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Note on Robins' conjecture in dimension four and higher

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Abstract. This article is motivated by a conjecture proposed by Sinai Robins in 2024. The conjecture asserts that two convex, centrally symmetric sets of positive measure that are not multi-tilers must coincide up to rigid motions if and only if their Fourier transforms agree on the lattice $\mathbb{Z}^d$. In this article, we disprove the conjecture by constructing explicit counterexamples in dimensions $d \geq 4$.

1 Introduction

For a function $f \in L^1(\mathbb{R}^d)$, its *Fourier transform* is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle x,\xi\rangle}\, dx, \quad \xi \in \mathbb{R}^d.$$

If $f \in L^2(\mathbb{R}^d)$, then, since $L^1(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, the Fourier transform extends naturally to $L^2(\mathbb{R}^d)$.

The Fourier transform is a fundamental mathematical tool that allows us to analyze functions and signals in terms of their frequency components. It decomposes a function into oscillatory modes, revealing its spectral structure and providing a powerful framework for studying differential equations and signal processing. For further details, see [2, 3, 11].

Let $A \subseteq \mathbb{R}^d$. The *characteristic function* (or indicator function) of A is defined by

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Thus, $\widehat{1_A}$ denotes the Fourier transform of the characteristic function of the set A.

The measurable set $A \subset \mathbb{R}^d$ is said to *k-tile* $\mathbb{R}^d$ with the translation set Λ if almost every point in $\mathbb{R}^d$ is covered exactly k times by the translates $A + \lambda$ for $\lambda \in \Lambda$, that is,

$$\sum_{\lambda\in\Lambda} 1_A(x-\lambda) = k \quad \text{a.e. in } \mathbb{R}^d. \tag{1.1}$$

A set with this property is called a *multi-tiler*. In the special case $k = 1$, we simply say that A *tiles* $\mathbb{R}^d$. Tilings and multi-tilings naturally arise in diverse areas, such as

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quasicrystals, Fourier analysis, and the theory of lattices. For further details, see [1, 4, 6, 7, 9] and the references therein.

In this article, we construct a counterexample to the following conjecture.

Conjecture 1.1 (Robins, 2024) *Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$ be two convex, central symmetric bodies. Suppose P, Q are not multi-tilers. Then $\widehat{1_{\mathcal{P}}}(\xi) = \widehat{1_{\mathcal{Q}}}(\xi)$ for all $\xi \in \mathbb{Z}^d$ if and only if $\mathcal{P} = \mathcal{Q}$ up to rigid motions, except on a set of measure zero.*

Next, we briefly describe the motivation behind the conditions in the conjecture. The Fourier transform at integer points is equivalent to the equality of the corresponding Fourier series. In particular, for two measures supported on $[0,1]^d$, equality at $\xi \in \mathbb{Z}^d$ implies that the measures themselves coincide. More generally, for two measures μ_1 and μ_2 with unbounded support, equality of their Fourier series implies that they agree on the factor space $\mathbb{R}^d/\mathbb{Z}^d$. In the case of measures uniformly distributed on convex bodies B_1 and B_2, this condition can be interpreted as a geometric or combinatorial property.

Furthermore, the class of convex and centrally symmetric bodies is a natural class of geometric objects to study, appearing in classical results, such as Minkowski's theorem and the second Minkowski theorem.

Finally, without the condition on multiple tilers, any two convex polytopes that tile $\mathbb{R}^d$ with the same lattice $\Lambda \subset \mathbb{Z}^d$, as described in Lemma 2.1, provide a natural source of counterexamples to the conjecture.

In this article, we only address one direction of the equivalence:

$$\widehat{1_{\mathcal{P}}}(\xi) = \widehat{1_{\mathcal{Q}}}(\xi) \text{ for all } \xi \in \mathbb{Z}^d \Rightarrow \mathcal{P} = \mathcal{Q} \text{ up to rigid motions.}$$

The main result of the article can be formulated as the following theorem.

Theorem 1.2 *Conjecture 1.1 is false for $d \geq 4$.*

The article is organized as follows. Section 2 presents the proof of Theorem 1.2 together with related remarks, while Section 3 states and proves a result illustrating the importance of convexity in Conjecture 1.1.

2 Proof of the main result

First, we recall a well-known result, which can be found, for example, in [8, Formula (1.5) in Lecture 1].

Lemma 2.1 *Let $Q \subset \mathbb{R}^d$ be a measurable set of positive and finite measure, and let $\Lambda \subset \mathbb{R}^d$ be a full-rank lattice. Then*

$$Q \text{ k-tiles } \mathbb{R}^d \text{ by translations with } \Lambda \iff \widehat{1_Q}(\xi) = 0 \text{ for all } \xi \in \Lambda^* \backslash \{0\},$$

where $\Lambda^* = \{ \xi \in \mathbb{R}^d : \xi \cdot \lambda \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \}$ is the dual lattice.

Let R be the rhombus with vertices

$$(-1,0),\ (0,3),\ (1,0),\ \text{and}\ (0,-3), \tag{2.1}$$

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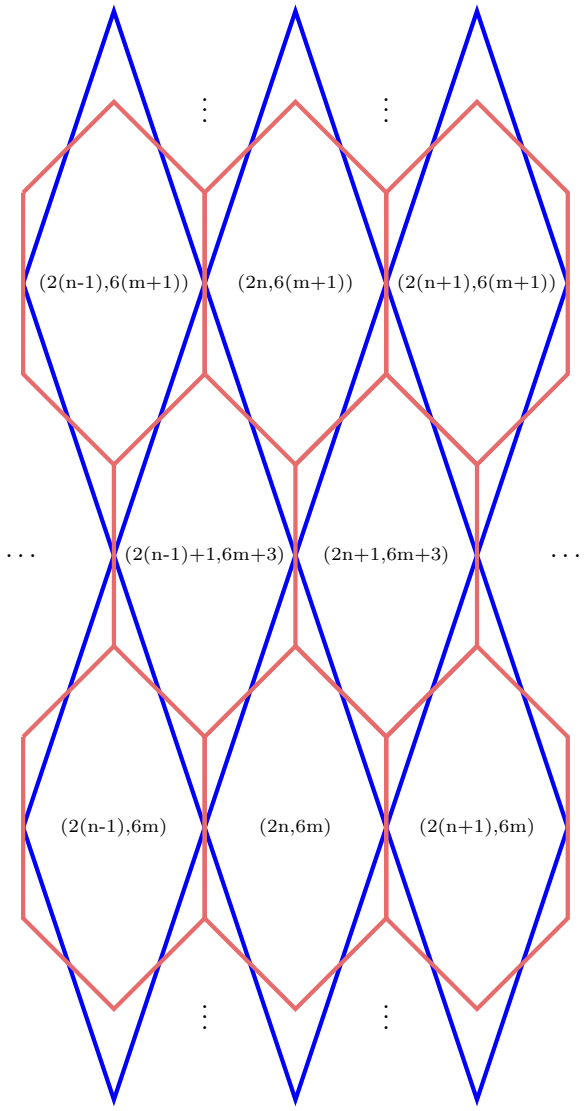


Figure 1: A tiling of $\mathbb{R}^2$ by the translational lattice Λ using the rhombus R (blue) and the hexagon H (red).

and let H be the hexagon with vertices

$$(-1,1),\ (0,2),\ (1,1),\ (1,-1),\ (0,-2),\ \text{and}\ (-1,-1). \tag{2.2}$$

Before proceeding to the proof of the main theorem, we state the following proposition.

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Proposition 2.2

$$\widehat{1}_R(\xi) = \widehat{1}_H(\xi) = \begin{cases} 6 & \text{if } \xi = (0,0) \\ 0 & \text{if } \xi \neq (0,0) \end{cases} \tag{2.3}$$

for all $\xi \in \mathbb{Z}^2$.

Proof Both R and H tile $\mathbb{R}^2$ by translations with the lattice

$$\Lambda = \bigcup_{j=0}^{1} \{(2n + j,\ 6m + 3j) : n, m \in \mathbb{Z}\} \text{ (see Figure 1)}.$$

Since $\Lambda \subset \mathbb{Z}^2$, it follows that $(\mathbb{Z}^2)^* \subset \Lambda^*$, so by Lemma 2.1, $\widehat{1}_R(\xi) = \widehat{1}_H(\xi) = 0$ for $\xi \in \mathbb{Z}^2 \backslash \{0\}$.

For $\xi = (0,0)$, a direct computation gives

$$\widehat{1}_R(0,0) = \int_R dx = 6 = \int_H dx = \widehat{1}_H(0,0). \qquad \blacksquare$$

Now, we are ready to prove the main result.

Proof of Theorem 1.2 Let $d \geq 4$. We consider two sets in $\mathbb{R}^2$, rhombus R with vertices (2.1) and hexagon H with vertices (2.2). Both R and H are centrally symmetric and convex. Moreover, the Fourier transforms of the characteristic functions of R and H satisfy (2.3).

Next, let $B \subset \mathbb{R}^{d-2}$ denote the unit ball centered at the origin. We define two sets: $\mathcal{P} = R \times B$ and $\mathcal{Q} = H \times B$. Since they are Cartesian products of centrally symmetric convex sets, both $\mathcal{P}$ and $\mathcal{Q}$ are themselves centrally symmetric and convex.

Note that if a convex body P admits a k-tiling, then P must be a polytope [5, Introduction]. Since neither $\mathcal{P}$ nor $\mathcal{Q}$ is a polytope, they cannot be multi-tilers.

Using (2.3), we see that for all $\xi_1 \in \mathbb{Z}^2$ and $\xi_2 \in \mathbb{Z}^{d-2}$, with $\xi = (\xi_1, \xi_2)$,

$$\widehat{1}_{\mathcal{P}}(\xi) = \widehat{1}_R(\xi_1)\widehat{1}_B(\xi_2) = \widehat{1}_H(\xi_1)\widehat{1}_B(\xi_2) = \widehat{1}_{\mathcal{Q}}(\xi).$$

Therefore, the conjecture is false for $d \geq 4$. $\blacksquare$

2.1 Remarks

(1) Conjecture 1.1 has not appeared in the published literature; however, it has been stated by Sinai Robins in several of his talks, for example, in "Multi-tilings and the geometry of numbers, using the Siegel–Bombieri approach,"[1] presented at the 1st Brazilian Workshop in Analytic Number Theory. A revised version of the conjecture will appear in [10].

(2) The proof of Theorem 1.2 remains valid if H and R are replaced by any other multi-tilers whose Fourier transforms agree on the lattice $\mathbb{Z}^2$.

(3) The method described in Section 3 does not apply for $d < 4$, since in this case, one of the sets must be one-dimensional. In dimension 1, however, all convex and centrally symmetric sets are intervals, and hence trivially multi-tile $\mathbb{R}$.

[1] https://www.youtube.com/watch?v=5NcevIR98Y (at 45:15).

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Moreover, since $|I| = \widehat{1_I}(0)$ for any interval $I \subset \mathbb{R}$, it is impossible to find two distinct intervals (up to rigid motions) whose Fourier transforms agree on the lattice $\mathbb{Z}$.

3 Counterexample to the conjecture without the Convexity Assumption

Theorem 3.1 *Conjecture 1.1 without the Convexity Assumption is false for* $d \geq 2$.

Proof We consider two sets in $\mathbb{R}$: $I = [0,1]$ and $J = [-1,-1/2) \cup [1/2,1]$. Note that I and J have measure 1 and tile $\mathbb{R}$ with the same lattice $\mathbb{Z}$. So, by Lemma 2.1, we conclude that

$$\widehat{1_I}(\xi) = \widehat{1_J}(\xi) = \begin{cases} 1 & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0 \end{cases} \tag{3.1}$$

for all $\xi \in \mathbb{Z}$.

Next, for $d \geq 3$, let $B \subset \mathbb{R}^{d-1}$ denote the unit ball centered at the origin. We define two sets: $\mathcal{P} = I \times B$ and $\mathcal{Q} = J \times B$. Since they are Cartesian products of centrally symmetric sets, both $\mathcal{P}$ and $\mathcal{Q}$ are themselves centrally symmetric. Note that $\mathcal{P}$ is a ball and $\mathcal{Q}$ is the union of two disjoint balls in $\mathbb{R}^d$, so neither of them can be a multi-tiler.

Using (3.1), we see that for all $\xi_1 \in \mathbb{Z}$ and $\xi_2 \in \mathbb{Z}^{d-1}$, with $\xi = (\xi_1, \xi_2)$,

$$\widehat{1_{\mathcal{P}}}(\xi) = \widehat{1_I}(\xi_1)\widehat{1_B}(\xi_2) = \widehat{1_J}(\xi_1)\widehat{1_B}(\xi_2) = \widehat{1_{\mathcal{Q}}}(\xi).$$

Thus, the conjecture without the Convexity Assumption is false for $d \geq 3$.

For the case $d = 2$, we use the same strategy; however, since a ball in one dimension is just a segment, we must instead consider a more intricate set that does not multi-tile $\mathbb{R}$. For example, we may take

$$B = [-4,-\pi] \cup [-e,e] \cup [\pi,4],$$

where e denotes Euler's number. This set is centrally symmetric and does not multi-tile $\mathbb{R}$. Thus, the argument for $d \geq 3$ can be adapted to this choice of B, and the conclusion follows. Therefore, the conjecture without the Convexity Assumption is false for $d \geq 2$. ∎

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