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Infinite iterated function systems with overlaps

SZE-MAN NGAI†‡ and JI-XI TONG§

† *College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China*

‡ *Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA*

(e-mail: smngai@georgiasouthern.edu)

§ *College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China*

(e-mail: jxtongmath@gmail.com)

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Abstract. We formulate two natural but different extensions of the weak separation condition to infinite iterated function systems of conformal contractions with overlaps, and study the associated topological pressure functions. We obtain a formula for the Hausdorff dimension of the limit sets under these weak separation conditions.

1. *Introduction*

Let X be a non-empty compact subset of $\mathbb{R}^d$, and I be a finite or countably infinite set. We call a family of distinct maps $\{S_i\}_{i\in I}$ an *iterated function system (IFS)* if each $S_i : X \to X$ is an injective contraction and, moreover, the family satisfies the following *uniform contractivity condition*: there exists $\rho \in (0, 1)$ such that

$$|S_i(x) - S_i(y)| \le \rho|x - y| \quad \text{for all } i \in I \text{ and } x, y \in X. \tag{1.1}$$

We call $\{S_i\}_{i\in I}$ an *infinite iterated function system (IIFS)* if I is countably infinite, and a *finite iterated function system (FIFS)* if I is finite.

IIFSs are known to have strictly more powerful descriptive power than FIFSs. In fact, Fernau [**5**] showed that in any separable metric space, every closed set is the fixed point of an IIFS, and moreover, there is a closed and bounded subset of a complete metric space that is the fixed point of an IIFS but not of any FIFS. Mauldin and Urbański [**11**] studied dimensions and measures associated with IIFSs of conformal contractions satisfying the open set condition. Their work revealed some interesting phenomena of IIFSs that are not observed for FIFSs. They also demonstrated the application of IIFSs to complex continued fractions.

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Although IIFSs satisfying the open set condition have been studied extensively, not much is known for those that have overlaps (i.e. not satisfying the open set condition). It is desirable to know to what extent results concerning FIFSs with overlaps can be generalized to IIFSs. This is a major motivation of this paper.

To describe our setup, we first introduce some notation. Let

$$I^0 := \{\emptyset\}, \qquad I^k := \{(i_1, \ldots, i_k) : i_1, \ldots, i_k \in I\} \quad \text{for } k \geq 1,$$
$$I^* := \bigcup_{k\geq 0} I^k, \qquad I^\infty := \{(i_1, i_2, \ldots) : i_j \in I \ \forall j \geq 1\}.$$

We call I^* the space of *finite words* in the *alphabets* $i \in I$, and I^∞ the *coding space*. For $\mathbf{i} = (i_1, \ldots, i_k) \in I^k$, we let $|\mathbf{i}| = k$ denote the *length* of $\mathbf{i}$, and write $S_\mathbf{i} := S_{i_1} \circ \cdots \circ S_{i_k}$ ($S_\emptyset$ is defined to be the identity). We also denote $\mathbf{i} = (i_1, \ldots, i_k)$ simply by $\mathbf{i} = i_1 \cdots i_k$. If $\mathbf{i} \in I^* \bigcup I^\infty$ and $n \geq 1$ does not exceed the length of $\mathbf{i}$, we denote by $\mathbf{i}|_n$ the word $i_1 \cdots i_n$ consisting of the first n alphabets of $\mathbf{i}$.

For any $E \subseteq \mathbb{R}^d$, we let $\mathcal{L}^d(E)$, $\dim_{\mathrm{H}}(E)$, $\dim_{\mathrm{P}}(E)$, $\dim_{\mathrm{B}}(E)$, $\mathcal{H}^\alpha(E)$, $\mathcal{P}^\alpha(E)$, $|E|$, E°, and ∂E denote, respectively, the d-dimensional Lebesgue measure, Hausdorff dimension, packing dimension, box dimension, α-dimensional Hausdorff measure, α-dimensional packing measure, diameter, interior, and boundary of E. We let $\#A$ denote the cardinality of a set A.

A major objective of this paper is to study the Hausdorff dimension of the limit set K. We are mainly interested in IIFSs of conformal contractions, which include the class of contractive similitudes. Let $V \subseteq \mathbb{R}^d$ be an open subset. Recall that a map $S : V \to V$ is *conformal* on V if, for each $x \in V$, $S'(x)$ is a similarity matrix. For such an S, we have $|\det S'(x)| = \|S'(x)\|^d$, where $\|S'(x)\| := \sup\{|S'(x)y| : y \in \mathbb{R}^d, |y| = 1\}$ is the operator norm of the matrix $S'(x)$.

Definition 1.1. Let $\emptyset \neq X \subset \mathbb{R}^d$ be compact, I a finite or countably infinite set, and $S_i : X \to X$, $i \in I$, conformal contractions. We call $\{S_i\}_{i\in I}$ an IFS of injective C^1 conformal contractions on X if each S_i can be extended to a C^1 injective conformal contraction on some bounded open connected neighborhood V of X so that (1.1) is satisfied on V and, furthermore,

$$0 < \inf_{x\in V} \|S_i'(x)\| \leq \sup_{x\in V} \|S_i'(x)\| < 1 \quad \text{for all } i \in I.$$

We write

$$r_\mathbf{i} := \inf_{x\in V} \|S_\mathbf{i}'(x)\|, \qquad R_\mathbf{i} := \sup_{x\in V} \|S_\mathbf{i}'(x)\|,$$
$$r_\sigma := r_\mathbf{i}, \qquad R_\sigma := R_\mathbf{i} \quad \text{if } \sigma = S_\mathbf{i} \quad \text{for some } \mathbf{i} \in I^*,$$
$$R = \sup_{i\in I} R_i. \tag{1.2}$$

The following inequalities will be used repeatedly (see [**1**, **10**]):

$$R_\mathbf{ij} \leq R_\mathbf{i} R_\mathbf{j}, \qquad r_\mathbf{ij} \geq r_\mathbf{i} r_\mathbf{j} \quad \text{for all } \mathbf{i}, \mathbf{j} \in I^*. \tag{1.3}$$

In §3 we will recall the definitions of the *bounded distortion property* (BDP), *open set condition* (OSC), and *cone condition* (CC) that appear in the following theorem. We

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will also define the *weak separation condition* (WSC) and the *weak* separation condition* (W*SC) for IIFSs. Mauldin and Urbański [**11**, Theorem 3.15] defined the topological pressure function

$$\widetilde{P}(s) := \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\mathbf{i}\in I^n} R_{\mathbf{i}}^s \tag{1.4}$$

and proved the following dimension result.

THEOREM 1.1. *Assume that the BDP, OSC, and CC hold, and let* $\tilde{\xi} := \inf\{s \geq 0 : \widetilde{P}(s) < 0\}$. *Then*

$$\dim_{\mathrm{H}}(K) = \tilde{\xi}.$$

In particular, if $\widetilde{P}(\tilde{\xi}) = 0$, *then* $\dim_{\mathrm{H}}(K) = \tilde{\xi}$.

We mention some of the anomalous phenomena exhibited by IIFSs. First, a limit set K need not be compact (see Example 2.2). Second, topological pressure functions need not have a zero. In fact, the effective domains of various topological pressures may be empty (see §4). Third, Moran [**12**] (see also [**11**, Examples 5.5 and 5.7]) showed that even for IIFSs of similitudes satisfying OSC, it is possible to have $\mathcal{H}^\alpha(K) = 0$, where $\alpha = \dim_{\mathrm{H}}(K)$. Fourth, Szarek and Wedrychowicz [**14**] showed that the OSC does not imply the strong open set condition (SOSC). Finally, Mauldin and Urbański [**11**, Examples 5.2, 5.6, and 5.7] showed that under the BDP and OSC, it is possible to have $\dim_{\mathrm{H}}(K) < \underline{\dim}_{\mathrm{B}}(K) \leq \dim_{\mathrm{P}}(K)$ and $\mathcal{P}^\alpha(K) = \infty$.

For FIFSs, it is proved in [**10**] that if the BDP and WSC hold, then $0 < \mathcal{H}^\alpha(K) < \infty$ where $\alpha = \dim_{\mathrm{H}}(K)$. Moreover, it is proved in [**1**] under these assumptions that $\dim_{\mathrm{H}}(K)$ is the zero of the (modified) topological pressure function

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\sigma\in\{S_{\mathbf{i}}:\mathbf{i}\in I^n\}} R_\sigma^s \quad (\text{where } I = \{1, \ldots, N\}). \tag{1.5}$$

We emphasize that the set $\{S_{\mathbf{i}} : \mathbf{i} \in I^n\}$ in (1.5) consists only of the distinct $S_{\mathbf{i}}$; in other words, the set of indices $\mathbf{i}$ that give rise to the same $S_{\mathbf{i}}$ contributes to only one term in the sum.

Associated with the weak separation conditions are various generalizations to IIFSs of topological pressure functions, $\underline{Q}$, $\overline{Q}$, Q, P^*, of the one in (1.5). We will also extend the one in (1.5) to IIFSs. The generalization to Q (and $\underline{Q}$, $\overline{Q}$) is obtained by summing R_σ^s over a different sequence of index sets (Definition 4.2), while that to P^* is obtained by taking the supremum over all pressure functions defined as in (1.5) by finite subsets of the given (infinite) index set (Definition 4.3). We will study these in §4. For each finite subset $J \subset I$, denote the limit set (i.e. attractor) of the FIFS $\{S_i\}_{i\in J}$ by K_J. Define

$$\begin{aligned} \xi &:= \inf\{s \geq 0 : P(s) < 0\}, & \xi^* &:= \inf\{s \geq 0 : P^*(s) < 0\}, \\ \underline{\zeta} &:= \inf\{s \geq 0 : \underline{Q}(s) < 0\}, & \overline{\zeta} &:= \inf\{s \geq 0 : \overline{Q}(s) < 0\}, \\ \alpha_J &:= \dim_{\mathrm{H}}(K_J), & \alpha^* &:= \sup\{\alpha_J : J \subset I \text{ is finite}\}. \end{aligned}$$

The following is our main result; it generalizes [**1**, Theorem 1.1] (see Definition 3.3 for definitions of WSC sets and regions).

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THEOREM 1.2. *Let K be the limit set of an IFS $\{S_i\}_{i\in I}$ of injective C^1 conformal contractions on a compact subset $X \subset \mathbb{R}^d$ with $X^\circ \neq \emptyset$ and let $\alpha = \dim_{\mathrm{H}}(K)$. Assume that the BDP and WSC hold.*

(a) *If K is a WSC set, then $\alpha = \underline{\zeta} = \overline{\zeta} = \alpha^* = \xi^* \leq \xi$.*

(b) *If a WSC region D satisfies the CC, then $\overline{D}$ is a WSC region. In particular, K is a WSC set and thus the conclusion of part* (a) *holds.*

Zerner [**15**] defined the growth dimension of an FIFS of contractive similitudes. We will extend the growth dimension to IIFSs in §5 and prove that, under suitable conditions, the growth and Hausdorff dimensions of the limit set coincide.

The rest of this paper is organized as follows. Section 2 studies some elementary properties of IIFSs and their limit sets. In §3 we define and study weak separation conditions for IIFSs. Topological pressure functions are studied in §4. Section 5 is devoted to the proof of the main dimension results. Some examples illustrating Theorem 1.2 are given in §6. Finally, we state some questions and comments in §7.

2. *IIFSs and their limit sets*

In this section we study IIFSs and their limit sets. Define a *coding map* $\pi : I^\infty \to X$ as follows. For each $\mathbf{i} \in I^\infty$, the uniform contractivity condition implies that the sequence of compact sets $\{S_{\mathbf{i}|_n}(X)\}_{n\geq 1}$ is decreasing with the diameters of $S_{\mathbf{i}|_n}(X)$ converging to zero. Thus,

$$\bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X) \tag{2.1}$$

is a singleton, which we will denote by $\pi(\mathbf{i})$. Following [**11**] and [**5**], we define the limit set and attractor of an IFS as follows.

Definition 2.1. Let $\{S_i\}_{i\in I}$ be an IFS of injective contractions on a non-empty compact subset $X \subset \mathbb{R}^d$. Define the limit set of the IFS as

$$K := \pi(I^\infty) = \bigcup_{\mathbf{i}\in I^\infty} \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X). \tag{2.2}$$

If K is closed, it is called the attractor or (fixed point) of the IFS.

It follows from Banach's fixed point theorem and the uniform contractivity that each IFS has a unique fixed point (see [**5**] and Proposition 2.3).

If the uniform contractivity condition (1.1) is not satisfied, then the infinite intersection in (2.1) need not be a singleton, as shown in the following example.

Example 2.1. Let $X = [0, 1]$ and define

$$S_1(x) = \frac{x}{2} + \frac{1}{2} \quad \text{and} \quad S_i(x) = \left(1 - \frac{1}{2^i}\right)x \quad \text{for } i \geq 2.$$

Note that (1.1) does not hold. Let $\mathbf{i} = (2, 3, \ldots)$. Then $\bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X)$ is the interval $[0, \prod_{i=2}^{\infty}(1 - 2^{-i})]$, which is not a singleton.

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The limit set K satisfies the natural invariance equality:

$$K = \bigcup_{i \in I} S_i(K). \tag{2.3}$$

Moreover, K is the attractor if and only if it satisfies $K = \overline{\bigcup_{i \in I} S_i(K)}$. Unlike FIFSs, K is not necessarily compact if I is infinite. Also, if K is not compact, it is not necessarily the unique non-empty bounded subset of $\mathbb{R}^d$ satisfying (2.3), as the following example shows.

Example 2.2. Let $X = [0, 1]$, $S_1(x) = x/2$, and $S_i(x) = x/2^i + \sum_{k=1}^{i-1} 1/2^j$ for $i \geq 2$. Then $K = [0, 1)$. Moreover, if we let $\widetilde{K} = K \setminus \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers, then $\widetilde{K}$ also satisfies the invariance equality (2.3).

The limit set K in (2.2) is a Souslin set and hence is measurable with respect to Hausdorff measure (see [**2**]). Recall that for an FIFS, K can also be expressed as the intersection of the unions $\bigcup_{\mathbf{i} \in I^n} S_{\mathbf{i}}(X)$. If $\{S_i\}_{i \in I}$ is *pointwise finite*, i.e. each point in X belongs to at most a finite number of the sets $S_i(X)$, then (see [**11**])

$$K = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in I^n} S_{\mathbf{i}}(X).$$

In this case, K is an $F_{\sigma\delta}$ set and is thus a Borel set.

PROPOSITION 2.3. *Let K be the limit set of an IFS $\{S_i\}_{i \in I}$ of injective contractions on a compact subset $X \subset \mathbb{R}^d$.*

(a) *If $\emptyset \neq D \subseteq X$ is invariant, then $K \subseteq \overline{K} \subseteq \overline{D}$.*

(b) *$\overline{K}$ is the unique attractor.*

Proof. For any invariant compact subset $H \subseteq X$, the argument applied to K, together with the uniform contractivity of $\{S_i\}_{i \in I}$, yields

$$K = \bigcup_{\mathbf{i} \in I^\infty} \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(H). \tag{2.4}$$

(a) For any $\mathbf{j} \in I^*$, $S_{\mathbf{j}}(\overline{D}) = \overline{S_{\mathbf{j}}(D)}$, and thus by (2.4),

$$K = \bigcup_{\mathbf{i} \in I^\infty} \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(\overline{D}) = \bigcup_{\mathbf{i} \in I^\infty} \bigcap_{n=1}^{\infty} \overline{S_{\mathbf{i}|_n}(D)} \subseteq \overline{D}.$$

(b) It suffices to notice that $\overline{K} = \overline{\bigcup_{i \in I} S_i(K)} = \overline{\bigcup_{i \in I} S_i(\overline{K})}$. □

We summarize the following well-known results for convenience.

PROPOSITION 2.4. *Let $\{S_i\}_{i \in I}$ be an IFS of C^1 injective conformal contractions on a compact subset $X \subset \mathbb{R}^d$ as in Definition 1.1.*

(a) *$R \leq \rho < 1$, where R is defined in (1.2).*

(b) *There exists a constant $c_2 > 0$ such that*

$$|S_{\mathbf{i}}(x) - S_{\mathbf{i}}(y)| \leq c_2 R_{\mathbf{i}} |x - y| \quad \text{for all } \mathbf{i} \in I^* \text{ and } x, y \in X.$$

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(c) *If, in addition, the BDP holds, then there exists* $c_2 > 0$ *such that*

$$c_2^{-1} R_{\mathbf{i}}|x - y| \le |S_{\mathbf{i}}(x) - S_{\mathbf{i}}(y)| \le c_2 R_{\mathbf{i}}|x - y| \quad \textit{for all } \mathbf{i} \in I^* \textit{ and } x, y \in X.$$

Proof. Part (a) can be proved by using the continuity of the S_i' and the mean-value theorem. Parts (b) and (c) follow from the derivations of [**11**, BDP.2–4]. □

Throughout the rest of the paper we let c_2 denote the constant in Proposition 2.4(c).

3. *Bounded distortion property, cone condition, and separation conditions*

In this section we recall several well-known geometric and separation properties and extend the weak separation condition to IIFSs. The bounded distortion property ensures that $r_{\mathbf{i}}$ and $R_{\mathbf{i}}$ are comparable, and as a consequence, various topological pressure functions are independent of the choice of the set V in Definition 1.1 (see §4).

Definition 3.1. An IFS $\{S_i\}_{i\in I}$ of injective C^1 conformal contractions as in Definition 1.1 is said to have the bounded distortion property (BDP) if there exists a constant $c_1 > 0$ such that, for any index $\mathbf{i} \in I^*$,

$$\frac{\|S_{\mathbf{i}}'(x)\|}{\|S_{\mathbf{i}}'(y)\|} \le c_1 \quad \text{for all } x, y \in V.$$

In particular,

$$r_{\mathbf{i}} \le R_{\mathbf{i}} \le c_1 r_{\mathbf{i}} \quad \text{for all } \mathbf{i} \in I^*. \tag{3.1}$$

Three sufficient conditions for the BDP are given in [**11**]. We will only need the following one: there exist constants $C \ge 1$ and $\alpha > 0$ such that

$$|\|S_i'(y)\| - \|S_i'(x)\|| \le C\|(S_i')^{-1}\|^{-1}|y - x|^{\alpha} \quad \text{for all } i \in I \text{ and } x, y \in V. \tag{3.2}$$

The open set condition for both FIFSs and IIFSs is well known (see [**6**, **3**, **11**]). Let $\{S_i\}_{i\in I}$ be an IFS as in Definition 1.1. A subset $U \subseteq X$ is said to be *invariant* under $\{S_i\}_{i\in I}$ if $\bigcup_{i\in I} S_i(U) \subseteq U$. Also, $\{S_i\}_{i\in I}$ satisfies the *open set condition* (OSC) if there exists a non-empty bounded invariant open set $U \subset X$, called an *OSC set*, such that $S_i(U) \cap S_j(U) = \emptyset$ for all $i \ne j$. For FIFSs, the weak separation condition is strictly weaker than the OSC [**8**, **10**, **1**]. As in [**11**], we will also need the cone condition.

Definition 3.2. $E \subset \mathbb{R}^d$ is said to satisfy the cone condition (CC) if there exist $\beta, h > 0$ such that, for every $x \in \partial E \subset \mathbb{R}^d$, there exists an open cone $C(x, u_x, \beta, h) \subset E^\circ$ with vertex x, direction vector u_x, central angle of Lebesgue measure β, and altitude h.

Roughly speaking, the CC ensures that each point $x \in E$ is covered by a bounded number of iterates of the form $S_{\mathbf{i}}(E)$, where $\mathbf{i}$ is taken from incomparable words in $\bigcup_{j\le n} I^j$ (see [**11**, Lemma 2.6]).

We now extend the weak separation condition in [**10**] to include IIFSs. For $0 < b < 1$, we let

$$\mathcal{I}_b := \{\mathbf{i} = (i_1, \ldots, i_n) : R_{\mathbf{i}} \le b < R_{i_1\ldots i_{n-1}}\} \quad \text{and} \quad \mathcal{A}_b := \{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_b\}.$$

Definition 3.3. Let $X \subset \mathbb{R}^d$ be a compact subset with $X^\circ \ne \emptyset$ and let $\{S_i\}_{i\in I}$ be an IFS of injective C^1 conformal contractions on X.

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(a) We say that $\{S_i\}_{i\in I}$ satisfies the weak separation condition (WSC) if there exist an invariant subset $D \subseteq X$ with $D^\circ \neq \emptyset$, called a WSC region, and a constant $\gamma \in \mathbb{N}$ such that

$$\sup_{x\in X} \#\{\tau \in \mathcal{A}_b : x \in \tau(D)\} \leq \gamma \quad \text{for all } b \in (0, 1). \tag{3.3}$$

(b) If $E \subseteq X$ is an invariant set and (3.3) holds with E replacing D, we call E a WSC set. Thus, any WSC set that has a non-empty interior is a WSC region.

Remark 3.1. If I is finite, this definition is equivalent to the one for an FIFS (see [**10**]). If I is infinite, by letting $U = D^\circ$ it is easy to see that the OSC implies the WSC. Moreover, examples in §6 show that the WSC is strictly weaker than the OSC.

PROPOSITION 3.2. *Let $\{S_i\}_{i\in I}$ be an IFS of C^1 injective conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ with $X^\circ \neq \emptyset$. If both the BDP and WSC hold, then*

$$\sum_{i\in I} R_i^d \leq \gamma c_1^d < \infty. \tag{3.4}$$

In particular, if I is infinite, then $\lim R_i = 0$.

Proof. For each $i \in I$, denote by $J_{S_i}(x)$ the Jacobian of S_i at x. By conformality, $|J_{S_i}(x)| = |\det S_i'(x)|$. If $R < b < 1$, then $\mathcal{A}_b = \{S_i\}_{i\in I}$. Let γ and D be as in the definition of the WSC. By combining the BDP and WSC, we have

$$\gamma \mathcal{L}^d(D) \geq \sum_{i\in I} \mathcal{L}^d(S_i(D)) = \sum_{i\in I} \int_D |J_{S_i}(x)|\, d\mathcal{L}^d(x) \geq \sum_{i\in I} c_1^{-d} R_i^d \mathcal{L}^d(D).$$

Since $0 < \mathcal{L}^d(D) < \infty$, the assertions follow immediately. □

We now describe another natural and useful extension of the WSC to IIFSs. For an index set I, we define, throughout the rest of this paper,

$$\mathcal{F} = \mathcal{F}(I) := \{J \subset I : J \text{ is finite}\}. \tag{3.5}$$

Definition 3.4. Let $\{S_i\}_{i\in I}$ be an IIFS of C^1 injective conformal contractions on a compact subset $X \subset \mathbb{R}^d$ with $X^\circ \neq \emptyset$. We say that $\{S_i\}_{i\in I}$ satisfies the weak* separation condition (W*SC) if, for any $J \in \mathcal{F}(I)$, the finite IFS $\{S_j\}_{j\in J}$ satisfies the WSC.

The following example shows that the W*SC is strictly weaker than the WSC.

Example 3.3. Let $X = [0, 1]$ and, for each $k \in \mathbb{N}$, let

$$S_{k,i}(x) := \frac{x}{2^k} + \frac{i}{2^k}, \quad i = 0, 1, \ldots, 2^k - 1.$$

That is, for each k, $S_{k,i}[0, 1]$, $i = 0, 1, \ldots, 2^k - 1$, is the union of all non-overlapping dyadic intervals in $[0, 1]$ with length $1/2^k$. Then $K = [0, 1]$ and the IIFS $\{S_{k,i} : i = 0, 1, \ldots, 2^k - 1,\ k \in \mathbb{N}\}$ satisfies the BDP and W*SC but not the WSC.

Proof. Each finite subcollection of $\{S_{k,i}\}$ is of finite type (see [**13**, Theorem 2.5]) and therefore satisfies the WSC (see [**10**]). Hence the W*SC holds. The BDP holds since each $S_{k,i}$ is a similitude. The WSC fails by Proposition 3.2, since $\sum_{i=1}^\infty R_i = \sum_{k=1}^\infty \sum_{i=0}^{2^k-1} 2^{-k} = \infty$. □

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4. *Topological pressure functions*

In this section we define and study three kinds of the topological pressure functions for IIFSs. We begin by generalizing those in (1.4) and (1.5) to IIFSs with overlaps. To simplify notation, for an IFS $\{S_i\}_{i\in I}$, let

$$\mathcal{S}_n = \mathcal{S}_n(I) := \{S_{\mathbf{i}} : \mathbf{i} \in I^n\}, \quad n \in \mathbb{N}.$$

In particular, $\mathcal{S}_1 = I$, since we assume that the IFS maps are distinct.

Definition 4.1. Let $\{S_i\}_{i\in I}$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subset \mathbb{R}^d$ as in Definition 1.1 with V being any open set as in that definition. Define the topological pressure function $P = P_V : \mathbb{R} \to [-\infty, \infty]$ as

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} R_\sigma^s,$$

if the limit exists. The upper and lower topological pressure functions are defined by using lim inf and lim sup, respectively. Define the effective domain of P as

$$\mathrm{dom}P := \{s \in \mathbb{R} : -\infty < P(s) < \infty\}.$$

We remark that if the OSC holds, then P is equivalent to $\widetilde{P}$. The following proposition shows that if the BDP holds then the definition of P is independent of V.

PROPOSITION 4.1. *Let $\{S_i\}_{i\in I}$, X, and V be as in Definition 4.1 and assume that the BDP holds. Then the definitions of $\underline{P}_V$, $\overline{P}_V$ and P are independent of the choice of V.*

Proof. For $x \in V$, define a pointwise analogue of $\overline{P}_V$ as follows:

$$\overline{P}_x(s) := \overline{\lim_{n\to\infty}} \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} \|\sigma'(x)\|^s.$$

For $s \geq 0$, using (3.1), we have

$$\frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} R_\sigma^s \geq \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} \|\sigma'(x)\|^s \geq \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} (c_1^{-1} R_\sigma)^s = \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n} R_\sigma^s + \frac{1}{n} \ln c_1^{-s}.$$

In particular, $\overline{P}_x(s) = \overline{P}_V(s)$. Reversing the inequalities proves the same results for $s < 0$. Now let V' be another bounded open set with the same properties as V. Letting $x \in X \subset V \cap V'$, we have $\overline{P}_V(s) = \overline{P}_x(s) = \overline{P}_{V'}(s)$, proving the independence of $\overline{P}_V(s)$ on V. The same holds for $\underline{P}_V$. □

Unlike FIFSs or IIFSs with the OSC, the existence of the limit defining the topological pressure $P(s)$ is less straightforward. We will establish some basic results in this regard by making use of the WSC. For each $n \in \mathbb{N}$ we define a relation $\sim_n$ on I^n so that two indices are related if the corresponding iterations are identical; more precisely,

$$j_1 \cdots j_n \sim_n i_1 \cdots i_n \quad \text{if } S_{j_1\cdots j_n} = S_{i_1\cdots i_n}.$$

It is clear that $\sim_n$ is an equivalence relation. We denote the equivalence class of $\mathbf{i} \in I^n$ by $[\mathbf{i}]$. We also use the notation $[\mathbf{i}]j := \{i_1 \cdots i_n j : i_1 \cdots i_n \in [\mathbf{i}]\}$. The following proposition gives bounds for the cardinality of $[\mathbf{i}]$ and that of $S_{\mathbf{i}}$ with $x \in S_{\mathbf{i}}(D)$.

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PROPOSITION 4.2. *Let $\{S_i\}_{i\in I}$ be an IFS of C^1 injective conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ with $X^\circ \neq \emptyset$. Assume that the WSC holds and let D, γ be as in the definition of the WSC. Then for every $n \geq 1$ and $\mathbf{i} \in I^n$,*

$$\#[\mathbf{i}] \leq \gamma^{(n-1)n/2} \quad \text{and} \quad \sup_{x\in X} \#\{S_{\mathbf{i}} \in \mathcal{S}_n : x \in S_{\mathbf{i}}(D)\} \leq \gamma^n. \tag{4.1}$$

Proof. We use induction on n. The case $n = 1$ is a direct consequence of the WSC and the distinctness of the S_i. Assume that (4.1) holds for some $n \geq 1$. Partition $[i_1 \cdots i_{n+1}]$ as

$$[i_1 \cdots i_{n+1}] = \bigcup \{[\mathbf{l}|_n] j_{n+1} : \mathbf{l} \in [i_1 \cdots i_{n+1}],\ [\mathbf{l}|_n] j_{n+1} \subseteq [i_1 \cdots i_{n+1}]\}.$$

Notice that if $j_1 \cdots j_n, k_1 \cdots k_n \in [\mathbf{l}|_n]$ and $j_1 \cdots j_n j_{n+1}, k_1 \cdots k_n k_{n+1} \in [i_1 \cdots i_{n+1}]$, then $j_{n+1} = k_{n+1}$. Hence,

$$\#[i_1 \cdots i_{n+1}] \leq \#\{[\mathbf{l}|_n] : \mathbf{l} \in [i_1 \cdots i_{n+1}]\} \cdot \sup \#[\mathbf{l}|_n]. \tag{4.2}$$

Next, by the invariance of D, for all $\mathbf{l} = \mathbf{l}|_n l_{n+1} \in [i_1 \cdots i_{n+1}]$, we have

$$S_{\mathbf{l}|_n}(D) \cap S_{i_1\cdots i_n}(D) \supseteq S_{\mathbf{l}|_n l_{n+1}}(D) \cap S_{i_1\cdots i_{n+1}}(D) = S_{i_1\cdots i_{n+1}}(D) \neq \emptyset.$$

Hence, by the induction hypothesis,

$$\#\{[\mathbf{l}|_n] : \mathbf{l} \in [i_1 \cdots i_{n+1}]\} \leq \sup_{x\in X} \#\{S_{\mathbf{l}|_n} \in \mathcal{S}_n : x \in S_{\mathbf{l}|_n}(D)\} \leq \gamma^n. \tag{4.3}$$

Since $\sup \#[\mathbf{l}|_n] \leq \gamma^{(n-1)n/2}$ by the induction hypothesis, the first inequality in (4.1) follows by combining this with (4.2) and (4.3).

To prove the second inequality in (4.1) for the case $n + 1$, we notice that

$$\begin{aligned} &\sup_{x\in X} \#\{S_{i_1\cdots i_{n+1}} : x \in S_{i_1\cdots i_{n+1}}(D)\} \\ &\quad \leq \sup_{x\in X} \#\{S_{i_1\cdots i_n} : x \in S_{i_1\cdots i_n}(D)\} \cdot \sup_{x\in X} \#\{S_i : x \in S_i(D)\} \\ &\quad \leq \gamma^{n+1}, \end{aligned}$$

where the first inequality follows from the facts that if $x \in S_{i_1\cdots i_{n+1}}(D) \cap S_{i'_1\cdots i'_{n+1}}(D)$ then $x \in S_{i_1\cdots i_n}(D) \cap S_{i'_1\cdots i'_n}(D)$ and, moreover, for each $S_{i_1\cdots i_n}$,

$$x \in S_{i_1\cdots i_n j_{n+1}}(D) \cap S_{i_1\cdots i_n i_{n+1}}(D) \Rightarrow S^{-1}_{i_1\cdots i_n}(x) \in S_{j_{n+1}}(D) \cap S_{i_{n+1}}(D).$$

This completes the proof. □

For $s \in \mathbb{R}$ and each integer $n \geq 1$, let

$$\psi_n(s) = \sum_{\sigma\in\mathcal{S}_n} R^s_\sigma, \quad F_n := \{s \in \mathbb{R} : \psi_n(s) < \infty\}, \quad \theta_n := \inf F_n, \quad \theta := \inf_{n\geq 1} \theta_n.$$

Clearly, $\psi_n(s)$ is a decreasing function of s and hence either $F_n = [\theta_n, \infty)$ or $F_n = (\theta_n, \infty)$. If I is infinite, then for all $n \geq 1$, $\psi_n(0) = \infty$ and thus $\theta_n \geq 0$.

PROPOSITION 4.3. *Assume the hypotheses of Proposition 4.2, and assume in addition that the BDP holds. Then, for any $s \geq 0$, $\psi_{n_0}(s) < \infty$ for some $n_0 \geq 1$ if and only if $\psi_n(s) < \infty$ for all $n \geq 1$. Thus, $F_n = F_1$ and $\theta_n = \theta_1$ for all $n \geq 1$.*

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Proof. For any $n \geq 1$, by using the BDP, (3.1), and Proposition 4.2,

$$\begin{aligned}\psi_1(s)^n &= \sum_{i_1,\ldots,i_n\in I} R_{i_1}^s \cdots R_{i_n}^s \leq c_1^{ns} \sum_{i_1,\ldots,i_n\in I} r_{i_1\cdots i_n}^s \leq c_1^{ns} \sum_{i_1,\ldots,i_n\in I} R_{i_1\cdots i_n}^s \\ &\leq c_1^{ns}\gamma^{(n-1)n/2} \sum_{\sigma\in\mathcal{S}_n} R_\sigma^s \quad \text{(Proposition 4.2)} \\ &= c_1^{ns}\gamma^{(n-1)n/2}\psi_n(s).\end{aligned}$$

Next, for all $n, k \in \mathbb{N}$ and any $s \geq 0$,

$$\psi_{n+k}(s) \leq \left(\sum_{\sigma\in\mathcal{S}_n} R_\sigma^s\right)\left(\sum_{\tau\in\mathcal{S}_k} R_\tau^s\right) = \psi_n(s)\psi_k(s). \tag{4.4}$$

In particular, for any $n \geq 1$, $\psi_n(s) \leq \psi_1(s)^n$. Thus, if $\psi_{n_0}(s) < \infty$ for some n_0, then $\psi_1(s) < \infty$ and hence $\psi_n(s) < \infty$ for all $n \geq 1$. □

We now prove some basic properties of the topological pressure P.

PROPOSITION 4.4. *Assume the same hypotheses as Proposition 4.3.*

(a) *For any fixed $s \in F_1$, the limit defining $P(s)$ exists and is given by*

$$P(s) = \inf\left\{\frac{1}{n}\ln\psi_n(s) : n \in \mathbb{N}\right\}.$$

(b) $\operatorname{dom}P = F_1$, $\psi_1(d) < \infty$, *and* $[d, \infty) \subseteq \operatorname{dom}P$.

(c) *P is strictly decreasing and convex on* $\operatorname{dom}P$, *and is continuous on* $(\operatorname{dom}P)^\circ$. *Moreover,* $\lim_{s\to\infty} P(s) = -\infty$.

(d) *If I is infinite, then $P(0) = \infty$.*

Proof. Since these properties are known for FIFSs, we will assume that I is infinite.

(a) It follows from (4.4) that, for each $s \in F_1$, $\{\ln\psi_n(s)\}$ is a subadditive sequence of real numbers. Moreover, for all $n \in \mathbb{N}$, $\psi_n(s) \geq (c_1^{-1}R)^{ns}$ and thus $P(s) > -\infty$. The assertion follows.

(b) That $\operatorname{dom}P = F_1$ follows directly from (a) and the fact that $F_n = F_1$ for all n. Next, from (3.4), we have $\psi_1(d) = \sum_{i\in I} R_i^d \leq \gamma c_1^d < \infty$, and thus $d \in \operatorname{dom}P$.

(c) To show the monotonicity of P, let $s \in F_1$ and $\delta > 0$. Then by (1.3),

$$\frac{1}{n}\ln\psi_n(s+\delta) \leq \frac{1}{n}\ln\sum_{\sigma\in\mathcal{S}_n} R_\sigma^s R^{n\delta} = \frac{1}{n}\ln\psi_n(s) + \delta\ln R.$$

Hence

$$P(s+\delta) \leq P(s) + \delta\ln R. \tag{4.5}$$

The convexity of P follows from Hölder's inequality. Thus P is continuous on $(\operatorname{dom}P)^\circ$. Finally, it is a direct consequence of (4.5) that $\lim_{s\to\infty} P(s) = -\infty$.

(d) Since $\#\mathcal{S}_n = \infty$ for all n, we have $P(0) = \infty$. This completes the proof. □

The following example shows that $\operatorname{dom}P$ can be empty in the absence of the WSC, even if the BDP holds.

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Example 4.5. Let $X = [0, 1]$ and $a \in (0, 1)$ a transcendental number. For each $i \in \mathbb{N} = I$, define $S_i : X \to X$ by

$$S_i(x) = \frac{x}{2} + a^i.$$

Then for all $n \geq 1$ and $s \geq 0$, $\psi_n(s) = \infty$ and $F_n = \emptyset$. Consequently, $\operatorname{dom} P = \emptyset$.

Proof. Since a is transcendental, for each $n \in \mathbb{N}$, the maps $S_{i_1 \cdots i_n} \in I^n$ are distinct. It follows that for all $n \in \mathbb{N}$ and $s \geq 0$, $\psi_n(s) = \sum_{i_1,\ldots,i_n \in I} R^s_{i_1\cdots i_n} = \sum_{i_1,\ldots,i_n \in I} (1/2^n)^s = \infty$. The assertion follows. □

We remark that [**11**, Example 5.3] shows that even if the OSC holds, the topological pressure function P need not have a zero on $\operatorname{dom} P$. Such IFSs are called *irregular* in [**11**].

We now define another topological pressure function that extends an analogous one in [**1**]. It is more suited to IFSs satisfying the WSC.

Definition 4.2. Let $\{S_i\}_{i \in I}$ and X be as in Definition 4.1. Define

$$Q(s) := \lim_{b \to 0^+} \frac{1}{-\ln b} \ln \sum_{\tau \in \mathcal{A}_b} R^s_\tau,$$

if the limit exists, and define $\underline{Q}(s)$, $\overline{Q}(s)$ by replacing the limit with lim inf and lim sup, respectively. The effective domains of $\underline{Q}$, $\overline{Q}$, Q are defined analogously to that of P.

We define an analogue of ψ_n as follows:

$$\phi_b(s) := \sum_{\tau \in \mathcal{A}_b} R^s_\tau.$$

It is clear that if $s \in \operatorname{dom} \underline{Q}$ then $s' \in \operatorname{dom} \underline{Q}$ for any $s' > s$. Hence, if $\operatorname{dom} \underline{Q} \neq \emptyset$, then it is an interval of the form $[a, \infty)$ or (a, ∞). The same holds for $\operatorname{dom} \overline{Q}$. Example 4.5 shows that these effective domains can be empty.

Example 4.6. Let $\{S_i\}_{i \in \mathbb{N}}$ be defined as in Example 4.5, and let $b \in (0, 1)$. Then for any $n \geq 1$, $\#\mathcal{A}_b = \infty$, and thus $\operatorname{dom} \underline{Q} = \operatorname{dom} \overline{Q} = \emptyset$.

PROPOSITION 4.7. *Assume the same hypotheses as Definition 4.2 and assume that* $\operatorname{dom} \underline{Q} \neq \emptyset$ *and* $\operatorname{dom} \overline{Q} \neq \emptyset$. *Then the following hold:*

(a) $\overline{Q}$ *is strictly decreasing and convex on* $\operatorname{dom} \overline{Q}$, *and hence is continuous on* $(\operatorname{dom} \overline{Q})^\circ$;

(b) $\lim_{s \to \infty} \overline{Q}(s) = -\infty$;

(c) $\overline{Q}(0) = \infty$ *if* I *is infinite.*

Analogous results, except for convexity, hold for $\underline{Q}$.

Proof. (a) Let $s \in \operatorname{dom} \overline{Q}$ and $\delta > 0$. Then

$$\overline{Q}(s + \delta) \leq \varlimsup_{\delta \to 0^+} \frac{1}{-\ln b} \ln \sum_{\tau \in \mathcal{A}_b} R^s_\tau b^\delta \leq \overline{Q}(s) - \delta < \overline{Q}(s). \tag{4.6}$$

Thus, $\overline{Q}$ is strictly decreasing. By applying Hölder's inequality, we see that it is convex on $\operatorname{dom} \overline{Q}$ and hence it is continuous on $(\operatorname{dom} \overline{Q})^\circ$.

Part (b) follows from (4.6) by letting $\delta \to \infty$. Part (c) follows from the fact that if I is infinite then $\#\mathcal{A}_b = \infty$ for all $b \in (0, 1)$. □

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Parts (a) and (b)(i) of the following proposition are analogues of Propositions 4.1 and 4.4. The proofs are similar; we omit the details. In part (b)(ii), by imposing the BDP and WSC, we compare the domains of P, $\underline{Q}$, and $\overline{Q}$.

PROPOSITION 4.8. *Assume the same hypotheses as Proposition 4.1.*

(a) *The definitions of* $\underline{Q}$ *and* $\overline{Q}$ *are independent of the choice of the set* V *in Definition 1.1.*

(b) *Assume in addition that the WSC holds. Then:*
   (i) $\underline{Q}(d) \le \overline{Q}(d) \le 0$*; in particular,* $[d, \infty) \subseteq \operatorname{dom} \underline{Q} \cap \operatorname{dom} \overline{Q}$*;*
   (ii) $\operatorname{dom} \overline{Q} \subseteq \operatorname{dom} \underline{Q} \subseteq \operatorname{dom} P$.

Proof. We only prove (b)(ii) for the case $I = \infty$. Assume without loss of generality that $I = \mathbb{N}$. Let $s > 0$ such that $\underline{Q}(s) < \infty$. Then there exists some $b \in (0, 1)$ such that

$$\phi_b(s) = \sum_{\tau \in \mathcal{A}_b} R_\tau^s < \infty. \tag{4.7}$$

Since $\lim\{R_i\}_{i\in I} \to 0$ (Proposition 3.2), only finitely many S_i do not belong to $\mathcal{A}_b$. Hence (4.7) implies that $\psi_1(s) = \sum_{i\in I} R_i^s < \infty$. This in turn implies that $\psi_n(s) < \infty$ for all $n \in \mathbb{N}$ (Proposition 4.3) and thus $P(s) < \infty$. We conclude that

$$\{s \ge 0 : \overline{Q}(s) < \infty\} \subseteq \{s \ge 0 : \underline{Q}(s) < \infty\} \subseteq \{s > 0 : P(s) < \infty\},$$

from which the result follows. □

Associated with the W*SC is another topological pressure function.

Definition 4.3. Let $\{S_i\}_{i\in I}$, X be as in Definition 4.1 and $\mathcal{F}$ be as in (3.5). For each $J \in \mathcal{F}$, let $P_J : \mathbb{R} \to \mathbb{R}$ be the topological pressure function for the FIFS $\{S_i\}_{i\in J}$, i.e.

$$P_J(s) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n(J)} R_\sigma^s.$$

Define the topological pressure function P^* and its effective domain respectively as

$$P^*(s) := \sup_{J\in\mathcal{F}} P_J(s) \quad \text{and} \quad \operatorname{dom} P^* := \{s \in \mathbb{R} : -\infty < P^*(s) < \infty\}.$$

For each $J \in \mathcal{F}(I)$, let P_J, $\underline{Q_J}$, $\overline{Q_J}$ be the corresponding topological pressure functions. It is known (see [**1**]) that if $\{S_i\}_{i\in J}$ satisfies the BDP and WSC, then α_J is the unique zero of P_J, $\underline{Q_J}$, and $\overline{Q_J}$. Moreover (see [**10**]),

$$0 < \mathcal{H}^{\alpha_J}(K_J) < \infty.$$

Remark 4.9.

(a) For each fixed $s \in \mathbb{R}$, $P_J(s)$ is monotone in J in the sense that $P_{J_1}(s) \le P_{J_2}(s)$ whenever $J_1 \subseteq J_2$.

(b) Example 4.5 shows that $\operatorname{dom} P^*$ can be empty. In fact, if we let $J = \{1, \ldots, N\}$, then for any $s \in \mathbb{R}$ and $n \in \mathbb{N}$, $\sum_{\sigma\in\mathcal{S}_n(J)} R_\sigma^s = N^n 2^{-ns}$. Thus, $P_J(s) = \ln(N/2^s)$, and it follows that $P^*(s) = \infty$.

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(c) If $\operatorname{dom}P^* \neq \emptyset$, then it is a half interval. In fact, if $s \in \operatorname{dom}P^*$, then for all $s' > s$, $s' \in \operatorname{dom}P^*$.

We summarize some elementary properties of P^* below.

PROPOSITION 4.10. *Let $\{S_i\}_{i\in I}$ and X be as in Definition 4.1.*

(a) *P^* is decreasing and convex on $\operatorname{dom}P^*$ and is thus continuous on $(\operatorname{dom}P^*)^\circ$.*

(b) *If the BDP holds, then the definition of P^* is independent of the choice of the set V in Definition 1.1.*

(c) *$P^*(d) \leq 0$ and $[d, \infty) \subseteq \operatorname{dom}P \subseteq \operatorname{dom}P^*$.*

Proof. Part (a) follows from the fact that, for each $J \in \mathcal{F}$, P_J is decreasing and convex (see [**1**]).

(b) Under the BDP, the definition of each P_J is independent of the choice of V (see [**1**]) and thus so is that of P^*.

(c) If both the BDP and WSC hold, then $P_J(d) \leq 0$ for all $J \in \mathcal{F}$ (see [**1**]) and thus $P^*(d) \leq 0$. Since $-\infty < P^*(s) \leq P(s)$ for any s, we have $\operatorname{dom}P \subseteq \operatorname{dom}P^*$. The first inclusion is in Proposition 4.4(b). □

The following example shows that Proposition 4.10(c) may fail without the WSC.

Example 4.11. Consider the well-known family of IFSs $\{S_1, S_2\}$ on $[0, 1]$ with

$$S_1(x) = rx, \quad S_2(x) = rx + (1 - r),$$

where we assume that $r \in (\frac{1}{2}, 1)$ is a transcendental number. Let $K = [0, 1]$ be the attractor. Applying $K = S_1(K) \cup S_2(K)$ repeatedly, we get

$$K = S_1(K) \cup S_2(S_1(K) \cup S_2(K)) = \cdots = S_1(K) \cup \bigcup_{k=2}^{\infty} S_{2^{k-1}1}(K).$$

Let $\tilde{S}_1(x) = S_1(x)$ and $\tilde{S}_k(x) = S_{2^{k-1}1}(x)$ for $k \geq 2$, and let $J_N = \{1, \ldots, N\}$. The limit set of the IIFS $\{\tilde{S}_i\}_{i\in J_N}$ on $[0, 1]$ is $K \setminus \{1\} = [0, 1)$. The WSC is not satisfied because r is a transcendental number. Since

$$P_{J_N}(1) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n(J_N)} R_\sigma = \lim_{n\to\infty} \frac{\ln(r + \cdots + r^N)^n}{n} = \ln \frac{r(1 - r^N)}{1 - r},$$

it follows that $P^*(d) \geq \ln(r/(1 - r)) > 1$.

Since $\dim_{\mathrm{H}}(K) = 1$, this example also shows that the conclusions of Theorem 1.2 may fail without assuming the WSC.

5. *Hausdorff dimension of limit sets*

We prove our main dimension results in this section. For FIFSs, it is known that $\alpha = \dim_{\mathrm{B}}(K) = \dim_{\mathrm{P}}(K)$, and $0 < \mathcal{H}^\alpha(K) \leq \mathcal{P}^\alpha(K) < \infty$ (see [**10**, Theorem 3.2]). However, as mentioned in §1, these may not hold for IIFSs. Nevertheless, our results show that under the BDP and WSC, the Hausdorff and growth dimensions of K are equal.

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LEMMA 5.1. *Let $\{S_i\}_{i\in I}$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subset \mathbb{R}^d$ with $X^\circ \neq \emptyset$. Assume that the BDP and WSC hold, and K is a WSC set. Then for all $b \in (0, 1)$,*

$$\sum_{\tau\in\mathcal{A}_b^J} R_\tau^{\alpha_J} \le c_1^{\alpha_J}\gamma \quad \text{for all } J \in \mathcal{F} \quad \text{and} \quad \sup_{J\in\mathcal{F}} \sum_{\tau\in\mathcal{A}_b^J} R_\tau^{\alpha^*} \le c_1^{\alpha^*}\gamma. \tag{5.1}$$

Proof. Let $J \in \mathcal{F}$. By assumption, each $x \in K \supseteq K_J$ is covered by no more than γ sets of the form $\tau(K) \supseteq \tau(K_J)$, where $\tau \in \mathcal{A}_b^J$. Hence, by using the BDP, the fact that K is a WSC set, and a change of variables (see [**4**, Lemma 2.8]), we have

$$\begin{aligned}\gamma\mathcal{H}^{\alpha_J}(K_J) &\ge \sum_{\tau\in\mathcal{A}_b^J} \mathcal{H}^{\alpha_J}(\tau(K_J)) = \sum_{\tau\in\mathcal{A}_b^J} \int_{K_J} |\tau'(x)|^{\alpha_J}\, d\mathcal{H}^{\alpha_J}(x)\\ &\ge \sum_{\tau\in\mathcal{A}_b^J} r_\tau^{\alpha_J}\mathcal{H}^{\alpha_J}(K_J) \ge \sum_{\tau\in\mathcal{A}_b^J} c_1^{-\alpha_J} R_\tau^{\alpha_J}\mathcal{H}^{\alpha_J}(K_J).\end{aligned}$$

Since $0 < \mathcal{H}^{\alpha_J}(K_J) < \infty$ (see [**10**]), the first inequality in (5.1) follows. The second one is a direct consequence. □

Proof of Theorem 1.2. We first establish the following:

(i) $\xi^* = \alpha^*$;

(ii) $\dim_{\mathrm{H}}(K) \le \underline{\zeta} \le \overline{\zeta}$;

(iii) $\dim_{\mathrm{H}}(K) \le \xi$.

To prove (i), we notice that, for any $J \in \mathcal{F}$, $P_J(\alpha^*) \le 0$ and hence $P^*(\alpha^*) \le 0$. Thus $\xi^* \le \alpha^*$. To prove the reverse inequality we suppose that $\xi^* < \alpha^*$. Then there would exist some $J \in \mathcal{F}$ such that $\xi^* < \alpha_J < \alpha^*$ and thus $P^*(s) \ge P_J(s) > 0$ for all $s \in (\xi^*, \alpha_J)$, contradicting the definition of ξ^*. It follows that $\xi^* \ge \alpha^*$.

To prove (ii), let $t > \underline{\zeta}$ and thus $\underline{Q}(t) < 0$. Then there exists a decreasing sequence $\{b_k\} \subset (0, 1)$ converging to 0 such that, for all $k \ge 1$,

$$\phi_{b_k}(t) < 1. \tag{5.2}$$

Since $K = \bigcup_{\tau\in\mathcal{A}_{b_k}} \tau(K)$ for all $k \ge 1$, $\{\tau(K) : \tau \in \mathcal{A}_{b_k}\}$ is a δ_k-cover of K, where $\delta_k = c_2 b_k |K|$ (see Proposition 2.4(b)). Combining this with (5.2), we have

$$\mathcal{H}^t_{\delta_k}(K) \le \sum_{\tau\in\mathcal{A}_{b_k}} |\tau(K)|^t \le c_2^t |K|^t \phi_{b_k}(t) \le c_2^t |K|^t.$$

It follows that $\mathcal{H}^t(K) < \infty$ and hence $\dim_{\mathrm{H}}(K) \le t$.

To prove (iii), notice that for $t > \xi$ we have $P(t) < 0$. Hence for all $n \in \mathbb{N}$ sufficiently large,

$$\sum_{\sigma\in\mathcal{S}_n} |\sigma(X)|^t \le \sum_{\sigma\in\mathcal{S}_n} c_2^t |X|^t R_\sigma^t \le c_2^t |X|^t e^{nP(t)/2} \to 0 \quad \text{as } n \to \infty.$$

Thus, $\mathcal{H}^t(K) = 0$, since the families $\{\sigma(X) : \sigma \in \mathcal{S}_n\}$ are $c_2 R^n |X|$-covers of K. Hence $\dim_{\mathrm{H}}(K) \le t$.

(a) Let $t > \alpha^*$ and $b \in (0, 1)$. Then

$$\phi_b(t) = \sup_{J\in\mathcal{F}} \sum_{\tau\in\mathcal{A}_b^J} R_\tau^t \le \sup_{J\in\mathcal{F}} \sum_{\tau\in\mathcal{A}_b^J} b^{t-\alpha^*} R_\tau^{\alpha^*} \le b^{t-\alpha^*} c_1^{\alpha^*}\gamma,$$

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where the last inequality follows from Lemma 5.1. This gives $\underline{Q}(t) \le \overline{Q}(t) \le \alpha^* - t < 0$ and hence $t \ge \overline{\zeta}$. Letting $t \searrow \alpha^*$ yields $\alpha^* \ge \overline{\zeta}$. Combining this with (i)–(iii), we have $\alpha^* \le \dim_{\mathrm{H}}(K) \le \underline{\zeta} \le \overline{\zeta} \le \alpha^* = \xi^*$, proving part (a).

(b) Let $b \in (0, 1)$ and $\tau \in \mathcal{A}_b$. Then by the WSC, for any $x \in X$,

$$\#\{\tau \in \mathcal{A}_b : x \in \tau(D)\} \le \gamma. \tag{5.3}$$

Now let $\tau \in \mathcal{A}_b$ and $x \in \tau(\partial D)$. Then by the CC, we let $C_{\tau^{-1}(x)} \subseteq D$ be a cone (open by definition) with angle β and vertex $\tau^{-1}(x)$. Since τ is conformal, $\tau(C_{\tau^{-1}(x)}) \cap B_\epsilon(x)$ approaches a cone as $\epsilon \to 0$. Hence the number of non-overlapping sets of the form $\tau(C_{\tau^{-1}(x)})$ is no more than w_{d-1}/β, where w_{d-1} is the $(d-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^d$. In particular, the maximum number C_d of non-overlapping sets of the form $\sigma(C_{\sigma^{-1}(x)})$, with $\sigma \in \mathcal{A}_b$, that intersect a fixed $\tau(C_{\tau^{-1}(x)})$ is no more than w_{d-1}/β. Hence, in view of the WSC, the total number of sets of the form $\tau(C_{\tau^{-1}(x)})$ is bounded by $\gamma C_d w_{d-1}/\beta$, which implies that $\#\{\tau \in \mathcal{A}_b : x \in \tau(\partial D)\} \le \gamma C_d w_{d-1}/\beta$. Combining this estimate with (5.3), we have

$$\#\{\tau \in \mathcal{A}_b : x \in \tau(\overline{D})\} \le (1 + C_d w_{d-1}/\beta)\gamma.$$

Consequently, $\overline{D}$ is a WSC region. Since $K \subseteq \overline{D}$, K is a WSC set. □

We do not know whether it is possible to have $\underline{Q}(\underline{\zeta}) < \overline{Q}(\overline{\zeta})$ under the assumptions of Theorem 1.2.

Zerner [**15**] defined the growth dimension of an FIFS of contractive similitudes as

$$\lim_{b \to 0^+} \frac{\ln \#\mathcal{A}_b}{-\ln b}.$$

Note that for an IIFS, $\#\mathcal{A}_b = \infty$ for any $b \in (0, 1)$ and thus a direct generalization of this definition fails. We extend the definition as follows.

Definition 5.1. Let $\{S_i\}_{i \in I}$ be an IIFS of injective conformal contractions on a compact subset $X \subset \mathbb{R}^d$. For $J \in \mathcal{F} = \mathcal{F}(I)$, let d_G^J be the growth dimension of the finite IFS $\{S_j\}_{j \in J}$. We define the growth dimension of $\{S_i\}_{i \in I}$ as

$$d_G := \sup_{J \in \mathcal{F}} d_G^J.$$

As a corollary of Theorem 1.2, we extend [**1**, Theorem 1.2] to IIFSs.

COROLLARY 5.2. *Let $\{S_i\}_{i \in I}$ be an IIFS of injective conformal contractions on a non-empty compact subset $X \subset \mathbb{R}^d$. Assume that the BDP holds.*

(a) $d_G \le \dim_{\mathrm{H}}(K)$.

(b) *If, in addition, $X^\circ \ne \emptyset$, $\{S_i\}_{i \in I}$ satisfies the WSC, and K is a WSC set, then $d_G = \dim_{\mathrm{H}}(K)$.*

Proof. (a) By [**1**, Theorem 1.2(a)], for all $J \in \mathcal{F}$, $d_G^J \le \dim_{\mathrm{H}}(K_J) \le \dim_{\mathrm{H}}(K)$. Thus, $d_G \le \dim_{\mathrm{H}}(K)$.

(b) By [**1**, Theorem 1.2(b)], for all $J \in \mathcal{F}$, $d_G^J = \dim_{\mathrm{H}}(K_J)$. Hence

$$d_G = \sup_{J \in \mathcal{F}} d_G^J = \sup_{J \in \mathcal{F}} \dim_{\mathrm{H}}(K_J) = \alpha^* = \dim_{\mathrm{H}}(K),$$

where the last equality follows from Theorem 1.2. □

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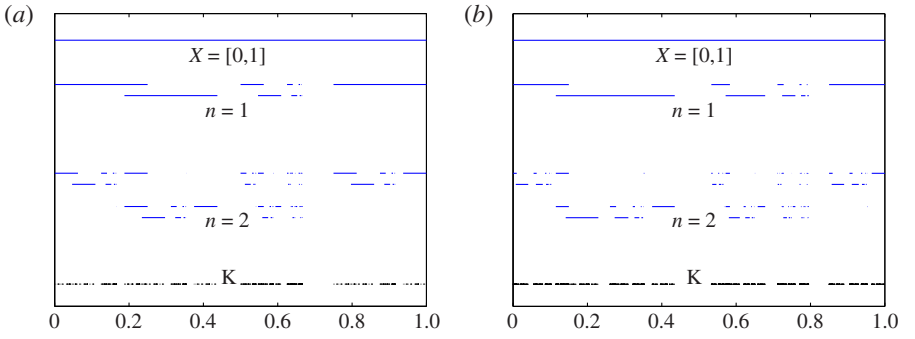


FIGURE 1. (a) First two iterations of $X = [0, 1]$ under the IIFS in Example 6.1, with $r = 1/4$ and $t = 2/3$. The limit set K is also shown. (b) Example 6.2, drawn with $a = 1/10$, $b = 1/20$, $r = 1/3$ and $t = 4/5$.

6. *Examples*

Example 6.1. Let $X = [0, 1]$, $0 < r < (2 - \sqrt{2})/2 \approx 0.292\,893\ldots$, $r(2 - r)/(1 - r) < t < 1 - r$, and K be the limit set of the following IIFS of similitudes:

$$S_1(x) = rx + 1 - r, \quad S_{2k}(x) = r^k x + t(1 - r^{k-1}),$$
$$S_{2k+1}(x) = r^k x + t(1 - r^{k-1}) + r^k(1 - r),$$

for any $k \geq 1$ (see Figure 1(a)). Then the OSC fails, while both the BDP and WSC hold, and X is a WSC region. Moreover, $\dim_{\mathrm{H}}(K) = \ln((2 - \sqrt{2})/2)/(-\ln r)$.

Proof. We first explain the conditions on r and t. In fact,

$$t = \lim_{k\to\infty} S_{2k}(1) = \lim_{k\to\infty} S_{2k+1}(1).$$

The conditions on r and t are necessary and sufficient to ensure that $t < S_1(0)$ and $S_{2k-1}(1) < S_{2k}(0)$ for all $k \geq 1$.

For any $k \geq 1$ and $\mathbf{i} \in I^*$, $S_{\mathbf{i}}S_{2k}S_1 = S_{\mathbf{i}}S_{2k+1}S_2$ and thus the OSC fails. It can be proved directly by induction on the length of the composition that all overlapping iterates can be expressed in the form $S_{\mathbf{i}}S_{2k}[0, 1] \cap S_{\mathbf{i}}S_{2k+1}[0, 1]$, with no other iterates $S_{\mathbf{j}}[0, 1]$ with $|\mathbf{j}| = |\mathbf{i}| + 1$ intersecting the union $S_{\mathbf{i}}S_{2k}[0, 1] \cup S_{\mathbf{i}}S_{2k+1}[0, 1]$. Thus it follows by induction on n with $r^n \leq b < r^{n-1}$ that $\#\{\tau \in \mathcal{A}_b : x \in \tau(X)\} \leq 2$ for any $b \in (0, 1)$.

To compute the dimension of K, we first notice that for each $N = 2n + 1$, the FIFS $\{S_i\}_{i=1}^N$ is of finite type (see [**13**, **9**]) with three neighborhood types: $\mathcal{T}_1 = [(S_1, 1)]$, $\mathcal{T}_2 = [(S_2, 1)]$, $\mathcal{T}_3 = [(S_3, 1)]$. Using the notation in [**9**], we have

$$\mathcal{T}_1 \longrightarrow \mathcal{T}_1(r) + \sum_{k=1}^{n} \mathcal{T}_2(r^k) + \sum_{k=1}^{n} \mathcal{T}_3(r^k), \quad \mathcal{T}_2 \longrightarrow \mathcal{T}_1(r) + \sum_{k=1}^{n} \mathcal{T}_2(r^k) + \sum_{k=1}^{n} \mathcal{T}_3(r^k),$$
$$\mathcal{T}_3 \longrightarrow \sum_{k=1}^{n} \mathcal{T}_2(r^k) + \sum_{k=1}^{n} \mathcal{T}_3(r^k).$$

By Theorem 1.2, $\dim_{\mathrm{H}}(K) = \alpha^* = \lim_{N\to\infty} \alpha_N$. Using the continuity of the spectral radius, we conclude that α^* is the unique number for which the spectral radius of the

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following weighted incidence matrix equals 1:

$$A_\alpha = \frac{r^\alpha}{1-r^\alpha}\begin{bmatrix} 1-r^\alpha & 1 & 1 \\ 1-r^\alpha & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Equivalently, α satisfies $2r^{3\alpha} - 6r^{2\alpha} + 5r^\alpha + 1 = 0$, which yields the dimension result. □

Each IIFS in the following four-parameter family consists of conformal maps that are not similitudes. It shows the abundance of such examples.

Example 6.2. Let $X = [0, 1]$, $a, b, r \in (0, 1)$ so that $1 - a - b > 0$, $c_1(a, b) = 2a - 2a^2 + b - 2ab$, and $c_2(a, b) = (1 - a)(1 - a - b)(a + b)$. Define

$$S_1(x) = ax^2 + bx, \quad S_2(x) = ax^2 + bx + 1 - a - b,$$
$$S_3(x) = ax^2 + c_1(a, b)x + c_2(a, b)$$
$$S_{2k}(x) = r^{k-1}S_1(x) + t(1 - r^{k-1}), \quad S_{2k+1}(x) = r^{k-1}S_3(x) + t(1 - r^{k-1}), \quad k \geq 1.$$

Assume that the following conditions hold:
(1) $|S_3'(1)| < 1$, i.e. $|2a(2 - a) + b(1 - 2a)| < 1$; this implies that $|S'(1)| < 1$;
(2) $S_3(1) < t < S_2(0)$, i.e. $a + c_1(a, b) + c_2(a, b) < t < 1 - a - b$;
(3) $S_3(1) < S_4(0)$, i.e. $0 < r < 1 - (a + c_1(a, b) + c_2(a, b))/t$.
Then the OSC fails, but the BDP and WSC hold with X being a WSC region (see Figure 1(b)).

Proof. By the choice of the coefficients c_1, c_2, we get $S_1 \circ S_2 = S_3 \circ S_1$ and thus the OSC fails. Condition (1) guarantees that the IIFS satisfies the conditions in Definition 1.1. Conditions (2) and (3) ensure that the WSC is satisfied; the proof is similar to that for Example 6.1. Finally, the BDP can be verified by using the condition in (3.2). We omit the details. □

7. *Questions and comments*

We state some questions and comments. First, we do not know whether the condition that K is a WSC set in Theorem 1.2 can be removed. Second, we would like to know whether the inequality $\dim_H(K) \leq \xi$ in Theorem 1.2 is an equality, and if not, we are interested in conditions guaranteeing that equality holds. Third, it would be of interest to have methods to compute the dimensions of the attractor $\overline{K}$. Finally, it is also interesting to study the analogues of self-conformal measures associated with IIFSs with overlaps and the validity of the multifractal formalism for these measures. The reader is referred to the recent paper by Käenmäki and Reeve [**7**] for some work in this direction.

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