Proc. Camb. Phil. Soc. (1969), **66**, 247
PCPS 66–30
Printed in Great Britain

On sunny collapsing

By J. F. P. HUDSON†
Department of Pure Mathematics, Cambridge

(*Received* 11 *July* 1968)

Introduction. The technique of sunny collapsing was introduced by Zeeman in (1) in order to prove the unknotting of PL ball pairs with co-dimension $\geqslant 3$. Since then the technique has been used a number of times. In each case the method was used to prove special cases of the Theorem in this paper. Apart from providing a more unified treatment of various applications of this one technique, the theorem has important applications to the theory of relative regular neighbourhoods.

Definitions. *Let $X \subset Y$ be a polyhedra (compact PL spaces). We say Y collapses to X, $Y \searrow X$, if there is a finite sequence of subpolyhedra $X = Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y$, such that for each i, $\overline{Y_i - Y_{i-1}}$ is a PL ball, B_i say, and $B_i \cap Y_{i-1}$ is a PL ball in the boundary of B_i of dimension one less than B_i.*

A polyhedron which collapses to a single point is called collapsible.

Theorem. *Let $X \subset Y \subset M$ be a polyhedra, M being a PL m-manifold. If $M \searrow X$, $Y \searrow X$, and $\dim(Y - X) \leqslant m-3$, then $M \searrow Y$.*

Corollary 1. *If X is a collapsible polyhedron in a PL ball B with $\dim B \geqslant \dim X + 3$, then $B \searrow X$.*

Definitions. *If $K, L \subset J$ are finite simplicial complexes, K is* link-collapsible on L *if, for every simplex A in $\overline{K-L} \cap L$, $|\mathrm{link}(A; \overline{K-L})|$ is collapsible.*

J link-collapses to K on L *if, for every simplex A in $\overline{K-L} \cap L$, $|\mathrm{link}(A, J)| \searrow |\mathrm{link}(A, \overline{K-L})|$.*

If $X, Y \subset M$ are polyhedra, we say X is link-collapsible on Y, *if there are triangulations $K, L \subset J$ of X, $Y \subset M$ such that K is link-collapsible on L.*

We say M link-collapses to X on Y *if there are triangulations $K, L \subset J$ of X, $Y \subset M$ such that J link-collapses to K on L.*

Corollary 2. *Let $X, Y \subset M$ be polyhedra, M being a PL m-manifold. Suppose that $Y \subset X \cap \partial M$, X is link-collapsible on Y and $\dim(X - Y) \leqslant m-3$. Then M link-collapses to X on Y.*

This last lemma plays an important part in the theory of relative regular neighbourhoods, see (4).

Definitions. *Let Q be a PL manifold, I the unit interval, and X a compact polyhedron in $Q \times I$. If (x,t) and $(x',t') \in Q \times I$, we say (c,t) is* directly below (x',t') *if $x = x'$ and $t < t'$.*

† This research was supported in part by the National Science Foundation Grant NSF GP-5609 and the Office of Naval Research Grant NONR (G)-0016-66.

https://doi.org/10.1017/S0305004100044935 Published online by Cambridge University Press

For any subset U of $Q \times I$, the shadow *of U is the set of points of $Q \times I$ directly below points of U. X* sunny collapses *to X_0 in $Q \times I$ if there are triangulations K of X, J of Q, such that K is linearly embedded in $J \times I$ [i.e. each simplex of K is linearly embedded in $\sigma \times I$ for some simplex $\sigma \in J$], and there is a sequence of elementary simplical collapses*

$$K = K_r \searrow K_{r-1} \searrow \dots \searrow K_0,$$

where $|K_0| = X_0$ and for each i, shadow $(K_i) \cap K \subset K_{i-1}$.

If $Y_0 \subset Y$ are polyhedra, Y_0 is said to be of local co-dimension $\geqslant c$ in Y if for any triangulation $K_0 \subset K$ of $Y_0 \subset Y$, and every simplex $\sigma \in K_0$ there is a simplex τ of K with $\sigma < \tau$ and $\dim \sigma \leqslant \dim \tau - c$.

For any continuous mapping $f: X \to Y$,

$$S'_r(f) = \{x \in X | f^{-1}fx \text{ contains at least } r \text{ points}\},$$

$$S_r(f) = \text{closure of } S'_r(f),$$

f is non-degenerate *if $S_\infty(f)$ is empty.*

LEMMA 1. *If Q is a PL manifold, X a polyhedron in $Q \times 1$, $p: Q \times I \to Q$ the projection on the first factor,* $\dim X = k < q$ *and* $\dim [X \cap (\partial Q \times I)] < k$, *then there is a level-preserving PL homeomorphism $h: Q \times I \to Q \times I$, arbitrarily close to the identity such that $S_2(p|hX)$ is a local co-dimension $\geqslant q - k$ in hX.*

Note. This implies that $p|hX$ is non-degenerate.

This lemma is a fairly standard type of general position lemma. A proof is given in (2).

LEMMA 2. *Let X be a polyhedron, $F: X \times I \to Q \times I$ a PL embedding with*

$$F^{-1}[(Q \times 0) \cup (\partial Q \times I)] = X \times 0.$$

Let $p: Q \times I \to Q$ and $\pi: X \times I \to X$ be the projections on the first factor. Suppose

(a) $S_2(pF)$ is of local co-dimension $\geqslant 2$ in $X \times I$, and

(b) $\pi | S_2(pF)$ is non-degenerate.

Then $F(X \times I)$ sunny collapses to $F(X \times 0)$ in $Q \times I$.

The proof of an essentially identical lemma is proved in (2). The proof is similar to the proof of Lemma 9 in (1). For this reason I give just a summary of the proof.

Summary of proof of Lemma 2. Triangulate $X \times I$, X so that $S_2(pF)$ is triangulated as a subcomplex of $X \times I$ and the projection $X \times I \to X$ is simplicial. If $|L| = S_2(pF)$, let βL be a subdivision of L such that $pF|L: \beta L \to Q$ is simplicial for some subdivision of Q. Now let X^{k-1} = the $(k-1)$-skeleton of X and construct 'blisters' over the top dimensional simplexes of βL as in (1). Let Y be the subset of $X \times I$ obtained from $X^{k-1} \times I$ by pushing in the blisters. Then $F(x \times I)$ sunny collapses to $F(Y)$ and $Y \cong X^{k-1} \times I$ is in a suitable position to apply induction on the dimension.

LEMMA 3. *If $X \subset Q \times I$, Q being compact, and X sunny collapses to*

$$X_0 = X \cap [(Q \times 0) \cup (\partial Q \times I)],$$

then $Q \times I$ collapses to $(Q \times 0) \cup (\partial Q \times I) \cup X$.

Proof identical with lemma 7 of (1) and the corresponding lemma in (2).

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LEMMA 4. *Let $B \subset Q \times I$, Q being compact, be an m-ball with*

$$B \cap [(Q \times 0) \cup (\partial Q \times I)] = F,$$

an $(m-1)$-ball in ∂B. If $m \leqslant q-2$, then $Q \times I$ collapses to $(Q \times 0) \cup (\partial Q \times I) \cup B$.

Proof. Let $\alpha\colon F \times I \to B$ be a PL homeomorphism with $\alpha(F \times 0) = F$. Applying Lemma 1 first to $Q \times I$ and then to $F \times I$, there are level-preserving PL homeomorphisms $h\colon F \times I \to F \times I$ and $k\colon Q \times I \to Q \times I$ such that $S_2(pk\alpha)$ is of local co-dimension $\geqslant 2$ in $F \times I$ and $\pi | hS_2(pk\alpha)$ is non-degenerate, p and π being the projections on to the first factors as before. Then if $\alpha' = k\alpha h^{-1}$, α' satisfies the conditions for Lemma 2, and $\alpha'(F \times I)$ sunny collapses to $\alpha'(F \times 0)$. So, by Lemma 3, $Q \times I$ collapses to

$$(Q \times 0) \cup (\partial Q \times I) \cup \alpha'(F \times I) = (Q \times 0) \cup (\partial Q \times I) \cup kB.$$

But PL homeomorphism preserves collapses, so, applying k^{-1}, $Q \times I$ collapses to $(Q \times 0) \cup (\partial Q \times I) \cup B$.

LEMMA 5. *Let $K \subset L \subset J$ be simplicial complexes (finite), with K full in J,* [i.e.

$$\sigma \in J, \dot{\sigma} \in K \Rightarrow \sigma \in K].$$

Let $K' \subset L' \subset J'$ be the barycentric first derived complexes. Let $N = N(K', J')$, the closed simplicial neighbourhood of K' in J'. Then $N \cup L'$ collapses to L'.

Proof. Let $A_1, A_2, \ldots, A_r$ be the simplexes of J–L meeting K in order of increasing dimension. Now $A_i \cap K = B_i$ say, a single face of A_i. So $N \cap A_i$ is a ball of the same dimension as A_i and $N \cap \dot{A}_i$ is a ball of one lower dimension. So $N \cap A_i \searrow N \cap \dot{A}_i$.

So

$$L \cup \bigcup_{j=1}^{i} (N \cap A_j) \quad \text{collapses to} \quad L \cup \bigcup_{j=1}^{i-1} (N \cap A_j).$$

This sequence gives the required collapse $N \cup L$ to L.

Proof of the main theorem. $X_0 \subset X \subset M$, $X \searrow X_0$ and $M \searrow X_0$. $\dim (X - X_0) \leqslant m-3$. Suppose $X = X_r \searrow X_{r-1} \searrow \ldots \searrow X_0$ is a sequence of elementary collapses. The theorem will be proved by induction on the number of elementary collapses. By the inductive hypothesis we can assume that $M \searrow X_{r-1}$. Triangulate M with X and X_{r-1} triangulated as sub-complexes, and let N be the second derived neighbourhood of X_{r-1} in M. Now N is a regular neighbourhood of X_{r-1} in M meeting the boundary regularly and M is a regular neighbourhood of X_{r-1} in M, so, by the 'regular neighbourhood annulus theorem' ((3), Corollary 1), closure $(M - N)$ is PL homeomorphic to the product Frontier $(N) \times I$. Let $Q =$ Frontier N. Now we have an elementary collapse of X to X_{r-1}, so $X_r = X_{r-1} \cup B$, where B is an s-ball, say, and $B \cap X_{r-1} = F$, a face of B (i.e. an $(s-1)$-ball in ∂B). Now $N \cap B$ is a second derived neighbourhood of F in B, so $B' =$ closure $(B - B \cap N)$ is also an s-ball, and $F' = B \cap$ Frontier N is a face of B'.

Now let $\alpha\colon \overline{M - N} \to Q \times I$ be the PL homeomorphism from above. This may be taken to send Frontier N on to $Q \times 0$. Since ∂Q has a collar neighbourhood in Q, we may construct a PL homeomorphism $k\colon Q \times I \to Q \times I$ which sends $Q \times 0$ to $(Q \times 0) \cup (\partial Q \times I)$,

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and $(\partial Q \times I) \cup (Q \times I)$ to $Q \times 1$. Let $\alpha' = k\alpha$: $\overline{M - N} \to Q \times I$. Then $\alpha' B'$ is a PL ball in $Q \times I$ satisfying the requirements of Lemma 4. So $Q \times I$ collapses on to

$$(Q \times 0) \cup (\partial Q \times I) \cup \alpha' B' \quad \text{and} \quad \overline{M - N} \searrow B' \cup \text{Frontier } N.$$

So $M \searrow N \cup B'$. But, by Lemma 5, $N \cup B' = N \cup X_r \searrow X_r$. This completes the inductive step. The induction starts trivially with $X_0 = X$.

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