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Quantitative aspects of speed-up and gap phenomena

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We show that, for any abstract complexity measure in the sense of Blum and for any computable function f (or computable operator F), the class of problems that are f-speedable (or F-speedable) does not have effective measure 0. On the other hand, for sufficiently fast growing f (or F), the class of non-speedable computable problems does not have effective measure 0. These results answer some questions raised by Calude and Zimand. We also give a quantitative analysis of Borodin and Trakhtenbrot's Gap Theorem, which corrects a claim by Calude and Zimand.

1. Introduction

In the early 1990s, Lutz started a program for a quantitative analysis of phenomena in structural complexity theory (Lutz 1992). This program, which attracted quite a large number of researchers, built on resource-bounded variants of Schnorr's effective measure (Schnorr 1973) and Mehlhorn's effective Baire category (Mehlhorn 1973), which allowed the application of these classical classification tools to the (countable) computable universe. While most of Lutz's program was devoted to complexity classes in the lower part of the intractable world, such as the exponential time classes, Calude and Zimand (1996) exploited these tools for the quantitative analysis of more general phenomena encountered in all of the common complexity measures, thereby taking up some earlier lines of research by Mehlhorn and others.

In particular, Calude and Zimand applied effective measure and category to a quantitative analysis of the fundamental theorems in Blum's abstract complexity theory (Blum 1967) such as the Speed-Up and Gap Theorems. For instance, they have shown that, in the sense of effective category, speedable sets are not rare, namely, for any complexity measure and for any effective operator F, the class of the computable sets that are F-speedable is not effectively meagre in the sense of Mehlhorn. Calude and Zimand (1996) asked whether this observation is also true in the sense of effective measure. Here we answer this question affirmatively by showing that (in any Blum space) the class of F-speedable problems does not have effective measure 0 in the sense of Schnorr (see Section 3).

We also show, however, that, for any sufficiently fast growing operator F, the class of the non-F-speedable problems does not have effective measure 0. In fact we prove a similar stronger result for the function version of the Speed-Up Theorem, and obtain the corresponding result for Mehlhorn's effective category instead of the effective measure (see Section 4). One interpretation of our results is that the effective measure and category

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concepts of Schnorr and Mehlhorn, respectively, are too coarse to give a complete quantitative analysis of the speed-up phenomena.

However, in the case of another important result in abstract complexity theory, namely the Gap Theorem, we can give a complete quantitative analysis. For sufficiently fast growing effective operators F, F-gaps are rare, that is, the class of the computable complexity bounds t at which F-gaps occur is:

— effectively meagre; and
— has effective measure 0

(in the Baire space). The first of these results corrects a claim in Calude and Zimand (1996), while the second answers another question they raised (see Section 5).

Before we present our results, we use the next section to give a short review of the required basic concepts and results of: Blum's abstract complexity theory (Blum 1967); effective measure theory; and the effective Baire category. For a more complete account of this material, see, for example, Zimand's recent monograph (Zimand 2006).

2. Preliminaries

2.1. *Abstract complexity measures*

An *abstract complexity measure* (or *Blum space* for short) is a pair (φ, Φ) of partially computable functions over the set of the non-negative integers, ω, where φ is a Gödel numbering of the unary partially computable functions over ω and Φ satisfies the following two conditions, called *Blum axioms*:

(B1) $\mathrm{dom}(\varphi) = \mathrm{dom}(\Phi)$
(B2) $\mathrm{graph}(\Phi) = \{(e, x, y) : \Phi_e(x) = y\}$ is computable.

In the following we will focus on the complexity of computable sets. We identify a set with its characteristic function and let REC denote the class of computable sets.

Among the most interesting phenomena that hold for all Blum spaces are the Speed-Up and Gap Theorems. Blum's Speed-Up Theorem says that there are computable problems without optimal solutions even if we measure the costs only up to a very large (for example, exponential) factor. Formally, given a computable function $f : \omega \times \omega \to \omega$, a computable set A is *f-speedable* with respect to Blum space (φ, Φ) if

$$(\forall e)(\varphi_e = A \Rightarrow (\exists e')(\varphi_{e'} = A \text{ and } f(x, \Phi_{e'}(x)) \leqslant_{a.e.} \Phi_e(x))) \tag{1}$$

(where $g \leqslant_{a.e.} h$ denotes the fact that $g(x) \leqslant h(x)$ for almost all x, and where we let $\Phi_e(x) = \infty$ if $\Phi_e(x)$ is undefined).

Theorem 1 (Speed-Up Theorem: function version (Blum 1967)). Let f be a computable function of type $\omega \times \omega \to \omega$ and let (φ, Φ) be a Blum space. There is a computable f-speedable set A, that is,

$$\mathrm{SPEED}_f^{(\varphi,\Phi)} = \{A \in \mathrm{REC} : A \ f\text{-speedable}\} \neq \varnothing.$$

As Meyer and Fischer have shown, the Speed-Up Theorem can be strengthened by replacing the computable function f by a *total effective operator* F, that is, a mapping

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of partially computable functions to partially computable functions such that totally computable functions are mapped to totally computable functions and $(e, x) \mapsto F(\varphi_e)(x)$ is partially computable. A computable set A is *F-speedable* with respect to (φ, Φ) if

$$(\forall e)(\varphi_e = A \Rightarrow (\exists e')(\varphi_{e'} = A \text{ and } F(\Phi_{e'})(x) \leqslant_{a.e.} \Phi_e(x))). \tag{2}$$

Theorem 2 (Speed-Up Theorem: operator version (Meyer and Fischer 1972)). Let F be a total effective operator and (φ, Φ) be a Blum space. There is a computable F-speedable set A, that is,

$$\mathrm{SPEED}_F^{(\varphi,\Phi)} = \{A \in \mathrm{REC} : A \ F\text{-speedable}\} \neq \varnothing.$$

The second fundamental observation on abstract complexity measures that we will discuss is Trakhtenbrot and Borodin's Gap Theorem, which states that for any total effective operator F, there is a computable cost function t such that an increase of the complexity bound t by the factor F will not allow the solution of any additional problems. For the formal statement let

$$\mathrm{C}_t^{(\varphi,\Phi)} = \{A \in \mathrm{REC} : (\exists e)(\varphi_e = A \text{ and } \Phi_e(x) <_{a.e.} t(x))\} \tag{3}$$

denote the (φ, Φ)-*complexity class* with *bound (name)* t.

Theorem 3 (Gap Theorem: operator version (Trakhtenbrot 1967 and Borodin 1972)). Let (φ, Φ) be a Blum space and F be a total effective operator such that, for every partially computable function ψ, we have $F(\psi)(n) \geqslant_{a.e.} \psi(n)$. There is a computable function t such that $\mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{F(t)}^{(\varphi,\Phi)}$.

In the literature there is also the following weaker function version of the Gap Theorem, which follows immediately from Theorem 3.

Theorem 4 (Gap Theorem: function version). Let (φ, Φ) be a Blum space and $h : \omega \to \omega$ be a total computable function such that $h(n) \geqslant n$ for almost all n. There is a computable function t such that

$$\mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{h(t)}^{(\varphi,\Phi)}. \tag{4}$$

In fact, for any computable function g, there is a computable function t such that $t(n) \geqslant_{a.e.} g(n)$ and such that (4) holds.

Finally, we will need the following technical result from abstract complexity theory, which states that two Blum spaces are always related in the following way.

Theorem 5 (Recursive Relatedness Theorem (Blum 1967)). Let (φ, Φ) and (ψ, Ψ) be two abstract complexity measures and let h be some computable translation function from φ to ψ (that is, $\varphi_e = \psi_{h(e)}$ for $e \geqslant 0$). Then there exists a strictly monotone 2-ary computable function g such that

$$(\forall e)(\Psi_{h(e)}(x) \leqslant_{a.e.} g(x, \Phi_e(x)) \text{ and } \Phi_e(x) \leqslant_{a.e.} g(x, \Psi_{h(e)}(x))). \tag{5}$$

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2.2. *Effective measure on the Cantor space*

In this section we outline the effective measure theory on the Cantor space, which we will need for our quantitative analysis of the Speed-Up Theorems. We identify a set A of non-negative integers with its characteristic sequence, so A may be viewed as an element of the Cantor space 2^ω. The classical Lebesgue measure on 2^ω can be described in terms of martingales (betting games). By restricting consideration to computable martingales only, this yields an *effective* measure on 2^ω (Schnorr 1973). For our purposes in developing this theory, it suffices to consider rational-valued martingales d of norm 1 (see, for example, Ambos-Spies and Mayordomo (1997)).

Definition 1 (Effective measure on 2^ω (Schnorr 1973 and Lutz 1992)).

(a) A *(rational-valued, normed) martingale* d is a function $d : 2^{<\omega} \to \mathbb{Q}_{\geqslant 0}$ such that $d(\lambda) = 1$ and d satisfies the *fairness condition*

$$d(w) = \frac{d(w0) + d(w1)}{2}. \tag{6}$$

A martingale d *succeeds* on a set $A \in 2^\omega$ if $limsup_{n\to\infty} d(A \upharpoonright n) = \infty$ (where $A \upharpoonright n = A(0)\dots A(n-1)$), and d *succeeds on a class* $\mathrm{C} \subseteq 2^\omega$ if d succeeds on all sets $A \in \mathrm{C}$.

(b) A class $\mathrm{C} \subseteq 2^\omega$ has *effective measure 0* if there is a *computable* martingale that succeeds on C, and C has *effective measure 1* if the complement of C has effective measure 0.

(c) A class C has *measure 0 in* REC if $\mathrm{C} \cap \mathrm{REC}$ has effective measure 0, and C has *measure 1 in* REC if the complement of C has measure 0 in REC.

Intuitively, a class C of computable sets is *small* if it has effective measure 0 (or, equivalently, measure 0 in REC), *non-small* if it does not have effective measure 0 (or, equivalently, does not have measure 0 in REC) and *large* if it has measure 1 in REC.

2.3. *Effective category on the Baire space*

We conclude these preliminaries with a short account of the effective Baire category on the Baire space, which will be the basis of our quantitative analysis of the Gap Theorem. By identifying a function $f : \omega \to \omega$ with the enumeration $f(0)f(1)f(2)\dots$ of its range, f may be viewed as an infinite sequence of non-negative integers, that is, as an element of the Baire space ω^ω. As in the case of sets A, we abbreviate the initial segment of the sequence f of length n by $f \upharpoonright n$, that is, $f \upharpoonright n = f(0)\dots f(n-1)$, and we let $\omega^{<\omega}$ denote the set of finite sequences over ω. For $\alpha, \beta \in \omega^{<\omega}$, we write $\alpha \sqsubseteq \beta$ if β extends α and $\alpha \sqsubset \beta$ if this extension is proper. Similarly, $\alpha \sqsubset f$ denotes the fact that the string α is an initial segment of the function f, that is, $\alpha = f \upharpoonright n$ for some n. We let

$$U_\alpha = \{f \in \omega^\omega : \alpha \sqsubset f\}$$

be the class of functions extending string α and call U_α the *cone* with *base* α.

The classical Baire category concept on ω^ω can be described in terms of extension functions, where a total function $f : \omega^{<\omega} \to \omega^{<\omega}$ is an *extension function* if $\alpha \sqsubseteq f(\alpha)$ for all strings $\alpha \in \omega^{<\omega}$. As Mehlhorn (1973) observed, we get an effective counterpart of the Baire category concept that allows a quantitative analysis of the class of the computable

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functions, FREC $= \{f : \omega \to \omega : f$ computable$\}$, if one considers only *computable* extension functions.

Definition 2 (Effective Baire category on ω^ω (Mehlhorn 1973)).

(a) A class $C \subseteq \omega^\omega$ is *effectively nowhere dense via* f if f is a computable extension function and, for all $\alpha \in \omega^{<\omega}$, $U_{f(\alpha)} \cap C = \varnothing$. We say C is *effectively nowhere dense* if C is effectively nowhere dense via some f.

(b) A class $C \subseteq \omega^\omega$ is *effectively meagre via* $f = \langle f_n : n \geqslant 0\rangle$ if the functions f_n, $n \geqslant 0$, are uniformly computable extension functions such that

$$C \subseteq \bigcup_{n \geqslant 0} C_n$$

for classes C_n, $n \geqslant 0$, where C_n is effectively nowhere dense via f_n. We say C is *effectively meagre* if C is effectively meagre via some f.

(c) A class $C \subseteq \omega^\omega$ is *effectively co-meagre* if the complement of C, $\overline{C} = \omega^\omega \setminus C$, is effectively meagre.

(d) A class $C \subseteq \omega^\omega$ is *meagre in* FREC if $C \cap$ FREC is effectively meagre. We say C is *co-meagre in* FREC if $\overline{C} \cap$ FREC is effectively meagre.

Mehlhorn has shown that the class FREC is not effectively meagre (whereas any *uniformly* computable class of computable functions is effectively meagre). This implies that a class C of computable functions that is co-meagre in FREC is not effectively meagre (but not any class $C \subseteq$ FREC that is not effectively meagre is co-meagre in FREC).

Intuitively, a class C of computable functions is *small* if it is effectively meagre (or, equivalently, meagre in FREC), *non-small* if it is not effectively meagre and *large* if it is co-meagre in FREC. So, by the above remarks, FREC itself is large, any large class $C \subseteq$ FREC is non-small, but there are non-small classes $C \subseteq$ FREC that are not large.

3. A measure-theoretic version of the Speed-Up Theorem

Our first result is the measure theoretic analogue of Calude and Zimand's quantitative analysis of the Speed-Up Theorem in category theory terms. It answers the central open question of Calude and Zimand (1996).

Theorem 6 (Operator speed-up: measure-theoretic version). Let F be a total effective operator and (φ, Φ) be a Blum space. Then the class of the F-speedable sets, $\mathrm{SPEED}_F^{(\varphi,\Phi)}$, does not have effective measure 0.

As a corollary, we obtain the corresponding statement for the function version of the Speed-Up Theorem.

Corollary 1 (Function speed-up: measure-theoretic version). Let f be a computable function of type $\omega \times \omega \to \omega$ and (φ, Φ) be a Blum space. Then the class of the f-speedable sets, $\mathrm{SPEED}_f^{(\varphi,\Phi)}$, does not have effective measure 0.

Proof of Theorem 6. Given a total effective operator F, a Blum space (φ, Φ) and a computable martingale d, we will construct an F-speedable set A on which d does not

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succeed. This proves that the class of F-speedable sets does not have measure 0 via d and, since d was chosen arbitrarily, does not have effective measure 0.

The set A will be constructed in stages $s \geqslant 0$, where at stage s, given $A \upharpoonright s$, we define $A(s)$. During the construction we will try to satisfy the (finitary) requirements

$$R_e : A = \varphi_e \Rightarrow \Phi_e(x) >_{a.e.} u_e(x) \tag{7}$$

for some appropriate sequence of uniformly computable functions $(u_e)_{e\in\omega}$. Then, in order to guarantee that A is F-speedable, it suffices to ensure that

$$(\forall e)(\exists e')(A = \varphi_{e'} \text{ and } F(\Phi_{e'})(x) \leqslant_{a.e.} u_e(x)). \tag{8}$$

Let us first assume that we are given the sequence of the functions u_e. Then the basic strategy for satisfying the requirements $R_e, e \geqslant 0$, is as follows. If there are infinitely many numbers x such that $\Phi_e(x) \leqslant u_e(x)$ (note that R_e is trivially met otherwise), then requirement R_e is met by diagonalisation, that is, by ensuring $A(x) \neq \varphi_e(x)$ for some such x. To be more precise, at any stage $s > e$ such that R_e has not been satisfied at any previous stage and such that $\Phi_e(s) \leqslant u_e(s)$, we say that requirement R_e *requires attention* at stage s. If R_e is the highest priority requirement that requires attention at stage s (that is, e is the least number e' such that $R_{e'}$ requires attention at stage s), we let $A(s) = 1 \dot{-} \varphi_e(s)$ and declare R_e to be *active* and *satisfied* at stage s. (If no requirement requires attention, we let $A(s) = 0$.)

Note that, by the Blum axioms, we can decide whether R_e requires attention at stage s, and if it does, then $\varphi_e(s)$ is defined, so $A(s) \neq \varphi_e(s)$ if R_e becomes satisfied at stage s. Moreover, any requirement will become active at most once.

So, by defining A according to the above strategy, A will be computable, and, by a straightforward induction, any requirement that is not trivially met will eventually become satisfied.

Note that the strategy described above works for every sequence of uniformly computable functions $u_e, e \geqslant 0$.

We will now describe how the basic strategy for meeting the requirements R_e can be modified to ensure that the martingale d does not succeed on A.

In order to guarantee that d does not succeed on A, we just need to meet the requirements

$$Q_s : d(A \upharpoonright s+1) \leqslant 1 \tag{9}$$

($s \geqslant 0$). Note that there is a trivial strategy for meeting these requirements: if we let $A(s) = 0$ if $d((A \upharpoonright s)0) \leqslant d((A \upharpoonright s)1)$ and $A(s) = 1$ otherwise, then, by the fairness property (6) of d, we have $d(A \upharpoonright s+1) \leqslant d(A \upharpoonright s)$. (If $A(s)$ is defined in this way, we say that $A(s)$ is *defined according to the basic Q-strategy*.) So, using this strategy, the value of $d(A \upharpoonright s)$ is non-increasing in s, and thus, by $d(\lambda) = 1$, all Q-requirements are met. Of course, this basic Q-strategy is too strict and must be relaxed in order to become compatible with our strategy for meeting the requirements R_e.

Recall that, for a requirement R_e that is not trivially met, there are infinitely many stages at which R_e may become satisfied by becoming active. (In the following, we call such a requirement *non-trivial*.) So it suffices to ensure that at one of these stages s, requirement R_e can become satisfied without hurting the corresponding Q-requirement Q_s. One might

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be tempted to allow the highest priority requirement R_e that requires attention at stage s to act if letting $A(s) = 1 \dot{-} \varphi_e(s)$ will not injure Q_s. But this procedure is not fair and may result in some of the non-trivial R-requirements never becoming satisfied. So we have to design a slightly more involved strategy, which will treat the individual R-requirements fairly and will guarantee that any non-trivial requirement will eventually become satisfied.

The idea of this refined strategy is roughly as follows: if requirement R_e is entitled to become satisfied at stage s, then, instead of satisfying requirement R_e (by letting $A(s) = 1 \dot{-} \varphi_e(s)$), we define $A(s)$ according to the basic Q-strategy, then (by (6)) the value of d will *strictly decrease* at stage s, that is, $d(A \upharpoonright s+1) < d(A \upharpoonright s)$ (since otherwise $d((A \upharpoonright s)0) = d((A \upharpoonright s)1)$ so there would be no reason to prevent R_e from becoming satisfied at stage s). Now, whenever R_e is entitled to act but not allowed to do so, the resulting decrease $d(A \upharpoonright s) - d(A \upharpoonright s+1)$ in the value of d is paid into an 'account' of R_e, and R_e will be allowed to become satisfied at a stage at which it is entitled to act provided the increase of d caused by this action is bounded by the current balance of R_e's account. This will work since a non-trivial requirement R_e that was never satisfied would be entitled to act infinitely often and the balance of its account would be unbounded. So, eventually, the balance will cover the growth of d caused by satisfying R_e (which is bounded by 1), and thus R_e will become satisfied.

This will complete the construction if it is carried out with a given sequence of uniformly computable functions u_e for which condition (8) holds. However, since the truth of this condition itself depends on the set A to be constructed, we will use the recursion theorem in the actual construction to obtain a sequence of functions u_e satisfying (8). In particular, instead of defining just one set A, we construct sets $A_{n,p,e}$ for all parameters $n, p = \langle p_1, p_2, p_3 \rangle, e \in \omega$. The construction is similar to what we described above, with the following changes that depend on the parameters:

— $A_{n,p,e}$ is constructed corresponding to the sequence $(u_i)_{i \in \omega}$ given by letting $u_i(x) = \varphi_n(\langle i, x \rangle)$.
— $A_{n,p,e}$ has the (p_1+1)-th binary string w_{p_1} as a prefix, so that the R- and Q-strategies described are only implemented after stage $|w_{p_1}|$.
— In the construction of $A_{n,p,e}$, we ignore the first e requirements $R_0, \ldots, R_{e-1}$ and the finitely many requirements R_i with $i \in D_{p_2}$, where D_{p_2} is the (p_2+1)-th finite set; these requirements never require attention.
— In the construction of $A_{n,p,e}$, the accounts of finitely many requirements $R_0, \ldots, R_k$ do not start with balance 0, but with balances $v_0(p_3), \ldots, v_k(p_3)$, where

$$\rho_{p_3} = (v_0(p_3), \ldots, v_k(p_3))$$

is the (p_3+1)-th finite sequence of non-negative rationals in a computable enumeration of all such sequences.

Of course, if some u_i is not total, then the construction of a corresponding set $A_{n,p,e}$ may get stuck at some stage s and never be completed. But for the correct parameter n, chosen via the recursion theorem, all u_i will be total. For this n, we will let $A = A_{n,0,0}$. For $e \neq 0$, remember that there is some stage s after which the requirements $R_0, \ldots, R_{e-1}$ will never become active. So, if we choose the parameter p so as to encode all necessary

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(finite) information about the construction of $A_{n,0,0}$ up to stage s, then $A_{n,p,e}$ will agree with $A_{n,0,0} = A$. These copies of A will give us the infinitely many indices of A we need to satisfy condition (8), and by our choice of n, this condition will indeed be true.

More formally, the construction of the sets $A_{n,p,e}$ is as follows. Fix $n, p, e \geqslant 0$ and p_1, p_2, p_3 such that $p = \langle p_1, p_2, p_3 \rangle$. Let $u_i(x) = \varphi_n(\langle i, x \rangle)$ and, for ease of notation, abbreviate $A_{n,p,e}$ by A. Then, using the concepts introduced above when describing the basic R- and Q-strategies, A is defined as follows.

For $s < |w_{p_1}|$, we let no requirement require attention at stage s and we let $A(s) = w_{p_1}(s)$. If R_i is the highest priority requirement R_j such that $j \geqslant e$, $j \notin D_{p_2}$, and R_j requires attention at stage $s \geqslant |w_{p_1}|$, we say that R_i is *entitled to act* at stage s and we call s an *i-stage* (and we call s a -1-*stage* if there is no such requirement).

If s is an i-stage but R_i does not become active at stage s or if s is a -1-stage, we define $A(s)$ according to the basic Q-strategy, that is, we let $A(s) = 0$ if $d((A \upharpoonright s)0) \leqslant d((A \upharpoonright s)1)$ and $A(s) = 1$ otherwise.

In order to decide whether the requirement that is entitled to act at stage s (if any) actually becomes active, we attach to any requirement R_i and any stage s a rational $b_i(s)$ (denoting the balance of R_i's account at the end of stage s) where

$$b_i(s) = v_i(p_3) + \sum_{s' < s,\ s'\ i\text{-stage}} d(A \upharpoonright s') - d(A \upharpoonright s' + 1)$$

if $s > 0$ and R_i has not been satisfied by stage s, and $b_i(s) = 0$ otherwise; here $\rho_{p_3} = (v_0(p_3), \ldots, v_k(p_3))$ is the $(p_3 + 1)$-th finite sequence of non-negative rationals and $v_i(p_3) = 0$ for $i > k$.

Now, if R_i is entitled to act at stage s and

$$d((A \upharpoonright s)(1 \dot{-} \varphi_i(s))) \leqslant d(A \upharpoonright s) + b_i(s-1), \tag{10}$$

we let $A(s) = 1 \dot{-} \varphi_i(s)$ and say that requirement R_i becomes *active* and *satisfied* at stage s.

This completes the actual construction of $A_{n,p,e}$. Since the construction is uniformly effective in the parameters n, p, e, the function $(\langle n, p, e \rangle, s) \mapsto A_{n,p,e}(s)$ is partially computable. Hence, for the Gödel numbering φ there exists a computable translation function h such that $\varphi_{h(n,p,e)} = A_{n,p,e}$ for all n, p, e.

In order to find an appropriate n as described above, consider the following partially computable function.

$$\psi(n, \langle e, x \rangle) = \begin{cases} 0 & \text{if } x \leqslant e \\ 0 & \text{if there exists } i \leqslant e \text{ with} \\ & \quad \Phi_n(\langle 0, i \rangle) \geqslant x \\ \max\{F(\Phi_{h(n,p,e+1)})(x) : p \leqslant x\} & \text{otherwise, if } \Phi_n(\langle j, y \rangle) \downarrow \\ & \quad \text{for all } y \leqslant x \text{ and } e < j \leqslant x \\ \uparrow & \text{if none of the above is true.} \end{cases}$$

By the second Blum axiom, the second condition can be effectively checked; so we see that ψ is partially computable (although the maximum in the third case does not

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necessarily exist, even if the third condition is true). The recursion theorem yields an n_0 with $\lambda x.\psi(n_0, x) = \varphi_{n_0}$.

Now, for the remainder of the proof, we let $A = A_{n_0,0,0}$ where we assume that $\langle 0,0,0\rangle = 0$, w_0 is the empty string, D_0 is the empty set, and the sequence ρ_0 of non-negative rationals is the empty sequence. Moreover, we let $u_e(x) = \varphi_{n_0}(\langle e, x\rangle) = \psi(n_0, \langle e, x\rangle)$.

In order to show that the set A has the required properties, we prove a series of claims. First we show that the partially computable functions u_e are total.

Claim 1. For each $e \geqslant 0$, u_e is total.

Proof. First consider u_0. In order to show a contradiction, we assume that u_0 is not total and fix i minimal such that $u_0(i) = \varphi_{n_0}(\langle 0, i\rangle)$ is undefined. Then $\Phi_{n_0}(\langle 0, i\rangle) = \infty$ by the first Blum axiom. Hence, for all $e \geqslant i$ and all x, in the definition of $u_e(x) = \psi(n_0, \langle e, x\rangle)$, one of the first two cases applies and thus $u_e(x) = 0$, and thus, in particular, u_e is total for all $e \geqslant i$. So, by our assumption that u_0 is not total, we may fix $e < i$ such that u_e is not total and u_j is total for all $j > e$. In order to get the desired contradiction, we will show that u_e is total, that is, that $u_e(x) = \psi(n_0, \langle e, x\rangle)$ is defined for all x.

Since $u_j(y) = \varphi_{n_0}(\langle j, y\rangle)$, it follows that $\varphi_{n_0}(\langle j, x\rangle)$ and hence $\Phi_{n_0}(\langle j, x\rangle)$ is defined for all $j > e$ and $y \geqslant 0$. This implies that in the definition of $\psi(n_0, \langle e, x\rangle)$, one of the first three cases applies. So it only remains to show that if Case 3 applies,

$$\max\{F(\Phi_{h(n_0,p,e+1)})(x) : p \leqslant x\}$$

is defined. To do this, by the first Blum axiom, it suffices to show that for any $p \geqslant 0$, $\varphi_{h(n_0,p,e+1)}$ is total. By $A_{n_0,p,e+1} = \varphi_{h(n_0,p,e+1)}$, this is true if $A_{n_0,p,e+1}$ is computable, that is, if all stages in the construction of $A_{n_0,p,e+1}$ are completed. Note that a stage s will not be completed only if, for some requirement R_j that may require attention, the question as to whether $\Phi_j(s) \leqslant u_j(s)$ cannot be decided since $u_j(s)$ is undefined. But, since in the construction of $A_{n_0,p,e+1}$ only requirements R_j with $j \geqslant e+1$ may require attention, by the totality of u_j for $j > e$, this will never happen.

Now, having shown that u_0 is total, totality of all the other u_e is immediate: for almost every x, we have $u_0(x) = \psi(n_0, \langle 0, x\rangle)$ must be defined by the third clause of the definition, which is only possible if $\Phi_{n_0}(\langle j, y\rangle)$ is defined for every $j > 0$ and $y \geqslant 0$. So, by the first Blum axiom, $\varphi_{n_0}(\langle j, y\rangle) = u_j(y)$ is also defined for $j > 0$ and $y \geqslant 0$. □

It follows from Claim 1 that the construction of $A_{n_0,0,0} = A$ will never get stuck. So, in particular, A is computable. Next we show that, in the construction of $A_{n_0,0,0}$, $A_{n_0,0,0} = A$ satisfies all requirements.

Claim 2. In the construction of $A_{n_0,0,0}$, all requirements are met.

Proof. By a straightforward induction on s, we have $b_e(s) \geqslant 0$ and

$$d(A \upharpoonright s+1) + \sum_{e=0}^{\infty} b_e(s) \leqslant 1 \tag{11}$$

in the construction of $A_{n_0,0,0}$. So, in particular, the requirements Q_s are met.

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Moreover, every requirement R_e is met and will be entitled to act at most finitely often. To see this by showing a contradiction, we fix e minimal such that R_e is not met or is entitled to act infinitely often. Then R_e will be non-trivial, never satisfied and, by the minimality of e, entitled to act infinitely often. It follows that $b_e(s)$ is non-decreasing in s: for the first e-stage s_0, $b_e(s_0) > 0$; and for any of the following e-stages s', we have $b_e(s') > 2b_e(s'-1)$ (by (6) and by failure of (10) for $s = s'$). So $b_e(s)$ is unbounded, which contradicts (11). □

By $A = A_{n_0,0,0}$, Claim 2 implies that (7) and (9) are satisfied. It remains to show that (8) holds.

Claim 3. For every parameter e there exists some parameter p such that $A_{n_0,p,e} = A$.

Proof. Let $e \in \omega$. Fix a stage s_e in the construction of $A_{n_0,0,0}$ such that no requirement R_i, $i < e$, will ever be entitled to act at any stage $s > s_e$. Let $w_{p_1} = A \upharpoonright (s_e + 1)$ and D_{p_2} be the finite set of indices of those requirements that are satisfied during stages $0, 1, \dots s_e$, and let ρ_{p_3} be the sequence $(b_0(s_e), b_1(s_e), \dots b_k(s_e))$ of the account balances of the first $k+1$ R-requirements in the construction of $A_{n_0,0,0}$ where k is chosen minimal such that $b_l(s_e) = 0$ for $l > k$. We claim that, for $p = \langle p_1, p_2, p_3\rangle$, we have $A_{n_0,p,e}(s) = A_{n_0,0,0}(s)$ for every s.

If $s < |w_{p_1}| = s_e + 1$, this is clear by the choice of w_{p_1}. Then, at stage $s_e + 1$ of the constructions of $A_{n_0,0,0}$ and $A_{n_0,p,e}$, where we define the values $A_{n_0,0,0}(s_e+1)$ and $A_{n_0,p,e}(s_e+1)$, respectively, in either construction the same requirement will be entitled to act. (Namely, by the choice of s_e, none of the requirements R_i with $i \in \{0, \dots, e-1\}$ or $i \in D_{p_2}$ that are not allowed to require attention in the construction of $A_{n_0,p,e}$ will require attention in the construction of $A_{n_0,0,0}$ after stage s_e, and, conversely, by the choice of D_{p_2}, no requirement that is already satisfied in the construction of $A_{n_0,0,0}$ will require attention in the construction of $A_{n_0,p,e}$.) Furthermore, by our choice of p_3, the balances of the accounts at stage s_e are equal in the two constructions. Consequently, the same requirement (if any) becomes active and satisfied. Hence, $A_{n_0,0,0}(s_e + 1)$ and $A_{n_0,p,e}(s_e + 1)$ agree. By induction, we see that the same analysis also holds for all stages $s > s_e$. □

Claim 4. A satisfies condition (8).

Proof. Given e, by Claim 3, we can choose $p \in \omega$ such that $A_{n_0,p,e+1} = A$. Let $e' = h(n, p, e+1)$. Then

$$\varphi_{e'} = \varphi_{h(n_0,p,e+1)} = A_{n_0,p,e+1} = A.$$

So it suffices to show

$$F(\Phi_{e'})(x) \leqslant_{a.e.} u_e(x). \tag{12}$$

Since, for any $j, y \geqslant 0$, we have $u_j(y) = \varphi_{n_0}(\langle j, y\rangle) = \psi(n_0, \langle j, y\rangle)$, it follows by Claim 1 and the definition of ψ that

$$u_e(x) = \psi(n_0, \langle e, x\rangle) = \max\{F(\Phi_{h(n_0,p,e+1)})(x) : p \leqslant x\} = \max\{F(\Phi_{e'})(x) : p \leqslant x\}$$

for almost all x, which obviously implies (12). □

This completes the proof of Theorem 6. □

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4. A refined quantitative view of the Speed-Up Theorem

Our measure theoretic version of the Speed-Up Theorem shows that in any Blum space, the class of F-speedable (and thus the class of f-speedable) sets does not have effective measure 0, and thus *is not small*. But is this class *large*? Using Definition 1 (c), we may formalise this question by asking whether $\mathrm{SPEED}_F^{(\varphi,\Phi)}$ has measure 1 in REC, that is, whether the class of the computable non-F-speedable sets has effective measure 0. Our next theorem shows that for sufficiently fast growing functions f (and thus for sufficiently fast growing operators F), the answer is negative.

Theorem 7. Let (φ,Φ) be a Blum space. There is a computable function f such that the class

$$\overline{\mathrm{SPEED}_f^{(\varphi,\Phi)}} \cap \mathrm{REC}$$

does not have effective measure 0, that is, such that $\mathrm{SPEED}_f^{(\varphi,\Phi)}$ does not have measure 1 in REC.

Proof. We will only sketch the proof, which requires some facts on time-bounded random sets on ω that can be easily established along the lines of the corresponding facts for time-bounded random sets of strings (in the setting of worst-case complexity) – see Ambos-Spies and Mayordomo (1997) for details.

By the Recursive Relatedness Theorem, it suffices to prove the theorem for the Blum space (φ,Φ) of the Turing computable functions together with time complexity. We will show that, for the exponential function $f(m,n) = 2^n$, the class of computable non-f-speedable sets in (φ,Φ) does not have effective measure 0.

For a computable function $t(n)$, a set A is $t(n)$-*random* if there is no martingale d that can be computed in time $t(n)$ and that succeeds on A. It is easy to check that a class C does not have effective measure 0 if and only if for any computable function t there is a $t(n)$-random set in C. Also note that, for any computable functions t and t' such that $t(n) \leqslant_{a.e.} t'(n)$, any $t'(n)$-random set is $t(n)$-random too.

Now, it is easy to show that there is a number $k \geqslant 1$ such that, for any non-decreasing time constructible function $t(n) \geqslant n$, there is a $t(n)^2$-random set A_t that can be computed in time $t(n)^k$ while, on the other hand, no $t(n)^2$-random set can be computed in time $t(n)$. So, the computable $t(n)^2$-random set A_t is not f-speedable for the exponential function $f(m,n) = 2^n$. Since any computable function is dominated by a non-decreasing time constructible function, it follows, by the above remarks on random sets and effective measure, that for any computable function t there is a computable non-f-speedable $t(n)$-random set, so $\overline{\mathrm{SPEED}_f^{(\varphi,\Phi)}} \cap \mathrm{REC}$ does not have effective measure 0. □

By a similar argument, we obtain the corresponding result in terms of Mehlhorn's effective Baire category on the Cantor space.

Theorem 8. Let (φ,Φ) be a Blum space. There is a computable function f such that the class of the computable non-f-speedable sets, $\overline{\mathrm{SPEED}_f^{(\varphi,\Phi)}} \cap \mathrm{REC}$, is not effectively meagre.

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So it seems that both the effective measure and effective category are too weak for deciding whether a *typical* computable set is speedable or not.

For a weaker property than speedability, namely almost-everywhere complexity, there is a similar situation in the case of the Baire category. Both the class of the computable a.e.-$t(n)$-complex sets and the class of the computable sets that are not a.e.-$t(n)$-complex are not effectively meagre in the sense of Mehlhorn (see, for example, Calude and Zimand (1996) and Mayordomo (1994), respectively). In this case, however, for a refinement of Mehlhorn's effective Baire category concept based on *partial* extension functions introduced by Ambos-Spies (see Ambos-Spies (1996) and Ambos-Spies and Reimann (1997)), the question of which case is the typical one can be resolved. Namely, for this stronger effective Baire category concept, the class of the computable sets that are not a.e.-$t(n)$-complex is effectively meagre (see Ambos-Spies (1996)). For the case of speedability, however, this refinement of Mehlhorn's effective Baire category concept will not solve the classification problem. In the Baire category concept of Ambos-Spies, for sufficiently large f, the class of the computable non-f-speedable sets is not effectively meagre. For details, see the second author's diploma thesis (Kräling 2008).

5. The Gap Theorem: the quantitative view

We conclude our quantitative analysis of the fundamental results of abstract complexity theory by looking at the Gap Theorems. Calude and Zimand (1996) gives a quantitative analysis of the operator version of these theorems in terms of Mehlhorn's effective Baire category on the Baire space (Mehlhorn 1973). They claim that, for any Blum space (φ, Φ) and for any total effective operator F satisfying $F(\psi) \geqslant_{a.e.} \psi$ for all partially computable functions ψ, the class

$$\mathrm{GAP}_F^{(\varphi,\Phi)} = \{t \in \mathrm{FREC} : \mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{F(t)}^{(\varphi,\Phi)}\}$$

is not effectively meagre (see Calude and Zimand (1996, Theorem 4.1) and Zimand (2006, Theorem 2.4.1)). So, in a topological sense, F-gaps are not uncommon. However, the proof of this result is erroneous. In fact, as we will show next, for any Blum space and for any sufficiently large effective operator F, the gap class $\mathrm{GAP}_F^{(\varphi,\Phi)}$ is effectively meagre, that is, for sufficiently large F, F-gaps are rare in the sense of the effective Baire category.

Theorem 9. Let (φ, Φ) be a Blum space. There is a total effective operator F, where $F(\psi)(n) \geqslant \psi(n)$ for all partially computable functions ψ and almost all numbers n, such that $\mathrm{GAP}_F^{(\varphi,\Phi)}$ is effectively meagre.

For the proof of Theorem 9 we will need the following observations.

Lemma 1.

(a) For any computable function $g : \omega \to \omega$, the class

$$\mathrm{C}_{a.e.\neq g} = \{t \in \mathrm{FREC} : \ t(n) \neq_{a.e.} g(n)\}$$

is effectively meagre.

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(b) The class

$$\mathrm{NULL}^{\infty} = \{t \in \mathrm{FREC} : (\exists^{\infty} n)(t(n) = 0)\}$$

is co-meagre in FREC.

Proof.

(a) Fix $g \in \mathrm{FREC}$. The classes

$$\mathrm{C}_m = \{t \in \mathrm{FREC} : (\forall n \geqslant m)(t(n) \neq g(n))\}$$

are effectively nowhere dense via the uniformly computable extension functions f_m given by

$$f_m(\alpha) = \alpha 0^m g(|\alpha| + m)$$

($m \geqslant 0$). Since

$$\mathrm{C}_{a.e.\neq g} = \bigcup_{m \geqslant 0} C_m,$$

this implies that $\mathrm{C}_{a.e.\neq g}$ is effectively meagre.

(b) This is immediate by Part (a) since, for the constant null function $g(n) = 0$,

$$\overline{\mathrm{NULL}^{\infty}} \cap \mathrm{FREC} = \mathrm{C}_{a.e.\neq g}.$$

□

Lemma 2. Let (φ, Φ) be a Blum space. Then the class

$$\mathrm{NE}^{(\varphi,\Phi)} = \{t \in \mathrm{FREC} : \mathrm{C}_t^{(\varphi,\Phi)} \neq \varnothing\}$$

is effectively meagre.

Proof. By the definition of the complexity classes $\mathrm{C}_t^{(\varphi,\Phi)}$, for any set $A \in \mathrm{C}_t^{(\varphi,\Phi)}$, there is an index e of A such that $t(n) >_{a.e.} \Phi_e(n) \geqslant 0$. So, for non-empty $\mathrm{C}_t^{(\varphi,\Phi)}$, $t(n) >_{a.e.} 0$. It follows that $\mathrm{NE}^{(\varphi,\Phi)} \cap \mathrm{NULL}^{\infty} = \varnothing$, so $\mathrm{NE}^{(\varphi,\Phi)}$ is effectively meagre by Lemma 1(b). □

Proof of Theorem 9. Let A be any computable set and e be an index of A in (φ, Φ). We define the total effective monotone operator F by letting $F(\psi)(n) = \max(\psi(n), \Phi_e(n) + 1)$. Then, for any computable time bound t, we have $A \in \mathrm{C}_{F(t)}^{(\varphi,\Phi)}$, so $\mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{F(t)}^{(\varphi,\Phi)}$ implies that $\mathrm{C}_t^{(\varphi,\Phi)} \neq \varnothing$. So $\mathrm{GAP}_F^{(\varphi,\Phi)}$ is contained in the class $\mathrm{NE}^{(\varphi,\Phi)}$, and is thus effectively meagre by Lemma 2. □

Note that Lemma 2 shows that a quantitative analysis of computable time bounds in terms of the effective Baire category is rather meaningless since the class of the computable time bounds of the non-trivial complexity classes is effectively meagre, and thus small.

Still, for completeness, we will add a few observations on the frequency of function gaps. Note that operator gaps might be much larger than function gaps, so the above argument in the proof of Theorem 9 does not apply in general to function gaps. In fact, as we will show, the answer to the question as to whether large function gaps are typical in the sense of the effective Baire category depends on the underlying Blum space.

Theorem 10.

(a) Let (φ, Φ) be a Blum space such that

$$(\forall e)(\Phi_e(n) \geqslant_{a.e.} n), \tag{13}$$

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and let $h : \omega \to \omega$ be a computable function such that $h(n) \geqslant_{a.e.} n$. Then

$$\mathrm{GAP}_h^{(\varphi,\Phi)} = \{t : \mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{h(t)}^{(\varphi,\Phi)}\}$$

is co-meagre in FREC, and hence not effectively meagre.

(b) Let (φ, Φ) be a Blum space such that, for some computable set A and some index e_0 of A,

$$(\forall n)(\Phi_{e_0}(n) = 0), \tag{14}$$

and let $h : \omega \to \omega$ be a computable function such that $h(n) > 0$ for all n. Then $\mathrm{GAP}_h^{(\varphi,\Phi)}$ is effectively meagre.

Note that, for any given Blum space (φ', Φ'), we get a Blum space (φ, Φ) satisfying (13) by letting $\varphi = \varphi'$ and by defining Φ by $\Phi_e(n) = \Phi'_e(n) + n$. A natural example of a Blum space (φ, Φ) satisfying (13) is the space complexity of online Turing machines if numbers are represented in unary.

A Blum space (φ, Φ) satisfying (14) for some computable set A and some index e_0 of A is obtained as follows. Given any Blum space (φ', Φ'), fix any computable set A and a φ'-index e_0 of A, and let (φ, Φ) be the Blum space where $\varphi = \varphi'$ and Φ is defined by

$$\Phi_e(n) = \begin{cases} 0 & \text{if } e = e_0 \\ \Phi'_e(n) & \text{otherwise.} \end{cases}$$

A natural example of a Blum space (φ, Φ) satisfying (14) is the time complexity of Turing machines. (Note that there is a Turing machine accepting all inputs immediately. So, for $A = \omega$, there is an index e_0 such that (14) holds.)

Proof of Theorem 10.

(a) By Lemma 1, it suffices to show that $\mathrm{NULL}^\infty \subseteq \mathrm{GAP}_h^{(\varphi,\Phi)}$. Note that, for any number n such that $n > h(0)$ and $t(n) = 0$, we have $t(n) < n$ and $h(t)(n) = h(t(n)) = h(0) < n$. So, for $t \in \mathrm{NULL}^\infty$, there are infinitely many numbers n such that $t(n) < n$ and $h(t(n)) < n$. By the choice of the Blum space (φ, Φ) this implies that $\mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{h(t)}^{(\varphi,\Phi)} = \varnothing$, so $t \in \mathrm{GAP}_h^{(\varphi,\Phi)}$.

(b) Fix a computable set A and an index e_0 such that (14) holds. By Lemma 1, it suffices to show that $\mathrm{NULL}^\infty \cap \mathrm{GAP}_h^{(\varphi,\Phi)} = \varnothing$, that is, that, for any function $t \in \mathrm{NULL}^\infty$, we have $\mathrm{C}_t^{(\varphi,\Phi)} \neq \mathrm{C}_{h(t)}^{(\varphi,\Phi)}$. But this is immediate since, for $t \in \mathrm{NULL}^\infty$, we have $\mathrm{C}_t^{(\varphi,\Phi)} = \varnothing$, whereas for any computable t, we have $A \in \mathrm{C}_{h(t)}^{(\varphi,\Phi)}$. □

This completes our quantitative analysis of the basic gap phenomena in the setting of Mehlhorn's Baire category. As we have already pointed out above, this analysis is not very meaningful since, by Lemma 2, the class of all computable functions t giving non-empty complexity classes $\mathrm{C}_t^{(\varphi,\Phi)}$ is effectively meagre, so non-trivial complexity bounds are rare. One might try to overcome this problem by replacing the requirement $\Phi_e(n) <_{a.e.} t(n)$ in the definition of the complexity class $\mathrm{C}_t^{(\varphi,\Phi)}$ by the weaker requirement $\Phi_e(n) \leqslant_{a.e.} t(n)$, that is, by considering the complexity classes

$${}^*\mathrm{C}_t^{(\varphi,\Phi)} = \{A \in \mathrm{REC} : (\exists e)(\varphi_e = A \text{ and } \Phi_e(x) \leqslant_{a.e.} t(x))\}$$

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instead of the classes $\mathrm{C}_t^{(\varphi,\Phi)}$ defined by (3). Then, in fact, the question as to whether

$$^*\mathrm{EMPTY}^{(\varphi,\Phi)} = \{t \in \mathrm{FREC} : {}^*\mathrm{C}_t^{(\varphi,\Phi)} = \varnothing\}$$

is effectively meagre depends on the underlying Blum space. For instance, one can easily show that for a Blum space (φ,Φ) as in part (a) of Theorem 10, $^*\mathrm{EMPTY}^{(\varphi,\Phi)}$ is not effectively meagre whereas, for a Blum space (φ,Φ) as in part (b) of this theorem, $^*\mathrm{EMPTY}^{(\varphi,\Phi)}$ is effectively meagre. Still, the above theorems on gaps (Theorems 9 and 10) remain true if we replace the complexity classes $\mathrm{C}_t^{(\varphi,\Phi)}$ by the modified classes ${}^*\mathrm{C}_t^{(\varphi,\Phi)}$. We will now demonstrate this for Theorem 9.

Theorem 11. Let (φ,Φ) be a Blum space. There is a total effective operator F, where $F(\psi)(n) \geqslant \psi(n)$ for all partially computable functions ψ and almost all numbers n, such that

$$^*\mathrm{GAP}_F^{(\varphi,\Phi)} = \{t \in \mathrm{FREC} : {}^*\mathrm{C}_t^{(\varphi,\Phi)} = {}^*\mathrm{C}_{F(t)}^{(\varphi,\Phi)}\}$$

is effectively meagre.

To prove Theorem 11, we need the following observation on almost everywhere complex sets.

Lemma 3. Let (φ,Φ) be a Blum space and A be a computable set such that

$$(\forall e)(A = \varphi_e \Rightarrow \Phi_e(n) \geqslant_{a.e.} n). \tag{15}$$

Then

$$\mathrm{M}_A^{(\varphi,\Phi)} = \{t \in \mathrm{FREC} : A \in {}^*\mathrm{C}_t^{(\varphi,\Phi)}\} \tag{16}$$

is effectively meagre.

Note that in any Blum space there are computable sets A satisfying (15). (For instance, any computable set A that is F-speedable for $F(\psi)(n) = \psi(n) + n$ has this property.)

Proof. By the choice of A, for any $t \in \mathrm{M}_A^{(\varphi,\Phi)}$, we have $t(n) \geqslant_{a.e.} n$. So, in particular, $\mathrm{M}_A^{(\varphi,\Phi)} \cap \mathrm{NULL}^\infty = \varnothing$. So $\mathrm{M}_A^{(\varphi,\Phi)}$ is effectively meagre by Lemma 1. □

Proof of Theorem 11. Fix a computable set A such that (15) holds and let e be an index of A in (φ,Φ). As in the proof of Theorem 9, we define the total effective operator F by $F(\psi)(n) = \max(\psi(n), \Phi_e(n) + 1)$. Then, for any computable time bound t, we have $A \in \mathrm{C}_{F(t)}^{(\varphi,\Phi)}$, so $\mathrm{C}_t^{(\varphi,\Phi)} = \mathrm{C}_{F(t)}^{(\varphi,\Phi)}$ implies that $t \in \mathrm{M}_A^{(\varphi,\Phi)}$. So $^*\mathrm{GAP}_F^{(\varphi,\Phi)} \subseteq \mathrm{M}_A^{(\varphi,\Phi)}$, and the claim follows by Lemma 3. □

We conclude with the remark that it is easy to repeat the analysis of the gap theorems that we have given above using the effective Baire category but in terms of effective measure. For any computable function $f : \omega \to \mathbb{Q}$ defining a non-vanishing measure on ω (that is, satisfying $f(n) > 0$ for all n and $\sum_{n\in\omega} f(n) = 1$), let μ_f be the corresponding effective product measure on the Baire space. Then, for any Blum space (φ,Φ), we have $\mu_f(\mathrm{GAP}_F^{(\varphi,\Phi)}) = 0$ for all sufficiently large effective functionals F. The proof is similar to the proof of Theorem 9. It is based on the analogue of Lemma 2 in terms of effective measure. The latter shows that, just as in case of the effective category, the analysis of

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computable time bounds in terms of effective measure is not very meaningful, since the class of the time bounds of the non-empty complexity classes has effective measure 0, so we have omitted the details here.

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