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Applications of the Melnikov method to twist maps in higher dimensions using the variational approach

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Abstract. We work with symplectic diffeomorphisms of the n-annulus $\mathbb{A}^n = T^*(\mathbb{R}^n/\mathbb{Z}^n)$. Using the variational approach of Aubry and Mather, we are able to give a local description of the stable (and unstable) manifold for a hyperbolic fixed point. We use this in order to get a Melnikov-like formula for exact symplectic twist maps. This formula involves an infinite series that could be computed in some specific cases. We apply our formula to prove the existence of heteroclinic orbits for a family of twist maps in $\mathbb{R}^4$.

1. *Introduction*

The qualitative description of the flow of a Hamiltonian system is related to the study of the iterations of exact symplectic twist maps. It is a well known fact that the time-δ map of the flow of a Hamiltonian vector field is symplectic and homotopic to the identity. In many cases we can find a δ small enough so that this time-δ map is a twist map [**4**].

The question of twist maps was originally investigated in manifolds of dimension two (see for example [**6, 7**]). Only recently has the question of higher dimensions been addressed [**1, 4, 5, 8**]. The theory of exact symplectic twist maps is related to the older question of stability of elliptic fixed points. So it is important to have examples of non-trivial twist maps that reflect new phenomena due to the the extra dimensions and can serve as model problems for the further development of the theory. The general definition of a twist map and many ideas related to it will be given in §2.

Melnikov's method [**12**] is used to find transversal intersections of stable and unstable manifolds. The standard setting is a time-periodic perturbation of an autonomous vector field having a completely degenerate homoclinic or heteroclinic connection. One computes the so called 'Melnikov integral', which is a function of a parameter along the unperturbed connection. Wherever this function has a simple zero, a transversal intersection will occur for the perturbed vector field.

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In this paper we present an extension of the Melnikov method to symplectic twist maps. We start with a map having a completely degenerate homoclinic or heteroclinic connection (also called a saddle connection) and perturb it by adding a small term to the generating function. The analog of the Melnikov integral is an infinite sum which is again a function of a parameter along the unperturbed connection. As before, a simple zero of this function corresponds to a transversal intersection for the perturbed map. In this way, we can produce examples of twist maps with heteroclinic orbits.

The method is stated precisely by the following theorem, which is proved in §4. The terminology used in the theorem is developed in §2. Specific applications are given in §3. In §4, we define the corresponding Melnikov function for twist maps and we are able to find an explicit formula. This is the main result of the paper.

THEOREM 1.1. (Main Theorem) *Let F_0 be a C^2 twist map with generating function S_0. Assume that F_0 has two hyperbolic fixed points (θ_1, r_1) and (θ_2, r_2) and a saddle connection between them. More precisely, we assume that there exists an open set $\mathcal{U} \subset \mathbb{R}^n$ and functions $h : \mathcal{U} \to \mathcal{U}$ and $\chi : \mathcal{U} \to \mathbb{R}^n$ such that*

$$\{(\theta, \chi(\theta)) : \theta \in \mathcal{U}\}$$

is an open invariant subset of

$$W^s(\theta_1, r_1) \cap W^u(\theta_2, r_2),$$

and $F(\theta, \chi(\theta)) = (h(\theta), \chi(h(\theta)))$. Let $P(\cdot, \epsilon)$ be a perturbation of S_0 such that:

(a) *for all ϵ small enough, $S_\epsilon = S_0 + \epsilon P(\cdot, \epsilon)$ is a twist generating function, with F_ϵ as the corresponding twist map;*

(b) $P(\theta_1, \theta_1, 0) = P(\theta_2, \theta_2, 0) = 0$;

(c) $$\left.\frac{d}{d\theta}\right|_{\theta=\theta_1} P(\theta, \theta, \epsilon) = \left.\frac{d}{d\theta}\right|_{\theta=\theta_2} P(\theta, \theta, \epsilon) = 0.$$

Then F_ϵ has fixed points at (θ_1, r_1^ϵ) and (θ_2, r_2^ϵ), where $r_i^\epsilon = r_i - \epsilon \partial_1 P(\theta_i, \theta_i, \epsilon)$ and the series

$$L(\theta) = \sum_{k=-\infty}^{\infty} P(h^k(\theta), h^{k+1}(\theta), 0)$$

converges absolutely and uniformly on compact subsets of $\mathcal{U}$ and is C^2 as a function of θ. Furthermore, if L has a non-degenerate critical point then the manifolds $W^s((\theta_1, r_1^\epsilon), F_\epsilon)$ and $W^s((\theta_2, r_2^\epsilon), F_\epsilon)$ intersect transversally for ϵ small enough.

An earlier result in this direction was found by Easton [**3**]. Nevertheless, he did not apply his formulas to any specific example. His result and ours are similar in that they both involve an infinite series evaluated along the unperturbed connection. They differ in that his formula is given in terms of the perturbation of the map, while ours is given in terms of the perturbation of the generating function. This is an important difference, because in most interesting cases we are not able to find explicitly the twist map generated by the generating function. For instance, our formula can be applied to the problem of billiards (see [**9**]). It is known that the twist map associated with an elliptic billiard table has a saddle connection. A small perturbation of the table (for example with a small

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quartic term) induces naturally a small perturbation of the generating function that can be computed explicitly. The corresponding perturbation of the twist map would be hard to manage.

In order to prove the main theorem above, we will first describe the stable and unstable manifolds using certain properties of the twist maps. The basic tool used in this paper is the variational approach exploited by Aubry [**2**] and Mather [**11**], which enables us to write an explicit formula for the stable and unstable manifolds in terms of the generating function.

In §3 we give applications of the formula in the main theorem. For example, we apply our theorem to the following situation. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a gradient function that is the lift of a torus map, i.e. we assume $Dh(\theta) = Dh(\theta)^T$ and $h(\theta + k) = h(\theta) + k$ for all $\theta \in \mathbb{R}^2$ and all $k \in \mathbb{Z}^2$. Then the following function is a twist generating function:

$$S_0(\theta, \Theta) = \frac{1}{2}(\Theta - \theta)^2 - \theta^2 + \int_0^1 \theta \cdot (h(t\theta) + h^{-1}(t\theta))\, dt.$$

If we let, for example, $h_{(\nu_1,\nu_2)}(\theta_1, \theta_2) = (g_{\nu_1}(\theta_1), g_{\nu_2}(\theta_2))$, where

$$g_\nu(\theta) = \theta - \frac{1}{\pi} \arctan\left(\frac{(1-\nu)\tan \pi\theta}{1 + \nu \tan^2 \pi\theta}\right)$$

and $0 < \nu < 1$, then we can show that the twist map generated by S_0 has a double saddle connection between the points $(0, 0, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 0, 0)$. Notice that g_ν satisfies $g_\nu^k = g_{\nu^k}$.

We apply the main theorem using $\theta_1 = (0, 0)$, $\theta_2 = (\frac{1}{2}, \frac{1}{2})$, $r_1 = r_2 = (0, 0)$, $\mathcal{U} = (0, \frac{1}{2}) \times (0, \frac{1}{2})$, $h = h_{(\nu_1,\nu_2)}|_{\mathcal{U}}$ and $\chi = \mathrm{id} - h^{-1}$. So far the twist map is a trivial product of two one-dimensional twist maps. However, after perturbation we produce a non-trivial twist map (with coupling) that has heteroclinic orbits. We are able to prove the following.

PROPOSITION 1.2. *Let* $S_\epsilon(\theta, \Theta) = S_0^{(\nu_1,\nu_2)}(\theta, \Theta) + \epsilon \sin 2\pi\theta_1 \sin 2\pi\theta_2$ *be the twist generating function of the twist map* F_ϵ. *Then for* $\epsilon > 0$ *small enough,* $(0, 0, 0, 0)$ *and* $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ *are hyperbolic fixed points for* F_ϵ *and their stable and unstable manifolds intersect transversally provided* $0 < \nu_1 < \frac{1}{4}$ *and* $0 < \nu_2 < \frac{1}{4}$.

Examples of this type could be used together with the shadowing lemma to show the existence of twist maps in higher dimensions that have a pointwise rotation set with non-empty interior.

2. *Twist maps on* $\mathbb{T}^n \times \mathbb{R}^n$

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus. $\mathbb{T}^n$ is a Lie group and therefore its cotangent bundle $T^*(\mathbb{T}^n) \to \mathbb{T}^n$ is trivial. We will call it *the n-annulus*, and write

$$\mathbb{A}^n = T^*(\mathbb{T}^n) = \mathbb{T}^n \times \mathbb{R}^n.$$

Give $\mathbb{A}^n$ the coordinates $(\tilde{\theta}, r) = (\tilde{\theta}_1, \ldots, \tilde{\theta}_n, r_1, \ldots, r_n)$. In these coordinates the Liouville 1-form looks like $\tilde{\lambda} = \sum_{i=1}^n r_i\, d\tilde{\theta}_i$, and the standard symplectic form is $\tilde{\omega} = -d\tilde{\lambda} = d\tilde{\theta} \wedge dr = \sum_{i=1}^n d\tilde{\theta}_i \wedge dr_i$.

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$\mathbb{A}^n$ has as a covering space, the Euclidean space $\mathbb{R}^{2n}$. We will use the coordinates $(\theta, r) = (\tilde{\theta}, r) \circ \kappa$ on $\mathbb{R}^{2n}$, where κ is the covering map. We will translate everything to the covering space. In this way we have that the Liouville 1-form is $\lambda = \kappa^* \tilde{\lambda}$, the symplectic structure is $\omega = \kappa^* \tilde{\omega} = -d\lambda$ and given a C^1 function $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^n$, we can lift it to a C^1 function of $\mathbb{R}^{2n}$ as follows:

$$\begin{array}{ccc} \mathbb{R}^{2n} & \xrightarrow{F} & \mathbb{R}^{2n} \\ \downarrow \kappa & & \downarrow \kappa \\ \mathbb{A}^n & \xrightarrow{\tilde{F}} & \mathbb{A}^n \end{array}$$

Definition 2.1. A C^1 map $\tilde{F}$ of $\mathbb{A}^n$ is called a *twist map* if:

(a) $\tilde{F}$ is homotopic to the identity;

(b) $\tilde{F}$ is exact symplectic, i.e. there exists a C^2 function $\tilde{S} : \mathbb{A}^n \to \mathbb{R}$ such that $\tilde{F}^* \tilde{\lambda} - \tilde{\lambda} = d\tilde{S}$ (such a function is called a *generating function*);

(c) if $(\Theta, R) = F(\theta, r)$ is a lift of $\tilde{F}$ to its covering space, then $\psi(\theta, r) = (\theta, \Theta)$ is a diffeomorphism of $\mathbb{R}^{2n}$.

Let F be the lift of a twist map. Condition (a) from the definition implies that F commutes with integer translations, i.e. for all $k \in \mathbb{Z}^n$, $F(\theta + k, r) = F(\theta, r) + (k, 0)$. If we let $\lambda = \kappa^* \tilde{\lambda}$, condition (b) implies that

$$F^* \lambda - \lambda = d(\tilde{S} \circ \kappa).$$

The diffeomorphism ψ given in condition (c) will allow us to get a useful expression. Apply $(\psi^{-1})^*$ on both sides of equation (2.1) and get $dS = R\, d\Theta - r\, d\theta$, where $S = \tilde{S} \circ \kappa \circ \psi^{-1}$. Notice that $dS(\theta, \Theta) = \partial_1 S(\theta, \Theta)\, d\theta + \partial_2 S(\theta, \Theta)\, d\Theta$, where ∂_1 denotes the gradient taken with respect to the first variable and ∂_2 denotes the gradient taken with respect to the second variable. So we can get the formulas

$$r = -\partial_1 S(\theta, \Theta), \quad R = \partial_2 S(\theta, \Theta).$$

These formulas will be the basis of our theory. In fact, we can start with generating functions as our objects of study and produce the corresponding twist diffeomorphisms.

Definition 2.2. A *twist generating function* is a C^2 function on $\mathbb{R}^{2n}$ satisfying:

(a) $S(\theta + k, \Theta + k) = S(\theta, \Theta)$ for all $k \in \mathbb{Z}^n$;

(b) the maps $(\theta, \Theta) \mapsto (\theta, -\partial_1 S(\theta, \Theta))$ and $(\theta, \Theta) \mapsto (\Theta, \partial_2 S(\theta, \Theta))$ are diffeomorphisms of $\mathbb{R}^{2n}$.

PROPOSITION 2.1. *There is a homeomorphism between the set of lifts of C^1 symplectic twist maps of $\mathbb{A}^n$ and the set of C^2 twist generating functions such that $S(0, 0) = 0$. Furthermore, given a twist generating function, the corresponding twist map is given by $F(\theta, r) = (\Theta, R)$, where the functions Θ and R satisfy: $r = -\partial_1 S(\theta, \Theta)$ and $R = \partial_2 S(\theta, \Theta)$.*

Some times, condition (b) of Definition 2.2 is hard to verify. However, we can use the following lemma instead (see [**5**]).

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LEMMA 2.2. *Let $S : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function such that:*
(a) $\det \partial_{12} S \neq 0$;
(b) $\sup_{(\theta,\Theta)\in\mathbb{R}^{2n}} \|(\partial_{12} S(\theta, \Theta))^{-1}\| = K < \infty$.
Then S satisfies condition (b) of Definition 2.2.

The following proposition is simple but useful.

PROPOSITION 2.3. *Let F be a twist map with generating function $S_F(\theta, \Theta)$. Then F^{-1} is also a twist map and its generating function is*

$$S_{F^{-1}}(\theta, \Theta) = -S_F(\Theta, \theta).$$

Unfortunately, it is not true in general that the composition of two twist maps is a twist map. In the rest of this section, we will describe several examples of twist maps. We have seen that working with twist maps is equivalent to working with lifts of twist maps. We will abuse the language and also refer to the lifts of twist maps as twist maps.

Herman example. An interesting example of a twist map, first described by Herman, is generated by the following function:

$$S(\theta, \Theta) = \tfrac{1}{2}(\Theta - g(\theta))^2,$$

where g is the lift of a C^2 diffeomorphism of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, i.e. $g \in \mathrm{Diff}(\mathbb{R}^n)$ and $g(\theta + k) = g(\theta) + k$ for all $k \in \mathbb{Z}^n$.

We can 'solve' the corresponding equations and get the twist map

$$F(\theta, r) = (\Theta, R) = (Dg(\theta)^{-T} r + g(\theta), Dg(\theta)^{-T} r),$$

where we denote $A^{-T} = (A^{-1})^T = (A^T)^{-1}$. Notice that the torus $\{r = 0\}$ is invariant. On this torus, the twist map obtains the form $F(\theta, 0) = (g(\theta), 0)$. In this way, we can consider any torus diffeomorphism as a restriction of a twist map to an invariant torus.

Completely integrable example. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that $\nabla\varphi$ has an inverse. Then

$$S(\theta, \Theta) = \varphi(\Theta - \theta) - \varphi(0)$$

satisfies the conditions in order to be the generating function of a twist map. The corresponding function is

$$F(\theta, r) = (\theta + (\nabla\varphi)^{-1}(r), r).$$

Notice that each torus $\{r = r_0\}$ is invariant and that the dynamics in each invariant torus correspond to a rotation. The notion of *completely integrable* is related to, but not the same as, the corresponding notion for Hamiltonian systems.

Perturbation of a twist generating function. We will try to perturb a twist generating function $S_0(\theta, \Theta)$ with a function $P(\theta, \Theta, \epsilon)$ in such a way that for ϵ small enough

$$S_\epsilon = S + \epsilon P(\cdot, \cdot, \epsilon)$$

is also a twist generating function. In order to show that this is possible, we can use the following lemma.

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LEMMA 2.4. *Let $S_0(\theta, \Theta)$ and $P(\theta, \Theta, \epsilon)$ be functions such that for all $k \in \mathbb{Z}^n$, $S_0(\theta + k, \Theta + k) = S_0(\theta, \Theta)$ and $P(\theta + k, \Theta + k, \epsilon) = P(\theta, \Theta, \epsilon)$. Assume that:*

(a) $\det \partial_{12} S_0 \neq 0$;

(b) $\sup_{(\theta,\Theta)\in\mathbb{R}^{2n}} \|(\partial_{12} S_0(\theta, \Theta))^{-1}\| = K_0 < \infty$;

(c) $\limsup_{\epsilon\to 0} \sup_{(\theta,\Theta)\in\mathbb{R}^{2n}} \|(\partial_{12} S_0(\theta, \Theta))^{-1} \partial_{12} P(\theta, \Theta, \epsilon)\| < \infty$.

Then there exists an ϵ_0 such that for all $0 \leq \epsilon < \epsilon_0$ the following function is a twist generating function:

$$S_\epsilon = S_0 + \epsilon P(\cdot, \cdot, \epsilon).$$

Proof. We can find an ϵ_0 such that

$$\epsilon \|(\partial_{12} S_0(\theta, \Theta))^{-1} \partial_{12} P(\theta, \Theta, \epsilon)\| < K_1 < 1.$$

for all $0 < \epsilon < \epsilon_0$ and $(\theta, \Theta) \in \mathbb{R}^{2n}$

We are going to prove that the conditions of Lemma 2.2 are satisfied. Write $\partial_{12} S_\epsilon(\theta, \Theta)$ in the following way:

$$\partial_{12} S_\epsilon(\theta, \Theta) = (\partial_{12} S_0)(I + \epsilon(\partial_{12} S_0)^{-1} \partial_{12} P).$$

Therefore, if $0 < \epsilon < \epsilon_0$ then $(I + \epsilon(\partial_{12} S_0)^{-1} \partial_{12} P)$ is invertible and

$$\|(I + \epsilon(\partial_{12} S_0(\theta, \Theta))^{-1} \partial_{12} P(\theta, \Theta, \epsilon))^{-1}\| < 1/(1 - K_1) < \infty$$

for each $(\theta, \Theta) \in \mathbb{R}^{2n}$. This implies that $\sup_{(\theta,\Theta)\in\mathbb{R}^{2n}} \|(\partial_{12} S_\epsilon(\theta, \Theta))^{-1}\| < K_0/(1-K_1) < \infty$, and this together with Lemma 2.2 implies what we want. □

As a trivial example, we have the family $S_\epsilon = S_0 + \epsilon(\Theta - \theta)^T A(\Theta - \theta)$, where S_0 satisfies the conditions of Lemma 2.2 and A is an $n \times n$ matrix. This family satisfies the conditions of Lemma 2.4 and, therefore, each S_ϵ is a twist generating function for small ϵ. This example has the extra property that if F_ϵ is the twist function generated by S_ϵ then the set of fixed points for F_ϵ is equal to the set of fixed points for F_0. Other trivial examples are $P(\theta, \Theta, \epsilon) = Q(\theta, \epsilon)$ and $P(\theta, \Theta, \epsilon) = Q(\Theta, \epsilon)$. If $S_\epsilon(\theta, \Theta) = S_0(\theta, \Theta) + \epsilon Q(\theta, \epsilon)$, then the corresponding family of twist maps will be $F_\epsilon(\theta, r) = F_0(\theta, r + \epsilon \partial_1 Q(\theta, \epsilon))$. If $S_\epsilon(\theta, \Theta) = S_0(\theta, \Theta) + \epsilon Q(\Theta, \epsilon)$ then the corresponding family of twist maps will be $F_\epsilon(\theta, r) = F_0(\theta, r) + (0, \epsilon \partial_1 Q(\Theta(\theta, r), \epsilon))$, where $F_0(\theta, r) = (\Theta(\theta, r), R(\theta, r))$.

3. *Applications of the main theorem*

3.1. *Trivial application.* We first consider the case of a perturbation with compact support. This has been a favorite candidate for proofs that involve *generic* perturbations.

Consider a twist generating function S_0 with twist map F_0. Assume that F_0 has a saddle connection between the points $(\theta_1, 0)$ and $(\theta_2, 0)$. Let h be the induced function on the saddle connection, as defined previously. Let $\mathcal{U}$ be the domain of h. Let $P(\theta)$ be a bump function with support $A = \operatorname{supp} P$ and assume that A is small in the sense that $A \subset \mathcal{U} + \mathbb{Z}^n$ and that the sets $A_k = h^k(A)$ are disjoint.

We compute the function L and obtain that the support of L in $\mathcal{U}$ is equal to $\cup_{k\in\mathbb{Z}} A_k$ and that

$$L|_{A_k} = P \circ h^{-k}.$$

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3.2. *Heteroclinic orbits in* $\mathbb{R}^4$. Let $g_\nu : \mathbb{R} \to \mathbb{R}$ be as in the introduction:

$$g_\nu(\theta) = \theta - \frac{1}{\pi} \arctan\left(\frac{(1-\nu)\tan \pi\theta}{1+\nu \tan^2 \pi\theta}\right).$$

Let $h_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$h_\mu(\theta_1, \theta_2) = (g_{\nu_1}(\theta_1), g_{\nu_2}(\theta_2)),$$

where $\mu = (\nu_1, \nu_2)$. Using h_μ we define $S_0^\mu : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$S_0^\mu(\theta, \Theta) = \frac{1}{2}(\Theta - \theta)^2 - \theta^2 + \int_0^1 \theta \cdot (h_\mu(t\theta) + h_\mu^{-1}(t\theta))\, dt.$$

Then it is possible to prove that S_0^μ is the generating function of a standard twist map F_0. In fact, S_0^μ is given by

$$S_0^\mu(\theta, \Theta) = \frac{1}{2}(\Theta - \theta)^2 - \theta^2 + \int_0^{\theta_1} (g_{\nu_1}(u) + g_{\nu_1}^{-1}(u))\, du + \int_0^{\theta_2} (g_{\nu_2}(u) + g_{\nu_2}^{-1}(u))\, du.$$

Notice that $(0, 0)$ is an attractor for h_μ and $(\frac{1}{2}, \frac{1}{2})$ is a repeller. Using this fact, we can also prove that the point

$$p_0 = ((0, 0), (0, 0))$$

and the point

$$p_1 = ((\tfrac{1}{2}, \tfrac{1}{2}), (0, 0))$$

have saddle connections. In other words, if we let $\mathcal{U} = (0, \frac{1}{2})^2$, then we have the saddle connection

$$W^s(p_0) \cap W^u(p_1) = \{(\theta, \theta - h_\mu^{-1}(\theta)) \in \mathbb{R}^4 : \theta \in \mathcal{U}\}$$

and the corresponding twist map F_0 restricted to this manifold satisfies

$$F_0(\theta, \theta - h_\mu^{-1}(\theta)) = (h_\mu(\theta), h_\mu(\theta) - h_\mu^{-1}(h_\mu(\theta))).$$

Using the same argument with h_μ^{-1}, we also have the connection

$$W^u(p_0) \cap W^s(p_1) = \{(\theta, \theta - h_\mu(\theta)) : \theta \in \mathcal{U}\}$$

and the formula

$$F_0(\theta, \theta - h_\mu(\theta)) = (h_\mu^{-1}(\theta), h_\mu^{-1}(\theta) - h_\mu(h_\mu^{-1}(\theta))).$$

In this way we are in the situation of applying our main theorem with $h = h_\mu$, $\chi = \mathrm{id} - h_\mu^{-1}$ for one intersection and $h = h_\mu^{-1}$, $\chi = \mathrm{id} - h_\mu$ for the other one.

We will perturb S_0 using the function $\sin 2\pi\theta_1 \sin 2\pi\theta_2$. Notice that this perturbation gives a true coupling, and is not just a product of one-dimensional maps. Using this perturbing function, we get the following.

PROPOSITION 3.1. *Let* $S_\epsilon(\theta, \Theta) = S_0^\mu(\theta, \Theta) + \epsilon \sin 2\pi\theta_1 \sin 2\pi\theta_2$ *be the twist generating function of the twist map* F_ϵ. *Then for* $\epsilon > 0$ *small enough,* p_0 *and* p_1 *are hyperbolic fixed points for* F_ϵ *and their stable and unstable manifolds intersect transversally provided* $0 < \nu_1 < \frac{1}{4}$ *and* $0 < \nu_2 < \frac{1}{4}$.

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Proof. Let $L_\mu : (0, \frac{1}{2})^2 \to \mathbb{R}$ be given by

$$L_\mu(\theta) = \sum_{k=-\infty}^{\infty} P(h_\mu^k(\theta)),$$

where $P(\theta) = \sin 2\pi\theta_1 \sin 2\pi\theta_2$, $\mu = (\nu_1, \nu_2)$ and h_μ as before.

Since the perturbation P satisfies the hypotheses of the main theorem, it is enough to show that the function L_μ has a non-degenerate critical point. We will show that $L'_\mu(\frac{1}{4}, \frac{1}{4}) = 0$ and $\det(L''_\mu(\frac{1}{4}, \frac{1}{4})) \neq 0$.

Notice that $\tan \pi g_\nu(\theta) = \nu \tan \pi\theta$. This implies that

$$L_\mu(\theta) = \bar{L}_\mu(\tan \pi\theta_1, \tan \pi\theta_2),$$

where $\bar{L}_\mu$ is the function $\bar{L}_\mu : (0, \infty)^2 \to \mathbb{R}$ given by

$$\bar{L}_\mu(z) = \sum_{k=-\infty}^{\infty} \frac{2\nu_1^k z_1}{1 + \nu_1^{2k} z_1^2} \frac{2\nu_2^k z_2}{1 + \nu_2^{2k} z_2^2}.$$

This function has the symmetry

$$\bar{L}_\mu(z_1, z_2) = \bar{L}_\mu\left(\frac{1}{z_1}, \frac{1}{z_2}\right)$$

and therefore $(1, 1)$ is a critical point for $\bar{L}_\mu$. It remains to show that for the open set of parameters $\{(\nu_1, \nu_2) : 0 < \nu_1 < \frac{1}{4}, 0 < \nu_2 < \frac{1}{4}\}$ such a critical point is non-degenerate.

We will show that the Hessian of $\bar{L}$ is non-singular. After considerable computation we get that

$$D^2\bar{L}_\mu(1, 1) = \begin{pmatrix} \alpha(\mu) & \gamma(\mu) \\ \gamma(\mu) & \beta(\mu) \end{pmatrix},$$

where

$$\alpha(\mu) = 1 + 8\sum_{k=1}^{\infty} \frac{(\nu_1 - 6\nu_1^{3k} + \nu_1^{5k})}{(\nu_1^{2k} + 1)^3} \frac{\nu_2^k}{1 + \nu_2^{2k}}$$

$$\beta(\mu) = 1 + 8\sum_{k=1}^{\infty} \frac{(\nu_2 - 6\nu_2^{3k} + \nu_2^{5k})}{(\nu_2^{2k} + 1)^3} \frac{\nu_1^k}{1 + \nu_1^{2k}}$$

and

$$\gamma(\mu) = 8\sum_{k=1}^{\infty} \frac{(\nu_1^k - \nu_1^{3k})}{(\nu_1^{2k} + 1)^2} \frac{(\nu_2^k - \nu_2^{3k})}{(\nu_2^{2k} + 1)^2}.$$

If $0 < \nu_1 < \frac{1}{4}$ and $0 < \nu_2 < \frac{1}{4}$ then we will show that $\alpha(\mu) > \gamma(\mu) > 0$ and that $\beta(\mu) > \gamma(\mu) > 0$ and this would imply the proposition. In order to prove these inequalities let

$$f(u, v) = \frac{16(3u^2 - u^4 - v^2 + 3u^2v^2)}{(1 + u^2)^3(1 + v^2)^2}.$$

It is clear that if $0 < \nu_1 < \frac{1}{4}$ and $0 < \nu_2 < \frac{1}{4}$ then $f(\nu_1^k, \nu_2^k) < 3$. On the other hand, we can show that

$$\alpha(\mu) - \gamma(\mu) = 1 - \sum_{k=1}^{\infty} f(\nu_1^k, \nu_2^k)\nu_1^k\nu_2^k$$

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and therefore

$$\alpha(\mu) - \gamma(\mu) > 1 - 3\frac{\nu_1\nu_2}{1-\nu_1\nu_2} > 0.$$

Reversing the order of ν_1 and ν_2, we prove that $\beta(\mu) > \gamma(\mu)$. If $0 < \nu_1 < \frac{1}{4}$ and $0 < \nu_2 < \frac{1}{4}$ it is clear that $\gamma(\mu) > 0$ □

3.3. *Herman's example.* In the first application, we will use the generating function defined by Herman. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism such that

$$\forall k \in \mathbb{Z}^n,\ g(\theta + k) = g(\theta)$$

with 0 and θ_0 as fixed points. We consider the twist map generated by

$$S_0(\theta, \Theta) = \tfrac{1}{2}(\Theta - g(\theta))^2.$$

Assume that 0 is an attracting fixed point and that θ_0 is a repelling fixed point for g. Assume also that the basin of attraction of 0 intersects the basin of repulsion of θ_0. This intersection is an open set $\mathcal{U}$. So we have that the stable manifold of $(0, 0)$ coincides with the unstable manifold of $(\theta_0, 0)$. More precisely, there is a function $\chi : \mathcal{U} \to \mathbb{R}^n$ such that

$$W^u(0,0) \cap W^u(\theta_0, 0) = \{(\theta, \chi(\theta)) : \theta \in \mathcal{U}\}.$$

In fact, we have that $\chi \equiv 0$ and $F(\theta, 0) = (g(\theta), 0)$. So we have the conditions to apply the Melnikov method. Pick any perturbation $P(\theta, \Theta, \epsilon)$ such that for $i = 1, 2$

$$P(\theta_i, \theta_i, \epsilon) = 0$$

and

$$\left.\frac{d}{d\theta}\right|_{\theta=\theta_i} P(\theta, \theta, \epsilon) = 0.$$

Then the Melnikov function is $M = \partial L/\partial\theta$, where

$$L(\theta) = \sum_{k=-\infty}^{\infty} P(g^k(\theta), g^{k+1}(\theta), 0)$$

and they are defined on $\mathcal{U}$. So if this function has a non-degenerate critical point, then the stable and unstable manifolds of $(0, 0)$ and $(\theta_0, 0)$ intersect transversally.

3.4. *Hamiltonian systems example.* The third application involves the twist map generated by the Hamiltonian

$$H(\theta, r) = \tfrac{1}{2}r^2 - \tfrac{1}{2}\sin^2 2\pi\theta.$$

Let φ^t be the flow generated by H. It is easy to see that there exists a $\delta > 0$ such that $F = \varphi^\delta$ is a twist map. In general it is hard to compute the map or the generating function explicitly. In this case, the flow restricted to the level set $H \equiv 0$ is $\varphi^t(\theta, \sin(\theta)) = (h^t(\theta), \sin(h^t(\theta)))$ and $\varphi^t(\theta, -\sin(\theta)) = (h^{-t}(\theta), -\sin(h^{-t}(\theta)))$, where $h^t(\theta) = (1/\pi)\arctan(e^{2\pi t}\tan\pi\theta) + \lfloor\theta\rfloor$ (we are considering the codomain of arctan to be $(0, \pi)$).

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We can consider F as the lift of a twist diffeomorphism of the annulus $\mathbb{T}^1 \times \mathbb{R}^1$. This twist map has fixed points at $(0,0)+\mathbb{Z}\times 0$ and $(\frac{1}{2},0)+\mathbb{Z}\times 0$. In fact, the stable and unstable manifolds of $(0,0)$ and $(\frac{1}{2},0)$ coincide at points such that $0<\theta<\frac{1}{2}$.

Let $\mathcal{U}=(0,\frac{1}{2})$ and $\chi:\mathcal{U}\to\mathbb{R}$ be $\chi(\theta)=\sin 2\pi\theta$. Then we have that

$$W^u(0,0)\cap W^s(\tfrac{1}{2},0)=\{(\theta,\chi(\theta)):\theta\in\mathcal{U}\}$$

and if $\theta\in\mathcal{U}$ then $F(\theta,\chi(\theta))=(h(\theta),\chi(h(\theta)))$, where

$$h(\theta)=h^{\delta}(\theta)=\frac{1}{\pi}\arctan(e^{2\pi\delta}\tan\pi\theta)+\lfloor\theta\rfloor.$$

So, even though we can not compute explicitly the generating function for the twist map just defined, we can still use the formulas described in §4. Let S_0 be the generating function of F and let $P(\theta,\Theta,\epsilon)$ be a perturbation of S_0 with all the appropriate conditions we have required in this paper. Let $S_\epsilon=S_0+\epsilon P(\cdot,\cdot,\epsilon)$ and F_ϵ be the resulting twist map. Then $(0,0)$ and $(\frac{1}{2},0)$ are hyperbolic fixed points for F_ϵ for ϵ small enough and the stable and unstable manifolds of these points under F_ϵ intersect provided that the following function has a non-degenerate critical point on $\mathcal{U}$:

$$L(\theta)=\sum_{k=-\infty}^{\infty}P(h^k(\theta),h^{k+1}(\theta),0).$$

Notice that this function can be extended to $\mathbb{R}\setminus(\{\frac{1}{2},0\}+\mathbb{Z})$.

The condition $L\circ h=L$ implies the interval $(\theta,h(\theta))$ (or $(h(\theta),\theta)$) has all the relevant information of L and it can be considered as having the topology of a circle. So L *has to have* a critical point in this interval. Of course, we can give a generic argument in order to prove that the critical point is non-degenerate and therefore the intersection is transverse. Nevertheless, our method was developed to give criteria for specific examples. We will show how the formula we have obtained can be applied.

Let

$$P(\theta)=\begin{cases}\tan^6\pi\theta-4\tan^4\pi\theta+4\tan^2\pi\theta, & |\tan\pi\theta|\le 1\\ \cot^2\pi\theta, & |\tan\pi\theta|\ge 1.\end{cases}$$

This function has all the properties we require on a perturbation in order to apply the main theorem for the saddle connection between 0 and $\frac{1}{2}$, i.e. P is C^2, P is invariant under integer translations and $P(0)=P(\frac{1}{2})=P'(0)=P'(\frac{1}{2})=0$.

We will be able to compute the function $L(\theta)=\sum_{-\infty}^{\infty}P(h^k(\theta))$, where h is as before. We will compute first on the interval $[\frac{1}{4},h(\frac{1}{4}))$, and extend to the interval $(0,\frac{1}{2})$ using the invariance of L under h. First we get that if $\theta\in[\frac{1}{4},h(\frac{1}{4}))$, then

$$L(\theta)=\frac{1}{\mu^6-1}\tan^6\pi\theta-\frac{4}{\mu^4-1}\tan^4\pi\theta+\frac{4}{\mu^2-1}\tan^2\pi\theta+\frac{\mu^2}{\mu^2-1}\cot^2\pi\theta,$$

where $\mu=e^{2\pi\delta}$.

In general, let $I_m=[h^m(\frac{1}{4}),h^{m+1}(\frac{1}{4}))$ then $L|_{I_m}=L|_{I_0}\circ h^{-m}$. This implies that for all $k\in\mathbb{Z}$

$$L|_{I_m}(\theta)=\frac{\mu^{-6m}}{\mu^6-1}\tan^6\pi\theta-\frac{4\mu^{-4m}}{\mu^4-1}\tan^4\pi\theta+\frac{4\mu^{-2m}}{\mu^2-1}\tan^2\pi\theta+\frac{\mu^{2+2m}}{\mu^2-1}\cot^2\pi\theta.$$

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Once we have such an expression for L, it is possible to prove that L has a non-degenerate critical point on the interval $(\frac{1}{4}, h(\frac{1}{4}))$ and therefore a numerable number of non-degenerate critical points on the interval $(0, \frac{1}{2})$.

4. *Proof of the main theorem*

Throughout this section, we will make use of the variational approach developed by Aubry and Mather [**2, 11**]. We start with a technical lemma.

LEMMA 4.1. *Let $h : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function with an attracting fixed point at θ_0 and such that $0 < \|Dh(\theta_0)\| \leq \lambda_0$ for some $\lambda_0 < 1$. Let $B(\theta_0) \subset \mathcal{U}$ be the basin of attraction of θ_0 and let $G : B(\theta_0) \to \mathbb{R}$ be a C^2 function. Then each of the following series converges uniformly and absolutely on every compact subset of $B(\theta_0)$:*

$$\Gamma_0(\theta) = \sum_{k=0}^{\infty} \{G(h^k(\theta)) - G(\theta_0)\}$$

$$\Gamma_1(\theta) = \sum_{k=0}^{\infty} \frac{\partial (G \circ h^k)}{\partial \theta}(\theta)$$

$$\Gamma_2(\theta) = \sum_{k=0}^{\infty} D^2(G \circ h^k)(\theta).$$

Furthermore, Γ_0 is C^2 and $\partial \Gamma_0 / \partial \theta = \Gamma_1$, $D^2\Gamma_0 = D\Gamma_1 = \Gamma_2$.

Proof. Let K be a compact subset of $B(\theta_0)$. We will show that Γ_0, Γ_1 and Γ_2 converge uniformly and absolutely on K.

There exists a neighborhood $\mathcal{W}$ of θ_0 in $B(\theta_0)$ and constants C_0 and $\lambda_0 < \lambda < 1$ such that for all $\theta \in \mathcal{W}$,

$$|G(\theta) - G(\theta_0)| < C_0|\theta - \theta_0|, \quad \left|\frac{\partial G}{\partial \theta}(\theta)\right| < C_0,$$
$$\|D^2 G(\theta)\| < C_0, \quad |h(\theta) - \theta_0| < \lambda |\theta - \theta_0|,$$
$$\|Dh(\theta)\| < \lambda, \quad \|D(Dh(\theta)^T) Dh(\theta)^{-T}\| \leq C_2.$$

We can find N big enough so that $h^N(K) \subset \mathcal{W}$. So without loss of generality we can assume that $K \subset \mathcal{W}$.

For the first series we have that

$$\sum_{k=0}^{\infty} |G(h^k(\theta)) - G(\theta_0)| \leq \sum_{k=0}^{\infty} C_0 |h^k(\theta) - \theta_0| < \frac{C_0 C_1}{1 - \lambda}$$

for some C_1 depending only on K. For the second series, we notice that for every $\theta \in \mathcal{W}$, $\|Dh^k(\theta)\| < \lambda^k$. Let

$$H_k(\theta) = \frac{\partial G \circ h^k}{\partial \theta}(\theta) = Dh^k(\theta)^T \frac{\partial G}{\partial \theta}(h^k(\theta)).$$

Then

$$\sum_{k=0}^{\infty} \|H_k(\theta)\| \leq \sum_{k=0}^{\infty} \lambda^k \left|\frac{\partial G}{\partial \theta}(h^k(\theta))\right| \leq \frac{C_0}{1 - \lambda}.$$

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Notice that $H_{k+1}(\theta) = Dh(\theta)^T H_k(h(\theta))$. Therefore

$$\begin{aligned} DH_{k+1}(\theta) &= D(Dh(\theta)^T)H_k(h(\theta)) + Dh(\theta)^T DH_k(h(\theta))Dh(\theta) \\ &= D(Dh(\theta)^T)Dh(\theta)^{-T} H_{k+1}(\theta) + Dh(\theta)^T DH_k(h(\theta))Dh(\theta). \end{aligned}$$

This implies that for all $\theta \in K$,

$$\|DH_{k+1}(\theta)\| \leq \|H_{k+1}(h(\theta))\| + \lambda^2 \|DH_k(h(\theta))\| < C_0^2 \lambda^{k+1} + \lambda^2 \|DH_k(h(\theta))\|,$$

and therefore

$$\|DH_{k+1}(\theta)\| < C_0^2(\lambda^{k+1} + \lambda^{k+2} + \lambda^{k+3} + \cdots) < \frac{C_0^2 \lambda^{k+1}}{1-\lambda}.$$

From the inequality above we conclude the convergence of Γ_2. Standard theorems on convergence of series give us the fact that $\partial\Gamma_0/\partial\theta = \Gamma_1$, $D^2\Gamma_0 = D\Gamma_1 = \Gamma_2$. □

Consider a C^2 twist map F with generating function S and suppose that F has a fixed point (θ_0, r_0). Using the stable manifold theorem and the fact that F is a twist map, we can conclude that locally the stable manifold of (θ_0, r_0) is the graph of a function $\chi : \mathcal{U} \to \mathbb{R}^n$, where $\mathcal{U}$ is an open set around $\theta_0 \in \mathbb{R}^n$. This implies the existence of a function $h : \mathcal{U} \to \mathbb{R}^n$ as smooth as F such that $F(\theta, \chi(\theta)) = (h(\theta), \chi(h(\theta)))$. The most interesting fact about this is that the function χ can be determined by the generating function S. The basic idea behind this lemma is well known by many authors (compare for example with formula (3.10) in [**10**]). Nevertheless, the proof for higher dimensions is due to Tabacman [**13**]. Here we give a slightly different proof.

LEMMA 4.2. *Let F be a C^2 twist map with generating function S. Suppose that (θ_0, r_0) is a hyperbolic fixed point for F. Then the local stable manifold of (θ_0, r_0) can be written as the graph of a function $\chi^s : \mathbb{R}^n \to \mathbb{R}^n$, i.e. there exists an open neighborhood $\mathcal{U}$ of θ_0 such that $W^s_{\text{loc}}((\theta_0, r_0)) = \{(\theta, \chi^s(\theta)) : \theta \in \mathcal{U}\}$. This induces a local diffeomorphism $h : \mathcal{U} \to h(\mathcal{U})$ such that $F(\theta, \chi^s(\theta)) = (h(\theta), \chi^s(h(\theta)))$.*

Furthermore, the series

$$Z^s(\theta) = -\sum_{k=0}^{\infty} \{S(h^k(\theta), h^{k+1}(\theta)) - S(\theta_0, \theta_0)\} \tag{1}$$

converges uniformly on compact subsets of $\mathcal{U}$ and $\chi^s = \partial Z^s/\partial\theta$.

Proof. The existence of h and χ^s is given by the stable manifold theorem. We can consider θ_0 as an attracting fixed point of h with basin of attraction $\mathcal{U}$. Using Lemma 3.1 we conclude that Z^s is C^2 and that we can compute its gradient differentiating term by term in the following way:

$$\begin{aligned} \frac{\partial Z^s}{\partial\theta}(\theta) &= -\sum_{k=0}^{\infty} \frac{\partial}{\partial\theta}\{S(h^k(\theta), h^{k+1}(\theta))\} \\ &= -\sum_{k=0}^{\infty} \{(Dh^k(\theta))^T \partial_1 S(h^k(\theta), h^{k+1}(\theta)) \\ &\quad + (Dh^{k+1}(\theta))^T \partial_2 S(h^k(\theta), h^{k+1}(\theta))\}. \end{aligned}$$

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Using the fact that $\partial_1 S(h^{k+1}(\theta), h^{k+2}(\theta)) + \partial_2 S(h^k(\theta), h^{k+1}(\theta)) = 0$ we conclude that

$$\frac{\partial Z^s}{\partial \theta}(\theta) = -\lim_{N\to\infty} \{\partial_1 S(\theta, h(\theta)) + (Dh^{N+1}(\theta))^T \partial_2 S(h^N(\theta), h^{N+1}(\theta))\}.$$

Notice that $\lim_{N\to\infty} \|(Dh^N(\theta))^T\| = 0$. This implies that for every $\theta \in \mathcal{U}$,

$$\lim_{N\to\infty} (Dh^{N+1}(\theta))^T \partial_2 S(h^N(\theta), h^{N+1}(\theta)) = 0$$

and therefore

$$\frac{\partial Z^s}{\partial \theta}(\theta) = -\partial_1 S(\theta, h(\theta)) = \chi^s(\theta).$$

□

The case of the unstable manifold is totally analogous to the previous case.

COROLLARY 4.3. *Let F be a C^2 twist map with generating function S. Suppose that (θ_0, r_0) is a hyperbolic fixed point for F. Then the local unstable manifold of (θ_0, r_0) can be written as the graph of a function $\chi^u : \mathbb{R}^n \to \mathbb{R}^n$, i.e. there exists an open neighborhood $\mathcal{U}$ of θ_0 such that $W^u_{\text{loc}}((\theta_0, r_0)) = \{(\theta, \chi^u(\theta)) : \theta \in \mathcal{U}\}$. This induces a local diffeomorphism $g : \mathcal{U} \to g(\mathcal{U})$ such that $F(\theta, \chi^u(\theta)) = (g^{-1}(\theta), \chi^u(g^{-1}(\theta)))$.*

Furthermore, the series

$$Z^u(\theta) = \sum_{k=0}^{\infty} \{S(g^{k+1}(\theta), g^k(\theta)) - S(\theta_0, \theta_0)\} \tag{2}$$

converges uniformly on compact subsets of $g(\mathcal{U})$ and $\chi^u = \partial Z^u / \partial \theta$.

Proof. It is enough to notice that

$$F^{-1}(\theta, \chi^u(\theta)) = (g(\theta), \chi^u(g(\theta)))$$

and that the generating function $S_{F^{-1}}$ of F^{-1} is

$$S_{F^{-1}}(\theta, \Theta) = -S(\Theta, \theta),$$

where S is the generating function of F. □

Notice that if we let $h = g^{-1}$ then the formula for Z^u obtains the form

$$Z^u(\theta) = \sum_{k=-1}^{-\infty} \{S(h^k(\theta), h^{k+1}(\theta)) - S(\theta_0, \theta_0)\}$$

and the series converges uniformly on compact subsets of $h^{-1}(\mathcal{U})$.

We will describe how the so-called *Melnikov method* can be applied to predict the transversal intersection between the stable and unstable manifold of two different fixed point of a twist map. We will be able to get a *Melnikov-like formula for twist maps* that could be used, for example, in numerical investigations of twist maps.

We start with the setting previously described in §1. Let P be a perturbation of a generating function S_0, such that each of the functions of the family

$$S_\epsilon(\theta, \Theta) = S_0(\theta, \Theta) + \epsilon P(\theta, \Theta, \epsilon)$$

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is a twist map. Let F_ϵ be the twist function generated by S_ϵ.

The first question we can ask is how the perturbation P will affect the fixed points of F_0. Under a simple condition on P, there is a one-to-one correspondence between the fixed points of F_0 and the fixed points of F_ϵ.

LEMMA 4.4. *Let S_0, P, S_ϵ, F_0 and F_ϵ be as above. Assume that the perturbation P is such that for all $\epsilon > 0$ small,*

$$\frac{d}{d\theta}\bigg|_{\theta=\theta_0} P(\theta, \theta, \epsilon) = 0. \tag{3}$$

Then (θ_0, r_0) is a fixed point for F_0 if and only if $(\theta_0, r_0 - \epsilon\partial_1 P(\theta_0, \theta_0, \epsilon))$ is a fixed point for F_ϵ.

Take F_0 as the starting function. We will assume that F_0 has two hyperbolic fixed points (θ_1, r_1) and (θ_2, r_2). In order to make things simpler suppose that $r_1 = r_2 = 0$ (this can always be done using a change of coordinates chosen properly). Furthermore, assume that P satisfies equation (3) at the points θ_1, θ_2. This implies that for ϵ small enough $(\theta_1, r_1 - \epsilon\partial_1 P(\theta_1, \theta_1, \epsilon))$ and $(\theta_2, r_2 - \epsilon\partial_1 P(\theta_2, \theta_2, \epsilon))$ are hyperbolic fixed points for F_ϵ. We will also assume that we have a saddle connection between (θ_1, r_1) and (θ_2, r_2). In other words assume that a connected submanifold of $W^s(\theta_1, r_1) \setminus (\theta_1, r_1)$ coincides exactly with a connected submanifold of $W^u(\theta_2, r_2) \setminus (\theta_2, r_2)$. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be maximal open sets such that

$$\{(\theta, \chi^s(\theta)) : \theta \in \mathcal{U}_1\} \subset W^s(\theta_1, r_1) \quad \text{and} \quad \{(\theta, \chi^u(\theta)) : \theta \in \mathcal{U}_2\} \subset W^u(\theta_2, r_2)$$

for some functions $\chi^s : \mathcal{U}_1 \to \mathbb{R}^n$ and $\chi^u : \mathcal{U}_2 \to \mathbb{R}^n$. The intersection assumption implies that $\mathcal{U}_1$ and $\mathcal{U}_2$ are non-disjoint and that $\theta_2 \in \bar{\mathcal{U}_1}$ and $\theta_1 \in \bar{\mathcal{U}_2}$. Let h_1 and h_2 be the induced maps on the stable and unstable manifolds of (θ_1, r_1) and (θ_2, r_2) respectively. Then $h_1|_{\mathcal{U}_1} = h_2|_{\mathcal{U}_2} = h$. So, we will concentrate on the set $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ and use formulas (1) and (2).

So we have that on $\mathcal{U}$,

$$\chi = \chi^s = \frac{\partial Z^s}{\partial\theta} = \chi^u = \frac{\partial Z^u}{\partial\theta},$$

where

$$Z^s(\theta) = -\sum_{k=0}^{\infty}\{S(h^k(\theta), h^{k+1}(\theta)) - S(\theta_1, \theta_1)\},$$

and

$$Z^u(\theta) = \sum_{k=-1}^{-\infty}\{S(h^k(\theta), h^{k+1}(\theta)) - S(\theta_2, \theta_2)\}.$$

We apply the previous results to the functions F_ϵ. Let $W^s((\theta_1, r_1^\epsilon), F_\epsilon)$ be the stable manifold of (θ_1, r_1^ϵ) under F_ϵ and $W^u((\theta_2, r_2^\epsilon), F_\epsilon)$ be the unstable manifold of (θ_2, r_2^ϵ) under F_ϵ. Then there exist (maximal) open sets $\mathcal{U}_1^\epsilon$, $\mathcal{U}_2^\epsilon$ and functions $\chi_\epsilon^s : \mathcal{U}_1^\epsilon \to \mathbb{R}^n$, $\chi_\epsilon^u : \mathcal{U}_2^\epsilon \to \mathbb{R}^n$ such that

$$W^s((\theta_1, r_1^\epsilon), F_\epsilon) = \{(\theta, \chi_\epsilon^s(\theta)) : \theta \in \mathcal{U}_1^\epsilon\}$$

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and

$$W^u((\theta_2, r_2^\epsilon), F_\epsilon) = \{(\theta, \chi_\epsilon^u(\theta)) : \theta \in \mathcal{U}_2^\epsilon\}.$$

By continuity, for each $\theta \in \mathcal{U}$ there is an $\epsilon_\theta > 0$ such that for all $0 < \epsilon < \epsilon_\theta, \theta \in \mathcal{U}_1^\epsilon \cap \mathcal{U}_2^\epsilon$. On the other hand, there exist diffeomorphisms $h_1^\epsilon : \mathcal{U}_1^\epsilon \to h_1^\epsilon(\mathcal{U}_1^\epsilon)$ and $h_2^\epsilon : (h_2^\epsilon)^{-1}(\mathcal{U}_2^\epsilon) \to \mathcal{U}_2^\epsilon$ such that

$$F_\epsilon(\theta, \chi_\epsilon^s(\theta)) = (h_1^\epsilon(\theta), \chi_\epsilon^s(h_1^\epsilon(\theta)))$$

on $\mathcal{U}_1^\epsilon$, and

$$F_\epsilon((h_2^\epsilon)^{-1}(\theta), \chi_\epsilon^u((h_2^\epsilon)^{-1}(\theta))) = (\theta, \chi_\epsilon^u(\theta))$$

on $\mathcal{U}_2^\epsilon$. We want to estimate the distance between the stable and unstable manifolds of (θ_1, r_1^ϵ) and (θ_2, r_2^ϵ) respectively, for ϵ small enough. For $\epsilon = 0$, this distance is identically zero. We will base our estimates on the following lemma.

LEMMA 4.5. *Let*

$$M(\theta) = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (\chi_\epsilon^u(\theta) - \chi_\epsilon^s(\theta)).$$

M is defined on $\mathcal{U}$ and if it has a simple zero, then the manifolds

$$W^s((\theta_1, r_1^\epsilon), F_\epsilon) \quad \textit{and} \quad W^u((\theta_2, r_2^\epsilon), F_\epsilon)$$

intersect transversally for ϵ small enough.

We can call this function the *Melnikov function for S*. We will be able to give a formula for M.

THEOREM 4.6. *Let $S_0, F_0, h, \chi, \mathcal{U}, P, S_\epsilon$ and F_ϵ be as above. Let M be the Melnikov function defined in Lemma 4.5.*

Assume that the perturbation P has the following properties:

(a) $P(\theta_1, \theta_1, 0) = P(\theta_2, \theta_2, 0) = 0$;

(b) $\dfrac{d}{d\theta}\bigg|_{\theta=\theta_1} P(\theta, \theta, \epsilon) = \dfrac{d}{d\theta}\bigg|_{\theta=\theta_2} P(\theta, \theta, \epsilon) = 0.$

Then F_ϵ has fixed points at (θ_1, r_1^ϵ) and (θ_2, r_2^ϵ), where $r_i^\epsilon = r_1 - \epsilon P_1(\theta_i, \theta_1, \epsilon)$ and the series

$$L(\theta) = \sum_{k=-\infty}^{\infty} P(h^k(\theta), h^{k+1}(\theta), 0)$$

converges absolutely and uniformly on compact subsets of $\mathcal{U}$, is C^2 as a function of θ and $M = \partial/\partial\theta L$.

Proof. The conditions on the derivatives and the fact that both points are hyperbolic imply that the series defined in L are absolutely convergent. Consider $\theta_0 \in \mathcal{U}$. Let $\epsilon_{\theta_0} > 0$ small enough such that for all $0 < \epsilon < \epsilon_{\theta_0}$, $\theta_0 \in \mathcal{U}_1^\epsilon \cap \mathcal{U}_2^\epsilon$.

On each $\theta \in \mathcal{U}_1^\epsilon \cap \mathcal{U}_2^\epsilon$ we have the pair of formulas

$$\chi_\epsilon^s = \frac{\partial}{\partial\theta} Z_\epsilon^s, \quad \chi_\epsilon^u = \frac{\partial}{\partial\theta} Z_\epsilon^u,$$

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where

$$Z^s_\epsilon(\theta) = -\sum_{k=0}^{\infty}\{S_\epsilon((h^s_\epsilon)^k(\theta), (h^s_\epsilon)^{k+1}(\theta)) - S_\epsilon(\theta_2, \theta_2)\}$$

and

$$Z^u_\epsilon(\theta) = \sum_{k=-1}^{-\infty}\{S_\epsilon((h^u_\epsilon)^k(\theta), (h^u_\epsilon)^{k+1}(\theta)) - S_\epsilon(\theta_1, \theta_1)\}.$$

So we have that

$$M(\theta) = \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}\frac{\partial}{\partial\theta}(Z^u_\epsilon(\theta) - Z^s_\epsilon(\theta)).$$

Interchanging the order of differentiation we get that

$$M(\theta) = \frac{\partial}{\partial\theta}\,\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}(Z^u_\epsilon(\theta) - Z^s_\epsilon(\theta)).$$

Differentiating we have that

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}(-Z^s_\epsilon(\theta)) &= \frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}\sum_{k=0}^{\infty}\{S_\epsilon(h^{s\,k}_\epsilon(\theta), (h^s_\epsilon)^{k+1}(\theta)) - S_\epsilon(\theta_2, \theta_2)\}\\ &= \sum_{k=0}^{\infty}\{S_1(h^k_0(\theta), h^{k+1}_0(\theta))\,\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^k(\theta))\\ &\quad + S_2(h^k_0(\theta), h^{k+1}_0(\theta))\,\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^{k+1}(\theta))\\ &\quad + P(h^k_0(\theta), h^{k+1}_0(\theta), 0)\},\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}(-Z^s_\epsilon(\theta)) &= \lim_{N\to\infty} S_2(h^N_0(\theta), h^{N+1}_0(\theta))\,\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^{N+1}(\theta))\}\\ &\quad + \sum_{k=0}^{\infty} P(h^k_0(\theta), h^{k+1}_0(\theta), 0).\end{aligned}$$

We need to show that

$$\lim_{N\to\infty}\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^{N+1}(\theta)) = 0.$$

For that purpose, it is enough to notice that

$$\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^{N+1}(\theta)) = Dh_0((h_0)^N(\theta))\,\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^N(\theta)) + \frac{\partial h^s_\epsilon}{\partial\epsilon}((h_0)^N(\theta))\bigg|_{\epsilon=0}.$$

Let K be a compact set $K \subset \mathcal{U}$. Pick a small $\varepsilon > 0$. Let N be big enough so that for all $\theta \in K$, $\|Dh_0((h_0)^N(\theta))\| < \lambda_0$, where $\lambda_0 < 1$, and

$$\left\|\frac{\partial h^s_\epsilon}{\partial\epsilon}((h_0)^N(\theta))\bigg|_{\epsilon=0}\right\| < \varepsilon.$$

This implies that

$$\left\|\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^{N+1}(\theta))\right\| < \lambda_0\left\|\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0}((h^s_\epsilon)^N(\theta))\right\| + \varepsilon$$

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and therefore for all $M > N$,

$$\left\| \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} ((h_\epsilon^s)^{N+1}(\theta)) \right\| < \frac{\varepsilon}{1-\lambda_0}.$$

In the same way we can prove that

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} (Z_\epsilon^u(\theta)) = \sum_{k=-1}^{-\infty} P(h_0^k(\theta), h_0^{k+1}(\theta), 0),$$

and therefore

$$M(\theta) = \frac{\partial}{\partial \theta}\left(\sum_{k=-1}^{-\infty} P(h_0^k(\theta), h_0^{k+1}(\theta), 0) + \sum_{k=0}^{\infty} P(h_0^k(\theta), h_0^{k+1}(\theta), 0) \right) = \frac{\partial}{\partial \theta} L(\theta).$$

An alternative way of proving the theorem is by first using a change of variables that would 'lower' the saddle connection to the manifold $\{r = 0\}$. By Lemma 4.2, there is a C^2 function defined on $\mathcal{U}$ and such that $\partial Z/\partial\theta = \psi(\theta)$, where ψ is a function whose graph is the saddle connection. Extend Z smoothly to all of $\mathbb{R}^n$ and also the function ψ. We perform the change of coordinates $(\theta, r) \mapsto (\theta, r + \psi(\theta))$.

If $F_0(\theta, r) = (\Theta(\theta, r), R(\theta, r))$ is the old twist function, then, on these new coordinates, the new twist function looks like

$$\tilde{F}_0(\theta, r) = F_0(\theta, r + \psi(\theta)) - \psi(\Theta(\theta, r))$$

with generating function

$$\tilde{S}(\theta, \Theta) = S(\theta, \Theta) + Z(\theta) - Z(\Theta).$$

This new generating function has the property that

$$S_1(\theta, h_0(\theta)) = S_2(\theta, h_0(\theta)) = 0$$

for all $\theta \in \mathcal{U}$. Since the Melnikov function remains the same for this function, this trick simplifies the proof considerably. □

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