Adv. Appl. Prob. **50,** 231–244 (2018)
*doi:*10.1017/apr.2018.11
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PERCOLATION RESULTS FOR THE CONTINUUM RANDOM CLUSTER MODEL

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Abstract

The continuum random cluster model is a Gibbs modification of the standard Boolean model with intensity $z > 0$ and law of radii Q. The formal unnormalised density is given by $q^{N_{\mathrm{cc}}}$, where q is a fixed parameter and N_{cc} is the number of connected components in the random structure. We prove for a large class of parameters that percolation occurs for large enough z and does not occur for small enough z. We provide an application to the phase transition of the Widom–Rowlinson model with random radii. Our main tools are stochastic domination properties, a detailed study of the interaction of the model, and a Fortuin–Kasteleyn representation.

Keywords: Gibbs point process; Widom–Rowlinson model; stochastic comparison; Fortuin–Kasteleyn representation

2010 Mathematics Subject Classification: Primary 60D05
Secondary 60G10; 60G55; 60G57; 60G60; 60K35; 82B21; 82B26; 82B43

1. Introduction

In this paper we are interested in the continuum random cluster model (CRCM), which is defined on a bounded window as a penalised Poisson–Boolean model with intensity $z > 0$ and law of radii Q. The unnormalised density is $q^{N_{\mathrm{cc}}}$, where $q > 0$ is a positive real number and N_{cc} denotes the number of connected components of the random closed set considered. In the infinite-volume regime, a global density is meaningless and a definition of the CRCM using Gibbs modifications is required. In this paper we prove the existence of a subcritical and a supercritical phase for the percolation of the CRCM.

The original random cluster model was introduced as a lattice model in the late 1960s by Fortuin and Kasteleyn in order to unify models of percolation such as Ising and Potts models. Properties and results about this model, such as existence of the random cluster model on infinite graphs, percolation, and phase transition properties can be found in [8] and [10]. In the continuum setting, the CRCM is directly linked to the well-known Widom–Rowlinson (WR) multi-type model by the so-called Fortuin–Kasteleyn (FK) representation (also known as the grey representation or random cluster representation). This representation was used in the 1990s to provide a new proof of the phase transition for the WR model [2], and for the more general class of continuum Potts models [6]. The CRCM has also received attention in the stochastic geometry and spatial statistics setting as an interacting random germ-grain model [14], [15]. For a suitable parameter q, the CRCM provides an optimum fit to the clustering of the real dataset.

Received 25 October 2016; revision received 5 October 2017.
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https://doi.org/10.1017/apr.2018.11 Published online by Cambridge University Press

The existence of the infinite-volume CRCM, defined through a standard Dobrushin–Lanford–Ruelle (DLR) formalism, was recently investigated in [4], in which the authors provided an existence result in cases such as $q < 1$ or unbounded radii.

Percolation refers to the existence of at least one unbounded (or infinite) connected component in the random structure. For the Poisson–Boolean model percolation is well understood, see [13], and we have the existence of a positive threshold z^* for the intensity, such that percolation occurs for $z > z^*$ and not for $z < z^*$. Due to its strong independence properties, the Poisson–Boolean model is sometimes irrelevant for applications in physics or biology.

In recent works [11], [17], percolation was proved to occur in the case of the Boolean model with deterministic radii and germs driven by a Gibbs point process. Both proofs rely strongly on the independent marking of the Boolean model and do not apply to the CRCM.

Moreover, in statistical mechanics, the percolation of the CRCM leads to the phase transition of the WR model. This was carried out for the case of deterministic radii in [2] and [6]. Percolation is also related to the uniqueness of the Gibbs measure and with the method of disagreement percolation, which is based on constructing a coupling with suitable properties; see [8] and [18].

The aim of this paper is to provide percolation results for the CRCM with parameters as general as possible. The two cases considered are

(C1) $q \geq 1$, $\int R^d Q(\mathrm{d}R) < \infty$ and

(C2) $q < 1$, $Q([0, R_0]) = 1$ for some R_0.

In both cases we prove the absence of percolation for small z and the existence of percolation for large z. Part of the result is trivially obtained using a stochastic domination result of Georgii and Küneth [7]. The challenging part, proving the existence of percolation for large z in (C1) and the absence of percolation for small z in (C2), relies on a discrete stochastic domination result proved by Liggett *et al.* [12]. This result is stated in Proposition 4.2. This method was successfully used in [3] to prove a percolation result for the continuum Quermass-interaction model.

The paper is organised as follows. Section 2 is devoted to the introduction of the model and the notations. Section 3.1 contains the main results of the paper, concerning percolation for the CRCM. In Section 3.2 we present a phase transition result for the WR model with random radii. This result is a nontrivial extension of the results of [2] and [6], relying on an infinite-volume FK representation. In Section 4 we introduce basic notions of stochastic domination and we prove, as a direct consequence of Theorem 1.1 of [7], the absence of percolation for small activities in the (C1) case and the percolation for large activities in the (C2) case. Section 5 is devoted to the proof of the percolation for large activities z in the (C1) case and Section 6 is devoted to the proof of the absence of percolation for small activities in the (C2) case. Finally, Appendix A contains a sketch of the proof of standard results involving the uniqueness of the unbounded cluster and the infinite-volume FK representation.

2. Notations and definitions

2.1. State space and reference measure

For d at least 2, we denote by S the space $\mathbb{R}^d \times \mathbb{R}^+$ endowed with the Borel σ-algebra. A configuration ω is a nonnegative integer-valued measure on S, which can be represented as $\omega = \sum_{i \in I} \delta_{(x_i, R_i)}$ for a finite or infinite sequence $(x_i, R_i)_{i \in I}$, with finite mass on $\Lambda \times \mathbb{R}^+$ for every bounded subset Λ of $\mathbb{R}^d$. The configuration set Ω is equipped with the classical

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σ-algebra $\mathcal{F}$ generated by the counting variables $\omega \mapsto \omega(\Gamma)$, where Γ is a bounded Borel subset of S. The configuration ω restricted to a subset Λ of $\mathbb{R}^d$ is defined by $\omega_\Lambda(\cdot) := \omega(\cdot \cap \Lambda \times \mathbb{R}^+)$ and $\mathcal{F}_\Lambda$ is the sub-σ-algebra of $\mathcal{F}$ generated by the counting variables $\omega \mapsto \omega(\Gamma)$, where Γ is a bounded subset of $\Lambda \times \mathbb{R}^+$. We write $(x, R) \in \omega$ if $\omega(\{(x, R)\}) > 0$. To each configuration ω is associated the germ-grain structure

$$L(\omega) = \bigcup_{(x,R)\in\omega} B(x, R),$$

where $B(x, R)$ is the Euclidean closed ball centred in x with radius R.

For a positive z and a probability measure Q on $\mathbb{R}^+$, $\pi^{z,Q}$ denotes the distribution on Ω of the homogeneous Poisson point process with spatial intensity z and with independent marks distributed by Q. For $\Lambda \subseteq \mathbb{R}^d$, $\pi^{z,Q}_\Lambda$ denotes the projection of $\pi^{z,Q}$ on $\Lambda \times \mathbb{R}^+$. The random closed set L under the law $\pi^{z,Q}$ is the so-called Poisson–Boolean model with intensity z and radii distribution Q.

2.2. Interaction

For every configuration ω, the connected components in $L(\omega)$ are defined as follows. Two points $x, y \in \mathbb{R}^d$ are in the same connected component of $L(\omega)$ if there is a finite number of points $(x_1, R_1), \ldots, (x_n, R_n)$ in ω such that

- $x \in B(x_1, R_1)$;
- $y \in B(x_n, R_n)$;
- for all $i \in \{1, \ldots, n-1\}$, $B(x_i, R_i) \cap B(x_{i+1}, R_{i+1}) \neq \varnothing$.

Remark 2.1. This connectivity is the usual one defined with the Gilbert graph. However, it is different from the topological one for some infinite configurations.

The random cluster interaction between the particles in a finite configuration ω is given by the unnormalised density $q^{N_{\mathrm{cc}}(\omega)}$, where $N_{\mathrm{cc}}(\omega)$ denotes the number of connected components of $L(\omega)$. This density is well defined for only finite configurations. As usual, for infinite configurations we define a local conditional density.

Definition 2.1. For a bounded subset Λ of $\mathbb{R}^d$, the Λ-local number of connected components of a configuration (finite or infinite) is given by

$$N^\Lambda_{\mathrm{cc}}(\omega) = \lim_{\Delta\to\mathbb{R}^d} (N_{\mathrm{cc}}(\omega_\Delta) - N_{\mathrm{cc}}(\omega_{\Delta\setminus\Lambda})),$$

where the limit, taken along any increasing sequence, is well defined; see [4, Proposition 2.1]. This quantity can be negative.

2.3. The CRCM

The CRCM is defined using a standard DLR formalism which requires that the probability measure satisfies equilibrium equations based on Gibbs kernels; see (2.2).

Definition 2.2. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a CRCM for parameters z, Q, and q (CRCM(z, Q, q)) if it is stationary and if, for every bounded subset Λ of $\mathbb{R}^d$ and every bounded measurable function f, we have

$$\int_\Omega f(\omega)\mathbb{P}(\mathrm{d}\omega) = \int_\Omega \int_\Omega f(\omega'_\Lambda + \omega_{\Lambda^c}) \frac{q^{N^\Lambda_{\mathrm{cc}}(\omega'_\Lambda+\omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \pi^{z,Q}_\Lambda(\mathrm{d}\omega'_\Lambda)\mathbb{P}(\mathrm{d}\omega), \tag{2.2}$$

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where $Z_\Lambda(\omega_{\Lambda^c}) := \int_\Omega q^{N_{\mathrm{cc}}^\Lambda(\omega'_\Lambda+\omega_{\Lambda^c})}\pi_\Lambda^{z,Q}(\mathrm{d}\omega'_\Lambda)$ is the partition function which is assumed to be nondegenerate.

Equivalently, for $\mathbb{P}$-almost every configuration ω, the conditional law of $\mathbb{P}$ given ω_{Λ^c} is absolutely continuous with respect to $\pi_\Lambda^{z,Q}$ with density $q^{N_{\mathrm{cc}}^\Lambda(.+\omega_{\Lambda^c})}/Z_\Lambda(\omega_{\Lambda^c})$.

These equations, for every bounded Λ, are known as the DLR equations. Existence and (non)uniqueness of the CRCM are standard but complex questions of statistical mechanics. In a recent paper [4], the existence was proved for a large class of parameters (z, Q, q) such as the $q < 1$ case or unbounded radii. In this paper we are interested in the two following cases, which we restate:

(C1) $q \geq 1$ and Q satisfying $\int R^d\, Q(\mathrm{d}R) < +\infty$;

(C2) $q < 1$ and Q satisfying $Q([0, R_0]) = 1$ for a given $R_0 > 0$.

For both cases, the existence of a CRCM was proved in [4].

3. Results

3.1. Percolation results

A configuration ω is said to percolate if $L(\omega)$ contains at least one unbounded connected component. We say a probability measure percolates (respectively, does not percolate) if the probability of the percolation event $\{\omega \text{ percolates}\}$ is 1 (respectively, 0). The first natural question concerns the number of infinite connected components, which is answered in Theorem 3.1.

Theorem 3.1. *For every CRCM, almost surely the number of unbounded connected components is at most* 1.

Proof. The proof of this result uses classical percolation techniques and is developed in Appendix A. □

The main question of this paper is the classical question: does percolation occur? One trivial case is when $Q = \delta_0$ and there is trivially no percolation since the CRCM is just a Poisson point process. In the following this case is omitted. The following theorems form the main results of this paper. Using these theorems we prove the existence, for both cases (C1) and (C2), of a subcritical phase for small activities z where no percolation occurs and a supercritical phase for large activities z where percolation occurs.

Theorem 3.2. *In the (C1) case,*

- *there exists $z_0 := z_0(Q, q, d)$ positive such that, for every $z < z_0$, no* CRCM(z, Q, q) *percolates;*
- *with the additional assumption $Q(\{0\}) = 0$, there exists $z_1 := z_1(Q, q, d)$ finite such that, for every $z > z_1$, every* CRCM(z, Q, q) *percolates.*

Theorem 3.3. *In the (C2) case,*

- *there exists $z_1 := z_1(Q, q, d)$ finite such that, for every $z > z_1$, every* CRCM(z, Q, q) *percolates;*
- *there exists $z_0 := z_0(Q, q, d)$ positive such that, for every $z < z_0$, no* CRCM(z, Q, q) *percolates.*

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The proofs of Theorems 3.2 and 3.3 are based on stochastic domination techniques which enable us to compare the CRCM with more simple models, namely, the Poisson–Boolean model and the Bernoulli percolation model. Proofs of the theorems can be found in Sections 4–6.

Remark 3.1. For both Theorems 3.2 and 3.3, we obtain the existence of $0 < z_0 \leq z_1 < +\infty$. The existence of a threshold, meaning that $z_0 = z_1$, is a natural and interesting question and remains open for the CRCM and many other models. Indeed, the only existing technique available to address this question is to prove stochastic monotonicity with respect to the parameter z. This is proved for continuum models using the result of Georgii and Küneth [7], which does not apply for the CRCM.

Remark 3.2. The assumption $Q(\{0\}) = 0$ in Theorem 3.2 is an artefact of the proof developed in Section 5. This assumption ensures the existence of all parameters introduced in the proof. It is our belief that this theorem could be generalised to the optimal assumption $Q(\{0\}) < 1$.

3.2. Phase transition for the WR model

In this section the WR model [19] is introduced and a phase transition result is exhibited as a direct consequence of Theorem 3.2. In this section we provide a nontrivial generalisation of the results of [2] and [6] to the case of unbounded radii. We also provide a more direct proof relying on an infinite-volume FK representation (see Proposition 3.1). Let q be an integer greater than 1. The WR model is defined on the space $\widetilde{\Omega}$ of coloured configuration $\widetilde{\omega}$ in $\widetilde{S} = \mathbb{R}^d \times \mathbb{R}^+ \times \{1, \dots, q\}$. The marks in $\{1, \dots, q\}$ are called colours. The definitions of the σ-field $\widetilde{\mathcal{F}}$ and the coloured Poisson point process $\widetilde{\pi}^{z,Q,q}$ with uniform colour marks are the natural extension of the definitions in Section 2 and are omitted. Let $\mathcal{A}$ be the event of a coloured configuration $\widetilde{\omega}$ such that any two balls of two different colours do not overlap.

Definition 3.1. A probability measure $\widetilde{\mathbb{P}}$ on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ is a WR model of parameters z, Q, q (WR(z, Q, q)) if it is stationary and if, for every bounded set Λ and every bounded measurable function f,

$$\int_{\widetilde{\Omega}} f \, \mathrm{d}\widetilde{\mathbb{P}} = \int_{\widetilde{\Omega}} \int_{\widetilde{\Omega}} f(\widetilde{\omega}'_\Lambda + \widetilde{\omega}_{\Lambda^c}) \frac{\mathbf{1}_{\mathcal{A}}(\widetilde{\omega}'_\Lambda + \widetilde{\omega}_{\Lambda^c})}{\widetilde{Z}_\Lambda(\widetilde{\omega}_{\Lambda^c})} \widetilde{\pi}^{z,Q,q}_\Lambda(\mathrm{d}\widetilde{\omega}'_\Lambda) \widetilde{\mathbb{P}}(\mathrm{d}\widetilde{\omega}).$$

For $z > 0$, $\int R^d Q(\mathrm{d}R) < +\infty$ and q is a positive integer, the existence is well known and can be seen as a consequence of the existence of the CRCM and the FK representation (see Proposition 3.1).

Theorem 3.4. *For every integer q greater than* 1 *and every probability measure Q satisfying $Q(\{0\}) = 0$ and $\int R^d Q(\mathrm{d}R) < \infty$, we have the existence of $\widetilde{z} > 0$ such that, for every $z > \widetilde{z}$, there exists at least q ergodic* WR(z, Q, q).

Proof. The proof of this theorem is based on an infinite-volume FK representation; see Proposition 3.1. □

Proposition 3.1. *Let $\mathbb{P}$ be a* CRCM$(z/q, Q, q)$. *We build a probability measure $\widetilde{\mathbb{P}}^1$ on $\widetilde{\Omega}$ by colouring each finite connected component independently and uniformly over the q colours $\{1, \dots, q\}$. If it exists, the (unique) infinite connected component is assigned the colour* 1.

Then the measure $\widetilde{\mathbb{P}}^1$ is a WR(z, Q, q). *Moreover if $\mathbb{P}$ is ergodic, the same holds for $\widetilde{\mathbb{P}}^1$.*

Proof. See Appendix A. It relies on the Georgii–Nguyen–Zessin (GNZ) equations satisfied by the WR model and the CRCM.

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Using Theorem 3.2, it follows that, for $z > qz_1$, every CRCM$(z/q, Q, q)$ percolates. Choose an ergodic $\mathbb{P}$, which is possible since every CRCM$(z/q, Q, q)$ is a mixing of ergodic CRCM$l(z/q, Q, q)$; see [5]. From Proposition 3.1 one can build q ergodic WR measures $\widetilde{\mathbb{P}}^1, \ldots, \widetilde{\mathbb{P}}^q$, which are distinct since the unbounded connected component does not have the same colour. This proves Theorem 3.4. □

4. Stochastic domination

Stochastic domination is central to the proofs of Theorems 3.2 and 3.3. We recall some basic definitions on stochastic domination. We say that ω is smaller than ω', and we write $\omega \leq \omega'$ if $\omega(\Gamma) \leq \omega'(\Gamma)$ for every Borel set Γ of S. An event A in $\mathcal{F}$ is said to be increasing if, for every $\omega \in A$ and every $\omega' \geq \omega$, we have $\omega' \in A$. One example of an increasing event is the percolation event $\{\omega \in \Omega \mid \omega \text{ percolates}\}$. Finally, if $\mathbb{P}$ and $\mathbb{P}'$ are two probability measures on Ω, we say that $\mathbb{P}'$ stochastically dominates $\mathbb{P}$, writing $\mathbb{P} \preceq \mathbb{P}'$ if $\mathbb{P}(A) \leq \mathbb{P}'(A)$ for every increasing event A. One way of proving stochastic domination between Gibbs point processes is by the use of Theorem 1.1 of [7], which relies on an inequality between the Papangelou intensities of the processes considered. This leads to the following proposition.

Proposition 4.1. *Let $\mathbb{P}$ be a* CRCM(z, Q, q).

- *If $q \geq 1$ then $\mathbb{P} \preceq \pi^{qz,Q}$;*
- *if $q < 1$ then $\pi^{qz,Q} \preceq \mathbb{P}$.*

Proof. The Papangelou intensity of a CRCM(z, Q, q) is $q^{1-k(X,\omega)}$, where $k(X, \omega)$ counts the number of connected components of $L(\omega)$ overlapping the ball $B(X) = B(x, R)$. Since the function k is nonnegative, an application of Theorem 1.1 of [7] directly yields the result. □

Since the percolation event is increasing, using Proposition 4.1 and the existence of subcritical and supercritical phases for the Poisson–Boolean model of radii law Q satisfying $\int R^d Q(\mathrm{d}R) < +\infty$, see [9], we obtain the following corollary proving the existence of a subcritical phase (respectively, a supercritical phase) for the CRCM in the (C1) case (respectively, (C2)).

Corollary 4.1. *Let $z_c(d, Q)$ be the percolation threshold of the Poisson point process $\pi^{z,Q}$. Then*

- *in the (C1) case, for every $z < z_c(d, Q)$ no* CRCM(z, Q, q) *percolates;*
- *in the (C2) case, for every $z > z_c(d, Q)$ each* CRCM(z, Q, q) *percolates.*

Remark 4.1. An application of Corollary 4.1 yields the existence of the constant z_0 of Theorem 3.2 and the constant z_1 of Theorem 3.3. In some cases, such as deterministic radii, we have $k((x, R), \omega) \leq \kappa(d)$ and Theorem 1.1 of [7] can be used to determine the existence of the others constants. This argument was used in [2] and [6].

In the general case, we do not have an upper bound for the quantity $k((x, R), \omega)$ and, therefore, Theorem 1.1 of [7] cannot be used to prove the remainder of Theorems 3.2 and 3.3.

To this end, we introduce Proposition 4.2, which provides a good stochastic domination between the law of a dependent family $(\xi_x)_{x\in\mathbb{Z}^d}$ of random variables in $\{0, 1\}$ and Π^p, the Bernoulli (with parameter p) product measure on $\{0, 1\}^{\mathbb{Z}^d}$. This type of argument has been used to prove percolation results for continuum models; see [3].

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Proposition 4.2. (Liggett *et al.* [12].) *Let $(\xi_x)_{x\in\mathbb{Z}^d}$ be a (dependent) family of $\{0,1\}$-valued random variables of joint law ν. Let $p \in [0,1]$ and assume that, for every vertex x,*

$$\nu(\xi_x = 1 \mid \xi_y, \|x-y\|_\infty > k) \geq p \quad \textit{almost surely},$$

where $\|\cdot\|_\infty$ is the infinite norm on $\mathbb{R}^d$ and k is a positive fixed constant.

Then $\Pi^{f(p)}$ is stochastically dominated by the distribution of $(\xi_x)_{x\in\mathbb{Z}^d}$, where f is a deterministic function depending on k such that $\lim_{p\to 1} f(p) = 1$.

5. Proof of Theorem 3.2: percolation for large activities z

We assume here that condition (C1) is satisfied: $q \geq 1$ and Q satisfying $\int R^d\, Q(\mathrm{d}R) < +\infty$. We also make the extra assumption that $Q(\{0\}) = 0$. Let $\mathbb{P}^z$ be a CRCM(z, Q, q). The idea of the proof is to construct a family $(\xi_x)_{x\in\mathbb{Z}^d}$ such that the conditional probability, as in Proposition 4.2, is as large as we need and such that if the family (ξ_x) percolates, the same holds for our model. To this end, we look into the probability of covering small cubes. Intuitively, if the family of covered cubes percolates then the same holds for the underlying configuration. Let $R_1 > 0$ be such that $Q_{R_1} := Q([0,R_1]) < 1/q$ and let Λ be the cube $[-R_1/2\sqrt{d}, R_1/2\sqrt{d}]^d$. To look into the probability of covering the cube Λ, for a radius R let C_R be the event of configurations ω such that ω_Λ contains at least one ball of radius larger than R.

We define the random variable $\xi := \xi(\omega)$ equal to 1 if $\Lambda \subseteq L(\omega_\Lambda)$ and 0 otherwise. To use Proposition 4.2, we need to prove that

$$\inf_{\omega_{\Delta^c}} \mathbb{P}^z(\xi = 1 \mid \omega_{\Delta^c}) \to 1, \qquad z \to \infty, \tag{5.1}$$

where Δ is a bounded set containing Λ and which will be defined shortly. By construction, we have

$$\mathbb{P}^z(\xi = 1 \mid \omega_{\Delta^c}) \geq \mathbb{P}^z(C_{R_1} \mid \omega_{\Delta^c}).$$

The goal now is to find a suitable bound for $\mathbb{P}^z(C^c_{R_1} \mid \omega_{\Delta^c})$, uniform with respect to ω_{Δ^c}, which tends to 0 when z grows to ∞. To control the quantity N^Λ_{cc}, we need to introduce a 'protective layer', hence, the Δ, and a good event B_z on $\Delta \setminus \Lambda$, which is defined below. We have

$$\begin{aligned}\mathbb{P}^z(C^c_{R_1} \mid \omega_{\Delta^c}) &= \mathbb{P}^z(C^c_{R_1} \cap B_z \mid \omega_{\Delta^c}) + \mathbb{P}^z(C^c_{R_1} \cap B^c_z \mid \omega_{\Delta^c}) \\ &\leq \underbrace{\mathbb{P}^z(B_z \mid \omega_{\Delta^c})}_{\text{(a)}} + \underbrace{\mathbb{P}^z(C^c_{R_1} \cap B^c_z \mid \omega_{\Delta^c})}_{\text{(b)}}.\end{aligned} \tag{5.2}$$

5.1. Bound for the quantity (b) of (5.2)

Thanks to the DLR equations (2.2) on Λ, we have

$$\begin{aligned}&\mathbb{P}^z(C^c_{R_1} \cap B^c_z \mid \omega_{\Delta^c}) \\ &\quad= \int_\Omega \int_\Omega \mathbf{1}_{C^c_{R_1}}(\omega''_\Lambda)\, \mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda}) \frac{q^{N^\Lambda_{\mathrm{cc}}(\omega''_\Lambda + \omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})}}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda)\mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}) \\ &\quad\leq \int_\Omega \int_\Omega \mathbf{1}_{C^c_{R_1}}(\omega''_\Lambda)\, \mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda}) \frac{q^{\#\omega''_\Lambda}}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda)\mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c})\end{aligned}$$

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$$= \int_\Omega \mathbf{1}_{C^c_{R_1}}(\omega''_\Lambda) q^{\#\omega''_\Lambda} \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda) \int_\Omega \frac{\mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda})}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c})$$

$$\leq \exp(-z|\Lambda|(1 - q\,Q_{R_1})) \int_\Omega \frac{\mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda})}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}), \tag{5.3}$$

where $\#\omega$ denotes the number of balls of a finite configuration ω and where the second line comes from the trivial inequality $N^\Lambda_{\mathrm{cc}}(\omega) \leq \#\omega_\Lambda$. From (5.3) one can understand the condition $Q_{R_1} < 1/q$. Now we need to state a precise definition of Δ and of B_z. To this end, let R_2 be a positive real number such that $Q_{R_2} := Q([0, R_2]) > q\,Q_{R_1}$, which is possible by the choice of R_1. We define $\Delta := [-1 - R_2 - R_1/2\sqrt{d}, 1 + R_2 + R_1/2\sqrt{d}]^d$ and we take ε such that $0 < \varepsilon < |\Lambda|(Q_{R_2} - q\,Q_{R_1})/\ln q$. Finally, take $r > 0$ such that $Q_r := Q([0, r]) < \varepsilon/q\mathrm{e}|\Delta \setminus \Lambda|$. The last thing to do is to define B_z as the event of configurations having many small balls centred inside $\Delta \setminus \Lambda$,

$$B_z = \{\omega \in \Omega \mid \#\{(x, R) \in \omega_{\Delta\setminus\Lambda},\ R \leq r\} \geq \lceil \varepsilon z \rceil\},$$

where $\lceil \cdot \rceil$ is the ceiling function. In the following lemma we provide a suitable lower bound for N^Λ_{cc}.

Lemma 5.1. *For $\omega \in C^c_{R_2} \cap B^c_z$, we have $N^\Lambda_{\mathrm{cc}}(\omega) \geq K - \lceil \varepsilon z \rceil$ for a constant K.*

Proof. We first realise that the worst case scenario occurs when $L(\omega_\Lambda)$ has a single connected component which intersects many connected components of $L(\omega_{\Lambda^c})$. Since $\omega \in C^c_{R_2}$, the radii of the balls of $L(\omega_\Lambda)$ are bounded by R_2. Note that the number of connected components $L(\omega_{\Lambda^c})$ intersecting $L(\omega_\Lambda)$ is smaller or equal to the number of disjoint balls in $L(\omega_{\Lambda^c})$ that can intersect $L(\omega_\Lambda)$. By the choice of Δ, a ball $B(x, R)$ in $L(\omega_{\Delta^c})$ which intersects $L(\omega_\Lambda)$ satisfies $R \geq 1$ and, hence, $|B(x, R) \cap \Delta| \geq c_1$ for a given positive constant c_1. In the same manner, any ball $B(x, R)$ of $L(\omega_{\Delta\setminus\Lambda})$ with $R > r$ satisfies $|B(x, R) \cap \Delta| \geq c_2$. So the number of such balls is at most $|\Delta|/\min(c_1, c_2) := c$. To obtain the bound on $N^\Lambda_{\mathrm{cc}}(\omega)$, we need to take into consideration just the balls of $\omega_{\Delta\setminus\Lambda}$ with radius smaller than r, whose number is not greater than $\lceil \varepsilon z \rceil - 1$ since $\omega \in B^c_z$. Hence, the result holds by taking $K = 2 - c$. □

To obtain an upper bound for the integral in (5.3), we consider the conditional probability

$$\begin{aligned}
&\mathbb{P}^z(C^c_{R_2} \cap B^c_z \mid \omega_{\Delta^c}) \\
&\quad = \int_\Omega \int_\Omega \mathbf{1}_{C^c_{R_2}}(\omega''_\Lambda)\, \mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda}) \frac{q^{N^\Lambda_{\mathrm{cc}}(\omega''_\Lambda + \omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})}}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda) \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}) \\
&\quad \geq \int_\Omega \int_\Omega \mathbf{1}_{C^c_{R_2}}(\omega''_\Lambda)\, \mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda}) \frac{q^{K - \lceil \varepsilon z \rceil}}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda) \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}) \\
&\quad = q^{K - \lceil \varepsilon z \rceil} \int_\Omega \mathbf{1}_{C^c_{R_2}}(\omega''_\Lambda) \pi^{z,Q}_\Lambda(\mathrm{d}\omega''_\Lambda) \int_\Omega \frac{\mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda})}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}) \\
&\quad = q^{K - \lceil \varepsilon z \rceil} \exp(-z|\Lambda|(1 - Q_{R_2})) \int_\Omega \frac{\mathbf{1}_{B^c_z}(\omega'_{\Delta\setminus\Lambda})}{Z_\Lambda(\omega'_{\Delta\setminus\Lambda} + \omega_{\Delta^c})} \mathbb{P}^z(\mathrm{d}\omega'_\Delta \mid \omega_{\Delta^c}).
\end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$\begin{aligned}
\mathbb{P}^z(C^c_{R_1} \cap B^c_z \mid \omega_{\Delta^c}) &\leq q^{\lceil \varepsilon z \rceil - K} \exp(-z|\Lambda|(Q_{R_2} - q\,Q_{R_1})) \\
&\leq q^{-K+1} \exp(-z|\Lambda|(Q_{R_2} - q\,Q_{R_1})) \exp(\varepsilon z \ln q).
\end{aligned} \tag{5.5}$$

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By the choice of R_1, R_2, and ε, we have $|\Lambda|(Q_{R_2} - qQ_{R_1}) - \varepsilon \ln q > 0$ and, therefore, the right-hand side of (5.5) decreases to 0 when z grows to ∞.

5.2. Bound for the quantity (a) of (5.2)

Now we need to control the quantity (a) of (5.2). Using Theorem 1.1 of [7], the probability measure $\pi_\Delta^{qz,Q}$ stochastically dominates $\mathbb{P}^z(\cdot \mid \omega_{\Delta^c})$, and since the event B_z is increasing, we have

$$\begin{aligned}\mathbb{P}^z(B_z \mid \omega_{\Delta^c}) &\le \pi_\Delta^{qz,Q}(B_z)\\ &= \exp(-zq|\Delta \setminus \Lambda|Q_r) \sum_{k \ge \lceil \varepsilon z \rceil} \frac{(zq|\Delta \setminus \Lambda|Q_r)^k}{k!}\\ &\le \frac{(zq|\Delta \setminus \Lambda|Q_r)^{\lceil \varepsilon z \rceil}}{\lceil \varepsilon z \rceil!}\\ &\sim \frac{\exp(\lceil \varepsilon z \rceil \ln(zq|\Delta \setminus \Lambda|Q_r \mathrm{e}/\lceil \varepsilon z \rceil))}{\sqrt{2\pi \lceil \varepsilon z \rceil}}, \qquad z \to \infty, \end{aligned} \tag{5.6}$$

where the last line comes from the Lagrange inequality and the Stirling formula. By the choice of parameters, we have

$$\frac{zq|\Delta \setminus \Lambda|Q_r \mathrm{e}}{\lceil \varepsilon z \rceil} \le \frac{zq|\Delta \setminus \Lambda|Q_r \mathrm{e}}{\varepsilon z} < 1,$$

hence, the right-hand side of (5.6) converges to 0 when z grows to ∞, and the convergence (5.1) is proven.

5.3. Construction of the dependent family $(\xi_x)_{x \in \mathbb{Z}^d}$

To each $x \in \mathbb{Z}^d$ we associate the small cube $\Lambda_x := (R_1/\sqrt{d})x \oplus \Lambda$ and the large cube $\Delta_x := (R_1/\sqrt{d})x \oplus \Delta$, where $\oplus$ denotes the standard Minkowski sum.

We define ξ_x as we did for $\xi := \xi_0$, meaning that $\xi_x(\omega)$ is equal to 1 if $\Lambda_x \subseteq L(\omega_{\Lambda_x})$, and 0 otherwise. Finally, take k such that if $\|x - y\|_\infty > k$ then $\Delta_x \cap \Delta_y = \varnothing$. For example, $k = 2 + 2\sqrt{d}R_2/R_1 + 2\sqrt{d}/R_1$ is valid. For every $x \in \mathbb{Z}^d$, since $\mathbb{P}^z$ is stationary, we have

$$\begin{aligned}\mathbb{P}^z(\xi_x = 1 \mid \xi_y, \|x - y\|_\infty > k) &= \mathbb{P}^z(\xi = 1 \mid \xi_y, \|y\|_\infty > k)\\ &= \mathbb{E}_{\mathbb{P}^z}[\mathbb{P}^z(\xi = 1 \mid \mathcal{F}_{\Delta^c}) \mid \xi_y, \|y\|_\infty > k]\\ &\ge \inf_{\omega_{\Delta^c}} \mathbb{P}^z(\xi = 1 \mid \omega_{\Delta^c}).\end{aligned}$$

However, using convergence (5.1), we have, for any $p \in (0, 1)$, the existence of $z_1(p)$ such that, for every $z > z_1(p)$ and every x,

$$\mathbb{P}^z(\xi_x = 1 \mid \xi_y, \|y\|_\infty > k) \ge p \quad \text{almost surely.}$$

Using Proposition 4.2, $\Pi^{f(p)}$ is stochastically dominated by the law of $(\xi_x)_{x \in \mathbb{Z}^d}$. The parameter p can be chosen such that $f(p)$ is larger than any given percolation threshold. But it is clear that if the family $(\xi_x(\omega))_{x \in \mathbb{Z}^d}$ percolates with respect to the cubic lattice, the same holds for the configuration ω. Therefore, if the activity z is large enough then $f(p)$ is larger than the percolation threshold of $\Pi^{f(p)}$ and $\mathbb{P}^z$ percolates. Theorem 3.2 is proved. □

Remark 5.1. The assumption $Q(\{0\}) = 0$ was used in the proof to ensure that constants R_1 and r are well defined. It seems possible to use the same construction for radii law Q having a small atom in 0, but since the construction of r depends on all the other constants introduced in the proof, it is impossible to derive from the proof presented a general condition in that case.

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6. Proof of Theorem 3.3: absence of percolation for small activities z

We assume here that condition (C2) is satisfied: $q < 1$ and there exists $R_0 > 0$ such that $Q([0, R_0]) = 1$. Let $\mathbb{P}^z$ be a CRCM(z, Q, q). Without loss of generality, we are stating the proof in the specific case $R_0 = 1$. The idea is, as in the proof of Theorem 3.2, to build a good family $(\xi_x)_{x\in\mathbb{Z}^d}$ in order to apply Proposition 4.2.

Let $\Lambda = [-\frac{1}{2}, \frac{1}{2}]^d$ and $\Delta = [-8, 8]^d$. We define $\xi := \xi(\omega)$ equal to 1 when $L(\omega) \cap \Lambda = \varnothing$, and 0 otherwise. As the radii are bounded by 1, ξ depends only on the restricted configuration ω_Δ.

We want to prove that

$$\inf_{\omega_{\Delta^c}} \mathbb{P}^z(\xi = 1 \mid \omega_{\Delta^c}) \to 1, \qquad z \to 0. \tag{6.1}$$

For a configuration ω, let $N_\xi(\omega)$ denote the number of connected components of $L(\omega_\Delta)$ which overlap Λ. The random variables ξ and N_ξ are strongly related since $\xi = 1$ if and only if $N_\xi = 0$.

Lemma 6.1. *There exists $\alpha > 0$ such that*

$$\int_\Omega N_\xi(\omega)\mathbb{P}^z(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c}) \le z|\Delta|q^{1-\alpha} \quad \textit{almost surely.}$$

This lemma is proved at the end of the section. First, let us see how Lemma 6.1 leads to the desired result. We have

$$\mathbb{P}^z(\xi = 0 \mid \omega_{\Delta^c}) = \mathbb{P}^z(N_\xi > 0 \mid \omega_{\Delta^c}) = \mathbb{P}^z\big(N_\xi > \tfrac{1}{2} \;\big|\; \omega_{\Delta^c}\big) \le 2z|\Delta|q^{1-\alpha},$$

where the last inequality comes from the conditional Markov inequality and Lemma 6.1. From this bound, which does not depend on ω_{Δ^c}, the uniform convergence (6.1) follows.

Now consider the family of variables $(\xi_x)_{x\in\mathbb{Z}^d}$ defined as $\xi := \xi_0$, meaning that $\xi_x(\omega) = 1$ if $L(\omega) \cap \Lambda_x = \varnothing$, where $\Lambda_x = x \oplus \Lambda$. Take $k = 17$ and let $\Delta_x := x \oplus \Delta$. For every vertex x, we have

$$\begin{aligned}\mathbb{P}^z(\xi_x = 1 \mid \xi_y, \|x - y\|_\infty > k) &= \mathbb{P}^z(\xi = 1 \mid \xi_y, \|y\|_\infty > k)\\ &= \mathbb{E}_{\mathbb{P}^z}[\mathbb{P}^z(\xi = 1 \mid \mathcal{F}_{\Delta^c}) \mid \xi_y, \|y\|_\infty > k]\\ &\ge \inf_{\omega_{\Delta^c}} \mathbb{P}^z(\xi = 1 \mid \omega_{\Delta^c}),\end{aligned}$$

and using the convergence (6.1), we have, for any $p \in (0, 1)$, the existence of $z_0(p)$ such that, for every $z < z_0(p)$ and every x,

$$\mathbb{P}^z(\xi_x = 1 \mid \xi_y, \|y\|_\infty > k) \ge p \quad \text{almost surely.}$$

Using Proposition 4.2, $\Pi^{1-f(p)}$ stochastically dominates the law of $(1 - \xi_x)_{x\in\mathbb{Z}^d}$. The parameter p can be chosen such that $1 - f(p)$ is lower than any given percolation threshold.

As in Section 5, the absence of percolation of the family $(1 - \xi_x(\omega))_{x\in\mathbb{Z}^d}$, with respect to the cubic lattice, leads to the absence of percolation in the configuration ω. Therefore, for small activities z, $\mathbb{P}^z$ does not percolate.

Proof of Lemma 6.1. We use the GNZ equations satisfied by the (conditional) probability measure $\mathbb{P}^z(\cdot \mid \omega_{\Delta^c})$. We first need the following lemma.

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Lemma 6.2. (GNZ equations.) *For every bounded measurable function F,*

$$\int_\Omega \sum_{X\in\omega_\Delta} F(X,\omega-\delta_X)\mathbb{P}(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c}) = z\int_\Omega\int_{\Delta\times\mathbb{R}^+} F(X,\omega)q^{1-k(X,\omega)}m(\mathrm{d}X)\mathbb{P}(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c}),$$

where $X=(x,R)$ is a marked point, where $m(\mathrm{d}X)=\mathcal{L}^d(\mathrm{d}x)Q(\mathrm{d}R)$ with $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$, and where $k(X,\omega)$ is the function defined in the proof of Proposition 4.1 as the number of connected components of $L(\omega)$ which overlap the ball $B(X)=B(x,R)$. In the integrals, ω stands for $\omega_\Delta+\omega_{\Delta^c}$.

Proof. The proof of this lemma is exactly the same as the proof of the well-known Slivnyak–Mecke formula and is omitted. □

To use the GNZ equations, let us define a function F such that $F((x,R),\omega)$ is equal to 1 if the following conditions are fulfilled, and 0 otherwise:

- $x\in\Delta$;
- the connected component of $B(x,R)$ in $L(\omega_\Delta+\delta_{(x,R)})$ intersects Λ;
- $B(x,R)$ is one of the balls of its connected component of $L(\omega_\Delta+\delta_{(x,R)})$ which minimise the quantity $k((x,R),\omega)$.

By the GNZ equations applied to the bounded function $F(X,\omega)q^{k(X,\omega)-1}$, we have

$$\begin{aligned}\int_\Omega \sum_{(x,R)\in\omega_\Delta} F((x,R),\omega-\delta_{(x,R)})q^{k((x,R),\omega-\delta_{(x,R)})-1}\mathbb{P}^z(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c}) &= z\int_\Omega\int_{\Delta\times\mathbb{R}^+} F((x,R),\omega)m(\mathrm{d}x,\mathrm{d}R)\mathbb{P}^z(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c})\\ &\le z|\Delta|. \end{aligned} \tag{6.2}$$

In the following, we find an upper bound α for the quantity $k((x,R),\omega-\delta_{(x,R)})$ when $F((x,R),\omega-\delta_{(x,R)})=1$. To this end, consider a ball $B(x,R)$ and let us count the maximum number of disjoint balls of radius not smaller than $R/2$ overlapping $B(x,R)$. By a volume argument, this quantity is no more than 4^d. To obtain this value we look just at the intersection of each disjoint ball with the ball $B(x,2R)$. The volume of each intersection is at least $(R/2)^d v_d$, where v_d is the volume of the unit ball in dimension d. This bound is not sharp but is enough for the following since it does not depend on R.

By contradiction, assume that in $L(\omega_\Delta)$ there is a connected component $L(\mathcal{C})$ (with this notation $\mathcal{C}$ is a restriction of ω_Δ) which overlaps Λ and such that $\min_{(x,R)\in\mathcal{C}}k((x,R),\omega-\delta_{(x,R)})\ge\alpha$.

Let $(x_1,R_1)\in\mathcal{C}$ such that $B(x_1,R_1)$ overlaps Λ. Such a ball exists by the hypothesis on $\mathcal{C}$ and we have $R_1\le 1$, and the distance between x_1 and Λ is no more than 1.

By the hypothesis $k((x_1,R_1),\omega-\delta_{(x_1,R_1)})\ge\alpha$, therefore, $B(x_1,R_1)$ is connected to at least one ball of radius smaller than $R_1/2\le\frac{1}{2}$. Let $B(x_2,R_2)$ be such a ball. The distance between x_2 and Λ is no more than $\frac{5}{2}$, which means that $(x_2,R_2)\in\mathcal{C}$. By the same argument, we can construct a sequence $(x_n,R_n)_n$ with $R_n\le 2^{-n+1}$, and the distance between x_n and Λ is bounded by some quantity increasingly converging to 4. Therefore, each (x_n,R_n) is in $\mathcal{C}$. But this is impossible since $\mathcal{C}$ is a finite configuration. So, in the left-hand side of (6.2), each (x,R) such that F is equal to 1 satisfies $k((x,R),\omega-\delta_{(x,R)})\le\alpha$.

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With this, we obtain, from (6.2), the following inequality:

$$\int_\Omega \sum_{(x,R)\in\omega_\Delta} F((x,R),\omega-\delta_{(x,R)})\mathbb{P}^z(\mathrm{d}\omega_\Delta \mid \omega_{\Delta^c}) \le z|\Delta|q^{1-\alpha}.$$

But $\sum_{(x,R)\in\omega_\Delta} F((x,R),\omega-\delta_{(x,R)})$ dominates $N_\xi(\omega)$ (since a connected component can be counted several times if it has several 'minimising balls'), and we obtain the result. □

Appendix A.

A.1. Proof of Theorem 3.1

Let $\mathbb{P}$ be a CRCM(z,Q,q). Since $\mathbb{P}$ is a mixture of ergodic CRCM(z,Q,q), see [5], we can assume without loss of generality that $\mathbb{P}$ is ergodic. We want to prove Theorem 3.1, which states that $\mathbb{P}$ has almost surely at most one infinite connected component. This is a classical question of percolation theory which has been addressed for many models. The proof involves a positive correlation inequality between specific events. In the literature, this positive correlation inequality was proved using the independence property [13], the Fortuin–Kasteleyn–Ginibre (FKG) inequalities, or the finite energy property [1]. Our model does not satisfy the independence property, and the FKG inequalities are not proved. In the next lemma we state a (weak) continuum version of the finite energy property.

Lemma A.1. *Let Λ be a bounded subset of $\mathbb{R}^d$. Let A be a event of $\mathcal{F}_\Lambda$ such that $\pi_\Lambda^{z,Q}(A)>0$. Let B be an event of $\mathcal{F}_{\Lambda^c}$ such that $\mathbb{P}(B)>0$. Then*

$$\mathbb{P}(A\cap B)>0.$$

Proof. Since $\mathbb{P}$ satisfies the DLR equations (2.2), we have

$$\mathbb{P}(A\cap B)=\int_\Omega\int_\Omega \mathbf{1}_A(\omega'_\Lambda)\,\mathbf{1}_B(\omega_{\Lambda^c})\frac{q^{N_{\mathrm{cc}}^\Lambda(\omega'_\Lambda+\omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})}\pi_\Lambda^{z,Q,q}(\mathrm{d}\omega'_\Lambda)\mathbb{P}(\mathrm{d}\omega).$$

We see that the function when integrated is positive, therefore the integral is positive and the lemma is proven. □

With Lemma A.1, one can carry out the rest of this classical proof of percolation theory. We refer the reader to [13, Section 3.6] for more details. □

A.2. Proof of Proposition 3.1

In this section $\mathbb{P}$ is a CRCM$(z/q,Q,q)$ and $\widetilde{\mathbb{P}}^1$ is the associated coloured measure as in Proposition 3.1. The coloration kernel is denoted by C^1, meaning that

$$\widetilde{\mathbb{P}}^1(\mathrm{d}\widetilde{\omega})=C^1(\mathrm{d}\widetilde{\omega}\mid\omega)\mathbb{P}(\mathrm{d}\omega).$$

To prove that $\widetilde{\mathbb{P}}^1$ is a WR(z,Q,q), we are going to use the GNZ equations satisfied by the CRCM and the WR model.

Lemma A.2. *A probability measure $\mathbb{P}$ is a* CRCM(z,Q,q) *if and only if, for every bounded measurable function G, we have*

$$\int_\Omega\sum_{X\in\omega}G(X,\omega-\delta_X)\mathbb{P}(\mathrm{d}\omega)=z\int_\Omega\int_S G(X,\omega)q^{1-k(X,\omega)}m(\mathrm{d}X)\mathbb{P}(\mathrm{d}\omega),$$

where m and k are defined in Lemma 6.2.

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A probability measure $\widetilde{\mathbb{P}}$ is a $\mathrm{WR}(z, Q, q)$ *if and only if, for every bounded measurable function F, we have*

$$\int_{\widetilde{\Omega}} \sum_{\widetilde{X}\in\widetilde{\omega}} F(\widetilde{X}, \widetilde{\omega} - \delta_{\widetilde{X}})\widetilde{\mathbb{P}}(\mathrm{d}\widetilde{\omega}) = z \int_{\widetilde{\Omega}} \int_{\widetilde{S}} F(\widetilde{X}, \widetilde{\omega})\, \mathbf{1}_{\mathcal{A}}(\widetilde{\omega} + \delta_{\widetilde{X}})\widetilde{m}(\mathrm{d}\widetilde{X})\widetilde{\mathbb{P}}(\mathrm{d}\widetilde{\omega}),$$

where $\widetilde{X} = (x, R, i)$ is a coloured marked point and where $\widetilde{m} = \mathcal{L}^d \otimes Q \otimes \mathcal{U}_q$, with $\mathcal{U}_q$ the uniform measure on the set $\{1, \dots, q\}$.

Proof. See [16] for the proof of this lemma for general Gibbs interactions. □

By the definition of $\widetilde{\mathbb{P}}^1$, we have

$$\begin{aligned}\int_{\widetilde{\Omega}} \int_{\widetilde{S}} & zF(\widetilde{X}, \widetilde{\omega})\, \mathbf{1}_{\mathcal{A}}(\widetilde{\omega} + \delta_{\widetilde{X}})\widetilde{m}(\mathrm{d}\widetilde{X})\widetilde{\mathbb{P}}^1(\mathrm{d}\widetilde{\omega}) \\ &= \int_{\Omega} \int_{S} \underbrace{\left(\int_{\widetilde{\Omega}} \sum_{k=1,\dots,q} F((X, i), \widetilde{\omega})\, \mathbf{1}_{\mathcal{A}}(\widetilde{\omega} + \delta_{(X,i)}) C^1(\mathrm{d}\widetilde{\omega} \mid \omega) \right)}_{G(X,\omega)} \frac{z}{q} m(\mathrm{d}X)\mathbb{P}(\mathrm{d}\omega).\end{aligned}$$

Since $\mathbb{P}$ satisfies the GNZ equations of the $\mathrm{CRCM}(z/q, Q, q)$, we have

$$\int_{\widetilde{\Omega}} \int_{\widetilde{S}} zF(\widetilde{X}, \widetilde{\omega})\, \mathbf{1}_{\mathcal{A}}(\widetilde{\omega} + \delta_{\widetilde{X}})\widetilde{m}(\mathrm{d}\widetilde{X})\widetilde{\mathbb{P}}^1(\mathrm{d}\widetilde{\omega}) = \int_{\Omega} \sum_{X\in\omega} G(X, \omega - \delta_X) q^{k(X,\omega-\delta_X)-1} \mathbb{P}(\mathrm{d}\omega).$$

In $G(X, \omega - \delta_X)$, the indicator function is equal to 1 if the following hold.

- The ball $B(X)$ is connected only to finite connected components of $L(\omega - \delta_X)$. In that case, for each colour i of X, there is only one good coloration among the $q^{k(X,\omega-\delta_X)}$ possible colorations of the $k(X, \omega - \delta_X)$ connected components.
- The ball $B(X)$ is connected to the only infinite connected component of $L(\omega - \delta_X)$. In that case, X can only take the colour 1 and there is only one good coloration among the $q^{k(X,\omega-\delta_X)-1}$ possible colorations since the infinite connected component is coloured 1.

In both cases, we find that

$$G(X, \omega - \delta_X) q^{k(X,\omega-\delta_X)-1} = \int F(\widetilde{X}, \widetilde{\omega} - \delta_{\widetilde{X}}) C^1(\mathrm{d}\widetilde{\omega} \mid \omega),$$

which completes the proof of Proposition 3.1. □

Acknowledgement

This work was supported in part by the Labex CEMPI through the grant ANR-11-LABX-0007-01.

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