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A Banach space that cannot be made into a BIP space

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1. A Banach space E over the complex field $\mathbf{C}$ is said to be a *Banach inner-product* (BIP) *space* if there exists a mapping $\langle .,. \rangle$ of $E \times E$ into $\mathbf{C}$ satisfying:

(i) $\langle x, x \rangle \geqslant 0$ $(x \in E)$ with equality only if $x = 0$;

(ii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ $(x, y \in E)$;

(iii) $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$ $(x, y, z \in E, \lambda \in \mathbf{C})$;

(iv) $\langle x, x \rangle \leqslant k^2 \| x \|^2$ $(x \in E)$,

where k is a fixed positive number. Thus $\langle .,. \rangle$ is an inner product on E, which induces a norm $\| . \|_1$ by the relation $\|x\|_1^2 = \langle x, x \rangle$ $(x \in E)$.

Condition (iv) implies that the inner-product topology is coarser than the original norm topology.

Suitable inner products can easily be constructed in most of the familiar Banach spaces. In particular, in any separable Banach space the existence of a complete biorthogonal system ((1), pp. 237–8) and Exercise 18 of (3), p. 451, together provide a suitable inner product. The purpose of this note is to exhibit a Banach space that cannot be made into a BIP space. An example is also given of a BIP space whose dual cannot be made BIP.

2. Let $B(I)$ denote the Banach space of all bounded complex-valued functions on the interval $I = [0, 1]$ of the real line, with the usual supremum norm. We assume that $B(I)$ is a BIP space, and we shall obtain a contradiction.

For $t_0 \in I$ we define the function x_{t_0} by the equations

$$x_{t_0}(t) = 1 \quad (t = t_0)$$
$$= 0 \quad (t \neq t_0).$$

Let $\langle .,. \rangle$ be an inner product in $B(I)$ satisfying conditions (i)–(iv) above. For $\alpha > 0$, let

$$T_\alpha = \{t \in I : \langle x_t, x_t \rangle \geqslant \alpha\}.$$

We claim that the set $T_{1/n}$ is infinite for some positive integer n. Since $\langle x, x \rangle = 0$ implies $x = 0$, we have

$$I = \bigcup_{n=1}^{\infty} T_{1/n}; \tag{1}$$

if $T_{1/n}$ were finite for all n, the right-hand side of (1) would be countable, whereas the left-hand side is not.

We now fix $\alpha > 0$ so that $T = T_\alpha$ is infinite. Let t_1, t_2 be distinct points of T, and write

$$y_1 = x_{t_1}, \quad y_2 = e^{i\theta_2} x_{t_2},$$

where θ_2 is chosen so that $\langle y_1, y_2 \rangle = e^{-i\theta_2} \langle x_{t_1}, x_{t_2} \rangle \geqslant 0$.

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Put $z_1 = y_1$, $z_2 = y_1 + 2^{-\frac{1}{2}} y_2$. This accomplishes the first stage of an inductive process. At the nth stage, we choose $t_n \in T$ distinct from $t_1, \ldots, t_{n-1}$, and θ_n so that if

$$y_n = e^{i\theta_n} x_{t_n},$$

then
$$\langle z_{n-1}, y_n \rangle \geqslant 0. \tag{2}$$

Finally, we put $z_n = z_{n-1} + n^{-\frac{1}{2}} y_n$.

Since the t_n are all distinct, we have, for $n < m$,

$$\begin{aligned} \|z_n - z_m\| &= \|(n+1)^{-\frac{1}{2}} e^{i\theta_{n+1}} x_{t_{n+1}} + \ldots + m^{-\frac{1}{2}} e^{i\theta_m} x_{t_m}\| \\ &= (n+1)^{-\frac{1}{2}}; \end{aligned}$$

thus (z_n) is a Cauchy sequence in $B(I)$, so that there exists $z \in B(I)$ such that

$$\|z_n - z\| \to 0 \quad (n \to \infty).$$

By condition (iv),
$$\langle z_n, z_n \rangle \to \langle z, z \rangle \quad (n \to \infty). \tag{3}$$

On the other hand, we can also show by induction that

$$\langle z_n, z_n \rangle \geqslant \langle y_1, y_1 \rangle + \tfrac{1}{2} \langle y_2, y_2 \rangle + \ldots + (1/n) \langle y_n, y_n \rangle.$$

This certainly holds for $n = 1$; we suppose it true for $1, 2, \ldots, n-1$. We then have, by (2),

$$\begin{aligned} \langle z_n, z_n \rangle &= \langle z_{n-1}, z_{n-1} \rangle + n^{-\frac{1}{2}} \langle z_{n-1}, y_n \rangle + n^{-\frac{1}{2}} \langle y_n, z_{n-1} \rangle + n^{-1} \langle y_n, y_n \rangle \\ &\geqslant \langle y_1, y_1 \rangle + \ldots + \frac{1}{n-1} \langle y_{n-1}, y_{n-1} \rangle + \frac{1}{n} \langle y_n, y_n \rangle. \end{aligned}$$

Since $y_r \in T$ for all r, it follows that

$$\langle z_n, z_n \rangle \geqslant \alpha \sum_{r=1}^{n} \frac{1}{r} \to \infty \quad (n \to \infty).$$

This contradicts (3), so that $B(I)$ is not a BIP space.

3. For our second example we take E to be the vector space of complex-valued functions x on $I = [0, 1]$ that are non-zero for at most a countable subset of I and are such that

$$\sum_{t \in I} |x(t)| < +\infty.$$

With the norm
$$\|x\| = \sum_{t \in I} |x(t)|.$$

E is a Banach space. It is a BIP space under the inner product

$$\langle x, y \rangle = \sum_{t \in I} x(t) \overline{y(t)} \quad (x, y \in E).$$

By (2), p. 29, Theorem 1, the dual of E is $B(I)$.

REFERENCES

(1) BANACH, S. *Théorie des opérations linéaires* (Warszawa, 1932).
(2) DAY, M. M. *Normed linear spaces* (Berlin–Göttingen–Heidelberg, 1958).
(3) ZAANEN, A. C. *Linear analysis* (Amsterdam, 1953).

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