Probability in the Engineering and Informational Sciences, **34**, 2020, 72–91.
doi:10.1017/S0269964818000359

ACTIVE REDUNDANCY ALLOCATION FOR COHERENT SYSTEMS WITH INDEPENDENT AND HETEROGENEOUS COMPONENTS

Rui Fang
Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China
E-mail: xmufr1987@hotmail.com

Xiaohu Li
Department of Mathematical Sciences,
Stevens Institute of Technology, Hoboken, NJ 07030, USA

This paper studies the allocation of active redundancies to coherent systems on the context that the base and redundancy components have mutual independent lifetimes. For systems with two symmetric components and systems with one component's minimal cut sets (minimal path sets) covering those of another, we derive sufficient conditions to compare the resultant system lifetimes. Some numerical examples are also presented to illustrate the theoretical results.

Keywords: majorization, minimal cut set decomposition, stochastic order, symmetric components

1. INTRODUCTION

In industrial engineering, allocating redundancy components has been proved to be the most effective and economic way to improve system reliability. Typically, an active redundancy is assigned in parallel to and starts to work along with the base component, and a cold standby redundancy starts to work once the base component breaks down and serves as a substitution. In practice, the active redundancy is usually used to avoid the unexpected system failure through enhancing base component reliability, while the cold standby is utilized to improve system reliability by extending the running time of the base component.

A reliability system is said to be coherent if it never fails upon replacing a failed component with a working one and the performance of each component is relevant to the system status. As a very large class, the coherent system includes many commonly used structures as its special cases. For example, the k-out-of-n system, the parallel–series system, and the series–parallel system, and so on. To improve system reliability and safety is always a primary goal in industrial engineering. Typically, two methods are usually adopted to enhance engineering reliability. One method is to allocate non-random resource to system components so as to boost their reliability. See Liyanage and Shanthikumar [14] for a

© Cambridge University Press 2018 0269-9648/18 $25.00

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

general framework of the resource allocation problem and Feng and Shanthikumar [4] for recent discussions on the optimal resource allocation when the unit response functions have different specific structures. The other one aims to improve system performance by allocating active redundancy components to system components. Following the second line of research, this study will address the impact of allocating redundancy components to coherent systems. In engineering practice, once the redundancies are assigned, it is impractical to change allocation strategy during the system operation. Thus, it is rather important to figure out the allocation strategy stochastically maximizing the system lifetime before the system is put into operation. Tackling the problem from the static aspect, El-Neweihi, Proschan and Sethuraman [3] considered allocating several active redundancies with independent lifetimes to parallel–series and series–parallel systems and presented the optimal allocation strategy in the sense of maximizing the system reliability at some fixed time point. Boland, El-Neweihi and Proschan [2] and Shanked and Shanthikumar [17] were among the first to deal with the allocation strategy for coherent systems by utilizing the stochastic order. Focusing on series and parallel systems, Boland et al. [2] considered allocating one single active redundancy or cold standby, and Shanked and Shanthikumar [17] studied the problem of allocating several active redundancies or cold standby ones. As for k-out-of-n systems, Hu and Wang [9] studied the allocation of several active redundancies in the scenario that all the base and redundancy components are of independent and identically distributed (i.i.d.) lifetimes. Later, by relaxing the i.i.d. lifetimes to mutually independent heterogeneous base component lifetimes and identically distributed redundancy lifetimes, Li and Ding [13] derived the optimal strategy to allocate multiple active redundancies to k-out-of-n systems. We refer readers to Li and Ding [12] for a comprehensive summary on active redundancy allocation before 2012. Recently, for a series system with i.i.d. or exponentially distributed base and redundancy component lifetimes, Zhao, Chan and Ng [20] and Zhao, Zhang and Chen [21] derived the optimal allocation strategy of one or several active redundancies in the sense of some stochastic orders.

In engineering practice, it is not uncommon to run into complex systems rather than the k-out-of-n structure, the parallel-series structure, or the series-parallel structure. For such systems, it is usually challenging to derive the globally optimal allocation strategy due to complex system structure and heterogeneous component lifetimes. Instead, several authors investigated the locally optimal allocation strategy concerning two components. Zhuang and Li [22] showed that under some conditions on the base and redundancy component lifetimes, the difference between numbers of active redundancies to two components in a k-out-of-n system should not be too large. For system symmetric with respect to (w.r.t.) two components, Fang and Li [5–7] investigated allocation strategies to assign active redundancies to these two base components in several settings of component lifetimes. As for two asymmetric components, Fang and Li [5–7] evaluated some allocation strategies' performance in enhancing system reliability by using the structure decomposition through minimal cut sets and minimal path sets, respectively.

Along this line of research, this paper aims to further explore the locally optimal allocation strategies to assign multiple active redundancies to coherent systems. In our setting, the base component lifetimes are independent but not necessarily identically distributed, the active redundancy component lifetimes are i.i.d. and independent of the base component lifetimes. For system symmetric w.r.t. two components, we explicitly derived the locally optimal way to assign active redundancies. Also, we develop the locally optimal allocation strategy for one component having minimal cut sets (minimal path sets) covering those of another component.

Recently, allocation strategies of active redundancies for coherent systems with two symmetric base components are typically studied in the literature. Fang and Li [5,7] had

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

a discussion on allocating one single active redundancy to coherent systems with two symmetric components in the context that the redundancy component lifetime is independent of the statistically dependent base component lifetimes. For matched active redundancies, Fang and Li [6] investigated the strategy to allocate redundancy components of dependent lifetimes to a coherent system with dependent base component lifetimes symmetric w.r.t. two specific ones. Owing to the complexity introduced by the statistical dependence, the concerned findings of Fang and Li [5–7] are all developed under the assumption that the base component lifetimes or the redundancy component lifetimes are arrayed in terms of some stochastic order. In this paper, to give prominence to the structure of coherent systems, we study base components with mutually independent random lifetimes. The present main results bring forth some sufficient conditions for the better allocation strategy of multiple active redundancies without any requirement of stochastically ascending base component lifetimes.

The rest of this paper proceeds as follows: In Section 2 we recall several concepts related to the discussion throughout the remaining sections. Section 3 presents an important decomposition of system reliability based on two symmetric components. For coherent systems symmetric w.r.t. two base components, Section 4 provides a general sufficient condition deciding a better allocation strategy, and Section 5 addresses the locally optimal allocation strategy. Section 6 develops the optimal allocation strategies for systems with the class of minimal cut (path) sets of one component covering that of another component. Finally, Section 7 illustrates the main theoretical results through several numerical examples.

2. SOME PRELIMINARIES

Before proceeding to the main results on allocation strategies, let us recall some concepts and technical lemmas to be utilized in developing the main results in the coming sections.

For a general coherent system, with no doubt, the structure plays the first important role in forming the optimal redundancy allocation strategy. Despite the great variety and diversity of coherent structures, the well-known minimal cut set and minimal path set decompositions provide us a very useful tool to handle active redundancy allocation to coherent systems. For a coherent system, a cut set comprises system components whose failures cause the system to fail, and a minimal cut set $\mathcal{C}$ is the one such that any of its subset is not a cut set anymore. Let $\mathscr{C}$ be the class of all minimal cut sets of the system, and $\mathscr{C}_i = \{\mathcal{C} \in \mathscr{C} : i \in \mathcal{C}\}$ be the subclass of minimal cut sets including component i, $i = 1, \ldots, n$. With component lifetimes $\boldsymbol{X} = (X_1, \ldots, X_n)$ a coherent system lifetime can be represented as

$$T(\boldsymbol{X}) = \min_{\mathcal{C} \in \mathscr{C}} \max_{i \in \mathcal{C}} X_i. \tag{1}$$

As a dual, a minimal path set $\mathcal{P}$ is a collection of components whose functioning ensures the system's functioning and the removal of one of them fails to guarantee the system's functioning. Specifically, let $\mathscr{P}$ be the class of all minimal path sets of the system, and $\mathscr{P}_i = \{\mathcal{P} \in \mathscr{P} : i \in \mathcal{P}\}$ be the subclass of minimal path sets including component i, $i = 1, \cdots, n$. Then, the system lifetime can also be reexpressed as

$$\max_{\mathcal{P} \in \mathscr{P}} \min_{i \in \mathcal{P}} X_i. \tag{2}$$

For many commonly used structures, the minimal cut sets and the minimal path sets are not difficult to identify and of bounded cardinality. For example, a k-out-of-n system has $\binom{n}{k}$ minimal cut sets each comprising k of the n components and has $\binom{n}{k-1}$ minimal path sets

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

each comprising $n-k+1$ of the n components. For more on decompositions (1) and (2) we refer readers to the monograph Barlow and Proschan [1].

In the present setting, all active redundancies are i.i.d., and thus heterogeneity of the allocation strategy, namely, the vector of the numbers of redundancies assigned to base components, plays a role in determining the redundant system reliability. Specifically, majorization will be utilized to characterize vector heterogeneity in this study.

DEFINITION 2.1: *The real vector* $\boldsymbol{x}=(x_1,\ldots,x_n)$ *is said to be* majorized *by another real one* $\boldsymbol{y}=(y_1,\ldots,y_n)$, *denoted as* $\boldsymbol{x}\preceq_m\boldsymbol{y}$, *if* $\sum_{i=1}^{j}x_{i:n}\geq\sum_{i=1}^{j}y_{i:n}$ *for* $j=1,\ldots,n-1$ *and* $\sum_{i=1}^{n}x_{i:n}=\sum_{i=1}^{n}y_{i:n}$, *where* $x_{1:n}<\cdots<x_{n:n}$ *is the increasing arrangement of* $x_1,\ldots,x_n$.

For integers $x_1\geq\cdots\geq x_n$ with $x_i>x_j$, the transformation $(x_1',\ldots,x_n')$ such that $x_i'=x_i-1$, $x_j'=x_j+1$ and $x_l'=x_l$ for $l\neq i,j$ is called a *transfer* from i to j in Folkman and Fulkerson [8]. The next lemma on majorization will be useful in deriving one of the main results. For more on majorization, we refer readers to Marshall, Olkin and Arnold [15].

LEMMA 2.2 (Lemma D.1 of Marshall, et al. [15]): *If* $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ *are integers and* $\boldsymbol{x}\preceq_m\boldsymbol{y}$, *then* $\boldsymbol{x}$ *can be derived from* $\boldsymbol{y}$ *by successively applying a finite number of transfers.*

Apart from the system structure and the heterogeneity of the allocation strategy, the nonhomogeneous component lifetimes also play a part in producing the optimal allocation strategy of active redundancies. In this study, we will employ stochastic order to describe the heterogeneity of base component lifetimes and to assess the redundant system reliability.

Denote F_i, f_i and r_i the probability distribution function, probability density function and reversed hazard rate function of random variable X_i, $i=1,2$, respectively.

DEFINITION 2.3: X_1 *is said to be smaller than* X_2 *in the*

(i) likelihood ratio *order, denoted as* $X_1\leq_{\mathrm{lr}}X_2$ *or* $F_1\leq_{\mathrm{lr}}F_2$, *if* $f_2(t)/f_1(t)$ *increases in* $t\in\mathbb{R}$;

(ii) usual stochastic *order, denoted as* $X_1\leq_{\mathrm{st}}X_2$ *or* $F_1\leq_{\mathrm{st}}F_2$, *if* $F_1(t)\geq F_2(t)$ *for all* $t\in\mathbb{R}$;

(iii) relative aging *order (in terms of the reversed hazard rate), denoted as* $X_1\lesssim_{\mathrm{rh}}X_2$, *if* $r_2(t)/r_1(t)$ *is increasing in* $t>0$.

For more on stochastic orders and their applications in reliability, one may refer to Shaked and Shanthikumar [18] and Li and Li [11], and we refer readers to Rezaei, Gholizadeh and Izadkhah [16] and Li and Li [10] for partial orders of relative aging.

Throughout the remaining sections, we assume that real vectors and random vectors are both nonnegative. We also assume the base component lifetime X_i has a distribution function F_i, the i.i.d. active redundancy lifetimes Y_i have distribution function G, and, all the base and redundancy component lifetimes are independent.

3. SYSTEM RELIABILITY DECOMPOSITION DUE TO SYMMETRY

Denote by $\tau_{i,j}(\boldsymbol{x})=(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$ the permutation to exchange the ith and jth coordinates of $\boldsymbol{x}=(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\in\mathbb{R}^n$, $1\leq i<j\leq n$. As per Fang and Li

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

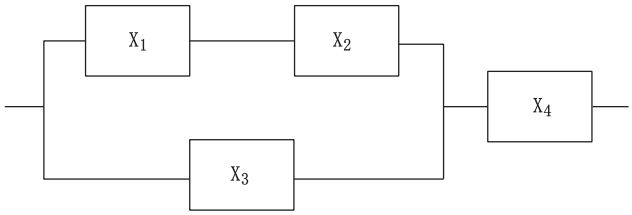


FIGURE **1.** Structure of the four-component system.

[5–7], a coherent system with lifetime function $T(\boldsymbol{x})$ in (1) is *symmetric* w.r.t. components $\{k,l\}$ such that $1 \le k \ne l \le n$ if

$$T(\boldsymbol{x}) = T(\tau_{k,l}(\boldsymbol{x})), \quad \text{for all vector } \boldsymbol{x} \text{ of component lifetimes.}$$

Succinctly speaking, *switching two symmetric components k and l of a system does not change the system lifetime function*. Evidently, a k-out-of-n system is symmetric w.r.t. any pair of its components, and Figure 1 presents a system with two symmetric components but not of k-out-of-n structure. In the literature, Li and Ding [13] investigated the optimal allocation of i.i.d. active redundancies to k-out-of-n systems. Naturally, an interesting question arises: What if the system is symmetric only w.r.t. some pair of its components, rather than any two components? In the following two sections, we will investigate sufficient conditions to compare the effect of allocation strategies and discuss the locally optimal way to allocate multiple active redundancies with i.i.d. lifetimes to systems with two symmetric components.

For coherent systems symmetric w.r.t. two components X_k and X_l, the next decomposition of system reliability function plays a vital role in deriving our main results in the sequel.

Conditioned by the event that components of both X_k and X_l are functioning at time t, the system reliability function

$$\begin{aligned}
\eta_{1|k,l}(t) &= \mathrm{P}\Big(T(\boldsymbol{X}) = \min_{\mathcal{C}\in\mathscr{C}} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k > t, X_l > t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}} X_i > t, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k > t, X_l > t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k > t, X_l > t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t\Big). \qquad \textbf{(3)}
\end{aligned}$$

Denote $\mathcal{C}_A = \mathcal{C}\setminus A$ for $A \subseteq \{1,\dots,n\}$. Conditioned by the functional mode of X_k and the failure mode of X_l at time t, the system gets the reliability function

$$\begin{aligned}
\eta_{2|k,l}(t) &= \mathrm{P}\Big(T(\boldsymbol{X}) = \min_{\mathcal{C}\in\mathscr{C}} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k > t, X_l \le t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}} X_i > t, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k > t, X_l \le t\Big)
\end{aligned}$$

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

$$= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i > t \,\Big|\, X_k > t, X_l \le t \Big)$$

$$= \mathrm{P}\Big(\min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i \Big\} > t \Big). \tag{4}$$

Owing to the system symmetry w.r.t. components k and l, it holds that

$$\mathrm{P}\Big(\min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i \Big\} > t \Big)$$

$$= \mathrm{P}\Big(\min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k\}}} X_i \Big\} > t \Big), \tag{5}$$

where the right-hand-side of (5) is just the conditional system reliability function given that X_l is working and X_k breaks down at time t. Thus,

$$\mathrm{P}(T(\boldsymbol{X}) > t \mid X_k \le t, X_l > t) = \eta_{2|k,l}(t).$$

In a similar manner, we also have

$$\begin{aligned}
\eta_{3|k,l}(t) &= \mathrm{P}\Big(T(\boldsymbol{X}) = \min_{\mathcal{C}\in\mathscr{C}} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k \le t, X_l \le t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}} X_i > t, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t \,\Big|\, X_k \le t, X_l \le t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k\}}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i > t, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k,l\}}} X_i > t \,\Big|\, X_k \le t, X_l \le t\Big) \\
&= \mathrm{P}\Big(\min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i > t, \min_{\mathcal{C}_j\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k\}}} X_i > t, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i > t, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k,l\}}} X_i > t\Big) \\
&= \mathrm{P}\Big(\min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_k\cup\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i, \min_{\mathcal{C}\in\mathscr{C}_k\setminus\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k\}}} X_i, \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}_{\{l\}}} X_i, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k\cap\mathscr{C}_l} \max_{i\in\mathcal{C}_{\{k,l\}}} X_i \Big\} > t \Big).
\end{aligned} \tag{6}$$

It is plain to check that

$$\eta_{1|k,l}(t) \ge \eta_{2|k,l}(t) \ge \eta_{3|k,l}(t), \quad \text{for any } t \ge 0. \tag{7}$$

With the help of conditional reliability function $\eta_{j|k,l}(t)$ for $j = 1, 2, 3$, we obtain an useful representation for the system reliability.

THEOREM 3.1: *If a coherent system is symmetric w.r.t. components* $\{k, l\}$*, then,*

$$\begin{aligned}
\mathrm{P}(T(\boldsymbol{X}) > t) &= \eta_{2|k,l}(t) + [\eta_{1|k,l}(t) - \eta_{2|k,l}(t)]\mathrm{P}(X_k > t, X_l > t) \\
&\quad + [\eta_{3|k,l}(t) - \eta_{2|k,l}(t)]\mathrm{P}(X_k \le t, X_l \le t), \quad \textit{for all } t \ge 0.
\end{aligned} \tag{8}$$

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

PROOF: For all $t \geq 0$, it holds that

$$\begin{aligned}
\mathrm{P}(T(\boldsymbol{X}) > t) &= \mathrm{P}(T(\boldsymbol{X}) > t \mid X_k > t, X_l > t)\mathrm{P}(X_k > t, X_l > t) \\
&\quad + \mathrm{P}(T(\boldsymbol{X}) > t \mid X_k > t, X_l \leq t)\mathrm{P}(X_k > t, X_l \leq t) \\
&\quad + \mathrm{P}(T(\boldsymbol{X}) > t \mid X_k \leq t, X_l > t)\mathrm{P}(X_k \leq t, X_l > t) \\
&\quad + \mathrm{P}(T(\boldsymbol{X}) > t \mid X_k \leq t, X_l \leq t)\mathrm{P}(X_k \leq t, X_l \leq t) \\
&= \eta_{1|k,l}(t)\mathrm{P}(X_k > t, X_l > t) + \eta_{3|k,l}(t)\mathrm{P}(X_k \leq t, X_l \leq t) \\
&\quad + \eta_{2|k,l}(t)[\mathrm{P}(X_k > t, X_l \leq t) + \mathrm{P}(X_k \leq t, X_l > t)] \\
&= \eta_{1|k,l}(t)\mathrm{P}(X_k > t, X_l > t) + \eta_{3|k,l}(t)\mathrm{P}(X_k \leq t, X_l \leq t) \\
&\quad + \eta_{2|k,l}(t)[1 - \mathrm{P}(X_k > t, X_l > t) - \mathrm{P}(X_k \leq t, X_l \leq t)] \\
&= \eta_{2|k,l}(t) + [\eta_{1|k,l}(t) - \eta_{2|k,l}(t)]\mathrm{P}(X_k > t, X_l > t) \\
&\quad + [\eta_{3|k,l}(t) - \eta_{2|k,l}(t)]\mathrm{P}(X_k \leq t, X_l \leq t),
\end{aligned}$$

and this completes the proof. ■

Let $\boldsymbol{r} = (r_1, \ldots, r_n)$ be the strategy allocating active redundancies $Y_{r_{i-1}+1}, \ldots, Y_{r_{i-1}+r_i}$ to the component X_i, $i = 1, \ldots, n$. By convention, set $r_0 \equiv 0$, $r_i = 0$ corresponds to no active redundancies assigned to X_i, and $r_1 + \cdots + r_n = m$ is such that

$$Y_1, \ldots, Y_{r_1}, Y_{r_1+1}, \ldots, Y_{r_1+r_2}, \ldots, Y_{r_1+\cdots+r_n}$$

are m i.i.d. redundancy lifetimes. Under the allocation strategy $\boldsymbol{r}$, denote

$$X_i(\boldsymbol{r}) = \max\{X_i, Y_{r_{i-1}+1}, \ldots, Y_{r_{i-1}+r_i}\}$$

the resultant ith component lifetime, $i = 1, \ldots, n$, and

$$T(\boldsymbol{X}, \boldsymbol{r}) = T(X_1(r_1), \ldots, X_n(r_n))$$

the resultant system lifetime, and $\eta_{j|k,l}(t, \boldsymbol{r})$ the resultant versions corresponding to $\eta_{j|k,l}(t)$ of (3), (4) and (6), $j = 1, 2, 3$. Let F_k, F_l, G be probability distribution functions of X_k, X_l and Y_1, respectively.

COROLLARY 3.2: *If a coherent system is symmetric w.r.t. components $\{k, l\}$, and lifetimes of base and redundancy components are mutually independent, then,*

$$\begin{aligned}
\mathrm{P}(T(\boldsymbol{X}, \boldsymbol{r}) > t) &= \eta_{1|k,l}(t, \boldsymbol{r})[1 - F_k(t)G^{r_k}(t)][1 - F_l(t)G^{r_l}(t)] \\
&\quad + \eta_{3|k,l}(t, \boldsymbol{r})F_k(t)F_l(t)G^{r_k+r_l}(t) + \eta_{2|k,l}(t, \boldsymbol{r})[F_k(t)G^{r_k}(t) \\
&\quad + F_l(t)G^{r_l}(t) - 2F_k(t)F_l(t)G^{r_k+r_l}(t)], \qquad \textbf{(9)}
\end{aligned}$$

for any allocation strategy $\boldsymbol{r}$ and all $t \geq 0$.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

PROOF: By (8), we have, for any allocation strategy $\boldsymbol{r}$,

$$\begin{aligned}
&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r})>t)\\
&\quad=\eta_{2|k,l}(t,\boldsymbol{r})+[\eta_{1|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})]\mathrm{P}(X_k(\boldsymbol{r})>t,X_l(\boldsymbol{r})>t)\\
&\qquad+[\eta_{3|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})]\mathrm{P}(X_k(\boldsymbol{r})\le t,X_l(\boldsymbol{r})\le t)\\
&\quad=\eta_{2|k,l}(t,\boldsymbol{r})+[\eta_{1|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})][1-F_k(t)G^{r_k}(t)][1-F_l(t)G^{r_l}(t)]\\
&\qquad+[\eta_{3|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})]F_k(t)F_l(t)G^{r_k+r_l}(t)\\
&\quad=\eta_{1|k,l}(t,\boldsymbol{r})[1-F_k(t)G^{r_k}(t)][1-F_l(t)G^{r_l}(t)]\\
&\qquad+\eta_{2|k,l}(t,\boldsymbol{r})[F_k(t)G^{r_k}(t)+F_l(t)G^{r_l}(t)-2F_k(t)F_l(t)G^{r_k+r_l}(t)]\\
&\qquad+\eta_{3|k,l}(t,\boldsymbol{r})F_k(t)F_l(t)G^{r_k+r_l}(t),\qquad\text{for all } t\ge 0,
\end{aligned}$$

and this is just the desired (9). ■

In the context that base and redundancy component lifetimes are independent, Li and Ding [13] obtained for k-out-of-n systems the decomposition of the reliability in Corollary 3.2. Since a k-out-of-n system is symmetric w.r.t. any pair of its components, Theorem 3.1 serves as a generalization of the result of Li and Ding [13].

4. ORDERING ALLOCATION STRATEGIES FOR SYMMETRIC COMPONENTS

Based on the decomposition developed in Section 3, one can evaluate the performance of strategies allocating i.i.d. active redundancies to coherent systems with two symmetric components. Here, we present one general sufficient condition to order allocation strategies.

THEOREM 4.1: *For systems with symmetric components $\{k,l\}$ and not necessarily homogeneous active redundancies, $T(\boldsymbol{X},\boldsymbol{r})\ge_{\mathrm{st}}T(\boldsymbol{X},\boldsymbol{s})$ for $\boldsymbol{r}$, $\boldsymbol{s}$ with $r_i=s_i$ for $i\notin\{k,l\}$ whenever*

$$\max\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}\ge_{\mathrm{st}}\max\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\},\ \ \min\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}\ge_{st}\min\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}. \tag{10}$$

PROOF: Owing to $r_i=s_i$ for $i\ne k,l$, it holds that $\eta_{j|k,l}(t,\boldsymbol{r})=\eta_{j|k,l}(t,\boldsymbol{s})$ for $j=1,2,3$. According to (8), we have, for any $t\ge 0$,

$$\begin{aligned}
&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r})>t)-\mathrm{P}(T(\boldsymbol{X},\boldsymbol{s})>t)\\
&\quad=[\eta_{1|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})][\mathrm{P}(X_k(\boldsymbol{r})>t,X_l(\boldsymbol{r})>t)-\mathrm{P}(X_k(\boldsymbol{s})>t,X_l(\boldsymbol{s})>t)]\\
&\qquad+[\eta_{3|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})][\mathrm{P}(X_k(\boldsymbol{r})\le t,X_l(\boldsymbol{r})\le t)-\mathrm{P}(X_k(\boldsymbol{s})\le t,X_l(\boldsymbol{s})\le t)].
\end{aligned}\tag{11}$$

In view of (10), we also have, for any $t\ge 0$,

$$\begin{aligned}
\mathrm{P}(\max\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}\le t)&=\mathrm{P}(X_k(\boldsymbol{r})\le t,X_l(\boldsymbol{r})\le t)\\
&\le\mathrm{P}(X_k(\boldsymbol{s})\le t,X_l(\boldsymbol{s})\le t)=\mathrm{P}(\max\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}\le t),\\
\mathrm{P}(\min\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}>t)&=\mathrm{P}(X_k(\boldsymbol{r})>t,X_l(\boldsymbol{r})>t)\\
&\ge\mathrm{P}(X_k(\boldsymbol{s})>t,X_l(\boldsymbol{s})>t)=\mathrm{P}(\min\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}>t).
\end{aligned}$$

Then, from $\eta_{1|k,l}(t,\boldsymbol{r})\ge\eta_{2|k,l}(t,\boldsymbol{r})\ge\eta_{3|k,l}(t,\boldsymbol{r})$ it follows immediately that the difference of (11) is non-negative, and this completes the proof. ■

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

Due to the decomposition (1), the two symmetric components must either belong to a common minimal cut set (namely, in parallel structure) or be in series structure and consist the system with the rest of the components. Because of the independence among component lifetimes, one just needs to focus on the subsystem formed by the two symmetric components when allocating redundancies to them. Actually, Theorem 4.1 confirms this point as follows: *for the system symmetric w.r.t. two components to be more reliable, one just resorts to the strategy simultaneously making both series and parallel systems of the two components more reliable in the sense of having a stochastically larger lifetime.*

Allocating active redundancies for two symmetric base components has been investigated in the literature. For example, Fang and Li [5,7] studied allocating a single redundancy to two symmetric components of a coherent system with stochastic arrangement increasing base component lifetimes independent of the redundancy component lifetime, and Fang and Li [6] investigated how to allocate matched redundancies with stochastic arrangement increasing lifetimes to a coherent system with dependent base components symmetric w.r.t. two specific ones. In the presence of the independence among component lifetimes, Theorem 4.1 brings forth a sufficient condition for a better allocation strategy without presuming any ordering among base component lifetimes.

5. ON THE OPTIMAL ALLOCATION STRATEGY FOR SYMMETRIC COMPONENTS

The ordering results in this section help to depict the possible form of the optimal allocation strategy for a system with two symmetric components.

THEOREM 5.1: *For symmetric components $\{k,l\}$ with $k<l$, if $X_k \geq_{\mathrm{st}} X_l$, then,*

$$T(\boldsymbol{X},\boldsymbol{r}) \geq_{\mathrm{st}} T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) \quad \Longleftrightarrow \quad r_k \leq r_l,$$

for any allocation strategy $\boldsymbol{r}$.

PROOF: Since $\eta_{j|k,l}(t,\boldsymbol{r})$ is independent of (r_k,r_l) for any allocation strategy $\boldsymbol{r}$, $j=1,2,3$, it holds that $\eta_{j|k,l}(t,\boldsymbol{r})=\eta_{j|k,l}\big(t,\tau_{k,l}(\boldsymbol{r})\big)$, for $j=1,2,3$. According to (9) of Corollary 3.2, we have

$$\begin{aligned}
&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r})>t)-\mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r}))>t)\\
&\quad=\eta_{1|k,l}(t,\boldsymbol{r})[1-F_k(t)G^{r_k}(t)][1-F_l(t)G^{r_l}(t)]\\
&\qquad-\eta_{1|k,l}(t,\tau_{k,l}(\boldsymbol{r}))[1-F_k(t)G^{r_l}(t)][1-F_l(t)G^{r_k}(t)]\\
&\qquad+\eta_{2|k,l}(t,\boldsymbol{r})[F_k(t)G^{r_k}(t)+F_l(t)G^{r_l}(t)-2F_k(t)F_l(t)G^{r_k+r_l}(t)]\\
&\qquad-\eta_{2|k,l}(t,\tau_{k,l}(\boldsymbol{r}))[F_k(t)G^{r_l}(t)+F_l(t)G^{r_k}(t)-2F_k(t)F_l'(t)G^{r_k+r_l}(t)]\\
&\qquad+\eta_{3|k,l}(t,\boldsymbol{r})F_k(t)F_l(t)G^{r_k+r_l}(t)-\eta_{3|k,l}(t,\tau_{k,l}(\boldsymbol{r}))F_k(t)F_l(t)G^{r_k+r_l}(t)\\
&\quad=\eta_{1|k,l}(t,\boldsymbol{r})[G^{r_l}(t)-G^{r_k}(t)][F_k(t)-F_l(t)]-\eta_{2|k,l}(t,\boldsymbol{r})[G^{r_l}(t)-G^{r_k}(t)][F_k(t)-F_l(t)]\\
&\quad=[\eta_{1|k,l}(t,\boldsymbol{r})-\eta_{2|k,l}(t,\boldsymbol{r})][G^{r_l}(t)-G^{r_k}(t)][F_k(t)-F_l(t)],\quad \text{for all } t\geq 0. \qquad \textbf{(12)}
\end{aligned}$$

Owing to (7) and $X_k \geq_{\mathrm{st}} X_l$, the $r_k \leq r_l$ is equivalent to the nonnegativity of (12). ■

For k-out-of-n systems and i.i.d. redundancy lifetimes, Li and Ding [13] proved in Theorem 1 that more redundancies should be allocated to the weaker base component

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

in the setting of independent component lifetimes. Later, You and Li [19] in their Theorem 4.7 generalized this result to base components with stochastic arrangement increasing lifetimes. The present Theorem 5.1 serves as a partial generalization of the finding in You and Li [19] to coherent systems with two symmetric components and i.i.d. base component lifetimes. According to Theorem 5.1, for systems symmetric w.r.t. two components having stochastically ordered lifetimes, allocating more redundancies to the less reliable one leads to a more reliable system. Thus, a question arises naturally: How close should the numbers of active redundancies allocated to two symmetric components be? The next theorem announces that the difference should not be too large.

THEOREM 5.2: *For symmetric components* $\{k,l\}$ *with* $k<l$, *if* $\max\{Y_1,X_l\}\geq_{\mathrm{st}} X_k$, *then,*

$$s_l-s_k\geq 2,\ r_k=s_k+1,\ r_l=s_l-1 \quad\Longrightarrow\quad T(\boldsymbol{X},\boldsymbol{r})\geq_{\mathrm{st}} T(\boldsymbol{X},\boldsymbol{s}),$$

for any allocation strategies $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *such that* $r_j=s_j$ *for* $j\neq k,l$.

PROOF: For active redundancies, it is plain that $\max\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}\overset{\mathrm{st}}{=}\max\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}$, where '$\overset{\mathrm{st}}{=}$' means both sides have the same distribution. According to Theorem 4.1, we just need to verify

$$\min\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}\geq_{\mathrm{st}}\min\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}. \tag{13}$$

In view of $s_l-s_k\geq 2$ and $\max\{Y_1,X_l\}\geq_{\mathrm{st}} X_k$, we have

$$F_k(t)-F_l(t)G^{s_l-s_k-1}(t)\geq F_k(t)-F_l(t)G(t)\geq 0,\quad \text{for any } t\geq 0,$$

and thus it holds that

$$\begin{aligned}
&\mathrm{P}(\min\{X_k(\boldsymbol{r}),X_l(\boldsymbol{r})\}>t)-\mathrm{P}(\min\{X_k(\boldsymbol{s}),X_l(\boldsymbol{s})\}>t)\\
&\quad=[1-F_k(t)G^{s_k+1}(t)][1-F_l(t)G^{s_l-1}(t)]-[1-F_k(t)G^{s_k}(t)][1-F_l(t)G^{s_l}(t)]\\
&\quad=F_k(t)G^{s_k}(t)+F_l(t)G^{s_l}(t)-F_k(t)G^{s_k+1}(t)-F_l(t)G^{s_l-1}(t)\\
&\quad=[1-G(t)]G^{s_k}(t)[F_k(t)-F_l(t)G^{s_l-s_k-1}(t)]\\
&\quad\geq 0,\qquad \text{for all } t\geq 0.
\end{aligned}$$

This verifies (13) and hence completes the proof. ■

Specifically, for k-out-of-n system with mutually independent base component lifetimes and i.i.d. redundancy component lifetimes, Zhuang and Li [22] obtained the same optimal allocation strategy to two components with lifetimes fulfilling the requirement in the above Theorem 5.2. According to Theorem 5.2, for coherent systems symmetric w.r.t. two components, the allocation strategy assigning the active redundancies as evenly as possible to the two symmetric components is the optimal one. As a consequence, the second corollary reveals the effect of heterogeneity between numbers of redundancies allocated to two symmetric components. The proof can be completed by using Lemma 2.2 and Theorem 5.2, and hence is omitted for brevity.

COROLLARY 5.3: *For a coherent system symmetric w.r.t. components* $\{k,l\}$ *with* $k<l$, *if* $\max\{Y_1,X_l\}\geq_{\mathrm{st}} X_k$, *then,*

$$(r_k,r_l)\preceq_m(s_k,s_l)\quad\Longrightarrow\quad T(\boldsymbol{X},\boldsymbol{r})\geq_{\mathrm{st}} T(\boldsymbol{X},\boldsymbol{s}),$$

for any allocation strategies $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *such that* $r_k\leq r_l$, $s_k\leq s_l$, *and* $r_j=s_j$ *for* $j\neq k,l$.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

6. ON ALLOCATION STRATEGIES FOR ASYMMETRIC COMPONENTS

Naturally, the results on allocation strategies in Sections 4 and 5 may not be suitable for coherent systems if the two concerned components are not symmetric anymore. In this section, we develop some results to evaluate allocation strategies for two components without symmetry from the viewpoint of minimal cut sets and minimal path sets, respectively.

The first one announces that more redundancies should be assigned to the component with the associated minimal cut sets covering those of the other.

THEOREM 6.1: *For components* $\{k,l\}$ *such that* $1 \le k < l \le n$, *if* $\mathscr{C}_k \subseteq \mathscr{C}_l$, *then,*

$$r_k \le r_l \quad \Longrightarrow \quad T(\boldsymbol{X},\boldsymbol{r}) \ge_{\mathrm{st}} T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})),$$

for any allocation strategy $\boldsymbol{r}$.

PROOF: The case of $r_k = r_l$ is trivial. Let us assume $r_k < r_l$. Under the strategy $\boldsymbol{r}$, set

$$Z_i = \max\{Y_{r_{i-1}+1},\dots,Y_{r_{i-1}+r_i}\}I(r_i>0), \tag{14}$$

where the indicator function $I(A)$ equals 1 whenever A is true and 0 otherwise, and denote $g_i(x) = r_i g(x) G^{r_i-1}(x)$ the density function of Z_i whenever $r_i > 0$, $i = 1,\dots,n$. Due to the independence among $Y_1,\dots,Y_m$, evidently $Z_1,\dots,Z_n$ are mutually independent. In view of $\mathscr{C}_k \subseteq \mathscr{C}_l$, by the minimal cut set decomposition (1) we have

$$\begin{aligned}
T(\boldsymbol{X},\boldsymbol{r}) &= \min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r}), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r}), \min_{\mathcal{C}\in\mathscr{C}_k} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r})\Big\} \\
&= \min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r}), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\{X_l(\boldsymbol{r}), \max_{i\in\mathcal{C},i\neq l} X_i(\boldsymbol{r})\}, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k} \max\{X_l(\boldsymbol{r}), X_k(\boldsymbol{r}), \max_{i\in\mathcal{C},i\neq k,l} X_i(\boldsymbol{r})\}\Big\} \\
&= \min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r}), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\{X_l, Z_l, \max_{i\in\mathcal{C},i\neq l} X_i(\boldsymbol{r})\}, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k} \max\{X_l, X_k, Z_l, Z_k, \max_{i\in\mathcal{C},i\neq k,l} X_i(\boldsymbol{r})\}\Big\}.
\end{aligned}$$

Similarly, it also holds that

$$\begin{aligned}
T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) &= \min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i(\tau_{k,l}(\boldsymbol{r})), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\{X_l(\tau_{k,l}(\boldsymbol{r})), \max_{i\in\mathcal{C},i\neq l} X_i(\tau_{k,l}(\boldsymbol{r}))\}, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k} \max\{X_l(\tau_{k,l}(\boldsymbol{r})), X_k(\tau_{k,l}(\boldsymbol{r})), \max_{i\in\mathcal{C},i\neq k,l} X_i(\tau_{k,l}(\boldsymbol{r}))\}\Big\} \\
&= \min\Big\{ \min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} X_i(\boldsymbol{r}), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\{X_l, Z_k, \max_{i\in\mathcal{C},i\neq l} X_i(\boldsymbol{r})\}, \\
&\qquad\qquad \min_{\mathcal{C}\in\mathscr{C}_k} \max\{X_l, X_k, Z_l, Z_k, \max_{i\in\mathcal{C},i\neq k,l} X_i(\boldsymbol{r})\}\Big\}.
\end{aligned}$$

Denote $\boldsymbol{z}_{\{k,l\}} = (z_1,\dots,z_{k-1},z_{k+1},\dots,z_{l-1},z_{l+1},\dots,z_n)$ and

$$\begin{aligned}
\varphi(u,v;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) = \min\Big\{ &\min_{\mathcal{C}\not\in\mathscr{C}_l} \max_{i\in\mathcal{C}} (x_i \vee z_i), \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\{x_l, u, \max_{i\in\mathcal{C},i\neq l} (x_i\vee z_i)\}, \\
&\min_{\mathcal{C}\in\mathscr{C}_k} \max\{x_l, x_k, u, v, \max_{i\in\mathcal{C},i\neq k,l} (x_i\vee z_i)\}\Big\}.
\end{aligned}$$

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

Since, for $u \le v$,

$$\begin{aligned}&\varphi(u,v;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) - \varphi(v,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}})\\ &\quad\overset{\text{sgn}}{=} \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\Big\{x_l, u, \max_{i\in\mathcal{C},i\neq l}(x_i\vee z_i)\Big\} - \min_{\mathcal{C}\in\mathscr{C}_l\setminus\mathscr{C}_k} \max\Big\{x_l, v, \max_{i\in\mathcal{C},i\neq l}(x_i\vee z_i)\Big\} \le 0,\end{aligned}$$

where $\overset{\text{sgn}}{=}$ means a common sign on both sides, by definition $u \le v$ implies $I(\varphi(u,v;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) \le I(\varphi(v,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)$. Note that $r_l > r_k > 0$ implies the increasing ratio $g_l(u)/g_k(u)$. For $r_l > r_k > 0$, it holds that,

$$\begin{aligned}&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t \,|\, \boldsymbol{X}=\boldsymbol{x}, \boldsymbol{Z}_{\{k,l\}} = \boldsymbol{z}_{\{k,l\}}) - \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t \,|\, \boldsymbol{X}=\boldsymbol{x}, \boldsymbol{Z}_{\{k,l\}} = \boldsymbol{z}_{\{k,l\}})\\ &\quad= \mathrm{P}(\varphi(Z_l,Z_k;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) - \mathrm{P}(\varphi(Z_k,Z_l;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)\\ &\quad= \iint_{u,v\ge 0} [I(\varphi(u,v;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) - I(\varphi(v,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)] g_l(u) g_k(v)\,\mathrm{d}u\,\mathrm{d}v\\ &\quad= \iint_{v\ge u\ge 0} [I(\varphi(u,v;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) - I(\varphi(v,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)][g_l(u)g_k(v) - g_l(v)g_k(u)]\,\mathrm{d}u\,\mathrm{d}v\\ &\quad\ge 0, \qquad \text{for any } t \ge 0.\end{aligned}$$

In view of $I(\varphi(u,0;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) \ge I(\varphi(0,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)$ for $u \ge 0$, we have, for $r_l > r_k = 0$,

$$\begin{aligned}&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t \,|\, \boldsymbol{X}=\boldsymbol{x}, \boldsymbol{Z}_{\{k,l\}} = \boldsymbol{z}_{\{k,l\}}) - \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t \,|\, \boldsymbol{X}=\boldsymbol{x}, \boldsymbol{Z}_{\{k,l\}} = \boldsymbol{z}_{\{k,l\}})\\ &\quad= \mathrm{P}(\varphi(Z_l,Z_k;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) - \mathrm{P}(\varphi(Z_k,Z_l;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)\\ &\quad= \int_0^{+\infty} [I(\varphi(u,0;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t) - I(\varphi(0,u;\boldsymbol{x},\boldsymbol{z}_{\{k,l\}}) > t)] g_l(u)\,\mathrm{d}u\\ &\quad\ge 0, \qquad \text{for any } t \ge 0.\end{aligned}$$

Now, by double expectation we reach $\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t) \ge \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t)$ for all $t \ge 0$. ■

Intuitively, the component having more minimal cut sets contributes more to the system reliability, and hence allocated with more active redundancies will benefit the system reliability more, which is confirmed by the finding of Theorem 6.1. When one component having minimal cut sets covering those of another, Theorem 6.1 shows that more active redundancies should be allocated to it. Under the same scenario, Fang and Li [7] in Theorem 2 considered allocating one active redundancy to two components without assuming independence among the component and redundancy lifetimes. They also showed that the active redundancy should go to the component having more minimal cut sets. Theorem 6.1 serves as a partial generalization of the finding of Fang and Li [7].

Let $\boldsymbol{r}^*$ be the allocation strategy with $r_k^* = 0$ and $r_l^* = m$. As a direct consequence of Theorem 6.1, we get the following corollary.

COROLLARY 6.2: *For components $\{k,l\}$ such that $1 \le k,l \le n$, if $\mathscr{C}_k \subseteq \mathscr{C}_l$, then,*

$$T(\boldsymbol{X},\boldsymbol{r}^*) \ge_{\text{st}} T(\boldsymbol{X},\boldsymbol{r}),$$

for any allocation strategy $\boldsymbol{r}$ with $r_k + r_l = m$ and $r_i = r_i^*$ for $i \ne k,l$.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

Based on Corollary 6.2, it is optimal to allocate all active redundancies for two components to the one with the associated class of minimal cut sets covering those of another.

As a dual to Theorem 6.1, the second result takes the viewpoint from the minimal path set.

THEOREM 6.3: *For components* $\{k,l\}$*, if* $\mathscr{P}_k \subseteq \mathscr{P}_l$ *and* $X_k \geq_{\mathrm{lr}} X_l$*, then,*

$$r_k \leq r_l \quad \Longrightarrow \quad T(\boldsymbol{X},\boldsymbol{r}) \geq_{\mathrm{st}} T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})),$$

for any allocation strategy $\boldsymbol{r}$*.*

PROOF: The case of $r_k = r_l$ is trivial. For $r_k < r_l$, in view of $\mathscr{P}_k \subseteq \mathscr{P}_l$ and (2), we have

$$\begin{aligned}
T(\boldsymbol{X},\boldsymbol{r}) &= \max\Big\{ \max_{\mathcal{P}\notin\mathscr{P}_l} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_l\setminus\mathscr{P}_k} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_k} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r})\Big\} \\
&= \max\Big\{ \max_{\mathcal{P}\notin\mathscr{P}_l} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_l\setminus\mathscr{P}_k} \min\{X_l\vee Z_l, \min_{i\in\mathcal{P},i\neq l} X_i(\boldsymbol{r})\}, \\
&\qquad \max_{\mathcal{P}\in\mathscr{P}_k} \min\{X_l\vee Z_l, X_k\vee Z_k, \min_{i\in\mathcal{P},i\neq k,l} X_i(\boldsymbol{r})\}\Big\}, \\
T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) &= \max\Big\{ \max_{\mathcal{P}\notin\mathscr{P}_l} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_l\setminus\mathscr{P}_k} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_k} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r})\Big\} \\
&= \max\Big\{ \max_{\mathcal{P}\notin\mathscr{P}_l} \min_{i\in\mathcal{P}} X_i(\boldsymbol{r}), \max_{\mathcal{P}\in\mathscr{P}_l\setminus\mathscr{P}_k} \min\{X_l\vee Z_k, \min_{i\in\mathcal{P},i\neq l} X_i(\boldsymbol{r})\}, \\
&\qquad \max_{\mathcal{P}\in\mathscr{P}_k} \min\{X_l\vee Z_k, X_k\vee Z_l, \min_{i\in\mathcal{P},i\neq k,l} X_i(\boldsymbol{r})\}\Big\},
\end{aligned}$$

where Z_i's are given in (14). Denote $\boldsymbol{w}_{\{k,l\}} = (\boldsymbol{x}_{\{k,l\}}, \boldsymbol{z}_{\{k,l\}})$, $\boldsymbol{W}_{\{k,l\}} = (\boldsymbol{X}_{\{k,l\}}, \boldsymbol{Z}_{\{k,l\}})$ and

$$\begin{aligned}
\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}}) = \max\{ &\max_{\mathcal{P}\notin\mathscr{P}_l} \min_{i\in\mathcal{P}}(x_i\vee z_i), \max_{\mathcal{P}\in\mathscr{P}_l\setminus\mathscr{P}_k} \min\{x\vee u, \min_{i\in\mathcal{P},i\neq l}(x_i\vee z_i)\}, \\
&\max_{\mathcal{P}\in\mathscr{P}_k} \min\{x\vee u, y\vee v, \min_{i\in\mathcal{P},i\neq k,l}(x_i\vee z_i)\}\}.
\end{aligned}$$

Denote f_k and f_l the density functions of X_k and X_l, respectively. For any $t>0$,

$$\begin{aligned}
&\mathrm{P}(\psi(X_l,X_k,u,v;\boldsymbol{w}_{\{k,l\}})>t) - \mathrm{P}(\psi(X_l,X_k,v,u;\boldsymbol{w}_{\{k,l\}})>t) \qquad \textbf{(15)} \\
&= \left(\iint_{y\leq x} + \iint_{y\geq x}\right)[I(\psi(x,y,v,u;\boldsymbol{w}_{\{k,l\}})\leq t) \\
&\quad - I(\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}})\leq t)]f_l(x)f_k(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{y\leq x}[I(\psi(x,y,v,u;\boldsymbol{w}_{\{k,l\}})\leq t) - I(\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}})\leq t)]f_l(x)f_k(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_{y\leq x}[I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}})\leq t) - I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}})\leq t)]f_l(y)f_k(x)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

$$
\begin{aligned}
&= \iint\limits_{y\le x} [I(\psi(x,y,v,u;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t)] f_l(x) f_k(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint\limits_{y\le x} I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t)[f_l(x)f_k(y) - f_l(y)f_k(x)]\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \iint\limits_{y\le x} [I(\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t)] f_l(x) f_k(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \iint\limits_{y\le x} I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t)[f_l(x)f_k(y) - f_l(y)f_k(x)]\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint\limits_{y\le x} [(I(\psi(x,y,v,u;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t)) \\
&\quad - (I(\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t))] f_l(x) f_k(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint\limits_{y\le x} [I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t)] \\
&\quad \times [f_l(x)f_k(y) - f_l(y)f_k(x)]\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

According to Lemma 4.1 of Fang and Li [6], for $u \le v$ and $y \le x$, it holds that $I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) \le I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t)$ and

$$
\begin{aligned}
&I(\psi(x,y,v,u;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) \\
&\quad \ge I(\psi(x,y,u,v;\boldsymbol{w}_{\{k,l\}}) \le t) - I(\psi(y,x,v,u;\boldsymbol{w}_{\{k,l\}}) \le t).
\end{aligned}
$$

Since $X_k \ge_{\mathrm{lr}} X_l$ implies $f_l(x)f_k(y) \le f_l(y)f_k(x)$ for any $x \ge y$, we have nonnegative (15) for $u \le v$ and $t \ge 0$. Moreover, $r_l > r_k > 0$ implies that $g_l(u)/g_k(u)$ is increasing and hence

$$
\begin{aligned}
&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t \,|\, \boldsymbol{W}_{\{k,l\}} = \boldsymbol{w}_{\{k,l\}}) - \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t \,|\, \boldsymbol{W}_{\{k,l\}} = \boldsymbol{w}_{\{k,l\}}) \\
&\quad = \iint\limits_{u,v\ge 0} [\mathrm{P}(\psi(X_l,X_k,u,v;\boldsymbol{w}_{\{k,l\}}) > t) - \mathrm{P}(\psi(X_l,X_k,v,u;\boldsymbol{w}_{\{k,l\}}) > t)] g_l(u) g_k(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \iint\limits_{v\ge u\ge 0} [\mathrm{P}(\psi(X_l,X_k,u,v;\boldsymbol{w}_{\{k,l\}}) > t) - \mathrm{P}(\psi(X_l,X_k,v,u;\boldsymbol{w}_{\{k,l\}}) > t)] \\
&\qquad\qquad \cdot [g_l(u)g_k(v) - g_l(v)g_k(u)]\,\mathrm{d}u\,\mathrm{d}v \\
&\quad \ge 0, \qquad \text{for any } t \ge 0.
\end{aligned}
$$

Also, as in (15), $X_k \ge_{\mathrm{lr}} X_l$ implies $\mathrm{P}(\psi(X_l,X_k,u,0;\boldsymbol{w}_{\{k,l\}}) > t) \ge \mathrm{P}(\psi(X_l,X_k,0,u;\boldsymbol{w}_{\{k,l\}}) > t)$ for any $u \ge 0$. Thus, for $r_l > r_k = 0$, it holds that

$$
\begin{aligned}
&\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t \,|\, \boldsymbol{W}_{\{k,l\}} = \boldsymbol{w}_{\{k,l\}}) - \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t \,|\, \boldsymbol{W}_{\{k,l\}} = \boldsymbol{w}_{\{k,l\}}) \\
&\quad = \int [\mathrm{P}(\psi(X_l,X_k,u,0;\boldsymbol{w}_{\{k,l\}}) > t) - \mathrm{P}(\psi(X_l,X_k,0,u;\boldsymbol{w}_{\{k,l\}}) > t)] g_l(u)\,\mathrm{d}u \\
&\quad \ge 0, \qquad \text{for any } t \ge 0.
\end{aligned}
$$

Therefore, by double expectation we reach $\mathrm{P}(T(\boldsymbol{X},\boldsymbol{r}) > t) \ge \mathrm{P}(T(\boldsymbol{X},\tau_{k,l}(\boldsymbol{r})) > t)$ for $t \ge 0$. ■

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

With the help of Theorem 6.3, we can explicitly obtain the optimal allocation for two components satisfying that minimal path sets of one component cover those of the other. Before presenting the theorem, let us build a lemma of independent interest. Recall that a bivariate function $f(x,y):\mathbb{R}^2\mapsto\mathbb{R}$ is said to be *totally positive of order 2* (TP2) if $f(x_1,y_1)f(x_2,y_2)\geq f(x_1,y_2)f(x_2,y_1)$ for any $x_1\leq x_2$ and $y_1\leq y_2$.

LEMMA 6.4: *If X is independent of the i.i.d. sequence Y_i's, then, for $n\geq m\geq 0$,*

$$X\lesssim_{\mathrm{rh}} Y_1 \quad\Longrightarrow\quad \max\{X,Y_1,\ldots,Y_n\}\geq_{\mathrm{lr}}\max\{X,Y_1,\ldots,Y_m\}.$$

PROOF: Note that $X\lesssim_{\mathrm{rh}} Y_1$ implies $\max\{X,Y_1\}\geq_{\mathrm{lr}} X$. Denote F, G distribution functions, and f, g probability densities of X and Y_i, respectively. It suffices to verify that $h_n(t)=f(t)G^n(t)+nG^{n-1}(t)F(t)g(t)$, the density of $\max\{X,Y_1,\ldots,Y_n\}$, is TP2 in $t\geq 0$ and $n\geq 1$.

For $x\leq y$ and $m\leq n$,

$$\begin{aligned}
&h_m(x)h_n(y)-h_n(x)h_m(y)\\
&\quad=[f(x)f(y)G^m(x)G^n(y)+mng(x)g(y)F(x)F(y)G^{m-1}(x)G^{n-1}(y)\\
&\qquad+nf(x)g(y)F(y)G^m(x)G^{n-1}(y)+mf(y)g(x)F(x)G^n(y)G^{m-1}(x)]\\
&\qquad-[f(x)f(y)G^m(y)G^n(x)+mng(x)g(y)F(x)F(y)G^{m-1}(y)G^{n-1}(x)\\
&\qquad+nf(y)g(x)F(x)G^m(y)G^{n-1}(x)+mf(x)g(y)F(y)G^n(x)G^{m-1}(y)]\\
&\quad=f(x)f(y)G^m(x)G^m(y)[G^{n-m}(y)-G^{n-m}(x)]\\
&\qquad+mng(x)g(y)F(x)F(y)G^{m-1}(x)G^{m-1}(y)[G^{n-m}(y)-G^{n-m}(x)]\\
&\qquad+f(x)g(y)F(y)G^m(x)G^{m-1}(y)[nG^{n-m}(y)-mG^{n-m}(x)]\\
&\qquad+f(y)g(x)F(x)G^m(y)G^{m-1}(x)[mG^{n-m}(y)-nG^{n-m}(x)]. \qquad \textbf{(16)}
\end{aligned}$$

Note that

$$\begin{aligned}
&[nG^{n-m}(y)-mG^{n-m}(x)]+[mG^{n-m}(y)-nG^{n-m}(x)]\\
&\quad=(n+m)[G^{n-m}(y)-G^{n-m}(x)]\geq 0,\\
&[nG^{n-m}(y)-mG^{n-m}(x)]-[mG^{n-m}(y)-nG^{n-m}(x)]\\
&\quad=(n-m)[G^{n-m}(y)-G^{n-m}(x)]\geq 0,
\end{aligned}$$

and hence

$$nG^{n-m}(y)-mG^{n-m}(x)\geq|mG^{n-m}(y)-nG^{n-m}(x)|. \qquad \textbf{(17)}$$

Moreover, $X\lesssim_{\mathrm{rh}} Y_i$ implies that $\frac{f(x)}{F(x)}/\frac{g(x)}{G(x)}=\frac{f(x)G(x)}{F(x)g(x)}$ decreases and hence

$$\frac{f(x)g(y)F(y)G^m(x)G^{m-1}(y)}{f(y)g(x)F(x)G^m(y)G^{m-1}(x)}=\left[\frac{f(x)G(x)}{F(x)g(x)}\right]\left[\frac{f(y)G(y)}{F(y)g(y)}\right]^{-1}\geq 1.$$

Along with (17) this implies that the sum of the last two terms of (16) is nonnegative. In light of the nonnegativity of the first two terms of (16), we conclude that (16) is nonnegative, and this verifies that $h_n(t)$ is TP2 in (t,n). ■

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

THEOREM 6.5: *For components* $\{k,l\}$ *with* $\mathscr{P}_k \subseteq \mathscr{P}_l$, *if* $\max\{X_l, Y_1, \ldots, Y_{m-1}\} \le_{\mathrm{lr}} X_k \lesssim_{\mathrm{rh}} Y_1$ *and* $X_l \lesssim_{\mathrm{rh}} Y_1$, *then,*

$$T(\boldsymbol{X}, \boldsymbol{r}^*) \ge_{st} T(\boldsymbol{X}, \boldsymbol{r}),$$

for any allocation strategy $\boldsymbol{r}$ *with* $r_k + r_l = m$ *and* $r_j = r_j^*$, $j \ne k, l$.

PROOF: Since $X_k \ge_{\mathrm{lr}} \max\{X_l, Y_1, \ldots, Y_{m-1}\}$, $X_k \lesssim_{\mathrm{rh}} Y_i$ and $X_l \lesssim_{\mathrm{rh}} Y_i$ for all $i = 1, \ldots, m$, according to Lemma 6.4, we have, for $m \ge r_l \ge r_k \ge 1$,

$$\max\{X_k, Y_1, \ldots, Y_{r_k-1}\} \ge_{\mathrm{lr}} X_k \ge_{\mathrm{lr}} \max\{X_l, Y_1, \ldots, Y_m\} \ge_{\mathrm{lr}} \max\{X_l, Y_1, \ldots, Y_{r_l}\}. \quad \textbf{(18)}$$

Consider the system with $r_k - 1$ active redundancies allocated to the component X_k and r_l allocated to the component X_l. For the remaining active redundancy with lifetime Y, the concerned scenario can be essentially described as allocating one active redundancy to two base components with lifetimes $\max\{X_k, Y_1, \ldots, Y_{r_k-1}\}$ and $\max\{X_l, Y_1, \ldots, Y_{r_l}\}$, when the minimal path sets of the latter base component cover those of the former.

By Theorem 6.3, (18) implies that the active redundancy Y should be assigned to the base component having more minimal path sets. Iterating the same procedure and using Theorem 6.3, (18) guarantees that compared with any allocation strategy with $1 \le r_k \le r_l$, assigning all the m active redundancies to the component X_l results in a more reliable system. That is, the desired allocation strategy $\boldsymbol{r}^*$ is indeed optimal. This completes the proof. ■

In the context that the class of minimal path sets of one base component covering that of another, Fang and Li [5,7] had a discussion on the possible form of the optimal allocation strategy of one active redundancy and the matched active redundancies in the presence of dependence, and it was found out that the single redundancy should be assigned to the base component having more minimal paths and the stronger redundancy should be allotted to the weaker base component having more minimal paths. However, lack of the explicit optimal allocation strategy retards the practice especially when a bunch of active redundancies are under consideration. To the best of our knowledge, Theorem 6.5 presents the first sufficient condition on the explicit optimal allocation strategy to two asymmetric base components in the setting of the mutually independent base and redundancy component lifetimes.

7. NUMERICAL EXAMPLES

To close this study, we illustrate the theoretical results through several numerical examples.

Based on components with independent lifetimes $\boldsymbol{X} = (X_1, \ldots, X_4)$ the system in Figure 1 gets the lifetime $T(\boldsymbol{X}) = \min\{\max\{\min\{X_1, X_2\}, X_3\}, X_4\}$. Assume that i.i.d. active redundancy lifetimes Y_1, Y_2, Y_3 have distribution function G and $X_1, \ldots, X_4, Y_1, \ldots, Y_3$ are mutually independent. Obviously, $T(\boldsymbol{X})$ is symmetric w.r.t. X_1 and X_2, $\mathscr{C}_1 \subseteq \mathscr{C}_3$ and $\mathscr{P}_3 \subseteq \mathscr{P}_4$. Under allocation strategy $\boldsymbol{r} = (r_1, \ldots, r_4)$ the system has reliability

$$\begin{aligned}&\mathrm{P}(T(\boldsymbol{X}, \boldsymbol{r}) > t)\\ &\quad = [1 - F_4(t)G^{r_4}(t)]\{1 - F_3(t)G^{r_3}(t)[1 - (1 - F_1(t)G^{r_1}(t))(1 - F_2(t)G^{r_2}(t))]\}.\end{aligned}$$

Let us consider the following five cases.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

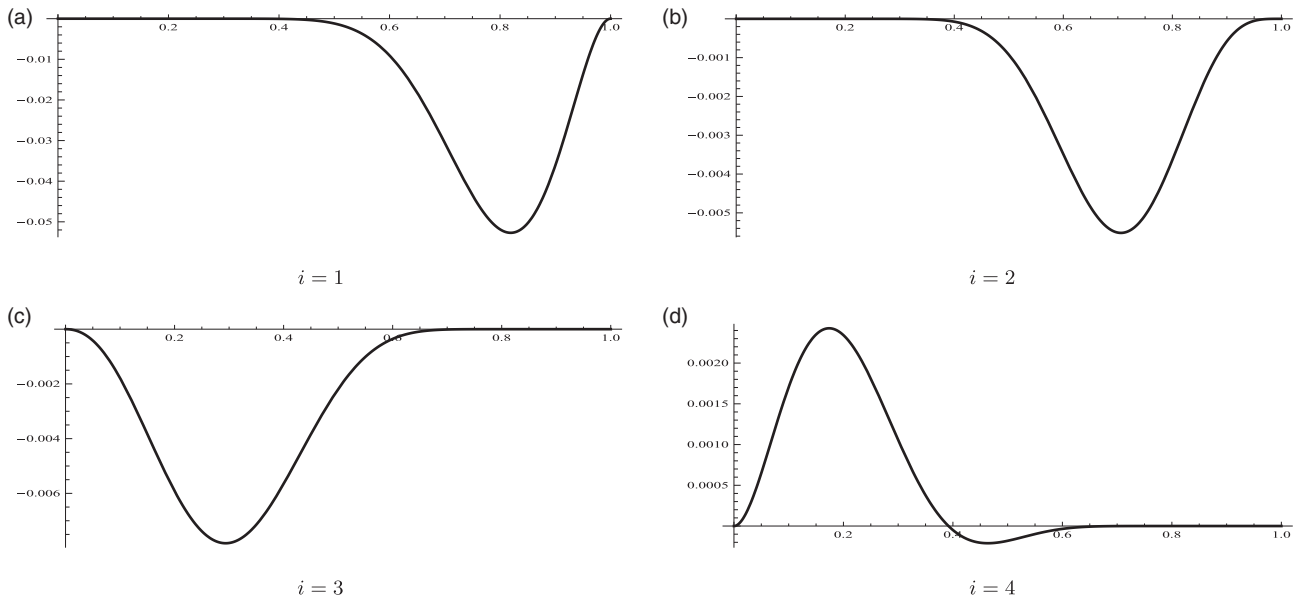


FIGURE **2.** Curves $\mathrm{P}(T(\boldsymbol{X}, \boldsymbol{r}_i) > (1-t)/t) - \mathrm{P}(T(\boldsymbol{X}, \boldsymbol{r}_i^*) > (1-t)/t)$, $i = 1, 2, 3, 4$.

Reversed hazard rate ratio of X_3 and Y_1 $j = 1$ (solid), $j = 2$ (dashed), $j = 3$ (dotted)

FIGURE **3.** Reversed hazard rate ratio and $\mathrm{P}(T(\boldsymbol{X}, \boldsymbol{s}_j) > (1-t)/t) - \mathrm{P}(T(\boldsymbol{X}, \boldsymbol{r}_5^*) > (1-t)/t)$.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

(i) All redundancies are allocated to X_1 and X_2. Set $G(x) = 1 - e^{-x}$ and $F_i(x) = 1 - e^{-(i+1)x}$, $i = 1, \ldots, 4$. Then, $Y_j \geq_{\mathrm{st}} X_1 \geq_{\mathrm{st}} X_2$ for $j = 1, 2, 3$. Consider the allocation strategies $\boldsymbol{r}_1 = (0, 3, 0, 0)$ and $\boldsymbol{r}_1^* = (1, 2, 0, 0)$, the non-positive difference in Figure 2a indicates $T(\boldsymbol{X}, \boldsymbol{r}_1) \leq_{\mathrm{st}} T(\boldsymbol{X}, \boldsymbol{r}_1^*)$, which confirms the result of Theorem 5.2.

(ii) All redundancies are allocated to X_1 and X_3. Set the same lifetime distributions in case (i). For $\boldsymbol{r}_2 = (1, 0, 2, 0)$ and $\boldsymbol{r}_2^* = (0, 0, 3, 0)$. The non-positive curve in Figure 2b tends to $T(\boldsymbol{X}, \boldsymbol{r}_2) \leq_{\mathrm{st}} T(\boldsymbol{X}, \boldsymbol{r}_2^*)$, consistent to the result of Corollary 6.2.

(iii) All redundancies are allocated to X_3 and X_4. Set $G(x) = e^{-(4x)^{-1}}$, $F_1(x) = e^{-2/x}$, $F_2(x) = e^{-3/x}$, $F_3(x) = e^{-5/x}$ and $F_4(x) = e^{-4/x}$. Clearly, $X_3 \lesssim_{\mathrm{rh}} Y_j$, $X_4 \lesssim_{\mathrm{rh}} Y_j$ for $j = 1, 2, 3$, and $X_3 \geq_{\mathrm{lr}} \max\{X_4, Y_1, Y_2, Y_3\}$. For strategies $\boldsymbol{r}_3 = (0, 0, 1, 2)$ and $\boldsymbol{r}_3^* = (0, 0, 0, 3)$, the difference curve in Figure 2c is non-positive, and this confirms $T(\boldsymbol{X}, \boldsymbol{r}_3) \leq_{\mathrm{st}} T(\boldsymbol{X}, \boldsymbol{r}_3^*)$, the result of Theorem 6.5.

(iv) All redundancies are allocated to X_3 and X_4. Set $G(x) = e^{-(4x)^{-1}}$ and $F_i(x) = e^{-(i+1)/x}$, $i = 1, \cdots, 4$. As in (iii), $X_3 \lesssim_{\mathrm{rh}} Y_j$ and $X_4 \lesssim_{\mathrm{rh}} Y_j$ for $j = 1, 2, 3$. But $X_3 \not\geq_{\mathrm{lr}} \max\{X_4, Y_1, Y_2\}$ violates the condition of Theorem 6.5. For strategies $\boldsymbol{r}_4 = (0, 0, 1, 2)$ and $\boldsymbol{r}_4^* = (0, 0, 0, 3)$, the difference curve is seen to cross the horizontal axis in Figure 2d, and this negates the usual stochastic order between $T(\boldsymbol{X}, \boldsymbol{r}_4)$ and $T(\boldsymbol{X}, \boldsymbol{r}_4^*)$. Therefore, the likelihood ratio order in Theorem 6.5 cannot be dropped off.

(v) All redundancies are allocated to X_3 and X_4. Set $G(x) = F_4(x) = (1 - e^{-x})^{1/3}$, $F_1(x) = F_2(x) = 1 - e^{-x}$ and $F_3(x) = e^{-1/x}$. Clearly, $X_4 \lesssim_{\mathrm{rh}} Y_j$ for $j = 1, 2, 3$, and the non-monotonic curve in Figure 3a verifies $X_3 \not\lesssim_{\mathrm{rh}} Y_j$. Moreover, $X_3 \geq_{\mathrm{lr}} \max\{X_4, Y_1, Y_2\}$ satisfies the condition in Theorem 6.5. For strategies $\boldsymbol{s}_1 = (0, 0, 3, 0)$, $\boldsymbol{s}_2 = (0, 0, 1, 2)$, $\boldsymbol{s}_3 = (0, 0, 2, 1)$ and $\boldsymbol{r}_5^* = (0, 0, 0, 3)$, the difference curves in Figure 3b are all beneath the horizontal axis, indicating that $T(\boldsymbol{X}, \boldsymbol{r}_5^*)$ is the stochastically largest and hence $\boldsymbol{r}_5^*$ is still the optimal one even though the relative aging order between the base component and redundancy lifetimes is violated.

Acknowledgements

The authors would like to thank the anonymous reviewers for their valuable comments, which helped improve the presentation of this article. Dr. Fang's research is supported by National Natural Science Foundation of China (11601303) and Guangdong Natural Science Foundation (2016A030310076).

References

1. Barlow, R.E. & Proschan, F. (1981). *Statistical theory of reliability and life testing: probability models.* Maryland: Silver Springs.
2. Boland, P.J., El-Neweihi, E., & Proschan, F. (1992). Stochastic order for redundancy allocation in series and parallel systems. *Advances in Applied Probability* 24: 161–171.
3. El-Neweihi, E., Proschan, F., & Sethuraman, J. (1986). Optimal allocation of components in parallel-series and series-parallel systems. *Journal of Applied Probability* 23: 770–777.
4. Feng, Q. & Shanthikumar, J.G. (2018). Arrangement increasing resource allocation. *Methodology and Computing in Applied Probability* 20: 935–955.
5. Fang, R. & Li, X. (2016). On allocating one active redundancy to coherent systems with dependent and heterogeneous components' lifetimes. *Naval Research Logistics* 63: 335–345.
6. Fang, R. & Li, X. (2017). On matched active redundancy allocation for coherent systems with statistically dependent component lifetimes. *Naval Research Logistics* 64: 580–598.
7. Fang, R. & Li, X. (2018). On active redundancy allocation for coherent systems - from the viewpoint of minimal cut decomposition. *Operations Reserach Letters* 46: 233–239.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press

8. Folkman, J.H. & Fulkerson, D.R. (1969) Edge colorings in bipartite graphs. In R.C. Bose & T.A. Dowling, (eds.), *Combinatorial mathematics and its applications*. Chapel Hill: University of North Carolina Press, pp. 561–577.
9. Hu, T. & Wang, Y. (2009). Optimal allocation of active redundancies in r-out-of-n systems. *Journal of Statistical Planning and Inference* 139: 3733–3737.
10. Li, C. & Li, X. (2016). Relative ageing of series and parallel systems with statistically independent and heterogeneous component lifetimes. *IEEE Transactions on Reliability* 65: 1014–1021.
11. Li, H. & Li, X (2013). *Stochastic orders in reliability and risk*. New York: Springer.
12. Li, X. & Ding, W. (2013). On allocation of active redundancies to systems: a brief review. In H. Li & X. Li, (eds.), *Stochastic orders in reliability and risk*. New York: Springer, pp. 235–254.
13. Li, X. & Ding, W. (2010). Optimal allocation of active redundancies to k-out-of-n systems with heterogeneous components. *Journal of Applied Probability* 47: 254–263.
14. Liyanage, L. & Shanthikumar, J.G. (1993). Allocation through stochastic schur convexity and stochastic transposition increasingness. *Stochastic Inequalities* 22: 253–273.
15. Marshall, A.W., Olkin, I., & Arnold, B.C (2011). *Inequalities: theory of majorization and its applications*. New York: Springer.
16. Rezaei, M., Gholizadeh, B., & Izadkhah, S. (2015). On relative reversed hazard rate order. *Communications in Statistics - Theory Methods* 44: 300–308.
17. Shaked, M. & Shanthikumar, J.G. (1992). Optimal allocation of resources to nodes of series and parallel systems. *Advances in Applied Probability* 24: 894–914.
18. Shaked, M. & Shanthikumar, J.G (2007). *Stochastic orders*. New York: Springer.
19. You, Y. & Li, X. (2014). On allocating redundancies to k-out-of-n reliability systems. *Applied Stochastic Models in Business and Industries* 30: 361–371.
20. Zhao, P., Chan, P., & Ng, H.K.T. (2012). Optimal allocation of redundancies in series systems. *European Journal of Operational Research* 220: 673–683.
21. Zhao, P., Zhang, Y., & Chen, J. (2017). Optimal allocation policy of one redundancy in a n-component series system. *European Journal of Operational Research* 257: 656–668.
22. Zhuang, J. & Li, X. (2015). Allocating redundancies to k-out-of-n systems with independent and heterogeneous components. *Communications in Statistics - Theory and Methods* 44: 5109–5119.

https://doi.org/10.1017/S0269964818000359 Published online by Cambridge University Press