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Regions of instability for non-twist maps

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Abstract. In this paper we consider an analog of the regions of instability for twist maps in the context of area preserving diffeomorphisms which are not twist maps. Several properties analogous to those of classical regions of instability are proved.

1. *Introduction and notation*

The objective towards which this article is directed is a good topological picture of the dynamics of C^r generic area preserving diffeomorphisms of surfaces. We address the case where the surface is either the annulus or S^2 and r is large ($r \geq 16$ is certainly sufficient). Even in this case our objective is an ambitious one. Despite some substantial progress, many fundamental questions remain unanswered. Perhaps the most important is the question of whether there is a C^r generic set of area preserving diffeomorphisms for which the periodic points are dense. It is not difficult to see that for generic diffeomorphisms this question is equivalent to the question of whether the union of all homoclinic points for all hyperbolic periodic orbits forms a dense set.

One of our main results, Theorem 9.15, asserts that for any $r \geq 16$ there is a C^r generic set of diffeomorphisms of S^2 with the property that the union of stable manifolds of all hyperbolic periodic points is a dense set. Of course, one can simultaneously have the union of all unstable manifolds of hyperbolic periodic points dense, but this does not imply the desired result about homoclinic points.

A guiding principle, in pursuing a topological understanding of area preserving diffeomorphisms, has been to establish analogs of the classical topological results about twist maps (primarily due to G. D. Birkhoff) in a context where the twist hypothesis is removed, but generic properties are assumed.

Our most successful result in this direction is the formulation of a generalization of Birkhoff's notion of 'regions of instability' and the proof that many of their key properties remain true for generic area preserving diffeomorphisms of the annulus or S^2. Our precise definition of region of instability for an annulus diffeomorphism is given in Definition 8.5,

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but in particular it is an essential open invariant annulus which contains no essential compact connected invariant sets which are periodic point free. In this sense it is very like the Birkhoff definition for a region of instability for a twist map which would require that it contain no essential invariant curves. For a generic twist map the two definitions coincide.

We show that inside any region of instability there is a hyperbolic periodic point z with the property that $K(z)$, the closure of $W^s(z)$, is essential (in the annulus) and contains both components of the frontier of the region of instability. There are many such 'essential' periodic points and we show in Theorem 8.6 and Corollary 8.7 that $K(z)$ is independent of the essential periodic point z and that we could have equally well defined it to be the closure of the unstable manifold $W^u(z)$. Further study of the dynamics on $K(z)$ is crucial for a good topological understanding of area preserving diffeomorphisms. In particular if, in the generic setting, homoclinic points of such a point z were always dense in $K(z)$ this would answer affirmatively the question of whether periodic points are generically dense for area preserving diffeomorphisms of the annulus.

One of the most striking results we establish for regions of instability is a generalization of known results (see Mather [**M2**]) for connecting orbits of twist maps. In Theorem 8.11 we show that in a region of instability there are always connecting orbits between the two components of its frontier. More precisely if the two components of the frontier are A and B, then there is an orbit whose α-limit set is in A and whose ω-limit set is in B, as well as one whose α-limit set is in B and whose ω-limit set is in A. While Mather's proof of the analogous result for twist maps uses variational techniques, he has also made seminal contributions to the topological methods on which our work is based (see [**M1, M3**]).

In the subsequent text, we write $\mathbb{T}^1$ for the one-dimensional torus and $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ for the infinite annulus. We denote by N (respectively S) the upper (respectively lower) end of $\mathbb{A}$ and $\widehat{\mathbb{A}}$ the end compactification of $\mathbb{A}$ which is a topological sphere, i.e. a topological set homeomorphic to the two-dimensional sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We consider the covering projection:

$$\begin{aligned} \pi : \mathbb{R}^2 &\to \mathbb{A} \\ (x, y) &\mapsto (x + \mathbb{Z}, y) \end{aligned}$$

of the universal covering space $\mathbb{R}^2$ and the deck transformation

$$\begin{aligned} T : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x + 1, y). \end{aligned}$$

We give the same name to the first projection

$$p_1 : (x, y) \mapsto x$$

defined on the plane or the annulus.

A set K of $\mathbb{A}$ is called an *essential continuum* if it is compact, connected, and if its complement has two unbounded components. We write U_K^+ for the upper one and U_K^- for

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the lower one. The sets

$$\widehat{U}_K^+ = U_K^+ \cup \{N\} \quad \text{and} \quad \widehat{U}_K^- = U_K^- \cup \{S\}$$

are simply connected domains of $\widehat{\mathbb{A}}$. If $\mathbb{A} \setminus K$ has no other component we will say that K is a *filled essential continuum*.

We say that a homeomorphism of $\mathbb{A}$ preserves area if it preserves the measure given by the volume form $dx \wedge dy$.

An open set U of $\mathbb{A}$ is called an *essential annulus* if it is homeomorphic to $\mathbb{A}$ and if the inclusion $i : U \hookrightarrow \mathbb{A}$ induces a one to one map

$$i_* : H_1(U, \mathbb{R}) \to H_1(\mathbb{A}, \mathbb{R})$$

on the first homology group. If U is bounded, its frontier has two connected components, we denote by $\partial^+ U$ (resp. $\partial^- U$) the upper (respectively the lower one).

We will consider

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

and

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

2. *Prime end theory*

Let U be a non-trivial simply connected domain of a topological sphere Σ (that means that the complement has at least two points). We can define the prime end compactification of U, introduced by Carathéodory [**C**], by adding the circle S^1. A very good exposition of the theory of prime ends in modern terminology and in the context we need can be found in the paper [**M3**] of John Mather. This paper also extends the theory to surfaces other than the plane or S^2 although we will not make use of this.

The prime end compactification can be defined purely topologically but has another significance if we put a complex structure on Σ. We can find a conformal map h between U and the open disk $\mathbb{D}$ and we put on $U \sqcup S^1$ the topology (which, up to homeomorphism of the resulting space, is independent of h) induced from the natural topology of $\overline{\mathbb{D}}$ by the bijection

$$\overline{h} : U \sqcup S^1 \to \overline{\mathbb{D}}$$

equal to h on U and to the identity on S^1.

If F is a homeomorphism of Σ which leaves U invariant, then the map $F_{|U}$ extends to a homeomorphism of the prime end compactification $U \sqcup S^1$ of U. If $F_{|U}$ is orientation preserving then its restriction to S^1 preserves orientation and we can define its rotation number which is an element of $\mathbb{T}^1$. We denote it by ρ_U and call it the *rotation number* of U.

Every homeomorphism F of $\mathbb{A}$ homotopic to the identity extends naturally to an orientation preserving homeomorphism $\widehat{F}$ of $\widehat{\mathbb{A}}$ which fixes the ends N and S. If K is an invariant essential continuum we write $-\rho_K^+$ (respectively ρ_K^-) for the rotation number of $\widehat{U}_K^+$ (respectively $\widehat{U}_K^-$) defined by $\widehat{F}$.

Every lift f of F to the plane induces naturally a lift of $F_{|U_K^+}$ to the universal covering space of U_K^+. This lift can be extended to the universal covering space of S^1 which admits a real rotation number $\tilde{\rho}_K^+$ such that $\tilde{\rho}_K^+ + \mathbb{Z} = \rho_K^+$. In the same way we can define $\tilde{\rho}_K^-$.

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By prime end theory we can compactify any invariant and bounded essential open annulus, by adding two copies C^+ and C^- of S^1 corresponding respectively to the upper and to the lower end of U. We will naturally get two rotation numbers ρ_U^+ and ρ_U^- and for a lift f two real rotation numbers $\tilde{\rho}_U^+$ and $\tilde{\rho}_U^-$.

Often, we will make use of the following result due to Cartwright and Littlewood [**CL**].

PROPOSITION 2.1. *Let F be an orientation preserving homeomorphism of a topological sphere Σ and U a non-trivial simply connected invariant domain. We suppose that:*

(i) *the rotation number $\rho_U = p/q + \mathbb{Z}$ is rational; and*

(ii) *there is a neighborhood of ∂U in $\overline{U}$ which doesn't contain the positive or the negative orbit of any wandering open set.*

Then ∂U contains a fixed point of F^q. There is an analogous result for the case that U is an open annulus.

Proof. Taking F^q instead of F, we can suppose that the rotation set ρ_U is equal to zero. It follows that there is a fixed point $z \in S^1$ of the extended homeomorphism in the prime end compactification of U.

From prime end theory we can find a sequence of open arcs $(\gamma_n)_{n=1}^{\infty}$ in U of diameter less than $1/n$ each of which extends to a closed arc in the prime end compactification, with end points which belong to S^1 and which are close to z, one on each side. Moreover, these arcs have the property that the closures $\overline{\gamma}_n$ converge to the point z in the prime end compactification as n goes to infinity.

Each of these arcs divides U into two simply connected domains. We denote by U_n the small one (i.e. z belongs to the closure of U_n in the prime end compactification). We wish to show that $F(\gamma_n) \cap \gamma_n \neq \emptyset$. If the arc γ_n does not meet its image under F, the fact that z is fixed implies that one of the inclusions $F(U_n) \subset U_n$ or $F^{-1}(U_n) \subset U_n$ is true. In the first case we can find an open set V contained in $U_n \setminus F(U_n)$ which is wandering and has a positive orbit contained in U_n. In the second case we can construct a wandering domain whose negative orbit is contained in U_n. Since these conditions contradict our hypothesis we conclude that $F(\gamma_n) \cap \gamma_n \neq \emptyset$.

This fact together with the fact that the diameters of the arcs γ_n tend to zero implies that any limit point in ∂U is a fixed point of F. □

Remark 2.2. We note that in the proof above we showed that for any fixed prime end z and any sequence of arcs γ_n defining it, we have $F(\gamma_n) \cap \gamma_n \neq \emptyset$. We will make use of this fact subsequently.

We deduce immediately the following corollaries of Proposition 2.1.

COROLLARY 2.3. *Let F be an orientation preserving homeomorphism of a topological sphere Σ and U a non-trivial simply connected invariant domain. We suppose that:*

(i) *the rotation number $\rho_U = p/q + \mathbb{Z}$ is rational; and*

(ii) *there is no wandering point in U.*

Then ∂U contains a fixed point of F^q.

COROLLARY 2.4. *Let F be an area preserving homeomorphism of $\mathbb{A}$ homotopic to the identity and K an invariant continuum. We suppose that $\rho_K^+ = p/q + \mathbb{Z}$ (respectively*

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$\rho_K^- = p/q + \mathbb{Z}$) is rational. Then K contains a fixed point of F^q, more precisely the fixed point is in ∂U_K^+ (respectively ∂U_K^-).

3. *Rotation number of invariant measures*

Let F be a homeomorphism of $\mathbb{A}$ homotopic to the identity and f a lift of F to $\mathbb{R}^2$. If μ is an invariant Borel probability measure we can define its rotation number

$$\rho(\mu) = \int_{\mathbb{A}} \phi \, d\mu,$$

where $\phi : \mathbb{A} \to \mathbb{R}$ is the function lifted by the function $p_1 \circ f - p_1$.

If O is a periodic orbit of F with q elements, there exists an integer $p \in \mathbb{Z}$ such that $f^q(z) = T^p(z)$ for every $z \in \pi^{-1}(O)$. We will say that O is a periodic orbit of *type* (p, q). The number p/q is called the rotation number of O, it is equal to the rotation number of the invariant measure

$$\mu = \frac{1}{q} \sum_{z \in O} \delta_z,$$

where δ_z is the Dirac measure concentrated on z.

If K is a non-empty compact invariant set, the set of Borel probability measures whose support is in K is a compact set in the weak topology and the subset $M(K)$ of the invariant measures is a non-empty convex and closed set. We can define the *rotation set* of K

$$\rho(K) = \{\rho(\mu) \mid \mu \in M_K\},$$

which is a non-empty segment $[\alpha, \beta]$ of $\mathbb{R}$. We have the following classical result.

PROPOSITION 3.1. *For every $\varepsilon > 0$ there exists $N \geq 0$ such that for every $z \in \pi^{-1}(K)$ and for every $n \geq N$ we have*

$$\alpha - \varepsilon < \frac{p_1 \circ f^n(z) - p_1(z)}{n} < \beta + \varepsilon.$$

Proof. Let us suppose for example that we can find a real number $\varepsilon > 0$, a sequence of integers $(n_k)_{k \geq 0}$ converging to $+\infty$ and a sequence $(z_k)_{k \geq 0}$ in $\pi^{-1}(K)$ such that

$$\frac{p_1 \circ f^{n_k}(z_k) - p_1(z_k)}{n_k} \geq \beta + \varepsilon.$$

We have the inequality

$$\int_{\mathbb{A}} \phi \, d\mu_k \geq \beta + \varepsilon$$

where

$$\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \delta_{F^i(\pi(z_{n_k}))}.$$

Any closure value μ of the sequence $(\mu_k)_{k \geq 0}$ belongs to $M(K)$ and satisfies the inequality

$$\int_{\mathbb{A}} \phi \, d\mu \geq \beta + \varepsilon.$$

We have a contradiction. □

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We obtain the following corollary which applies for example when K is an invariant essential simple closed curve.

COROLLARY 3.2. *If $\rho(K) = \{\alpha\}$ the sequence*

$$\frac{p_1 \circ f^n - p_1}{n}$$

converges uniformly on $\pi^{-1}(K)$ to the constant function α.

In the same way we can define the rotation number of an invariant probability measure when F is a homeomorphism of the closed annulus $\mathbb{T}^1 \times [0, 1]$ homotopic to the identity, for a lift f to $\mathbb{R} \times [0, 1]$. The rotation set of f is the set of the rotation numbers of all the invariant measures. It is a segment which contains the rotation number of the two boundary curves of the annulus. We will use the following result proved in [**F1**].

THEOREM 3.3. *Let F be an area preserving homeomorphism of the closed annulus $\mathbb{T}^1 \times [0, 1]$ homotopic to the identity and f a lift of F to $\mathbb{R} \times [0, 1]$. For every rational number ρ contained in the rotation set of f, there exists a periodic orbit of type (p, q), where $\rho = p/q$ and p and q are relatively prime.*

4. *Indices*

Let F be an orientation preserving homeomorphism of a topological sphere Σ and D a closed disk of Σ (i.e. a set homeomorphic to $\overline{\mathbb{D}}$). If there is no fixed point of F in ∂D, we can define the index $i(F, D)$ of D (see [**D**] for example). This index is equal to the sum of the Lefschetz indices $i(F, z)$ of the fixed points z contained in D if there are a finite number of fixed points.

If we can find a finite family of disjoint closed disks $(D_k)_{1 \leq k \leq n}$ such that every fixed point of F is contained in the interior of one these disks, we have the Lefschetz formula:

$$\sum_{k=1}^{n} i(F, D_k) = 2.$$

If U is a non-trivial simply connected domain of Σ with a non-zero rotation number, we can find a closed disk $D \subset U$ whose interior contains all the fixed points in U and such that $i(F, D) = 1$. We deduce that U contains at least one fixed point.

5. *Invariant essential continua and periodic orbits*

We begin with a useful extension lemma.

LEMMA 5.1. *Let F be a homeomorphism of $\mathbb{A}$ homotopic to the identity and f a lift of F to $\mathbb{R}^2$. We suppose that K is an invariant essential continuum and that $\tilde{\rho}_K^+ > 0$. Then there exists a homeomorphism F' of $\mathbb{A}$ homotopic to the identity and an essential simple closed curve $C^+ \subset U_K^+$ such that:*

(i) *the map F' coincides with F on the complement of a closed set contained in U_K^+;*

(ii) *the curve C^+ is invariant by F' and its rotation number is > 0 (for the lift f' which coincides with f on $\pi^{-1}(K)$);*

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(iii) *the invariant set* $U_K^+ \cap \overline{U_{C^+}^-}$ *of* F' *contains no fixed points of rotation number* 0*;*
(iv) *the map* F' *is area preserving if* F *is.*

Proof. We can find a complex structure on $\widehat{\mathbb{A}}$ which defines the same differential structure on $\mathbb{A}$. Let us consider a conformal map h between the simply connected domain $\widehat{U}_K^+$ and $\mathbb{D}$ and the maps

$$\begin{aligned} r : U_K^+ &\to]0, 1[\\ z &\mapsto |h(z)| \end{aligned}$$

and

$$\begin{aligned} \theta : U_K^+ &\to S^1 \\ z &\mapsto \frac{h(z)}{|h(z)|}. \end{aligned}$$

Also, we consider a lift

$$\tilde{\theta} : \pi^{-1}(U_K^+) \to \mathbb{R}$$

of θ to the universal covering space and the map

$$\tilde{r} : \pi^{-1}(U_K^+) \to]0, 1[$$

given by $\tilde{r}(z) = r(\pi(z))$.

For every $t \in]0, 1[$ we consider the essential simple closed curve $C_t = r^{-1}(\{t\})$ of U_K^+. Let us fix $t_0 < 1/2$ such that the curve $C^+ = C_{t_0}$ is above the image of $C_{1/2}$. We can find a homeomorphism between the closed annulus A_0 delimited by $C_{1/2}$ and C_{t_0} and the closed annulus A_1 delimited by $F(C_{1/2})$ and C_{t_0}, which is equal to F on $C_{1/2}$ and to the identity on C_{t_0}. We denote by F_1 its extension to the infinite annulus which is equal to F below $C_{1/2}$ and to the identity above C_{t_0}. If F is area preserving the fact that K is invariant tells us that the two annuli A_0 and A_1 have the same area and we can choose the homeomorphism F_1 to be area preserving.

We write f_1 for the lift of F_1 which coincides with f on $\pi^{-1}(K)$. Using the condition $\tilde{\rho}_K^+ > 0$, we can find $\varepsilon \in]0, 1/4[$ such that

$$\tilde{r}(z) > 1 - 2\varepsilon \quad \Rightarrow \quad \tilde{\theta} \circ f_1(z) > \tilde{\theta}(z).$$

Using an argument of compactness, we can find $M > 0$ such that

$$\tilde{r}(z) \geq t_0 \quad \Rightarrow \quad \tilde{\theta} \circ f_1(z) \leq \tilde{\theta}(z) + M.$$

Consider a C^∞ decreasing map

$$\lambda :]0, 1[\to [0, +\infty[$$

equal to zero on $[1 - \varepsilon, 1[$ and let us define the map F_2 equal outside of U_K^+ to the identity and on U_K^+ to the time one map of the Hamiltonian flow of $\lambda \circ r$ for the usual symplectic

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structure on $\mathbb{A}$. That means the flow defined by the equations

$$x' = \lambda'(r(x, y))\frac{\partial r}{\partial y}(x, y)$$
$$y' = -\lambda'(r(x, y))\frac{\partial r}{\partial y}(x, y).$$

The map $F' = F_1 \circ F_2$ satisfies the conditions (i) and (iv) of the lemma. It also fixes the curve C^+. We can choose the function λ such that its satisfies the condition (iii) and the second part of the condition (ii).

Indeed, if f_2 is the lift of F_2 equal to the identity on $\pi^{-1}(K)$, we have

$$\tilde{\theta} \circ f_2(z) \geq \tilde{\theta}(z) \quad \text{and} \quad \tilde{r} \circ f_2(z) = \tilde{r}(z)$$

for every $z \in \pi^{-1}(U_K^+)$, so we have the relation

$$\tilde{r}(z) > 1 - 2\varepsilon \quad \Rightarrow \quad \tilde{\theta} \circ f'(z) > \tilde{\theta}(z).$$

If we choose the function $|\lambda'|$ sufficiently large on $[r_0, 1 - 2\varepsilon]$ we will get

$$\tilde{r}(z) \in [r_0, 1 - 2\varepsilon] \quad \Rightarrow \quad \tilde{\theta} \circ f'(z) > \tilde{\theta}(z). \qquad \square$$

We have a similar result in the cases where $\tilde{\rho}_K^+ < 0$, $\tilde{\rho}_K^- > 0$ and $\tilde{\rho}_K^- < 0$.

The main result of this section is the following.

PROPOSITION 5.2. *Let F be an area preserving homeomorphism of $\mathbb{A}$ homotopic to the identity and f a lift of F to $\mathbb{R}^2$. If K is an invariant essential continuum which does not contain any periodic orbit of F, then:*

(i) *the set K is a filled essential continuum;*

(ii) *there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $\rho(K) = \{\alpha\}$;*

(iii) $\tilde{\rho}_K^+ = \tilde{\rho}_K^- = \alpha$.

Proof. First, we prove assertion (i) using the results of §§2 and 4. Suppose that $\mathbb{A} \setminus K$ has a bounded component U_0. It is periodic and we can suppose it fixed by taking an iterate of F instead of F. Consider the extension $\widehat{F}$ of F to the end compactification $\widehat{\mathbb{A}}$.

If U is any invariant bounded component of $\mathbb{A} \setminus K$ then it is topologically an open disk and must contain a fixed point since F is area preserving. The fact that K does not contain any fixed points implies that the set of invariant bounded components of $\mathbb{A} \setminus K$ is finite, since otherwise there would be a sequence of fixed points with limit in K. Moreover we can find a finite family $(D_k)_{1 \leq k \leq n}$ of closed disks, one in each fixed component of $\widehat{\mathbb{A}} \setminus K$, such that every fixed point of $\widehat{F}$ is contained in the interior of one these disks and such that $i(\widehat{F}, D_k) = 1$ for every k. There are at least three components: $\widehat{U}_K^+$, $\widehat{U}_K^-$ and U_0. This contradicts the Lefschetz formula.

Next, we prove assertions (ii) and (iii). Using Proposition 2.1, we know that the rotation numbers $\tilde{\rho}_K^+$ and $\tilde{\rho}_K^-$ are irrational. Let us consider a rational number p/q. Applying the extension lemma to F^q and to the lift $f^q \circ T^{-p}$, one can construct an area preserving homeomorphism F' of $\mathbb{A}$ which coincides with F^q on a neighborhood of K and which fixes two essential simple curves $C^+ \subset U_K^+$ and $C^- \subset U_K^-$ such that:

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- there is no fixed point of rotation number 0 in $U_K^+ \cap \overline{U_{C^+}^-}$;
- there is no fixed point of rotation number 0 in $U_K^- \cap \overline{U_{C^-}^+}$;
- the rotation numbers of C^+ and $q\tilde{\rho}_K^+ - p$ have the same sign;
- the rotation numbers of C^- and $q\tilde{\rho}_K^- - p$ have the same sign.

Using the fact that K is filled and does not contain any periodic orbit we deduce that the map F' has no fixed point of rotation number 0 in the closed annulus defined by the curves C^+ and C^-. From Theorem 3.3 we deduce that:

- the rotation set $\rho(K)$ does not contain p/q;
- the numbers $\tilde{\rho}_K^-$ and $\tilde{\rho}_K^+$ are not separated by p/q.

The conclusion follows immediately. □

Definition 5.3. Such a set will be called an irrational invariant essential continuum and $\alpha = \rho(K)$ its rotation number.

In the same way, we can prove the following result.

PROPOSITION 5.4. *Let F be an area preserving homeomorphism of $\mathbb{A}$ homotopic to the identity, f a lift of F to $\mathbb{R}^2$ and K an invariant essential continuum. For every rational number ρ contained in $\rho(K)$, there exists a periodic orbit of type (p, q) contained in K, where $\rho = p/q$ and p and q are relatively prime.*

Proof. We give a proof by contradicting the assumption that F has no periodic points of type (p, q) in K. If this is assumed then F has no periodic points of type (p, q) in some neighborhood V_0 of K since the periodic points of type (p, q) form a closed set. It then follows from Proposition 2.1 that $\tilde{\rho}_K^+ \neq p/q$ and $\tilde{\rho}_K^- \neq p/q$.

Using Lemma 5.1 we construct an area preserving map F', which coincides with F^q on some neighborhood $V \subset V_0$ of K, which leaves invariant two essential simple closed curves $C^+ \subset U_K^+$ and $C^- \subset U_K^-$, and which has no fixed points of rotation 0 in $\pi^{-1}(V)$ for the lift equal to $f^q \circ T^{-p}$ on $\pi^{-1}(K)$. By Theorem 3.3, there exists a fixed point of F' of rotation number 0 in the annulus B delimited by C^+ and C^-. Since it is not in K and not in V this fixed point must lie in a bounded invariant component of $B \setminus K$ and hence a bounded invariant component of $A \setminus K$.

Let $p : B_0 \to B$ be a finite covering map of degree greater than the absolute value of rotation number of any periodic point of F'. Then there is a lift $G : B_0 \to B_0$ of F' such that $z \in B_0$ is a fixed point of G if and only if $p(z)$ is a fixed point of F' of rotation number zero. Let $K_0 = p^{-1}(K)$ so that G has no fixed points in K_0. Since any component of $B \setminus K$ containing a fixed point of F' of rotation number zero is an invariant open topological disk it follows that any component of $B_0 \setminus K_0$ containing a fixed point of G is also an invariant open topological disk.

As in the proof of Proposition 5.2, the fact that K_0 and the boundary of B_0 do not contain any fixed points of G implies that the set of bounded components of $B_0 \setminus K_0$ containing fixed points is finite. Moreover we can find a finite family $(D_k)_{1 \leq k \leq n}$ of closed disks, one in each component of $B_0 \setminus K_0$ containing such a fixed point, such that every fixed point of G is contained in the interior of one these disks and such that $i(G, D_k) = 1$ for every k. There is at least one such disk since there was at least one fixed point of F' of rotation number zero. This again contradicts the Lefschetz formula for $G : B_0 \to B_0$. □

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6. *Generic properties of diffeomorphisms of the sphere*

We consider a smooth surface Σ diffeomorphic to S^2 and a volume form ω on it. For every integer $r \geq 1$ we write $\mathrm{Diff}^r_\omega(\Sigma)$ for the set of C^r diffeomorphisms which preserves ω. We put on that set the natural C^r-topology. If z is a periodic point of period q of $F \in \mathrm{Diff}^r_\omega(\Sigma)$, we have three possibilities:

- the eigenvalues of $DF^q(z)$ are both equal to 1 or -1: the point z is *degenerate*;
- the eigenvalues of $DF(z)$ are two different conjugate complex numbers of modulus 1: the point z is *elliptic*;
- the eigenvalues λ and μ of $DF(z)$ are real and satisfy $|\lambda| < 1 < |\mu|$: the point z is *hyperbolic* or a *saddle point*.

The *stable (respectively unstable) branches* of a hyperbolic periodic point z are the two components of $W^s(z) \setminus \{z\}$ (respectively $W^u(z) \setminus \{z\}$) for the topology induced by a parameterization of $W^s(z)$ (respectively $W^u(z)$). Let us recall that a *saddle connection* is a branch Γ which is a stable branch of a hyperbolic periodic point z and an unstable branch of a hyperbolic periodic point z' (perhaps equal to z).

LEMMA 6.1. *Suppose F is an area preserving diffeomorphism of a connected open subset U of S^2 and $K \subset U$ is a compact connected F-invariant set. If Γ is a branch of a hyperbolic periodic point of F then either $\Gamma \subset K$ or Γ is entirely contained in a single component of $U \setminus K$.*

Proof. Suppose Γ has non-empty intersection with both K and $U \setminus K$. Let U_0 be a component of $U \setminus K$ which contains a point of Γ. Then U_0 is a periodic domain since every component of the complement of $U \setminus K$ is. There is an open arc $\alpha \subset \Gamma \cap U_0$ whose endpoints are in K.

We note that if $i > 0$, then $F^i(\alpha)$ and α are disjoint since $F^i(\alpha)$ cannot be a subset of α (as α is in a branch) and $F^i(\alpha)$ cannot contain an endpoint of α.

Let V be the component of $S^2 \setminus K$ which contains U_0. Since V is simply connected α separates it into two non-empty simply connected domains V_1 and V_2 each of which has non-empty intersection with U_0. It is not difficult to show that this implies α separates U_0 into two connected open sets $U_1 = U_0 \cap V_1$ and $U_2 = U_0 \cap V_2$.

Choose $k > 0$ so that $F^k(U_0) = U_0$. The map F is area preserving, so $U_1 \times U_2$ is a non-wandering open set for $F^k \times F^k$. We deduce that there exists an integer $n \geq 1$ such that $F^{nk}(U_1) \cap U_1 \neq \emptyset$ and $F^{nk}(U_2) \cap U_2 \neq \emptyset$. Moreover the closure in U_0 of $F^{nk}(U_1)$ is not contained in U_1, it meets the boundary α. Similarly α meets the closure of $F^{nk}(U_2)$ in U_0. We deduce that α meets its image by F^{nk}, which is impossible. We conclude it is not possible for Γ to have non-empty intersection with both K and its complement. If Γ lies in the complement of K it must be in a single component because it is connected. □

We will say that a periodic point z of period q is *Moser stable* if it admits a fundamental system of neighborhoods which are closed disks D such that $F_{\partial D}$ is minimal.

Next, we present the following result of Mather [**M1**].

THEOREM 6.2. (Mather) *Let $F \in \mathrm{Diff}^r_\omega(\Sigma)$ such that:*

- *there is no degenerate periodic point;*
- *every elliptic periodic point is Moser stable; and*
- *there are no saddle connections.*

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Then, we know that:

(i) *the rotation number of any non-trivial simply connected periodic domain is irrational;*

(ii) *the branches of a hyperbolic periodic point have the same closure.*

Proof. First, we will prove assertion (i) by contradiction. If assertion (i) is false then by replacing F with an iterate we can assume that there is an invariant simply connected domain U with rotation number zero. This implies that the extension of F to the prime end compactification of U has a fixed point z on the boundary.

From the theory of prime ends and the proof of Proposition 2.1 we know that there is a sequence of open arcs $(\gamma_n)_{n=1}^{\infty}$ in U, each of which is the interior of a closed arc with endpoints in ∂U, which define the prime end z. Moreover, we can assume the sequence of diameters of (γ_n) tends to zero as n tends to infinity, so by choosing a subsequence we can assume the closures of the arcs converge to a point p in ∂U. We observed (see Remark 2.2) that $F(\gamma_n) \cap \gamma_n \neq \emptyset$ for each n and hence p is a fixed point of F.

We note first that this fixed point p cannot be a Moser stable point. This is because a Moser stable point is in the interior of an arbitrarily small F invariant disk on the boundary of which F is minimal. Thus the boundary of such a disk must intersect the complement of U and hence, by minimality, the boundary is disjoint from U. This would imply that the entire disk is disjoint from U which is impossible.

It follows that the point p must be a hyperbolic fixed point. By the Hartman–Grobman Theorem there is a neighborhood V of p on which there are topological co-ordinates (x, y) with p at the origin and in which $F(x, y) = (2x, y/2)$. We will use these local coordinates on V and refer to the four *local branches* at p by which we mean the intersection of V with the positive and negative x and y axes.

The contradiction we wish to derive will follow from the following lemma.

LEMMA 6.3. *Replacing the sequence $\{\gamma_n\}$ with a subsequence, we may conclude that there is a neighborhood $W \subset V$ of p such that one component of $U \cap W$ contains the arcs γ_n for n sufficiently large, and this component consists of one of the following:*

(a) *the intersection of W with a single open quadrant in the (x, y) co-ordinates; or*

(b) *the intersection of W with two adjacent open quadrants and the local branch between them; or*

(c) *the intersection of W with three open quadrants and the two local branches between them; or*

(d) *the complement in W of p and a single local branch.*

Proof. First, we observe that if γ is a local branch of p in V then by Lemma 6.1 γ must be entirely contained in U or disjoint from U. If it is contained in U there is an $x \in \gamma$ such that a neighborhood of the interval $[x, F(x)]$ is in U. The collection of all iterates of this neighborhood contain the intersection of a smaller neighborhood W_0 with the two open quadrants in (x, y) co-ordinates of which γ is part of the boundary. Thus if one point of a local branch is in U then, perhaps in a smaller neighborhood, the entire branch and the open quadrants on either side of it are in U. We will make repeated use of this property.

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The first use of this fact is to observe that it is not possible for all four local branches at p to be in U as this would imply that there is a neighborhood W of p such that $W \setminus \{p\} \subset U$. This would mean that $p \in U$ which is not the case.

Again, we use the fact that if a local branch intersects U then (perhaps in a smaller neighborhood) the entire branch and the open quadrants on either side of it are in U. This implies that either one of properties (b)–(d) above holds or there is a single quadrant such that the interiors of infinitely many of the arcs γ_n are in that quadrant, but the two branches which bound this quadrant are not in U. We complete the proof of the lemma by showing that this last alternative implies that (a) holds.

We can suppose, without loss of generality, that it is the open first quadrant

$$Q = \{(x, y) \mid x > 0,\ y > 0\}$$

which contains the interiors of infinitely many of the γ_n and the positive x and y axes are in the complement of U. By choosing a subsequence we may assume every γ_n is in the first quadrant. We need only show that there is a neighborhood $W \subset V$ of p such that $W \cap Q$ is entirely in U.

To do this, for each small $\varepsilon > 0$ we consider the arc Γ_ε made up of the two line segments $\{(x, y) = (t, \varepsilon) \mid 0 \le t \le \varepsilon\}$ and $\{(x, y) = (\varepsilon, t) \mid 0 \le t \le \varepsilon\}$. These two segments form two sides of a rectangle R_ε the other sides of which lie in the x and y axes. This rectangle contains infinitely many of the arcs γ_n and in particular points of U. Also if ε is sufficiently small we may assume this rectangle is disjoint from γ_1 and in particular cannot contain all of U. It follows that the arc Γ_ε has non-empty intersection with U.

Next, we want to show there is an open sub-arc β of Γ_ε with the property that it is in U but its endpoints are in the complement of U, and that $F(\beta) \cap \beta \neq \emptyset$. To see this we note that Γ_ε separates γ_1 and γ_N for a large N. Hence some component β of $\Gamma_\varepsilon \cap U$ must separate γ_1 and γ_N in U. Clearly β is an open sub-arc of Γ_ε with endpoints in the complement of U.

Let V_1 and V_2 be the components of $U \setminus \beta$ which contain γ_1 and γ_N respectively. The fact that $F(\gamma_1) \cap \gamma_1 \neq \emptyset$ and $F(\gamma_N) \cap \gamma_N \neq \emptyset$ implies $F(V_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. Since F is area preserving $F(V_i)$ cannot be a proper subset of V_i. We conclude that $F(\beta) \cap \beta \neq \emptyset$.

Since $\beta \subset \Gamma_\varepsilon$ and $F(\Gamma_\varepsilon) \cap \Gamma_\varepsilon$ consists of the single point $(x, y) = (\varepsilon, \varepsilon/2)$ we conclude that both $(\varepsilon, \varepsilon/2)$ and its inverse image $(\varepsilon/2, \varepsilon)$ are in β. Thus the arc J_ε made of the two line segments $\{(x, y) = (t, \varepsilon) \mid \varepsilon/2 \le t \le \varepsilon\}$ and $\{(x, y) = (\varepsilon, t) \mid \varepsilon/2 \le t \le \varepsilon\}$ is entirely in U. Similarly for any $\delta < \varepsilon$ we have $J_\delta \subset U$ and indeed for any $n \in \mathbb{Z}$, $F^n(J_\delta) \subset U$. This easily implies that $R_\varepsilon \cap Q \subset U$, which means that property (a) holds. □

Next, we observe that this lemma and the assumption that (i) is false do lead to a contradiction. This is because from the lemma we see that there is a neighborhood of the prime end z corresponding to p in the prime ends of U which consists of accessible prime ends corresponding to points in either stable or unstable branches of z. It also follows that there are finitely many fixed prime ends and they all have this property. We can conclude that any non-fixed prime end e which is between the fixed prime ends z_1 and z_2 must correspond to a point on a branch of p_1 and a branch of p_2 where p_i is the fixed point in

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∂U corresponding to the prime end z_i. Since this is true for the entire interval of prime ends between z_1 and z_2 there is a saddle connection between p_1 and p_2 which contradicts our hypothesis. We conclude that (i) of the theorem must hold.

Property (ii) of the theorem is a consequence of (i) and Lemma 6.1. As before, by taking an iterate we may assume the point in question, p, is fixed. The closure of one branch of this point is a compact connected invariant set K. From Lemma 6.1 we know that another branch of p is either a subset of K or disjoint from it. It cannot be disjoint since if U denotes the component of the complement of K which contains γ then U would be an invariant simply connected domain and the fixed point p would be an accessible point of ∂U. This would mean that the rotation number of U is rational and hence property (i) would be contradicted.

We conclude $\gamma \subset K$. Since we have shown any branch of p is a subset of the closure of any other, property (ii) follows. □

We remark that for $r \geq 16$ the set of diffeomorphisms satisfying the hypotheses of Theorem 6.2 is generic in $\mathrm{Diff}^r_\omega(\Sigma)$. The high smoothness appears to be required to have all elliptic points be Moser stable. The best published result of which we are aware, [**Do**], requires $r \geq 16$ for this.

Now, we recall a result of Robinson [**R2**] and Pixton [**P**]. It was first proved by Robinson in the case of fixed points and generalized by Pixton to periodic points (see also Oliveira [**O**] or Arnaud [**A**]).

THEOREM 6.4. *For every $r \geq 1$, there exists a generic set G_r in $\mathrm{Diff}^r_\omega(\Sigma)$ such that for every F in G_r the following properties are satisfied:*

(i) *if z is a hyperbolic periodic point, any stable branch Γ of z has non-empty transversal intersection with any unstable branch Γ' of z. Moreover $\overline{\Gamma} = \overline{\Gamma'}$.*

(ii) *if z and z' are two distinct hyperbolic periodic points such that z' belongs to the closure of a branch of z, then any stable (respectively unstable) branch Γ of z meets transversely any unstable (respectively stable) branch Γ' of z'. Moreover $\overline{\Gamma} = \overline{\Gamma'}$.*

We make two comments about these theorems.

- Every periodic connected and compact set which contains a hyperbolic periodic point, and which is not a single point, contains all the branches of that point.
- Any hyperbolic periodic point has a fundamental system of neighborhoods which are closed disks whose boundary is contained in $W^s(z) \cup W^u(z)$. This property and the classical λ-lemma (see Theorem 11.1 of Robinson [**R1**]) implies the condition (ii) of Pixton's theorem.

7. *Essential and inessential hyperbolic periodic points*

In this section we consider an area preserving C^1-diffeomorphism of S^2 such that:

(i) there are no degenerate periodic points;

(ii) every elliptic periodic point is Moser stable; and

(iii) the intersection of any branches of hyperbolic periodic points are transverse and any stable and any unstable branch of the same periodic point have non-empty intersection.

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If $\mathbb{A}_0$ is a connected open subset of S^2 whose complement has exactly two components, neither of which is a single point, we will call it an *annular domain.* It is homeomorphic to an open annulus. If $\mathbb{A}_0$ is an F-invariant annular domain then we may speak of an *essential* continuum $K \subset \mathbb{A}_0$ meaning one which separates the two components of the complement of $\mathbb{A}_0$.

Mather's Theorem and Pixton's Theorem immediately imply the following corollaries.

PROPOSITION 7.1. *For a diffeomorphism F of S^2 satisfying (i)–(iii) above we have the following properties:*

(1) *every periodic simply connected domain has an irrational prime end rotation number;*
(2) *every periodic annular domain has both prime end rotation numbers irrational;*
(3) *if $\mathbb{A}_0$ is an invariant annular domain and $K \subset \mathbb{A}_0$ is an invariant essential continuum, the rotation numbers ρ_K^+ and ρ_K^- are irrational;*
(4) *every stable branch Γ^s and every unstable branch Γ^u of a hyperbolic periodic point have a non-empty transversal intersection and we have $\overline{\Gamma^s} = \overline{\Gamma^u}$;*
(5) *if z is a hyperbolic periodic point and Γ^s and Γ^u are a stable and an unstable branch of z then there is a fundamental system of neighborhoods of z which are closed disks whose boundary is contained in $\Gamma^s \cup \Gamma^u$;*
(6) *a periodic compact connected set, which is not a single point, and which contains a hyperbolic periodic point, contains the branches of that point;*
(7) *if the closure of a branch of a hyperbolic periodic point z contains a hyperbolic periodic point z', then every stable (respectively unstable) branch Γ of z meets transversely any unstable (respectively stable) branch Γ' of z' and we have $\overline{\Gamma} = \overline{\Gamma'}$.*

Let $F : \mathbb{A}_0 \to \mathbb{A}_0$ be an area preserving diffeomorphism of an annular domain which extends continuously to $\operatorname{cl}\mathbb{A}_0 \subset S^2$. We consider a hyperbolic periodic point z for F. We will assume the generic property that each stable branch of z has non-empty intersection with each unstable branch and all intersections are transverse.

Definition 7.2. If there is an essential closed curve in $\mathbb{A}_0$ consisting of a finite number of segments of branches of z we will say that z is essential, otherwise we say that z is inessential.

It is immediate that a hyperbolic periodic orbit O consists of points which are all essential or all inessential.

PROPOSITION 7.3. *Suppose $z \in \mathbb{A}_0$ is a hyperbolic fixed point of F. Let $U(z)$ be the union of all closed topological disks in $\mathbb{A}_0$ whose boundaries consist of finitely many segments of the branches of z. Then:*

(i) *$U(z)$ is an open, connected, and F-invariant set which contains z and its branches;*
(ii) *if z is inessential then $U(z)$ is simply connected;*
(iii) *if z is essential then $U(z)$ is annular and essential in $\mathbb{A}_0$;*
(iv) *if $V \subset \mathbb{A}_0$ is any other open connected F-invariant set which is either simply connected or annular and essential then either $U(z) \subset V$, $V \subset U(z)$, or $U(z) \cap V = \emptyset$.*

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Proof. Let z be a hyperbolic fixed point of F and let $U(z)$ be the union of all closed topological disks in $\mathbb{A}_0$ whose boundaries consist of finitely many segments of the branches of z. Clearly $F(U(z)) = U(z)$. Property 5 of Proposition 7.1 implies z is in the interior of $U(z)$. Since any closed segment on a branch of z can be mapped by a power of F into a small neighborhood of z we see that these segments are in the interior of one of the closed disks whose union forms $U(z)$. It follows that any of these closed disks is contained in the union of the interiors of finitely many other such disks, so $U(z)$ is also equal to the union of the interiors of these disks and is open. This proves (i).

Next, we show (ii). This is equivalent to showing that z inessential implies the complement of $U(z)$ in S^2 is connected. If this is not the case then there is a simple closed curve α in $U(z)$ which separates two components, call them K_1 and K_2, of the complement of $U(z)$. Since $U(z)$ is the union of the interiors of closed disks whose boundaries are branch segments, the curve α is covered by the interiors of finitely many such disks. Let V denote the union of these open disks and let β be the frontier of the component of $S^2 \setminus V$ containing K_1. Then β is a simple closed curve which separates K_1 and K_2 and consists of finitely many segments of branches of z. If β were essential in $\mathbb{A}_0$ then z would be essential. Hence we know β bounds a disk in $\mathbb{A}_0$ which must contain one of K_1 or K_2. This would imply that one of K_1 or K_2 is in $U(z)$ which is a contradiction. Hence we have shown that z inessential implies that $U(z)$ is simply connected.

If z is essential then a similar argument shows that the complement of $U(z)$ has exactly two components so (iii) follows. More precisely, the fact that z is essential implies $U(z)$ is not simply connected so $S^2 \setminus U(z)$ has at least two components and in particular there are distinct components K_1 and K_2 each of which contains one of the components of $S^2 \setminus \mathbb{A}_0$. If (iii) does not hold then the complement of $S^2 \setminus U(z)$ has at least three connected components: K_1 and K_2 and one more K_3 which is a subset of $\mathbb{A}_0$. There is a simple closed curve α in $U(z)$ which separates K_3 from $K_1 \cup K_2$ and hence from $S^2 \setminus \mathbb{A}_0$. As above we can find β_0 a simple closed curve which separates K_3 and $S^2 \setminus \mathbb{A}_0$ and which consists of finitely many segments of branches of z. Since β_0 does not separate the two components of $S^2 \setminus \mathbb{A}_0$ it bounds a closed disk in $\mathbb{A}_0$. The interior of this disk contains K_3 but is also in $U(z)$, a contradiction. We conclude that $S^2 \setminus U(z)$ has only two components.

Finally, we show (iv). If $U(z)$ does not intersect both V and the complement of V then either $U(z) \subset V$ or $U(z) \cap V = \emptyset$. But if $U(z)$ does intersect both V and its complement then either V is entirely contained in the interior of a disk whose boundary consists of branch segments (in which case $V \subset U(z)$) or V contains part of a branch of z. If V contains a point of a branch of z then it contains the entire branch by Lemma 6.1 and hence all branches of z. Thus if D is any closed disk in $\mathbb{A}_0$ whose boundary consists of branch segments then the boundary of D is in V and inessential in $\mathbb{A}_0$. This boundary cannot be essential in V as V is either simply connected or annular and essential in $\mathbb{A}_0$. Hence the boundary of D is contractible in V. Since $D \subset \mathbb{A}_0$ we conclude that $D \subset V$. Since this is true for any D we have shown that $U(z) \subset V$. □

PROPOSITION 7.4. *Suppose $z \in \mathbb{A}_0$ is an essential hyperbolic fixed point of F. Then the union of any stable branch and any unstable branch of z contains an essential simple closed curve and a point in each component of the complement of this curve.*

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Proof. From the definition of essential hyperbolic fixed point we know that there is an essential curve β in $\mathbb{A}_0$ consisting of segments of branches of z which intersect transversely. The fact that any stable branch of the fixed point has non-empty transverse intersection with any unstable branch allows us to deduce that this curve can be chosen to lie in the union of any pair of branches, one stable and one unstable. Let α_0 be the frontier of one of the components of the complement of β which intersects one component of $S^2 \setminus \mathbb{A}_0$ and let α be the frontier of the component of the complement of α_0 which contains the other component of $S^2 \setminus \mathbb{A}_0$. Then α consists of segments of branches of z which intersect transversely and is simple. Since it separates the two components of $S^2 \setminus \mathbb{A}_0$ it is essential. Since α is in the interior of $U(z)$ the branches of z contain points from each component of the complement of α. □

8. *Regions of instability*

We begin with the following result which improves Proposition 5.4 in the generic case.

PROPOSITION 8.1. *Suppose F is an area preserving diffeomorphism of S^2 with an invariant annular domain $\mathbb{A}_0$ and P is the set of periodic points of F with period q. Assume:*

(i) *the set P is empty or finite and all points of P are non-degenerate;*

(ii) *every elliptic periodic point in P is Moser stable, and the invariant, F minimal Jordan curves bounding neighborhoods of elliptic periodic points have rotation numbers which are not constant in any neighborhood of the elliptic periodic point; and*

(iii) *the intersection of any branches of hyperbolic periodic points in P are transverse and any two branches of the same periodic point have non-empty intersection.*

Let $K \subset \mathbb{A}_0$ be an invariant essential continuum. If p is relatively prime to q and $\rho = p/q$ is in $\rho(K)$, then P is not empty and there exists an essential hyperbolic periodic orbit in $K \cap P$ with rotation number ρ. The analogous result holds if $A = \mathbb{T}^1 \times [0,1]$ and $F : A \to A$ has no periodic points on the boundary of A and satisfies (i)–(iii).

Proof. Taking F^q instead of F and $f^q \circ T^{-p}$ instead of f we can suppose that $\rho = 0$. Replacing $F : \mathbb{A}_0 \to \mathbb{A}_0$ by a lift to a finite cover we may assume that there are no fixed points except those which project to fixed points with rotation number 0. Note that a continuum is essential if and only if a lift to a finite cover is essential. Hence it suffices to prove the result for this lift since it has an essential continuum which projects to K. By abuse of notation we refer to the lift as $F : \mathbb{A}_0 \to \mathbb{A}_0$.

Consider $\widehat{F}$ the extension of F to $\widehat{\mathbb{A}}$ the end compactification of $\mathbb{A}_0$, i.e. the compactification obtained by collapsing each of the two components of $S^2 \setminus \mathbb{A}_0$ to a point, obtaining a space $\widehat{\mathbb{A}}$ homeomorphic to S^2. We will denote the two points added in this compactification by z^+ and z^-. If U^+ and U^- are the components of $\widehat{\mathbb{A}} \setminus K$ containing z^+ and z^- respectively then they are simply connected $\widehat{F}$-invariant domains. We can apply Theorem 6.2 to show these domains have irrational rotation number despite the fact that at the points z^+ and z^- the homeomorphism $\widehat{F}$ is not smooth and neither is the invariant measure. The proof of part (i) of Theorem 6.2 is still valid.

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The theory of prime ends allows us to conclude that there is a closed disk $D^+ \subset U^+$ which contains all the fixed points of $\widehat{F}$ which are in U^+ and such that $i(\widehat{F}, D^+) = 1$. We can find $D^- \subset U^-$ with the analogous property.

By Proposition 7.3 to every inessential hyperbolic fixed point $z \in \mathbb{A}_0$, we can associate an invariant simply connected domain $U(z)$. The same proposition implies there are a finite set of such domains which are disjoint and which contain all inessential fixed points of F (all of which, by the remarks above, can be assumed to have rotation number zero). Moreover they are either disjoint from the annular regions $U^+ \setminus \{z^+\}$ and $U^- \setminus \{z^-\}$ or contained in them. In each such domain $U(z)$ which is disjoint from $U^+ \cup U^-$ we may choose a closed disk $D \subset U(z)$ which contains all the fixed points in $U(z)$ and such that $i(F, D) = 1$. We write $\{D_i\}_{1 \le i \le k}$, $k \ge 0$, for the family of disks obtained in this way.

Next, we consider the finite family $\{z_j\}_{1 \le j \le l}$, $l \ge 0$, of fixed points of F which are outside all the disks $\{D_i\}$. By the Lefschetz formula, we have

$$\sum_{j=1}^{l} i(F, z_j) + \sum_{i=1}^{k} i(F, D_i) + i(\widehat{F}, D^+) + i(\widehat{F}, D^-) = 2,$$

and we can write

$$\sum_{j=1}^{l} i(F, z_j) + k = 0.$$

Proposition 5.4 tells us that one of the integers k or l is ≥ 1. We know that $i(F, z_j) = \pm 1$ for every j and that $i(F, z_j) = 1$ if z_j is elliptic. We deduce that $l \ge 1$ and that there exists a fixed point z_j such that $i(F, z_j) \le 0$. This point has to be hyperbolic. It is essential because it is not contained in the disks D_i.

The analogous result for $A = \mathbb{T}^1 \times [0, 1]$ follows from similar but easier reasoning. □

COROLLARY 8.2. *Suppose F is an area preserving diffeomorphism of S^2 with an invariant annular domain $\mathbb{A}_0$ and satisfies the hypothesis of Proposition 8.1 for every q. Then every invariant essential continuum K in $\mathbb{A}_0$ which is contained in the frontier of a connected open set U is irrational.*

Proof. If K is not irrational (see Definition 5.3) it contains a periodic point z. By Proposition 8.1 we can suppose that it is an essential hyperbolic periodic point. The set K contains an essential simple closed curve C (in $W^s(z) \cup W^u(z)$), and a point in each component of the complement of C by Proposition 7.4. We deduce that the open set U meets the two components of $\mathbb{A} \setminus C$ and does not meet C. The contradiction comes from the connectedness of U. □

The results above motivate the following definition.

Definition 8.3. Suppose F is an area preserving diffeomorphism of an annular domain $\mathbb{A}_0$ and $P(q)$ is the set of periodic points of F with period q. Assume that for all q:

(i) the set $P(q)$ is empty or finite and all points of P are non-degenerate;

(ii) every elliptic periodic point in $P(q)$ is Moser stable, and the invariant, F minimal, Jordan curves bounding neighborhoods of an elliptic periodic point z have rotation numbers which are not constant in any neighborhood of z; and

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(iii) the intersection of any branches of hyperbolic periodic points in $P(q)$ are transverse and any two branches of the same periodic point have non-empty intersection.

Then F will be called *Moser generic*.

We remark that for $r \geq 16$ the set of Moser generic diffeomorphisms form a residual subset of $\mathrm{Diff}^r_\omega(\Sigma)$. As mentioned above, the high smoothness appears to be required to have all elliptic points be Moser stable and the best published result of which we are aware is [**Do**] which requires $r \geq 16$.

PROPOSITION 8.4. *Let F be a Moser generic area preserving diffeomorphism of S^2 with an invariant annular domain and let $\check{F} : \check{\mathbb{A}} \to \check{\mathbb{A}}$ be the extension to the prime end compactification of this domain. Then we have the following:*

(i) *the union $\mathcal{K}$ of the irrational invariant essential continua for $\check{F}$ is closed;*

(ii) *the set of the irrational invariant essential continua for $\check{F}$ is closed in the Hausdorff topology and the function $K \mapsto \rho(K)$ is continuous on that set;*

(iii) *the connected components of $\mathcal{K}$ are the maximal irrational invariant essential continua.*

Proof. First, we prove assertion (ii). Suppose that the sequence $(K_n)_{n\geq 0}$ of irrational invariant essential continua converges to K. This set is an essential invariant continuum. Let us suppose that it is not irrational. It contains an essential hyperbolic periodic point z, an essential simple closed curve $C \subset W^s(z) \cup W^u(z)$, a point above that curve and a point below. For n big enough, K_n also contains a point in each component of the complement of C. We deduce that K_n meets the curve and hence must contain the point z. We have a contradiction to the assumption that K_n is irrational.

Let us suppose that the sequence $(\alpha_n)_{n\geq 0}$ converges to α, where α_n is the rotation number of K_n. We can choose for any $n \geq 0$ an invariant measure $\mu_n \in M(K_n)$. Any limit point μ of the sequence $(\mu_n)_{n\geq 0}$ belongs to $M(K)$ and its rotation number is $\rho(K)$. It is also equal to α. We deduce easily the continuity of the map $K \mapsto \rho(K)$ from this argument.

Assertion (i) is straightforward. If $(z_n)_{n\geq 0}$ is a sequence in $\mathcal{K}$, we can choose for every $n \geq 0$ an irrational invariant essential continuum K_n which contains z_n. Any limit point of the sequence $(K_n)_{n\geq 0}$ is an irrational invariant essential continuum which contains z.

Assertion (iii) is also very simple. Any connected component of $\mathcal{K}$ is closed, invariant, essential and does not contain any periodic orbit. So it is irrational. Of course, it is maximal among the irrational invariant essential continua. □

The complement of $\mathcal{K}$ is an open set $\mathcal{U}$ which contains all the periodic orbits of F. Every connected component of $\mathcal{U}$ is an invariant essential open annulus.

Definition 8.5. Suppose F is a Moser generic area preserving diffeomorphism of S^2 with an invariant annular domain $\mathbb{A}_0$. Let $\mathcal{K}$ denote the union of the irrational invariant essential continua in $\mathbb{A}_0$. Any invariant component of the complement of $\mathcal{K}$ in $\mathbb{A}_0$ will be called a *region of instability*.

Let U be a region of instability and $\check{U} = U \sqcup C^+ \sqcup C^-$ the prime end compactification of U, where C^+ is the upper boundary circle and C^- the lower one. Let us write $\check{F}$ for

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the extension of $F_{|U}$ to $\check{U}$. We know that the rotation numbers defined on C^+ and on C^- are irrational. Likewise, in S^2 we will denote by $\partial^+ U$ and $\partial^- U$ the frontiers of the two components of the complement of U.

THEOREM 8.6. *Let F be a Moser generic area preserving diffeomorphism of S^2 with an invariant annular domain $\mathbb{A}_0$ and let U be a region of instability in $\mathbb{A}_0$. We have the following results.*

(i) *The rotation set of $\check{U}$ is a segment $\rho(U)$ of length > 0.*

(ii) *For every rational number $\rho \in \rho(U)$, there exists an essential hyperbolic periodic orbit of type (p, q) in U, where $\rho = p/q$ and p and q are relatively prime.*

(iii) *Let $X = K(z)$ denote the closure in $\overline{U}$ of a branch of an essential hyperbolic periodic point z. Then X is independent of the essential hyperbolic periodic point z and of the branch. It is a subset of any invariant essential continuum contained in $\overline{U}$ which is not a subset of $\partial^+ U$ or $\partial^- U$.*

(iv) *The set $\check{X} = K(z)$ defined to be the closure in $\check{U}$ of a branch of z is independent of the essential hyperbolic periodic point z and of the branch. It is a subset of any invariant essential continuum contained in $\check{U}$ which is distinct from C^+ and from C^-.*

Proof. First we prove assertions (i) and (ii). The set $\overline{U}$ is an invariant essential continuum which is not included in $\mathcal{K}$. So it contains a periodic point. This periodic point belongs to U because $\partial^+ U$ and $\partial^- U$ are irrational invariant essential continua. The rotation numbers of C^- and C^+ being irrational, the set $\rho(U)$ is a segment of length > 0. The extension of F to the prime end compactification of U satisfies the hypothesis given at the beginning of §7 and we can apply Proposition 8.1 to the essential continuum $\check{U}$ itself: for every rational number $\rho \in \rho(U)$ there exists in $\check{U}$ an essential hyperbolic periodic point of type (p, q), where $\rho = p/q$ and p and q are relatively prime. But this point belongs to U and not to the boundary curves.

Now, we prove assertion (iii). Let K be an essential invariant continuum contained in $\overline{U}$ and not a subset of $\partial^+ U \cup \partial^- U$. The boundary of U_K^+ is an irrational invariant essential continuum. It cannot meet U (since it is irrational) and must be contained in ∂U. More precisely it is contained in $\partial^+ U$ because it is connected and because K is not a subset of $\partial^- U$. In the same way the boundary of U_K^- is contained in $\partial^- U$. Hence we conclude that K meets the two boundary components of U. If z is an essential hyperbolic periodic point, we can find an essential curve contained in $W^s(z) \cup W^u(z)$. The set K meets that curve and must contain $K(z)$. In particular, if z' is another essential hyperbolic periodic point contained in U the set $K(z')$ contains $K(z)$.

The proof of assertion (iv) is exactly the same. □

We deduce immediately the following result about periodic points.

COROLLARY 8.7. *Let z and z' be two essential hyperbolic periodic points of F. The following conditions are equivalent:*

(i) *the points z and z' are in the same region of instability;*

(ii) *we have $K(z) = K(z')$;*

(iii) *we have $W^u(z) \cap W^s(z') \neq \emptyset$.*

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Next, we consider a result that generalizes a property of regions of instability of twist maps.

THEOREM 8.8. *Let U be a region of instability. For any neighborhood W of $\partial^+ U$ in $\overline{U}$ whose closure does not contain $\partial^- U$, the connected component of $\bigcap_{k\in\mathbb{Z}} F^{-k}(\overline{W})$ which contains $\partial^+ U$ is equal to $\partial^+ U$.*

Proof. The connected component K of $\bigcap_{k\in\mathbb{Z}} F^{-k}(\overline{W})$ which contains $\partial^+ U$ is an invariant essential continuum and the boundary of U_K^- is an irrational invariant essential continuum. This boundary is connected, does not meet U and does not contain $\partial^- U$: it is contained in $\partial^+ U$. We deduce that $K = \partial^+ U$. □

We obtain the following corollaries.

COROLLARY 8.9. *Let U be a region of instability. In any neighborhood W of $\partial^+ U$ in $\overline{U}$ there exists a periodic orbit. Also in W there is a homoclinic point of any essential periodic point in U.*

Proof. We write z^+ for the upper end of U and z^- for the lower end. Let us consider the extension $\widehat{F}$ of $F_{|U}$ to the end compactification $\widehat{U}$. The end z^+ is a fixed point of $\widehat{F}$ and the sequence $(i(\widehat{F}^k, z^+))_{k\geq 1}$ of Lefschetz indices is well defined and constant equal to 1. Indeed, the rotation number of C^+ is irrational. Let us consider a closed disk D which contains z^+ in its interior and which is contained in $(W \setminus \partial U) \sqcup \{z^+\}$. The connected component of $X = \bigcap_{k\in\mathbb{Z}} \widehat{F}^k(D)$ which contains z^+ is equal to z^+ by Theorem 8.8. So, we can find a closed disk $D' \subset D$ which contains z^+ in its interior and such that $\partial D' \cap X = \emptyset$. We deduce that the set $\bigcap_{k\in\mathbb{Z}} \widehat{F}^k(D')$, which is contained in X, does not meet the boundary of D'. Such a set is called an isolating block. To such a set we can associate a sequence of integers

$$a_k = \left(i\left(\widehat{F}^k, \bigcap_{0\leq i\leq k} f^{-i}(D') \right) \right)_{k\geq 1}$$

and this sequence is non-positive for infinitely many values of k (see [**F2**] or [**LY**]). The integer a_k is equal to the sum of the Lefschetz indices (for $\widehat{F}^k$) of all the fixed points of $\widehat{F}^k$ whose $\widehat{F}$ orbit lies entirely in D' if there are a finite number of such points. For any value k such that $a_k \leq 0$ there exists a fixed point of $\widehat{F}^k$ distinct from z^+, whose orbit lies entirely in D'.

To prove that there is a homoclinic point in W we consider an arbitrary essential fixed point z of F^q and let C be an essential simple closed curve in U consisting of segments of $W^s(z)$ and $W^u(z)$. We define

$$B_n = \bigcup_{k=-n}^{n} F^{-k}(C)$$

and let A_n be the closure of the component of the complement of B_n in $\overline{U}$ which contains $\partial^+ U$. We note that $F(B_n) \subset B_{n+1}$ so $A_{n+1} \subset F(A_n)$. From this it follows that $\bigcap_{n\geq 0} A_n \subset \bigcap_{n\geq 0} F(A_n) = F\left(\bigcap_{n\geq 0} A_n\right)$ and hence that $X = \bigcap_{n\geq 0} A_n$ is forward invariant under F^{-1} (and similarly under F). We conclude X is invariant and from Theorem 8.8 that $X = \partial^+ U$ and since the family $\{A_n\}$ is a nested sequence of compact

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sets we know that $A_n \subset W$ for n sufficiently large. The frontier of A_n consists of finitely many segments of $W^s(z)$ and $W^u(z)$ and hence must contain a homoclinic point of z. □

Remark 8.10. In the case where F is a twist map, the Birkhoff theory tells us that any irrational invariant essential continuum is the graph of Lipschitz map $\psi : \mathbb{T}^1 \to [0, 1]$. In particular, we do not need the prime end theory. We have a natural order on the set of these curves ($K < K'$ if K' is above K) and the function $K \mapsto \rho(K)$ is strictly monotone.

Using variational methods, Mather [**M4**] (also see [**M2**]) proved that there exists a point $z \in U$ whose α-limit set is contained in $\partial^+ U$ and whose ω-limit set is contained in $\partial^- U$, if U is a region of instability of a twist map. There exists a topological proof of this result [**L**] which uses the Birkhoff theory for positively and negatively invariant essential continua (also see [**Ha**]). Now, we prove that this result remains valid for generic area preserving diffeomorphisms without the requirement of the twist condition.

PROPOSITION 8.11. *Let U be a region of instability. There exists a point x in U such that $\omega(x) \subset \partial^+ U$ and $\alpha(x) \subset \partial^- U$.*

Proof. If C is an essential simple closed curve in U and if N is the component of its complement (in $\overline{U}$) which contains $\partial^+ U$, a well-known result of Birkhoff (see [**Bi2**, Ch. VIII, §8]) implies that the component of the compact set

$$\bigcap_{k=0}^{\infty} F^{-k}(N)$$

containing $\partial^+ U$, must meet C. This component, X, is a positively invariant set containing the set $\partial^+ U$ but not equal to it.

Theorem 8.8 tells us that the globally invariant set $\bigcap_{n\geq 0} F^n(X)$ is reduced to $\partial^+ U$. So, for any point $w \in X$, we have $\omega(w) \subset \partial^+ U$. Theorem 8.8 tells us that the connected globally invariant set $cl\big(\bigcup_{n\geq 0} F^{-n}(X)\big)$ meets $\partial^- U$. If $z \in U$ is an essential hyperbolic periodic point, Proposition 7.4 says there exists an essential closed curve contained in the union of any unstable branch of z and any stable branch. If n is large enough, the set $F^{-n}(X)$ must meet this curve. We deduce that X meets both unstable branches of z. More precisely, there exists an open interval γ of $W^u(z)$, which contains z, is disjoint from X, and whose boundary points belong to X. This arc is contained in a component V of $S^2 \setminus X$ and separates V into two components (because X is connected) which are connected components of $S^2 \setminus (\gamma \cup X)$.

Similarly we can construct a negatively invariant compact set Y in $U \cup \partial^- U$, containing the set $\partial^- U$ but not equal to it. For any point $w \in Y$ we have $\alpha(y) \subset \partial^- U$. Also as before, Y meets both branches of $W^s(z)$.

Consider two points $w_1, w_2 \in Y$, one in each branch of $W^s(z)$. For large n the points $F^n(w_1)$, $F^n(w_2)Y$ will be close to z, one in each component of $V \setminus \gamma$. Hence the connected set $F^n(Y)$ meets the two components of $V \setminus \gamma$. The set $F^n(Y)$ does not intersect γ, so it intersects X. Any point of intersection of X and $F^n(Y)$ satisfies the proposition. □

Among the periodic orbits that we can find in a region of instability there are particular orbits, called *well-ordered orbits* or *Birkhoff orbits*. Assertion (ii) of Theorem 8.6 can be

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improved: the periodic orbit can be chosen among the Birkhoff orbits. There is a natural generalization of Birkhoff orbits in the general case which is the notion of *unlinked orbit*. Boyland [**Bo**] proved that for a homeomorphism of the closed annulus the existence of a periodic orbit of type (p, q) implies the existence of an unlinked periodic orbit of the same type. Moreover there is a Lefschetz formula for these orbits. With these remarks it is not difficult to improve assertion (ii) of Theorem 8.6: the periodic orbit can be chosen among the unlinked orbits.

In the case of a twist map there exists in a region of instability U a well-ordered invariant Cantor set of rotation number ρ for every irrational number in the interior of the rotation set of U. Such a set is called an Aubry–Mather set. Using symbolic dynamics and the heteroclinic intersections between periodic points of rotation number smaller and bigger than ρ, is it possible to find an invariant (or an unlinked invariant) compact set of rotation number ρ in the general case (see [**H**])?

The following result is closely related to a result of Boyland [**Bo**].

PROPOSITION 8.12. *Let F be a Moser generic area preserving diffeomorphism of S^2 with an invariant annular domain $\mathbb{A}_0$ and let K be an essential continuum which is the closure of one branch of a hyperbolic fixed point z of F. Let $\tilde{z}$ be a point in $\pi^{-1}(z)$ in the universal cover $\widetilde{\mathbb{A}}$ of $\mathbb{A}_0$ and let $f : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be the lift which fixes $\tilde{z}$. If $\widetilde{K} \subset \widetilde{A}$ is $\pi^{-1}(K)$ and $\rho(\tilde{z}, f) \in int\, \rho(\widetilde{K}, f)$ then $\widetilde{K}$ is the closure of any branch of $\tilde{z}$, i.e. each branch of $\tilde{z}$ in $\widetilde{A}$ is dense in $\widetilde{K}$.*

Proof. Let Γ be a branch of $\tilde{z}$ for the lift f. We can assume without loss of generality that it is an unstable branch. Since z is essential there is a $k \neq 0$ such that Γ intersects $W^s(T^k(\tilde{z}))$ transversely. The λ-lemma then implies the closure of Γ contains the closure of $W^u(T^k(\tilde{z}))$. The λ-lemma also implies that the set J consisting of all values of k for which Γ intersects $W^s(T^k(\tilde{z}))$ is closed under addition.

Another straightforward λ-lemma argument and the fact that the rotation number of z is in the interior of the rotation set for K shows that J contains both positive and negative elements. From this we may conclude that J is a subgroup of $\mathbb{Z}$, say the subgroup $d\mathbb{Z}$.

If we replace $\mathbb{A}_0$ and F by a d-fold cover of $\mathbb{A}_0$ and a lift of F to it then clearly nothing has changed in what we want to prove except that now d has been replaced by 1, i.e. we may assume without loss of generality that $d = 1$ or that $J = \mathbb{Z}$. Hence we may assume that the closure of Γ contains the closure of $W^u(T^k(\tilde{z}))$ for every k.

Now let $\tilde{y}$ be an element of $\widetilde{K}$ and let $y = \pi(\tilde{y})$. The fact that $W^u(z, F)$ is dense in K implies that for any $\varepsilon > 0$ there is a point x of $W^u(z, F)$ which is within ε of y. Equivalently, there is an integer k and a point $\tilde{x}$ of $W^u(T^k(\tilde{z}, f))$ which is within ε of $\tilde{y}$. Hence Γ is dense in $\widetilde{K}$. □

9. *Generic diffeomorphisms of the sphere*

In this section we consider a C^r-diffeomorphism F of S^2 which preserves a volume form ω. We will further assume F is Moser generic (see Definition 8.3).

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For every integer $q \geq 1$ we write $\mathrm{Hyp}_q(F)$ (respectively $\mathrm{Ell}_q(F)$) for the set of hyperbolic (respectively elliptic) fixed point of F^q. We denote by

$$\mathrm{Hyp}(F) = \bigcup_{q \geq 1} \mathrm{Hyp}_q(F)$$

the set of hyperbolic (respectively elliptic) periodic points and

$$\mathrm{Ell}(F) = \bigcup_{q \geq 1} \mathrm{Ell}_q(F)$$

the set of elliptic periodic points.

For every integer $q \geq 1$, we denote by $\mathcal{K}_q$ the set of non-empty continua which are (globally) invariant by F^q and $\mathcal{K}_q^* \subset \mathcal{K}_q$ the set of continua which do not consist of a single fixed point of F^q. We denote by $\mathcal{K} = \bigcup_{q \geq 1} \mathcal{K}_q$ the set of periodic continua and $\mathcal{K}^* = \bigcup_{q \geq 1} \mathcal{K}_q^*$ the set of non-trivial periodic continua.

We recall a property we will use in this section: a closed set $K \subset S^2$ is connected if and only if the complement $U = S^2 \setminus K$ is simply connected.

From our earlier results we have the two following properties.

PROPOSITION 9.1. *Any continuum $K \in \mathcal{K}^*$ which contains a hyperbolic periodic point z contains the four branches of z.*

PROPOSITION 9.2. *Any continuum $K \in \mathcal{K}_q$ which does not contain a periodic point separates the sphere into two connected and simply connected components which are F^q-invariant. Moreover the rotation set of the continuum consists of a single irrational number (defined up to sign).*

A periodic continuum without any periodic orbit will be called a *periodic irrational continuum* and we will define its *rotation number* which is an element of $\mathbb{T}^1$ defined up to sign.

Let K and K' be two non-disjoint periodic irrational continua and write U and V for the components of $S^2 \setminus K$ and U' and V' for the components of $S^2 \setminus K'$. The set $K \cup K'$ is a periodic irrational continuum and the set

$$S^2 \setminus (K \cup K') = (U \cap U') \sqcup (U \cap V') \sqcup (V \cap U') \sqcup (V' \cap U')$$

has exactly two connected components by Proposition 9.2. Each set U and V contains a periodic point and cannot be contained in K': we deduce that one of the sets $U \cap U'$ or $U \cap V'$ is non-empty as is one of the sets $V \cap U'$ and $V \cap V'$. Doing the same thing for U' and V' we can suppose (exchanging U and V if necessary) that the two components of $S^2 \setminus K \cup K'$ are $U \cap U'$ and $V \cap V'$ and that $U \cap V' = V \cap U' = \emptyset$. The sets U and U' contain the same periodic points as do the sets V and V'. The open sets $U \cup U'$ and $V \cup V'$ are connected and simply connected, because $U \cap U'$ and $V \cap V'$ are non-empty and connected. Thus the sets $U \cup U'$ and $V \cup V'$ are the components of the complement of the irrational continuum $K \cap K'$.

By induction we can prove the analogous result for more than two periodic irrational continua. More precisely, if $(K_i)_{1 \leq i \leq n}$ is a family of periodic irrational continua such that the union is connected, the union is a periodic irrational continuum. Its complement has two components $\bigcap_{1 \leq i \leq n} U_i$ and $\bigcap_{1 \leq i \leq n} V_i$ where U_i and V_i are the components

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(well chosen) of $S^2 \setminus K_i$. The set $\bigcap_{1\le i\le n} K_i$ is a periodic irrational continuum and the components of its complement are $\bigcup_{1\le i\le n} U_i$ and $\bigcup_{1\le i\le n} V_i$.

Let K be a periodic irrational continuum whose interior O is not empty and consider the two components U and V of $S^2 \setminus K$. The periodic irrational continua ∂U and ∂V do not intersect. This is because the sets U, V and O are in three separate components of $S^2 \setminus (\partial U \cup \partial V)$ and if ∂U meets ∂V their union is an invariant irrational continuum whose complement has only two components. Thus the set O must be an open annulus.

Let Γ^s be a stable branch of $z \in \mathrm{Hyp}_q(F)$ and Γ^u an unstable one. Then, the two branches intersect, they have the same closure $K(z) \in \mathcal{K}_q$ and there is a fundamental system of neighborhoods of z which are Jordan domains with a boundary contained in $\Gamma^s \cup \Gamma^u$. We know that the equality $K(z) = K(z')$ occurs if and only if any stable branch of z meets any unstable branch of z'. We know that there is no strict inclusion $K(z) \subset K(z')$.

Let us prove the following fundamental result.

PROPOSITION 9.3. *If $(U_i)_{i\in I}$ is a family of periodic simply connected open sets, the union $U = \bigcup_{i\in I} U_i$ is simply connected.*

Proof. It is sufficient to prove this result in the case where I is finite and the U_i are connected. In fact, appealing to induction, it is sufficient to prove that the union of two invariant simply connected domains U and U' is simply connected.

The proof is obvious if one of the complements $S^2 - U$ or $S^2 - U'$ is empty or reduced to a point. We will suppose that none of these situations occur. The boundaries ∂U and $\partial U'$ are irrational invariant continua and we can write

$$S^2 = U \sqcup \partial U \sqcup V = U' \sqcup \partial U' \sqcup V'$$

where V and V' are some invariant simply connected domains.

If the sets ∂U and $\partial U'$ meet, their union is an irrational invariant continuum and the complement

$$S^2 \setminus (\partial U \cup \partial U') = (U \cap U') \sqcup (U \cap V') \sqcup (V \cap U') \sqcup (V \cap V')$$

has exactly two components, by Proposition 9.2. At least one of the sets $V \cap U'$ or $V \cap V'$ is not empty (because $\partial U'$ has no interior and cannot contain V). We deduce that $U \cap U'$ is either empty or connected and that $U \cup U'$ is simply connected.

If ∂U is contained in U', then one of the open sets U or V does not meet the connected set $S^2 \setminus U'$ and is contained in U'. In the first case $U \cup U' = U'$ is simply connected, in the second case we have

$$S^2 \setminus U = \partial U \cup V \subset U'$$

so $U \cup U' = S^2$ is simply connected.

If ∂U is contained in V', then one of the open sets U or V does not meet the connected set $S^2 \setminus V'$ and is contained in V'. In the first case U and U' are disjoint and $U \cup U'$ is simply connected, in the second case we have

$$S^2 \setminus U = \partial U \cup V \subset V' \subset S^2 \setminus U'$$

and $U \cup U' = U$ is simply connected. □

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COROLLARY 9.4. *Let U and U' be two disjoint periodic simply connected domains. If $(U_i)_{i\in I}$ is the family of all periodic domains disjoint from U and U', then the complement $K(U,U')$ of*

$$U \cup U' \bigcup_{i\in I} U_i$$

is the smallest periodic continuum such that U and U' are components of its complement. Moreover $K(U,U')$ is invariant if U and U' are invariant.

Proof. By Proposition 9.3, we know that $K(U,U')$ is connected. It is clear that U and U' are connected components of $S^2 \setminus K(U,U')$. If K is a periodic continuum such that U and U' are components of $S^2 \setminus K$, then the other components are in the family $(U_i)_{i\in I}$, so K contains $K(U,U')$. To see that $K(U,U')$ is invariant if U and U' are we observe that $\bigcup_{i\in I} U_i$ is F-invariant in that case. We still have to prove that K is periodic, but this follows by considering F^q where q is a common period for U and U' and noting that $K(U,U')$ is the same for F and F^q, so our previous remark implies $K(U,U') = F^q(K(U,U'))$. □

PROPOSITION 9.5. *Let U and U' be two disjoint periodic simply connected domains. If $S^2 \setminus (U \cup U')$ contains a periodic point, so does $K(U,U')$. Moreover every periodic point $z \in K(U,U')$ is hyperbolic.*

Proof. We can suppose that U and U' are invariant. If $K(U,U')$ has no periodic point it is an irrational continuum, so the components of $K(U,U')$ are exactly U and U' and we deduce that $K(U,U') = S^2 \setminus (U \cup U')$.

Let us prove now the second part of the proposition. There is no periodic point in ∂U and in $\partial U'$ because these two sets are periodic irrational continua. So every elliptic periodic point $z \in S^2 \setminus (U \cup U')$ has a periodic simply connected neighborhood, bounded by a Jordan curve, which meets neither U nor U'. We deduce that $z \notin K(U,U')$. □

We can improve this result.

PROPOSITION 9.6. *Let U and U' be two disjoint invariant simply connected domains. Consider a fixed point $z \in U$, a fixed point $z' \in U'$ and a lift f of $F_{|S^2\setminus\{z,z'\}}$ to the universal covering space of the annulus $S^2 \setminus \{z,z'\}$. Let $p \in \mathbb{Z}$ and $q \geq 1$ relatively prime. If $S^2 \setminus (U \cup U')$ contains a periodic point of type (p,q), so does $K(U,U')$.*

Proof. The set Z of periodic points of type (p,q) in $S^2 \setminus (U \cup U')$ is finite. If it is disjoint from $K(U,U')$, there exists a smallest finite family $(U_i)_{1\leq i\leq n}$ of components of $S^2 \setminus K(U,U')$ which covers Z. Every U_i is F^q-invariant and every fixed point of F^q in U_i belongs to Z. The contradiction comes from the Lefschetz formula

$$0 = \sum_{z\in Z} i(F^q, z) = \sum_{i=1}^{n} i(F^p, U_i) = n. \qquad \square$$

We can be more precise.

PROPOSITION 9.7. *Let U and U' two disjoint invariant simply connected domains. For any periodic hyperbolic point $z \in K(U,U')$, the set $K(z)$ (defined to be the closure of*

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a branch of z) is invariant and separates U and U'. Moreover if Γ^s is a stable branch of z and Γ^u an unstable one, there is a Jordan curve which separates U and U' and which is contained in $\Gamma^u \cup \Gamma^s$.

Proof. The periodic continuum $K(z)$ is contained in $K(U, U')$. If it does not separate U and U', the component of $S^2 \setminus K(z)$ which contains U and U' is a periodic simply connected domain V and $z \notin \overline{V}$ since z has a fundamental system of neighborhoods whose boundaries are in the branches of z. We deduce that the component of $S^2 \setminus \overline{V}$ which contains z is a periodic domain disjoint from U and U'. It is disjoint from $K(U, U')$ by definition of this set so we have a contradiction.

Let us prove now that $K(z)$ is invariant by showing that the orbit of z is contained in $K(z)$. Consider the components V and V' of $S^2 \setminus K(z)$ which contain respectively U and U'. The set ∂V is an F^q-invariant irrational continuum and meets its image because it separates U and U' and F is area preserving. We deduce that $K = \bigcup_{0 \le k < q} F^k(\partial V)$ is an F-invariant irrational continuum. The orbit of z does not meet the component $\bigcap_{0 \le k < q} F^k(V)$ of $S^2 \setminus K$ which contains U, it is contained in the other component which is disjoint from V. Similarly the orbit does not meet V'. It does not meet any other component of $S^2 \setminus K(z)$ because it is contained in $K(U, U')$. We have proven that it is contained in $K(z)$.

We deduce that any stable branch of z' in the orbit of z meets any unstable branch of z. We deduce from this fact and from the λ-lemma the end of the proposition in the case where z is a periodic point of period $q \ge 2$. But if z is a fixed point then we choose a hyperbolic periodic point $w \in K(z)$ of period at least two and observe that $K(z) = K(w)$. The fact that $W^s(w)$ intersects Γ^u transversely and $W^u(w)$ intersects Γ^s transversely allows us to apply the λ-lemma once again and obtain a simple Jordan curve separating U and U' which is a subset of $\Gamma^u \cup \Gamma^s$. □

COROLLARY 9.8. *Any non-trivial invariant continuum K which is not irrational contains an invariant continuum $K(z)$, for some $z \in \mathrm{Hyp}(F)$.*

Proof. If the complement of K has at least two invariant domains, this is a consequence of Propositions 9.5 and 9.7. In the other case the Lefschetz formula tells us that there exists at least one fixed point in K. If K contains a hyperbolic fixed point, the result is obvious. If K contains an elliptic fixed point z, we can find two minimal invariant curves C_1 and C_2 of different irrational rotation number surrounding z in a neighborhood of that point. These curves meet K and are contained in it. By the Poincaré–Birkhoff Theorem, we can apply Proposition 9.5 and 9.7 to the domain U containing z bounded by the closest curve and to the domain U' which does not contain z and bounded by the other curve. There exists $z' \in \mathrm{Hyp}(F)$, such that $K(z')$ is invariant and a simple curve separating U and U' and contained in $W^u(z') \cup W^s(z')$. This curve meets the connected set K and we have $K(z') \subset K$. □

COROLLARY 9.9. *The invariant continua $K(z)$, $z \in \mathrm{Hyp}(F)$, are the minimal elements in the set of the non-trivial invariant continua which are not irrational.*

Proof. This is an immediate consequence of Corollary 9.8 and of the fact that there is no strict inclusion $K(z) \subset K(z')$, for $z, z' \in \mathrm{Hyp}(F)$. □

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PROPOSITION 9.10. *The set of irrational invariant continua and continua* $\{z\}$, $z \in \mathrm{Ell}_1(F)$, *is closed in the Hausdorff topology. The union of all these sets is closed.*

Proof. Recall that the set $\mathcal{K}_1$ consists of invariant continua. The result is a consequence of Corollary 9.8 and of the following facts:

- the set $\mathcal{K}_1$ is closed in the Hausdorff topology;
- any continuum $\{z\}$, $z \in \mathrm{Hyp}_1(F)$, is isolated in $\mathcal{K}_1$;
- for any $z \in \mathrm{Hyp}(F)$, the set of continua $K \in \mathcal{K}_1$ which contain $K(z)$ is open in $\mathcal{K}_1$. □

Remark 9.11. By Birkhoff's theory, if z is a elliptic fixed point and V a small neighborhood of z, there is a neighborhood of $\{z\}$ in the set defined in Proposition 9.10 which consists of $\{z\}$ and of the invariant curves surrounding z and contained in V. If z' is another fixed point, the set which consists of $\{z\}$, $\{z'\}$ and of the irrational invariant continua which separate z and z' is closed.

PROPOSITION 9.12. *The set consisting of irrational invariant continua, of invariant sets* $K(z)$, *for* $z \in \mathrm{Hyp}(F)$, *and of continua* $\{z\}$, $z \in \mathrm{Ell}_1(F)$, *is closed in the Hausdorff topology.*

Proof. Indeed, any convergent sequence $K(z_n)$, $z_n \in \mathrm{Hyp}(F)$, of invariant continua converges either to a trivial continuum $\{z\}$, $z \in \mathrm{Ell}_1(F)$, or to an irrational invariant continuum, or to an invariant continuum which contains an invariant set $K(z)$, $z \in \mathrm{Hyp}(F)$. In the last situation the set $K(z)$ is contained in $K(z_n)$, for $n \geq n_0$ big enough. The sequence $(K(z_n))_{n \geq n_0}$ is constant and equal to $K(z)$. □

A periodic domain U of period $q \geq 1$ will be called a *strictly periodic* domain if the domains $F^k(U)$, $0 \leq k < q$ are all disjoint. Note that the closures also are disjoint. Indeed if ∂U meets $\partial U'$ where $U' = F^k(U)$, $0 < k < q$, the union of these two sets is an F^q-invariant irrational continuum. Writing $V = S^2 \setminus \overline{U}$ and $V' = S^2 \setminus \overline{U'}$, the components of the complement must be $U \cap V' = U$ and $U' \cap V = U'$ (because U and U' are disjoint) and we must have $V \cap V' = \emptyset$. This is possible only if $q = 2$, and $F(U) = U'$, $F(U') = U$ and $S^2 = \overline{U} \cup \overline{U'}$. But that leads to a contradiction: since there is no fixed point in $U \cup U'$ there must (by the Lefschetz Theorem) be a fixed point in $\partial U \cup \partial U'$.

PROPOSITION 9.13. *The union* X_1 *of the irrational invariant continua, of the invariant sets* $K(z)$, $z \in \mathrm{Hyp}(F)$ *and of the elliptic fixed points is an invariant continuum. Its complement is the union of the simply connected strictly periodic domains of period* $q \geq 2$.

Proof. The fact that it is closed is a consequence of Proposition 9.12. We prove that it is connected by showing that any component U of $S^2 \setminus X_1$ is simply connected.

First, we observe that U is not invariant. If it were invariant, its closure would be an invariant continuum which is not irrational (because $U \not\subset X_1$). It would contain an invariant continuum $K(z)$, $z \in \mathrm{Hyp}(F)$, which is disjoint from U (because $U \cap X_1 = \emptyset$). A contradiction arises from the fact that there is a fundamental system of neighborhoods of z whose boundary is in $W^s(z) \cup W^u(z)$.

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Next, we prove that U is simply connected. Any Jordan curve γ contained in U is disjoint from its image. The map F being volume preserving, one of the two domains bounded by γ is disjoint from its image. Any component of X_1 being invariant, we deduce that this domain is disjoint from X_1 and contained in U. We conclude that U is simply connected. We have proven that any component of $S^2 \setminus X_1$ is a strictly periodic simply connected domain of period $q \geq 2$.

Conversely, a simply connected strictly periodic domain U of period $q \geq 2$ does not contain any fixed points. It cannot meet any fixed continuum $K(z)$, $z \in \mathrm{Hyp}(F)$ since otherwise it will meet $W^s(z)$ and the closure $\overline{U}$ which is disjoint from its image will contain $K(z)$.

Finally, we must show that U cannot meet any irrational invariant continuum K. If it did then we could not have $K \subset U$ so $K \cap \partial U \neq \emptyset$. It follows that ∂U is a periodic irrational continuum and by our remarks above that $K' = K \cup \partial U \cup F(\partial U)$ is a periodic irrational continuum. Since K has no interior, the complement of K' would necessarily have at least three components since it would contain a point from U, a point from $F(U)$ and a point in neither of them. Each of these would be in different components of $S^2 \setminus K'$. This contradicts Proposition 9.2. □

We will write E for the union of the periodic irrational continua and of the elliptic periodic points, H for the union of the $K(z)$, $z \in \mathrm{Hyp}(F)$ and I for the complement of $E \cup H$. So we have a decomposition

$$S^2 = E \cup H \sqcup I.$$

PROPOSITION 9.14. *We have the following properties:*

(i) *the sets E and H are F_σ;*

(ii) *we have $I \subset \partial E = \partial H$;*

(iii) *the set $E \cup H$ is connected and dense;*

(iv) *a point z belongs to I if and only if there exists a decreasing sequence $(U_n)_{n\geq 0}$ of strictly periodic simply connected domains which contain z such that the sequence of periods tends to the infinity.*

Proof. We can write $E = \bigcup_{q\geq 1} E_q$ and $H = \bigcup_{q\geq 1} H_q$, where E_q is the union of the F^q-invariant irrational continua and of the elliptic fixed points of F^q and H_q the union of the sets $K(z)$, $z \in \mathrm{Hyp}(F)$, which are F^q-invariant. Each set E_q is closed and each set H_q is an F_σ, moreover $E_q \cup H_q$ is closed, connected and contains $H_1 \cup E_1$. The first assertion is clear as is the connectedness of $E \cup H$. The last assertion is an immediate consequence of Proposition 9.13. The density of $E \cup H$ follows from the inclusion $I \subset \overline{E}$ that we will prove now. For any decreasing sequence $(U_n)_{n\geq 0}$ of strictly periodic domains which contains $z \in I$ such that the sequence of periods tends to infinity, the volume of U_n tend to zero. So ∂U_n which is contained in E meets a given neighborhood of z if n is big enough.

Since $I \subset \overline{E}$ we deduce that $\partial H \subset \partial E$. If we show $I \subset \overline{H}$ then it follows that $\partial E \subset \partial H$ and we can conclude assertion (ii). If z_0 is in I then by (iv) $z \in U$ where U is a strictly periodic simply connected domain of high period (and hence of small measure).

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Given $\varepsilon > 0$ we may choose U so that any point of U is within ε of ∂U. Also, we know that $z \in U' \subset U$ where U' is a strictly periodic simply connected domain of higher period than U. The annular domain $U \setminus \overline{U'}$ is invariant under some iterate of F and contains periodic points, so we may apply Proposition 8.1 to its prime end compactification to conclude that it has an essential periodic point z_0. Then $K(z_0)$ separates z and ∂U. It follows that there must be a point of $K(z_0) \subset H$ within ε of z and hence $I \subset \overline{H}$. □

THEOREM 9.15. *There exists a dense G_δ, $\mathcal{G}$, in* $\mathrm{Diff}^r_\omega(S^2)$ *such that the interior of E is empty if $F \in \mathcal{G}$. The union of stable manifolds of hyperbolic periodic points is dense in S^2 for F in $\mathcal{G}$.*

Proof. It is sufficient to prove that among the generic diffeomorphisms studied in this section (i.e. the Moser generic diffeomorphisms, see 8.3), the ones which do not have any periodic open annulus without periodic points are generic. For such a map the interior of the sets E_q, $q \geq 1$, is empty as is the interior of the union E. We deduce the second part of the proposition from the inclusion $I \cup \partial E \subset \partial H$.

Thus it suffices to prove that there is a generic set of diffeomorphisms in $\mathrm{Diff}^r_\omega(S^2)$ which have no periodic open annulus which contains no periodic points. Consider an open connected and simply connected subset $V \subset S^2$. We will first show that for a dense open subset of $\mathrm{Diff}^r_\omega(S^2)$ the set V is not contained in a periodic open annulus which contains no periodic points.

Consider the set of diffeomorphisms $\mathcal{G}_V \subset \mathrm{Diff}^r_\omega(S^2)$ defined by the property that $F \in \mathcal{G}_V$ if and only if there are two disjoint simple closed curves C_1 and C_2 contained in one component of $\bigcup_{n\in Z} F^n(V)$ with the property that there are hyperbolic periodic points of F in each of the three components of $S^2 \setminus (C_1 \cup C_2)$. It is clear that if $F \in \mathcal{G}_V$ then V is not contained in a periodic open annulus which contains no periodic points since C_1 and C_2 would have to be contained in such an annulus and this is impossible. We will show that $\mathcal{G}_V$ is dense and open.

The fact that $\mathcal{G}_V$ is open is clear since for any diffeomorphism G close to F we know $C_1 \cup C_2$ is contained in one component of $\bigcup_{n\in Z} G^n(V)$ because $C_1 \cup C_2$ is compact. Also, if G is close to F it will still have hyperbolic periodic points in the three components of $S^2 \setminus (C_1 \cup C_2)$. So $\mathcal{G}_V$ is open.

To see that $\mathcal{G}_V$ is dense consider any $F_0 \in \mathrm{Diff}^r_\omega(S^2)$. We can approximate it arbitrarily closely by a Moser generic diffeomorphism F. We will show either $F \in \mathcal{G}_V$ or it can be approximated by an element of $\mathcal{G}_V$. Consider $U = \bigcup_{n\in Z} F^n(V)$. If U contains a periodic point of F then it contains infinitely many hyperbolic periodic points since for Moser generic diffeomorphisms hyperbolic periodic points are not isolated and elliptic periodic points are in the closure of hyperbolic periodic points. It is thus clear that if U contains any periodic points we may choose simple closed curves C_1 and C_2 contained in one component of U such that each component of $S^2 \setminus (C_1 \cup C_2)$ contains a hyperbolic periodic point. Thus in this case $F \in \mathcal{G}_V$.

So we may assume U contains no periodic points. We choose a periodic component U_0 of U, say of period q, so $F^q(U_0) = U_0$. If the complement of U_0 has three or more components containing periodic points we can, as before, choose C_1 and C_2 showing $F \in \mathcal{G}_V$. It is not possible for the complement of U_0 to have only one component

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containing periodic points, since the complement of that component would be a simply connected F^q invariant domain with no periodic points.

All that remains is the case that the complement of U_0 has exactly two components containing periodic points. We choose a simple closed curve C in U_0 which separates these two components. The complement of these two components is an open F^q invariant annulus W containing U_0 and in which C is essential.

We choose a small annular neighborhood U_1 of C in U_0. Perturbing F by the time t map of an area preserving C^∞ flow ϕ_t supported in the interior of U_1 we may continuously vary the mean rotation number of $F^q \circ \phi_t = (F \circ \phi_t)^q$ on the annulus W. However a result of [**F3**] says that if this number is rational there must exist a periodic point. Clearly if $t > 0$ is small the perturbation is C^r close to F. One point on the new periodic orbit must lie in the interior of U_1 since that is where the perturbation is supported. By a slight further perturbation we obtain G close to F with a hyperbolic periodic point in the interior of U_1. Choosing C_1 and C_2 to be the boundary components of U_1 we observe that $G \in \mathcal{G}_V$.

Thus we have shown that $\mathcal{G}_V$ is open and dense. If we now choose a countable basis $\{V_n\}$ for the topology of S^2 consisting of open disks then

$$\mathcal{G} = \bigcap_{n \geq 0} \mathcal{G}_{V_n}$$

is the desired dense G_δ. □

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