V}| \ll \frac{2\pi\hbar v}{R} = \frac{\pi\hbar^2}{mR^2} 2kR. \tag{18.41}$$

The first condition (18.40) is generally stronger than the second, since if it is satisfied, the approximation is valid for all bombardment energies. On the other hand, the second condition is always satisfied for sufficiently high energies, so that kR is as large as necessary; hence the Born approximation is considered to be a high-energy approximation.

Exercise E18.1. Scattering by a 1D delta potential

A particle is scattered by a one-dimensional delta potential at the origin. Find the wave function, the scattering amplitude and the energy of the bound state.

Solution. Assuming that the beam hits the scattering potential $K\delta(z)$ from the left of the z-axis, the wave function is given by the 1D version of Eq. (18.22) with $k = p/\hbar$,

$$\psi(z) = e^{ikz} + \frac{2m}{\hbar^2} \int_{-\infty}^{+\infty} G(z, z')V(z')\psi(z')dz', \tag{18.42}$$

where $G(z, z')$ is the outgoing Green function of the operator $\left(d^2/dx^2\right) + k^2$,

$$G(z) = \int_{-\infty}^{+\infty} \frac{e^{ik'z}}{k'^2 - k^2 + i\varepsilon} dk' = -\frac{i}{2k} e^{ik|z|}. \tag{18.43}$$

Therefore,

$$\begin{aligned} \psi(z) &= e^{ikz} - \frac{imK}{\hbar^2 k} \int_{-\infty}^{+\infty} e^{ik|z-z'|} \delta(z')\psi(z')dz' \\ &= e^{ikz} - \frac{imK}{\hbar^2 k} \psi(0) e^{ik|z|} = e^{ikz} - \frac{imK}{\hbar^2 k + imK} e^{ik|z|}. \end{aligned} \tag{18.44}$$

This result shows that for $k \to 0$ virtually all the particles are reflected by the potential and the transmission coefficient goes to zero. Instead, for $k \to \infty$ the scattering potential becomes transparent.

To construct the scattering amplitude for the 1D case we consider the limit $|z| \to \infty$ and write, with $k' = k\,(z/|z|)$,

$$\psi(z) = e^{ikz} + e^{ik|z|} f(k', k), \tag{18.45}$$

$$f(k', k) = -\frac{im}{\hbar^2 k} \int_{-\infty}^{+\infty} e^{-ikz'} V(z')\psi(z')dz'. \tag{18.46}$$

Comparing with (18.44), we obtain for the dispersion by the delta potential

$$f(k', k) = -\frac{imK}{\hbar^2 k + imK}. \tag{18.47}$$

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This scattering amplitude has a pole at $p = \hbar k = -imK/\hbar$, corresponding to the energy

$$E = p^2/2m = -\frac{mK^2}{2\hbar^2}. \tag{18.48}$$

Therefore the delta potential has a single bound state with energy E given by Eq. (18.48).

The result just derived, which is an example of the general property that establishes that the poles of the scattering amplitude for negative energies correspond to bound states, can be obtained with the simple methods used in Chapter 5. To do this, it is enough to write the wave function in the form

$$\psi(z) = Ae^{ikz} + Be^{-ikz}, \quad z < 0,$$
$$\psi(z) = Ce^{ikz}, \quad z \geq 0.$$

From the continuity conditions at the origin of the wave function and its derivative (its discontinuity at $z = 0$ is obtained by integrating the Schrödinger equation in a small interval around the origin) we arrive at

$$\frac{B}{A} = \frac{mK}{i\hbar^2 k - mK}, \quad \frac{C}{A} = \frac{i\hbar^2 k}{i\hbar^2 k - mK}.$$

For a bound state the energy is negative; in this case $\hbar k = \sqrt{-2m|E|} \equiv i\alpha$, where $\alpha = \sqrt{2m|E|} \geq 0$. The scattering amplitude is then (compare with Eq. (18.47))

$$\frac{B}{A} = -\frac{mK}{\hbar\alpha + mK}$$

and has a single pole at $\alpha = -mK/\hbar$, in agreement with Eq. (18.48); see also problem P5.5.

18.2.3 Scattering by a central potential: Rutherford cross section

A general expression for the scattering amplitude for central potential problems is obtained by performing the angular integrals in Eq. (18.29). With the z-axis of the reference frame along the vector $\boldsymbol{q}$, $\boldsymbol{q} \cdot \boldsymbol{r} = qr\cos\theta$ and we get

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int_0^\infty r^2 dr \int_0^\pi \sin\theta' d\theta' \int_{-\pi}^{\pi} d\varphi' V(r) e^{iqr\cos\theta'}$$
$$= -\frac{m}{\hbar^2}\int_0^\infty r^2 V(r) dr \int_{-1}^{1} e^{iqrx} dx, \tag{18.49}$$

or

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty rV(r)\sin qr dr, \tag{18.50}$$

which is the desired result.

Let us consider the case, which is particularly important in atomic physics, where V represents a shielded Coulomb potential between the charges $Z_1 e$ and $Z_2 e$. The shielding effect in an atom is due to the fact that the orbital electrons partially shield

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or insulate the nucleus, as discussed in Section 16.3. A quantum evaluation shows that this effect can be taken into account by introducing the shielding factor $e^{-\alpha r}$, where α depends on the structure of the atom. Substituting $V_0 R = Z_1 Z_2 e^2$ and $R = \alpha^{-1}$ in (18.53), we obtain, for the Coulomb scattering cross section,

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZ_1Z_2e^2R^2}{\hbar^2}\right)^2 \frac{1}{(1+4k^2R^2\sin^2\theta/2)^2}. \tag{18.51}$$

For sufficiently high energies, $kR \gg 1$ and this expression becomes independent of R (for θ not too small),

$$\frac{d\sigma}{d\Omega} = \frac{Z_1^2 Z_2^2 m^2 e^4}{4p^4 \sin^4 \theta/2} \quad (\theta \neq 0). \tag{18.52}$$

Since this result no longer depends on $\alpha = R^{-1}$, we can apply it to the limiting case $\alpha = 0$, that is, to the Coulomb potential. Therefore, outside the cone $\theta < 1/kR = \alpha/k \to_{\alpha\to 0} 0$, Eq. (18.52) gives the angular distribution produced by a Coulomb potential at sufficiently high energies $kR = k/\alpha \gg 1$. This is precisely the formula derived by Rutherford for the scattering cross section of α particles by heavy atoms using purely classical methods; for this reason it is called the *Rutherford cross section*. This happy coincidence was made possible by the fact that the quantum formula (18.52) is independent of $\hbar$.

The Coulomb potential is long-range, so the preceding methods are only valid for high energies. However, the Coulomb scattering problem has an exact solution in a parabolic coordinate system, which once again leads to the formula (18.52). We will not go into this problem here, but refer the student to the literature on the subject.

Exercise E18.2. Scattering by a Yukawa potential

The Yukawa potential

$$V = V_0 \frac{e^{-r/R}}{r/R} = V_0 R \frac{e^{-r/R}}{r} \tag{18.53}$$

is important for at least two reasons: first, it represents the static approximation to the nuclear interaction potential, and second, it describes – with a simple change of parameters – the partially shielded Coulomb potential (and in the limit $R \to \infty$ the Coulomb potential itself). Apply the previous results to the study of the dispersion produced by the Yukawa potential.

Solution. Applying the general formula (18.50) to the potential (18.53), we obtain

$$f(\theta) = -\frac{2m}{\hbar^2 q} V_0 R \int_0^\infty e^{-r/R} \sin qr dr = -\frac{2mV_0}{\hbar^2} \frac{R^3}{1+q^2R^2}. \tag{18.54}$$

For the differential cross section we thus get

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{a^2}{(1+q^2R^2)^2}, \tag{18.55}$$

where we introduced the abbreviation

$$a = \frac{2mV_0R^3}{\hbar^2}. \tag{18.56}$$

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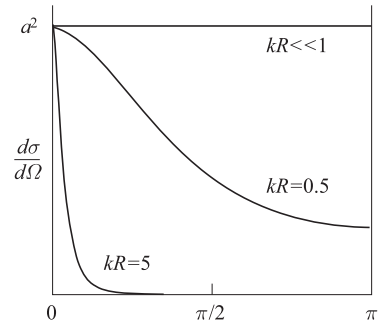


Figure 18.6 Differential cross section for a short-range potential as a function of the scattering angle and the parameter kR. At very low energies the cross section is isotropic, while at very high energies the particles are only slightly dispersed. The maximum value of the differential cross section is a^2.

Fig. 18.6 shows how the differential cross section varies as a function of the scattering angle, for different values of kR; these results exemplify the characteristic general behavior of short-range potentials, as discussed earlier. Using formula (18.10) for q we see that the total cross section is

$$\begin{aligned}\sigma &= 2\pi \int_0^\pi d\theta \sin\theta \frac{a^2}{(1+k^2R^2\cdot 4\sin^2\theta/2)^2} \\ &= 2\pi a^2 \int_0^2 \frac{dz}{(1+2k^2R^2z)^2} = \frac{4\pi a^2}{1+4k^2R^2}.\end{aligned} \tag{18.57}$$

At low energies σ has the approximate value $4\pi a^2$, which shows that $2a$ plays the role of an effective scattering radius; at high energies σ is inversely proportional to the bombardment energy, $E \sim k^2$, as expected.

18.3 Form Factors

In the previous analysis we considered the scattering potential to be point-like, but many important sources of interaction have a more or less extended structure, which means that the scattering potential has a significant spatial distribution. This is the case, for example, of the interaction of an electron (which we shall continue to consider as point-like) with an atom which has an essentially point-like positive nucleus, but whose orbital electrons produce an average charge density that may extend over a few Bohr radii and consequently affect the motion of the electron in a different way from more or less localized charges.

Recall that Rutherford used the scattering of α particles by heavy atoms to test Thomson's model of the atom as a distributed positive charge with electrons embedded in it, and concluded that the model was invalid. We will see in what follows how we can treat this type of situation, apply the results to the Rutherford case, and determine when the distributed charge of the electrons matters and when it does not.

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We will look at the electrostatic case for the sake of simplicity. Considering a point charge e (the projectile) interacting with a distributed electric charge of density $e'\rho(\boldsymbol{r})$, where the density ρ is normalized to unity, $\int \rho d^3r = 1$, the interaction potential is

$$V(r) = ee' \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3r', \tag{18.58}$$

where the integral extends over the volume of the extended charge. Substituting in Eq. (18.29) (with an obvious change in the names of the variables), we get

$$f(\theta) = -\frac{mee'}{2\pi\hbar^2} \int\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3r d^3r'. \tag{18.59}$$

This expression is conveniently rewritten with the change of variable $\boldsymbol{r} - \boldsymbol{r}' = \boldsymbol{R}$,

$$f(\theta) = -\frac{mee'}{2\pi\hbar^2} \int \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{R}}}{R} d^3R \int \rho(\boldsymbol{r}') e^{i\boldsymbol{q}\cdot\boldsymbol{r}'} d^3r'. \tag{18.60}$$

The second integral,

$$F(\boldsymbol{q}) \equiv \int \rho(\boldsymbol{r}) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} d^3r, \tag{18.61}$$

which is the Fourier transform of the charge distribution of the target, is generically called the *form factor* (in the present case it is the electric form factor). If $\rho(\mathbf{r})$ has spherical symmetry, the form factor depends only on q; we will assume that this is the case in what follows. The scattering amplitude is then

$$f(\theta) = -\frac{mee'}{2\pi\hbar^2} F(q) \int \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{r}}}{r} d^3r. \tag{18.62}$$

To express this result in terms of the scattering amplitude produced by a point-like potential, we note that ee'/r is the interaction potential between two point charges e and e', so the previous result can be rewritten as (cf. Eq. (18.29))

$$f(\theta) = F(q) f_{\text{point}}(\theta). \tag{18.63}$$

It is clear that this result is general in the sense that it gives the relationship between the point scattering amplitude and the corresponding amplitude for the problem with structure. It follows that the differential scattering cross section of these two problems is generally related as follows,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{ext}} = |F(q)|^2 \left.\frac{d\sigma}{d\Omega}\right|_{\text{point}}. \tag{18.64}$$

This very important result shows that by comparing the scattering produced by a target with structure (which determines $d\sigma/d\Omega|_{\text{ext}}^{\text{exp}}$) and the theoretical cross section calculated for a point target (which requires prior knowledge of the interaction potential), we can determine the form factor (or, where appropriate, the form factors) of the potential source distribution, that is, we can study the internal structure of the target–projectile system, given the interaction (see problem P18.8).

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18.3.1 Atomic form factor

As an example we will study the dispersion of electrons produced by an atom. The atomic charge density can be written as the sum of a point-like contribution from the nucleus $Ze_0\delta(\boldsymbol{r})$ and the electronic contribution that corresponds to a charge $-Ze_0$ distributed over the entire volume of the atom, with density $-e_0 Z\rho_{\rm elect}(r)$, so that

$$Ze_0\rho(r) = Ze_0[\delta(\boldsymbol{r}) - \rho_{\rm elect}(r)]. \tag{18.65}$$

The atomic (electric) form factor is then

$$F(q) = \int d^3r[\delta(\boldsymbol{r}) - \rho_{\rm elect}(r)]e^{i\boldsymbol{q}\cdot\boldsymbol{r}} = 1 - F_{\rm elect}(q). \tag{18.66}$$

The form factor associated with the spatial density of electrons is given by

$$F_{\rm elect} = \int d^3r\rho_{\rm elect}(r)e^{i\boldsymbol{q}\cdot\boldsymbol{r}}, \tag{18.67}$$

where $\rho_{\rm elect} = |\psi|^2$ is the spatial density of orbital electrons normalized to unity,

$$\int \rho_{\rm elect}(r)d^3r = 1. \tag{18.68}$$

The scattering cross section of a charged particle by an atom is then given by

$$\frac{d\sigma}{d\Omega} = |1 - F_{\rm elect}(q)|^2 \left(\frac{d\sigma}{d\Omega}\right)_{\rm Rutherford}. \tag{18.69}$$

We will analyze two limiting cases of this expression. First, consider q small, $qa \ll 1$, where a is the atomic radius. This can be either because the bombardment energy is small $ka \ll 1$, or for very small angles, $\theta \ll \hbar/mav \equiv v_0/v$, where $v_0 = \hbar/ma = \alpha c$ can be interpreted as a characteristic speed of the orbital motion of the electrons; in particular for $\theta \ll 1$, $qa \approx ka\theta \ll 1$, $\theta \ll 1/ka$. In this case we can expand the exponential in Eq. (18.67) into a power series, keeping only the first few terms, to obtain

$$\begin{aligned} F_{\rm elect} &= \int \rho_{\rm elect}(r)[1 + i\boldsymbol{q}\cdot\boldsymbol{r} - \tfrac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{r})^2 + \dots]\,d^3r = \\ &= 1 + \frac{i\boldsymbol{q}}{Ze}\cdot\boldsymbol{d} - \tfrac{1}{6}q^2\int r^2\rho_{\rm elect}(r)\,d^3r, \end{aligned} \tag{18.70}$$

where $\boldsymbol{d}$ is the electric dipole moment of the atom, which we should take to be zero in the general case. In the third integral we took into account that for an isotropic distribution,

$$\overline{(\boldsymbol{q}\cdot\boldsymbol{r})^2} = \overline{q^2z^2} = \tfrac{1}{3}q^2\overline{(x^2+y^2+z^2)} = \tfrac{1}{3}q^2\overline{r^2}. \tag{18.71}$$

Therefore, if r_0 is the mean square value of the distance of the electrons from the atomic nucleus, that is,

$$r_0^2 \equiv \int r^2\rho_{\rm elect}(r)d^3r, \tag{18.72}$$

we get up to second order

$$F_{\rm elect} = 1 - \tfrac{1}{6}q^2r_0^2. \tag{18.73}$$

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We see that at low energies the form factor of the electron distribution is very close to unity. This is because the orbital electrons have time to "average" the effects of the projectile, producing a result close to what they would produce if they were concentrated at the origin. Substituting into Eq. (18.69), we get

$$\frac{d\sigma}{d\Omega} = \left(\tfrac{1}{6}q^2 r_0^2\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \qquad (qr_0 \ll 1). \tag{18.74}$$

We see that the scattering cross section is considerably reduced by the orbiting electrons, whose effects tend to cancel out that of the nucleus. Introducing here the value of the Rutherford cross section given by Eq. (18.52) and the value of q given by Eq. (18.10), the angular dependence of both factors cancels out and we obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{Zme_0^2 r_0^2}{3\hbar^2}\right)^2. \tag{18.75}$$

This result shows that studying the small-angle scattering of slow electrons gives information about the spatial distribution of electrons in a complex atomic or molecular system.

The case $qa \gg 1$ occurs at high energies ($ka \gg 1$) and scattering angles greater than v_0/v. In this case the exponential in Eq. (18.67) oscillates rapidly and the integral becomes small compared to unity; neglecting it, Eq. (18.66) gives $F(q) = 1$, which leads to

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{\text{Rutherford}}. \tag{18.76}$$

In this case the spatial structure of the atom is not manifested in the scattering, that is, only the nuclear point charge is effective in the scattering. This was precisely the case in Rutherford's experiments.

For the H atom we can proceed without any approximation. In the ground state

$$\rho_{\text{elect}}(r) = \frac{1}{\pi a_0^3} e^{-2r/a_0}, \tag{18.77}$$

and the electronic form factor is

$$F_{\text{elect}}(q^2) = \frac{1}{\pi a_0^3} \int e^{-2r/a_0 + i\boldsymbol{q}\cdot\boldsymbol{r}} d^3r = \frac{1}{\left(1 + \tfrac{1}{4}q^2 a_0^2\right)^2}. \tag{18.78}$$

Substituting in (18.69) and simplifying, we get

$$\frac{d\sigma}{d\Omega} = \left(\frac{Zme^2 a_0^2}{\hbar^2}\right)^2 \left(\frac{1 + \tfrac{1}{8}a_0^2 q^2}{1 + \tfrac{1}{4}a_0^2 q^2}\right)^2. \tag{18.79}$$

This cross section is weakly dependent on q and thus on the scattering angle.

18.4 Partial-Wave Expansion. Introducing the *S* Matrix

Using Born's method, we have obtained an approximate expression for the scattering amplitude. We will now present a method, called *partial waves*, which allows us

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to obtain an exact expression for $f(\theta, \varphi)$. The basic idea is to decompose this function into an infinite sum, each term of which is associated with a given value of the orbital momentum. The method is quite complicated when it comes to applying it to noncentral potentials (e.g., the problem of particle scattering by a crystalline lattice), so in practice it is mainly used with central potentials, which is the only case we will consider here.

In Sections 5.4 and 6.2 we studied particle scattering by a 1D potential and showed that a significant effect of the potential is a phase shift with respect to the wave in the absence of the scattering potential. Here we will show that this is true for any potential and draw some general conclusions for central potentials. It is highly recommended to read Section 12.1 in preparation for this exposition.

Since, by hypothesis, the potential is short-range, the term $V(r)u$ in the Schrödinger equation is negligible in the asymptotic region $r \to \infty$ and the equation for a given l reduces to the free-particle equation,

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - k^2\right) u = 0, \quad r \to \infty, \tag{18.80}$$

the solution of which is

$$R(r) = \frac{u(r)}{r} = A_l j_l(kr) + B_l n_l(kr). \tag{18.81}$$

We rewrite this in the equivalent form

$$R = \alpha_l [j_l(kr) \cos \delta_l - n_l(kr) \sin \delta_l]. \tag{18.82}$$

To investigate the behavior of this solution at infinity we use the asymptotic expressions for the spherical Bessel and Neumann functions (see Section A.1 of the Appendix A). For $\rho \gg l$ we have

$$j_l(\rho) \approx \frac{1}{\rho} \sin\left(\rho - \frac{\pi}{2} l\right), \tag{18.83}$$

$$n_l(\rho) \approx -\frac{1}{\rho} \cos\left(\rho - \frac{\pi}{2} l\right). \tag{18.84}$$

Therefore, in the asymptotic region we get

$$u = rR \approx \frac{\alpha_l}{k} \left[\sin\left(kr - \frac{\pi}{2} l\right) \cos \delta_l + \cos\left(kr - \frac{\pi}{2} l\right) \sin \delta_l\right], \tag{18.85}$$

that is,

$$u(kr) \approx \frac{\alpha_l}{k} \sin\left(kr - \frac{\pi}{2} l + \delta_l\right). \tag{18.86}$$

In the absence of the scattering potential we have $\delta_l = 0$ and

$$u_0(kr) = \frac{A}{k} \sin\left(kr - \frac{\pi}{2} l\right). \tag{18.87}$$

Comparing the last two expressions we see that apart from a global factor α_l / A, the potential produces a *dephasing* of the radial wave function, equal to δ_l. For this reason, δ_l is called the *phase shift (or dephasing) of the wave* l (i.e., of the l component of the wave function).

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The integration constants A_l and B_l must be determined from the continuity conditions of the solution, that is, the proposed solution for $r \to \infty$ must be smoothly tied to the regular solution at the origin. In terms of an amplitude C_l and the phase δ_l, the solution is

$$R(r) = \frac{C_l}{kr} \sin\left(kr - \frac{\pi}{2}l + \delta_l\right), \quad kr \to \infty. \tag{18.88}$$

In the absence of the scattering potential, (18.88) holds for all r, including the origin. Since the solution must be regular and $n_l(0)$ is infinite, in this case we must take $B_l = 0$, thus $\delta_l = \arctan(-B_l/A_l) = 0$. In other words, δ_l represents the phase shift produced by the scattering potential on the l component.

Suppose that the incident beam has axial symmetry with respect to the z-axis. In this case the wave function does not depend on the azimuthal angle (which is equivalent to taking $m_l = 0$ for all l), the spherical harmonics reduce to Legendre polynomials, and we can write the asymptotic solution in the form

$$\psi = \sum_{l=0}^{\infty} \frac{C_l}{kr} \sin\left(kr - \frac{\pi}{2}l + \delta_l\right) P_l(\cos\theta). \tag{18.89}$$

This expression is an expansion in Legendre polynomials of the previously constructed solution, Eq. (18.8); equating them and isolating the scattering amplitude, we obtain

$$f(\theta) = -re^{ik(z-r)} + \sum_l \frac{C_l}{k} e^{-ikr} \sin\left(kr - \frac{\pi}{2}l + \delta_l\right) P_l(\cos\theta). \tag{18.90}$$

To determine the coefficients C_l we observe that they must be chosen so that the right-hand side of this equation does not depend on r, that is, it reduces to a function of θ. To introduce this condition, we have to express the plane wave e^{ikz} in terms of Legendre polynomials, which we can do with the help of the equation

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta). \tag{18.91}$$

Since we are only interested in the value of this expression for $r \to \infty$, we can write instead of the spherical Bessel function its asymptotic value given by $j_l(kr) \simeq \frac{1}{kr}\sin(kr - \frac{\pi}{2}l)$, which gives

$$e^{ikz} = \sum_l i^l (2l+1) \frac{\sin\left(kr - \frac{\pi}{2}l\right)}{kr} P_l(\cos\theta), \quad kr \to \infty. \tag{18.92}$$

Introducing this expansion in (18.90) and writing the trigonometric functions in terms of exponentials, we obtain

$$\begin{aligned} f(\theta) = {} & \frac{1}{2ik} \sum_l [C_l e^{i\delta_l} - i^l(2l+1)] e^{-i\frac{\pi}{2}l} P_l(\cos\theta) \\ & - \frac{1}{2ik} e^{-2ikr} \sum_l \left[C_l e^{-i\delta_l} - i^l(2l+1)\right] e^{i\frac{\pi}{2}l} P_l(\cos\theta). \end{aligned} \tag{18.93}$$

In order to remove the dependence on r the second sum must be canceled, which gives

$$C_l = i^l(2l+1)e^{i\delta_l}. \tag{18.94}$$

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Substituting this value in the same Eq. (18.93) we obtain, taking into account that $i^l e^{-i\frac{\pi}{2}l} = (ie^{-i\frac{\pi}{2}})^l = 1$,

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta) \equiv \sum_{l=0}^{\infty} f_l(\theta). \tag{18.95}$$

This important result, which is exact, forms the basis for all further analysis in terms of phase shifts. It can be expressed in various forms, which are obtained by using one of the following equivalent expressions

$$\frac{e^{2i\delta_l}-1}{2i} = e^{i\delta_l}\sin\delta_l = a_l = kf_l = \frac{S_l(k)-1}{2i}. \tag{18.96}$$

Both a_l and f_l are referred to interchangeably as the amplitude of the partial wave l; $S_l(k)$ is the element l of the scattering matrix S and is given by[5]

$$S_l(k) = e^{2i\delta_l}; \tag{18.97}$$

obviously $SS^\dagger = 1$, so S is unitary. A complete knowledge of the phases δ_l (or, equivalently, of the partial amplitudes a_l, or of the S matrix) as functions of energy, specifies the scattering amplitude $f(\theta)$ and this in turn allows us, in principle, to determine the scattering potential (e.g., by means of an inverse Fourier transformation if we use the Born approximation). In practice, however, this program requires overcoming major difficulties. This is a very simple example of the S matrix as a scattering matrix, the theory of which is highly relevant to more advanced studies.

18.4.1 The optical theorem

An expression for the total (elastic) scattering cross section can be obtained from Eq. (18.95). We have from (18.35) that

$$\sigma = \frac{2\pi}{k^2}\int_0^\pi d\theta\sin\theta\sum_{l,l'}(2l+1)(2l'+1)a_l^* a_{l'}P_l(\cos\theta)P_{l'}(\cos\theta); \tag{18.98}$$

taking into account the orthogonality of the Legendre functions

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}, \tag{18.99}$$

we obtain the formula

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{18.100}$$

Since the maximum value of l satisfies $l_{\max} \leq kR$, we see that the total cross section has the upper bound

$$\sigma_{\max} \leq \frac{4\pi}{k^2}\sum_{l=0}^{l_{\max}}(2l+1) = \frac{4\pi}{k^2}(l_{\max}+1)^2 \lesssim 4\pi R^2. \tag{18.101}$$

[5] An example of the use of the S matrix to describe elastic dispersion in a 1D problem was seen in Section 5.3.

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From (18.95) it follows that

$$\mathrm{Im}\, f(0) = \frac{1}{k}\sum_l (2l+1)\sin^2\delta_l P_l(1) \tag{18.102}$$

and since $P_l(1) = 1$ for all l, comparing with (18.100), we obtain

$$\sigma = \frac{4\pi}{k}\mathrm{Im}\, f(0). \tag{18.103}$$

This relation between the total cross section and the imaginary part of the forward scattering amplitude is called the *optical theorem* and is more general than what the previous derivation suggests (it is valid, e.g., for inelastic processes if σ represents the total section $\sigma = \sigma_{\mathrm{elast}} + \sigma_{\mathrm{inel}}$). This theorem is so called because it goes back to the optical work of Lord Rayleigh; in QM it was established in 1932 by E. Feenberg (1906–1977), but it is also known as the Bohr–Peierls–Placzek relation.

The meaning of the optical theorem can be easily understood by calculating the total flux of particles through a spherical surface in the asymptotic region centered on the origin of the coordinates (see problem P18.13),

$$J = -\frac{i\hbar}{2m}\int_\Omega (\psi^*\nabla\psi - \psi\nabla\psi^*)r^2 d\Omega = V\left(\sigma - \frac{4\pi}{k}\mathrm{Im}\, f(0)\right) \quad (r\to\infty). \tag{18.104}$$

Since the process is elastic, the total number of particles is conserved and their net flux through the surface must vanish; this immediately leads to the optical theorem. On the other hand, if the phase shifts δ_l were not real, the optical theorem would not be satisfied as demonstrated here; we conclude that the unitary property of the scattering matrix S is a consequence of the flux conservation, as can be shown directly (problem P18.12).

From Eq. (18.95) it follows that the scattering amplitude can be written in the general form

$$f(\theta) = A_0 + A_1\cos\theta + A_2\cos^2\theta + \ldots = \sum_{n=0}^{\infty} A_n\cos^n\theta, \tag{18.105}$$

so the differential cross section will have an analogous structure,

$$\frac{d\sigma}{d\Omega} = \sum_{n=0}^{\infty} B_n\cos^n\theta, \tag{18.106}$$

which is a sum of terms produced by each value of l plus all possible interference terms. From the analysis of the experimental differential cross section, it is possible to determine the coefficients B_n of this expansion (which must be done for each energy value!) and, from them, the phase shifts as a function of energy. It is these latter data that have to be interpreted theoretically, which makes their theoretical determination a problem of prime importance. The problem is generally very complex and various methods have been developed to solve it, including numerical integration of the Schrödinger equation.

In particular, when formulas such as those given by the Born approximation are acceptable, it is possible to obtain theoretical expressions for the phase shifts for any

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model potential. For example, if we accept that the expression (18.29) is satisfactory, then integrating with respect to the angular variables for a central potential, we obtain

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty rV(r)\sin qr dr \tag{18.107}$$

(in this formula the scattering angle enters through q). To give it the form of Eq. (18.95) we have to expand $f(\theta)$ into Legendre polynomials, which can be done using the formula taken from the theory of Bessel functions,

$$\frac{\sin qr}{qr} = \sum_{l=0}(2l+1)j_l^2(kr)P_l(\cos\theta) \quad \left(q \equiv 2k\sin\frac{\theta}{2}\right). \tag{18.108}$$

Inserting this expression into the previous one gives

$$f(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty dr r^2 V(r)\sum_l (2l+1)j_l^2(kr)P_l(\cos\theta). \tag{18.109}$$

If we now compare the coefficients of P_l of this expression and (18.95), we obtain

$$\begin{aligned} a_l = e^{i\delta_l}\sin\delta_l &= -\frac{2mk}{\hbar^2}\int_0^\infty V(r)j_l^2(kr)r^2 dr \\ &= -\frac{\pi m}{\hbar^2}\int_0^\infty V(r)J_{l+1/2}^2(kr)r dr, \end{aligned} \tag{18.110}$$

where we have used the relation between cylindrical and spherical Bessel functions, Eq. (A.62),

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}}J_{l+1/2}(\rho). \tag{18.111}$$

For small phases it follows from (18.96) that $a_l \simeq \delta_l$, which gives an expression for the phase shifts when they are small,

$$\delta_l = -\frac{2mk}{\hbar^2}\int_0^\infty r^2 j_l^2(kr)V(r)dr. \tag{18.112}$$

This result allows us to verify that for an attractive potential ($V < 0$) the phase shifts are positive, while for repulsive potentials ($V > 0$), the δ_l take negative values.

Exercise E18.3. Scattering by an impenetrable sphere

Calculate the differential cross section for particles scattered by a perfectly rigid and impenetrable sphere of radius a.

Solution. We can model the scattering effect of the sphere with the central potential

$$V(r) = \begin{cases} \infty, & r \le a, \\ 0, & r > a. \end{cases} \tag{18.113}$$

The condition of total impenetrability can alternatively be expressed as the boundary condition corresponding to total reflection, $\psi(r)|_{r=a} = 0$. The radial Schrödinger equation is

$$R'' + \frac{2}{r}R' + \left(k^2 - \frac{l(l+1)}{r^2}\right)R = 0, \quad k^2 = \frac{2mE}{\hbar^2}, \quad r \ge a, \tag{18.114}$$

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and $R = 0$ for $r < a$. The solution can be written in terms of the spherical Bessel functions. In this case it is convenient to take $j_l(kr)$ and $h_l^{(1)}(kr)$ as independent solutions; the spherical Hankel function of the first kind (see Section A.1),

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(1)}(x), \tag{18.115}$$

behaves asymptotically as an outgoing spherical wave, as follows from (A.55),

$$h_l^{(1)}(kr) \longrightarrow_{kr\to\infty} (-i)^{l+1} \frac{e^{ikr}}{kr}. \tag{18.116}$$

This is precisely the property that makes this function especially useful in scattering problems. Therefore, we write for $r > a$,

$$\psi = \sum_{l=0}^{\infty} \Big((2l+1) i^l j_l(kr) + C_l h_l^{(1)}(kr) \Big) P_l(\cos\theta). \tag{18.117}$$

The coefficient of the first term has been chosen appropriately to reproduce the incident plane wave, as will become clear later. The boundary condition on the surface of the sphere can only be satisfied if the coefficient of each Legendre polynomial vanishes separately at $r = a$. This means that each of the equations

$$(2l+1) i^l j_l(ka) + C_l h_l^{(1)}(ka) = 0 \tag{18.118}$$

must be satisfied separately; from them we obtain the value of the coefficients C_l. Substituting in the function ψ of Eq. (18.117), we obtain

$$\begin{aligned} \psi &= \sum_{l=0}^{\infty} (2l+1) i^l \left[j_l(kr) - \frac{j_l(ka)}{h_l^{(1)}(ka)} h_l^{(1)}(kr) \right] P_l(\cos\theta) \\ &= e^{ikz} - \sum_{l=0}^{\infty} (2l+1) i^l \frac{j_l(ka)}{h_l^{(1)}(ka)} h_l^{(1)}(kr) P_l(\cos\theta). \end{aligned} \tag{18.119}$$

The second equality was obtained using the expansion of the plane wave into spherical waves and justifies the previous choice of the coefficients of the j_l functions. This is an exact solution of the Schrödinger equation whose asymptotic behavior is given by

$$\psi(r,\theta)|_{r\to\infty} = e^{ikz} + \frac{i}{k} \frac{e^{ikr}}{r} \sum_{l=0}^{\infty} (2l+1) \frac{j_l(ka)}{h_l^{(1)}(ka)} P_l(\cos\theta). \tag{18.120}$$

Comparing this expression with the general form of the asymptotic solution of the scattering problem, Eq. (18.8), we obtain the following exact expression for the scattering amplitude by an impenetrable sphere,

$$f(\theta) = \frac{i}{k} \sum_{l=0}^{\infty} (2l+1) \frac{j_l(ka)}{h_l^{(1)}(ka)} P_l(\cos\theta), \tag{18.121}$$

which gives for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1) \frac{j_l(ka)}{h_l^{(1)}(ka)} P_l(\cos\theta) \right|^2 \tag{18.122}$$

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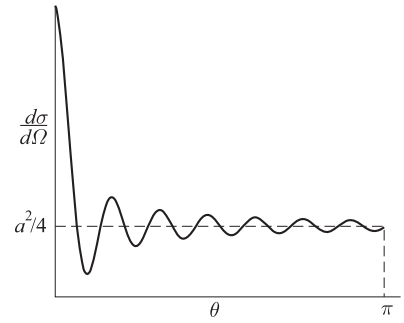


Figure 18.7 Differential cross section for a rigid sphere as a function of the scattering angle.

and for the total cross section (see Eq. (18.100))

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left| \frac{j_l(ka)}{h_l^{(1)}(ka)} \right|^2 . \tag{18.123}$$

The terms that make up the differential cross section combine to give a series of maxima and minima as the angle θ varies. For $ka \ll 1$ there is a very sharp maximum in the forward direction and rapidly decreasing oscillations around the classical differential section $a^2/4$, as shown in Fig. 18.7. Since $j_l(ka)/h_l^{(1)}(ka) \sim (ka)^{2l+1}$, the dominant contribution comes from the s-wave. In this case, and as usual at low energies, the scattering by the short-range central potential is isotropic,

$$\frac{d\sigma}{d\Omega} = a^2 \tag{18.124}$$

and the total cross section is four times the classical cross section of a perfectly rigid sphere or twice the total cross section of a perfectly absorbing sphere,

$$\sigma = 4\pi a^2. \tag{18.125}$$

18.5 *Resonant Scattering

When the potential is attractive and allows the existence of bound states in the continuum, a resonance phenomenon can occur at low energies, which manifests itself as a significant increase in the scattering cross section around the resonance energy. This phenomenon is very important and is due to the formation of states that are bound by the potential but unstable. The topic is part of a more advanced course; however, because of its importance, the main results are presented here without derivation.

In the region of a resonance, one of the phases, let's say δ_l, changes rapidly with energy by the amount π (and passes through the angle $\pi/2$, where its tangent becomes

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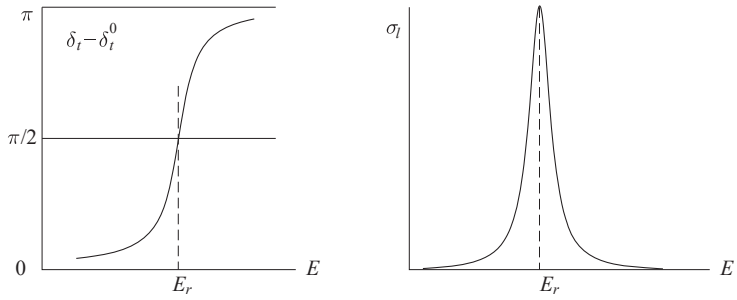


Figure 18.8 The phase shift δ_l and the cross section σ as functions of the energy E, close to a resonance.

infinite). If the value of this δ_l very far from resonance is denoted by δ_l^0, it can be written approximately as

$$\tan(\delta_l - \delta_l^0) = \frac{\Gamma_l/2}{E_r - E}. \tag{18.126}$$

The parameter Γ_l measures the width of the resonance and E_r is the resonance energy, that is, the energy at which the difference $\delta_l - \delta_l^0$ takes the value $\pi/2$. At this energy $\text{sen}^2\,(\delta_l - \delta_l^0)$ reaches its maximum value of 1, so that the cross section, which is approximately given by the dominant orbital momentum term l according to Eq. (18.100),

$$\sigma \simeq \frac{4\pi}{k^2}(2l+1)\text{sen}^2(\delta_l - \delta_l^0) = \frac{4\pi}{k^2}(2l+1) \quad \text{for} \quad E = E_r, \tag{18.127}$$

also reaches a maximum which is normally very sharp. The characteristic behavior of δ_l and σ around a resonance is shown in Fig. 18.8.

A formula for the scattering amplitude can be derived using the general equation (18.95) and assuming that the other phases are small, the result being

$$f(\theta) = f_l^0(\theta) - \frac{2l+1}{k}\frac{\Gamma_l/2}{E - E_r + i\Gamma_l/2}e^{2i\delta_l^0}P_l(\cos\theta) \tag{18.128}$$

where $f_l^0(\theta)$ is the scattering amplitude away from the resonance. This expression clearly separates the *external* or *potential scattering amplitude* (the nonresonant contribution) from the *internal* or *resonant scattering amplitude*. Since δ_l^0 is usually also small (see Fig. 18.9), the amplitude in the resonance region is often written in the form

$$f(\theta) = -\frac{2l+1}{k}\frac{\Gamma_l/2}{E - E_r + i\Gamma_l/2}P_l(\cos\theta). \tag{18.129}$$

The total cross section is then given by

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l = \frac{4\pi}{k^2}(2l+1)\frac{g_l^2}{1+g_l^2},$$

that is,

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\frac{\Gamma_l^2/4}{(E - E_r)^2 + \Gamma_l^2/4}. \tag{18.130}$$

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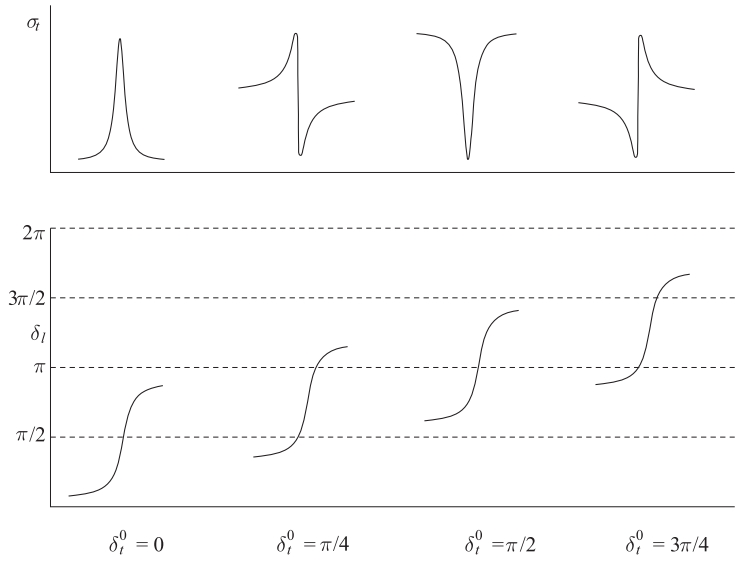


Figure 18.9 a) At energies close to resonance, the phase changes abruptly by π and the differential cross section reaches a peaked maximum when the background phase is small. (b) However, for large values of δ^0, there is no resonance; there may even be antiresonance.

This expression is a particular case of the Breit–Wigner formula for the resonant cross section. It shows that when $|E - E_r| = \Gamma_l/2$, the cross section is half its maximum; therefore, Γ_l measures the width of the resonance curve and is called the *width of the resonance*. Resonance curves of the form of Eq. (18.130) as a function of E are called *Lorentzian*.[6]

The preceding results admit of an interesting interpretation. We saw in Chapter 12 that the asymptotic solution of the Schrödinger equation for $r \to \infty$ has the general form

$$u = A(E)e^{-\alpha r} + B(E)e^{\alpha r}, \tag{18.131}$$

where $\alpha \equiv -ik = \sqrt{-2mE}/\hbar$ is real for bound states ($E < 0$) and imaginary for dispersive states. Rewriting this solution in the usual form of scattering theory, as

$$u = C(E)\sin(kr + \delta_l), \tag{18.132}$$

[6] A Lorentzian distribution has no moments after the second (they are all infinite). In mathematics, the Lorentzian distribution is written in the form

$$P(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

and is known as the Cauchy distribution. It is usual to characterize this distribution by its width Γ,

$$P_\Gamma(x) = \frac{1}{\pi}\frac{\Gamma}{x^2+\Gamma^2}.$$

However, its scale invariance

$$P_\Gamma(\lambda x)d(\lambda x) = P_{\Gamma/\lambda}(x)dx,$$

combined with the absence of moments after the second, imply the *nonexistence of a natural scale* associated with this distribution, so the fluctuations of x occur equally at all scales.

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gives $e^{2i\delta_l} = -\frac{A}{B}$. Therefore, the amplitude f_l can be written in the form (see Eq. (18.96))

$$f_l = \frac{e^{2i\delta_l} - 1}{2ik} = -\frac{1}{2ik}\frac{A+B}{B}. \tag{18.133}$$

When the energy is negative (i.e., for bound states), α is positive and the second term of Eq. (18.131) diverges, so it is necessary to take $B(E) = 0$. But from (18.133) we see that the amplitude f_l has a single pole at $B = 0$, that is, the scattering amplitude, considered as a function of the energy, has single poles at the energies corresponding to the bound states.[7]

Returning now to resonant scattering, we see from Eq. (18.129) that $f(\theta, E)$ has a single pole for the *complex* energy

$$E = E_r - \frac{i}{2}\Gamma_l. \tag{18.134}$$

This suggests interpreting a resonance as a bound state, but with positive energy and metastable, meaning that it decays "spontaneously" with a relatively long lifetime. This can be seen by remembering that the stationary wave function is multiplied by $e^{-iEt/\hbar}$, which in this case is $e^{-iEt/\hbar} = e^{-iE_rt/\hbar}\, e^{-\Gamma t/2\hbar}$, so that the probability density *decreases* with time as $\rho \sim e^{-\Gamma t/\hbar}$; the state becomes depopulated, with a lifetime given by

$$T = \frac{\hbar}{\Gamma}. \tag{18.135}$$

In other words, $\Gamma/\hbar$ is the decay probability of the resonance per unit time.

The conclusion is that a resonance can be interpreted as an unstable composite particle, that is, as a quasi-stationary state of the target particle and the projectile, with well-defined quantum numbers (rest mass $E_r c^{-2}$, spin l, etc.) and whose stability is measured by Γ. Based on these results, a resonance is often formally defined as a single pole in the scattering amplitude (or in the S matrix) for a complex energy.

Exercise E18.4. Scattering by a spherical barrier at low energies

Compare the total scattering cross section produced by a uniform spherical barrier of radius a and height V_0 calculated by means of an expansion in partial waves, with that obtained in the Born approximation at low energies ($ka \ll 1$).

Solution. The elastic scattering cross section at low energies is essentially given by the s-wave contribution,

$$\sigma = 4\pi a^2 \left(1 - \frac{\tanh Ka}{Ka}\right)^2, \tag{18.136}$$

where $K = \sqrt{2mV_0}/\hbar$. On the other hand, the Born approximation gives the following amplitude for this interaction potential, using Eq. (18.50) with $\sin qr \simeq qr$,

$$f(\theta) = -\frac{2mV_0}{\hbar^2 q}\int_0^a qr^2 dr = -\frac{2mV_0 a^3}{3\hbar^2}, \tag{18.137}$$

[7] A similar argument was found in Chapter 5 from the properties of the transmission coefficient.

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so the cross section predicted by this approximation is

$$\sigma = 2\pi \int_0^{\pi} |f(\theta)|^2 \text{sen}\theta d\theta = \pi \left(\frac{4mV_0 a^3}{3\hbar^2} \right)^2 = \pi \left(\frac{2}{3} K^2 a^3 \right)^2 . \tag{18.138}$$

For the Born approximation to be applicable, it must be true that $\sigma \ll \pi a^2$; from (18.138) we see that this is the case only if $4mV_0 a^2/3\hbar^2 = \frac{2}{3}K^2a^2 \ll 1$, that is, if $Ka \ll 1$. When this condition is met, we can approximate $\tanh Ka \simeq Ka - \frac{1}{3}(Ka)^3$ and Eq. (18.136) reduces to

$$\sigma = 4\pi a^2 \cdot \left(\tfrac{1}{3} K^2 a^2 \right)^2 = \pi \left(\tfrac{2}{3} K^2 a^3 \right)^2 , \tag{18.139}$$

which agrees with the Born approximation, as expected.

Problems

P18.1 Derive the relations (18.1) and (18.2) that allow the transition between the laboratory and CM reference frames in a two-particle problem.

P18.2 Derive Eq. (18.3) which relates the number of particles per unit area in both reference frames.

P18.3 Prove that for a collision process of the type $P_1 + P_2 \rightarrow P_3 + P_4$, which can be elastic or inelastic so that the quantity $Q \equiv E_{\text{kyn.final}} - E_{\text{kyn.initial}}$ can be different from zero, in Eq. (18.3) the factor $\gamma = m_1/m_2$ must be replaced by

$$\gamma = \left(\frac{m_1}{m_2} \frac{m_3}{m_4} \frac{E_r}{E_r + Q} \right)^{1/2} , \quad E_r = \tfrac{1}{2} m_{\text{red}} v^2 . \tag{18.140}$$

P18.4 A particle of mass m collides elastically with a particle of mass M at rest (in the laboratory). Suppose that the scattering has spherical symmetry in the CM frame. What is the angular distribution of the target particles in the laboratory frame?

P18.5 A beam of particles with density $n_0\,\text{part/cm}^2$ is fired at a target with total effective cross section σ. Show that if the thickness of the target is d, the beam that crosses it comes out with density $n = n_0 e^{-\mu d}$, where $\mu = N\sigma$ is the so-called linear attenuation coefficient and N is the number of collision centers that the target contains in the unit volume.

P18.6 Show that the differential scattering cross section in the Born approximation due to a spherical barrier of radius R and constant height V_0 is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2} \right)^2 \frac{(\text{sen}qR - qR\cos qR)^2}{q^6} . \tag{18.141}$$

P18.7 For the Ramsauer–Townsend effect to occur (see Section 5.3.2), the cross section must vanish at very small bombardment energies (which can be assumed to be zero). Using a uniform spherical well as a model for the interaction

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potential, find the value that should be assigned to the product a^2V_0. *Hint*: Consider at least the s and p waves.

P18.8 Find the differential cross section for the case where both the target and the projectile have structure.

P18.9 Protons with energy of 0.3 MeV are scattered by a thin sheet of aluminum. The number of backscattered protons is observed to be 0.96 times the predicted value for a Coulomb potential. Interpreting this discrepancy as due to shielding effects and assuming that the corresponding potential change appreciably affects only the s wave, determine the change in the value of δ_0 . Indicate whether the interaction is attractive or repulsive.

P18.10 Consider scattering by a uniform spherical well of radius a and (large) depth V_0. Show that the condition for bound states to exist is

$$k_1 a = \left(n + \tfrac{1}{2}(l+1)\right)\pi, \tag{18.142}$$

where $k_1^2 = 2m(E + V_0)/\hbar^2$.

P18.11 By solving the radial Schrödinger equation with the scattering potential as a perturbation, derive Eq. (18.110) for the phase shift when it is small,

$$\delta_l = -\frac{\pi m}{\hbar^2}\int_0^\infty rV(r)J_{l+1/2}^2(kr)dr. \tag{18.143}$$

Note that with this procedure we avoid using (18.109).

P18.12 Show that the condition: incident wave flux = outgoing wave flux, implies that $|\widehat{S}|^2 = 1$.

P18.13 Show that the flux J through a spherical surface of radius $r \to \infty$ is given by

$$J = v\left(\sigma - \frac{4\pi}{k}\mathrm{Im}\, f(0)\right). \tag{18.144}$$

P18.14 Determine the scattering amplitude and the differential cross section in Born's first approximation for the potential $V_0\delta(r - R)$.

P18.15 Find the differential and total cross section in Born's first approximation for the Gaussian potential $V_0 \exp(-\alpha^2 r^2)$.

P18.16 Particles of mass m are incident on a central potential of the form

$$V(r) = \begin{cases} \infty, & r < a, \\ -V_0, & a \le r \le 2a, \\ 0, & r > 2a, \end{cases} \tag{18.145}$$

$V_0 > 0$. Show that the cross section σ is such that

$$\lim_{v\to 0}\sigma = 4\pi\left[k_0^{-1}\tan(k_0 a) - 2a\right]^2, \tag{18.146}$$

with $k_0 = \left[2mV_0/\hbar^2\right]^{1/2}$.

P18.17 The scattering amplitude for a certain interaction is given by

$$f(\theta) = \frac{1}{k}\left(e^{ika}\mathrm{sen}\, ka + 3ie^{2ika}\cos\theta\right), \tag{18.147}$$

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where a is the characteristic length of the interaction potential and k is the wave number of the incident particles. Find the differential cross section for s waves for this interaction.

P18.18 Determine the phase shifts δ_l produced by the potential $V(r) = g/r^2$. Find the effective differential section and verify that it is inversely proportional to the energy.

Bibliographical Notes

A very useful reference on scattering theory is Newton (1982).

The textbook by Afnan (2011) contains the exact solution of the Coulomb scattering problem in parabolic coordinates.

An extended modern account of scattering theory, including the field of ultracold matter that has been of so much interest in recent years, is Friedrich (2013).

A reference of historical value on the important Fermi experiments on meson-nucleon systems is Fermi (1955).

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