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Regularity of the variance in quenched CLT for random intermittent dynamical systems

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Abstract. We study random dynamical systems composed of Liverani–Saussol–Vaienti maps with varying parameters, without any mixing assumptions on the base space of random dynamics. We establish a quenched central limit theorem and identify conditions under which the associated limit variance varies continuously and differentiably with respect to perturbations of the random dynamics. Our arguments rely on recent results on statistical stability and linear response for random intermittent maps established by Dragičević *et al* [*J. Lond. Math. Soc.* **111** (2025), e70150].

Key words: linear response, random dynamical systems, intermittent maps, variance
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1. *Introduction*

In their seminal paper [31], Liverani, Saussol, and Vaienti introduced a now famous class of intermittent maps (called LSV maps) on the unit interval. Trajectories of these nonuniformly expanding maps behave chaotically on a large part of the state space, but when they visit small neighborhoods of the neutral fixed point at the origin, it takes a long time before they return to the chaotic regime. We stress that the study of statistical properties of such maps turned out to be a challenging problem that resulted in the development of new techniques. For example, it turned out that such maps exhibit only polynomial decay of correlations, while uniformly expanding (or hyperbolic) maps possess exponential decay of correlations. We refer to [3] for a detailed survey.

In recent years, there has been a growing interest in the study of statistical properties of non-stationary (or non-autonomous) dynamical systems formed by sequential or random compositions of LSV maps with varying parameters. Notable contributions to their study include estimates on memory loss and decay of correlations (see, for example, [1, 5, 6,

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26–28]), various limit laws (see [**2**, **4**, **11**, **24**, **32**, **33**, **36**, **37**] and references therein), and linear response [**8**, **17**].

In this paper, we first establish the quenched central limit theorem (CLT) for random compositions of LSV maps with varying parameters. Previous results on the quenched CLT in this setting include [**32**, **33**] for independent and identically distributed (i.i.d.) compositions and [**36**] for ergodic compositions. Our theorem (see Theorem 3), which applies to ergodic compositions, also shows an 'unexpected' CLT in the spirit of Gouëzel [**22**]: for a class of special observables that vanish at the neutral fixed point, the CLT holds in a parameter range where it is not expected for a generic regular observable. We deduce the quenched CLT from an abstract result due to Kifer [**25**, Theorem 2.3.] which, to our knowledge, has not previously been applied in this context. This result can be seen as a natural development in the recent line of research devoted to quenched limit theorems for broad classes of random dynamical systems that avoid any mixing assumptions for the base system (see [**14**, **15**, **18**, **23**]).

Our main objective is to study the regularity of the limit variance in Theorem 3 under perturbations of the random intermittent dynamics. More precisely, Theorem 5 (respectively Theorem 9) gives conditions under which the variance varies continuously (respectively differentiably) under suitable perturbations. Although both results consider perturbations of the same type, Theorem 5 applies within the parameter range where correlations are summable, similar to the CLT in Theorem 3, whereas Theorem 9 requires a more restrictive parameter range. Consequently, Theorem 5 is not implied by Theorem 9.

We note that there exist several results in the literature devoted to the regularity of variance in the CLT for various classes of deterministic dynamics [**7**, **10**, **21**, **35**], among which the recent work of Sélley [**35**] dealing with LSV maps is closest to the setting of our paper. In the context of random dynamics, the regularity of the variance in the quenched CLT was studied for random uniformly hyperbolic dynamics [**20**], certain examples of random partially hyperbolic systems [**12**], and for broad classes of non-uniformly expanding random systems [**19**], which do not include random LSV maps (see below for more details).

Consequently, Theorem 9 can be seen as a random counterpart of the main result of [**35**], although we rely on completely different techniques. More precisely, the arguments in [**35**] rely on inducing, while our arguments are based on the recent statistical stability and linear response results for random intermittent maps established in [**17**], together with the arguments developed in the proof of [**19**, Theorem 12]. However, we stress that there are non-trivial differences between the proof of [**19**, Theorem 12] and Theorem 9, as [**19**, Theorem 12] requires (and its proof uses) that the assumptions of [**19**, Theorem 2] hold with $\mathcal{B}_w = C^0$, $\mathcal{B}_s = C^1$, and $\mathcal{B}_{ss} = C^3$. In our setting, [**19**, (4)] holds with $\mathcal{B}_w = L^1(m)$ (see (A.2) for $h = 1$). The same comment applies for other conditions in [**19**, Theorem 2].

Thus, it is necessary to combine the approach from [**19**] with the cone techniques used in the study of LSV maps initiated in [**31**].

We emphasize that the proof of Theorem 9 actually requires a certain improvement of the linear response result of [**17**] (see Theorem 6 and Remark 7) and that contrary to [**35**], it yields an explicit formula for the derivative of the variance (see Remark 10). However, Theorem 9 does require (in comparison with Theorems 3 and 5) an additional constraint

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for the range of parameter values of the LSV maps. Unfortunately, we currently do not see a possible way to eliminate this constraint. We refer the reader to Remark 10 for further discussion on this issue.

Convention. Throughout this work, C and D will denote positive constants whose precise values may change from one appearance to the next. Dependence on parameters such as α will be indicated with subscripts, e.g. $C = C_\alpha$, as needed.

2. *Quenched central limit theorem for random LSV maps*

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is an arbitrary probability space equipped with an invertible $\mathbb{P}$-preserving transformation $\sigma\colon \Omega \to \Omega$. We assume that $\mathbb{P}$ is ergodic. Moreover, m will denote the Lebesgue measure on the unit interval $[0, 1]$. We recall that for each $\gamma \in (0, 1)$, the associated Liverani–Saussol–Vaienti (LSV) map (introduced in [31]) is given by

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma), & 0 \le x < \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Moreover, for $\gamma \in (0, 1)$ and an observable ϕ, we set

$$N_\gamma \phi := g_\gamma' \phi \circ g_\gamma,$$

where g_γ denotes the inverse of the restriction of T_γ to $[0, \frac{1}{2}]$. Finally, we will also denote $v_\gamma(x) := \partial_\gamma T_\gamma(x) = 2^\gamma x^{1+\gamma} \log(2x)$ and

$$X_\gamma := v_\gamma \circ g_\gamma = 2^\gamma g_\gamma^{1+\gamma} \log(2g_\gamma).$$

For each $\omega \in \Omega$, let $T_\omega\colon [0, 1] \to [0, 1]$ be an LSV map with the parameter $\beta(\omega) \in (0, 1)$. As usual, we set

$$T_\omega^n := T_{\sigma^{n-1}\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega, \quad \omega \in \Omega,\ n \in \mathbb{N}.$$

In the same manner, denoting by $\mathcal{L}_\omega\colon L^1(m) \to L^1(m)$ the transfer operator associated to T_ω and m, we set

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega, \quad \omega \in \Omega,\ n \in \mathbb{N}.$$

We also set $\mathcal{L}_\omega^0 := \mathrm{Id}$ for $\omega \in \Omega$. In the following, we will assume that $\omega \mapsto \beta(\omega)$ is measurable and that

$$\alpha := \operatorname{esssup}_{\omega\in\Omega} \beta(\omega) < 1.$$

It is proved in [17, Proposition 9] that the cocycle of maps $(T_\omega)_{\omega\in\Omega}$ admits a unique random absolutely continuous invariant measure (a.c.i.m.) μ on $\Omega \times [0, 1]$, which can be identified with a family of probability measures $(\mu_\omega)_{\omega\in\Omega}$ on $[0, 1]$ such that

$$T_\omega^* \mu_\omega = \mu_{\sigma\omega} \quad \text{for } \mathbb{P}\text{-almost every (a.e.) } \omega \in \Omega.$$

Moreover, $d\mu_\omega = h_\omega\, dm$ with $h_\omega \in \mathcal{C}_* \cap \mathcal{C}_2$ for some $a > 1$, where $\mathcal{C}_* = \mathcal{C}_*(a)$ and $\mathcal{C}_2 = \mathcal{C}_2(b_1, b_2)$ are cones as in [17, §2.2], that is, $\mathcal{C}_*$ consist of $\phi \in C^0(0, 1] \cap L^1(m)$ such that $\phi \ge 0$, ϕ decreasing, $X^{\alpha+1}\phi$ increasing (where X denotes the identity map), and

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$$\int_0^x \phi(t)\,dt \le ax^{1-\alpha}\int_0^1 \phi(t)\,dt, \quad x \in (0,1].$$

Moreover, $\mathcal{C}_2$ consists of all $\phi \in C^2(0,1]$ so that

$$\phi(x) \ge 0, \quad |\phi'(x)| \le \frac{b_1}{x}\phi(x) \quad \text{and} \quad |\phi''(x)| \le \frac{b_2}{x^2}\phi(x), \quad x \in (0,1].$$

We stress that the parameters a, b_1, and b_2 depend only on α. Throughout the paper, we will often use the following simple result.

If $h \in \mathcal{C}_*$ and $\varphi \in C^1[0,1]$, then by [30, Lemma 3.3] (see also [33, Remark 2.5] and [32, Lemma 3.4]),

$$\varphi h = g_1 - g_2 \tag{1}$$

for $g_1, g_2 \in \mathcal{C}_*$ with $\|g_i\|_{L^1(m)} \le C_\alpha \|\varphi\|_{C^1}\|h\|_{L^1(m)}$ for $i = 1, 2$. Moreover, if $h \in C(0,1]$, then $g_i \in C^1(0,1]$.

PROPOSITION 1. *There exists $C_\alpha > 0$ such that*

$$\|\mathcal{L}_\omega^n(\varphi h)\|_{L^1(m)} \le C_\alpha n^{-1/\alpha+1}\|\varphi\|_{C^1} \cdot \|h\|_{L^1(m)} \tag{2}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \in \mathbb{N}$, and $\varphi \in C^1[0,1]$, $h \in \mathcal{C}_*$ with $\int_0^1 \varphi h\,dm = 0$.

Proof. The result follows by combining (A.1) with (1). □

PROPOSITION 2. *Suppose that $\psi \in C^1(0,1]$ satisfies $\int_0^1 \psi\,dm = 0$ and, for $x \in (0,1]$,*

$$|\psi(x)| \le C_0 x^{-\gamma} \quad \text{and} \quad |\psi'(x)| \le C_1 x^{-\gamma-1},$$

where $0 < \gamma \le \alpha$ and $C_0, C_1 > 0$. Then, there exists $C = C_{a,\alpha,\gamma} > 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^n(\psi)\|_{L^1(m)} \le C(C_0 + C_1)n^{-1/\alpha+\gamma/\alpha} \tag{3}$$

holds for $n \in \mathbb{N}$. In particular, if $\varphi \in C^1[0,1]$ satisfies $\int_0^1 \varphi h_\omega\,dm = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, and

$$|\varphi(x)| \le C_\varphi x^\gamma \quad \text{for all } x \in [0,1],$$

where $\gamma \ge 0$, then there exists a constant $C = C_{a,\alpha,\gamma} > 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^n(\varphi h_\omega)\|_{L^1(m)} \le C(C_\varphi + \|\varphi\|_{C^1})n^{1-(1+\min\{\gamma,\alpha\})/\alpha} \tag{4}$$

holds for $n \in \mathbb{N}$.

Proof. Denote by $\mathcal{C}_*(a;\gamma)$ the cone obtained by replacing α with γ in the definition of $\mathcal{C}_*$. It follows from [17, Lemma 3] or [29, Lemma 3.3] that (note that although $F \in C^2(0,1]$ is assumed in [17, Lemma 3], the proof only uses $F \in C^1(0,1]$)

$$\psi(x) = \psi(x) + \lambda x^{-\gamma} - \lambda x^{-\gamma} = \psi_1 - \psi_2,$$

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where $\psi_i \in \mathcal{C}_*(a;\gamma)$, provided that

$$\lambda \geq (C_0 + C_1)\max\left\{2, \frac{1}{\gamma}, \frac{1}{a-1}\right\}.$$

If $\int_0^1 \psi_1\, dm = 0$, then $\psi_1(x) = 0$ for m-a.e. $x \in [0,1]$ and (3) holds trivially. Otherwise, let $\tilde{\nu}_i$ denote the probability measure with density $\tilde{\psi}_i = \psi_i / \int_0^1 \psi_i\, dm \in \mathcal{C}_*(a;\gamma)$. Then, $\tilde{\nu}_i$ is regular in the sense of [26, Definition 3.5]; see [26, Proposition 3.14]. Moreover, the first entry time

$$\tau_\omega(x) = \inf\{n \geq 1 \,:\, T_\omega^n(x) \in [1/2, 1]\}$$

satisfies

$$\tilde{\nu}_i(\tau_\omega \geq n) \leq C_{a,\gamma,\alpha} n^{-1/\alpha+\gamma/\alpha}.$$

In the terminology of [26], this means that $\tilde{\nu}_i$ has tail bound $C_{a,\gamma,\alpha} n^{-1/\alpha+\gamma/\alpha}$. Note that

$$\|\mathcal{L}_\omega^n(\psi)\|_{L^1(m)} = \frac{\lambda}{1-\gamma}\|\mathcal{L}_\omega^n(\tilde{\psi}_1 - \tilde{\psi}_2)\|_{L^1(m)}.$$

Now, (3) follows from [26, Theorem 3.8], and (4) is a straightforward consequence of (3) using $h_\omega \in \mathcal{C}_* \cap \mathcal{C}_2$. □

We are now in a position to state and prove our quenched central limit theorem for random compositions of LSV maps.

THEOREM 3. *Let $\Phi\colon \Omega \times [0,1] \to \mathbb{R}$ be a measurable map such that $\varphi_\omega := \Phi(\omega, \cdot) \in C^1[0,1]$ for $\omega \in \Omega$ and that*

$$\omega \mapsto \|\varphi_\omega\|_{C^1} \in L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{5}$$

Moreover, assume that one of the following two conditions holds:

(i) $\alpha < 1/2$;

(ii) $\alpha \geq 1/2$ *and, for $x \in [0,1]$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$|\varphi_\omega(x)| \leq C_\omega x^\gamma \quad \text{and} \quad \int_0^1 \varphi_\omega h_\omega\, dm = 0 \tag{6}$$

hold with $\gamma > 2\alpha - 1$, where $\omega \mapsto C_\omega \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\frac{\sum_{i=0}^{n-1}(\varphi_{\sigma^i\omega} \circ T_\omega^i - \int_0^1 \varphi_{\sigma^i\omega}\, d\mu_{\sigma^i\omega})}{\sqrt{n}} \to_d \mathcal{N}(0, \Sigma^2) \quad \text{on } ([0,1], \mu_\omega),$$

where

$$\Sigma^2 := \int_\Omega \int_0^1 \varphi_{\omega,0}^2 h_\omega\, dm\, d\mathbb{P}(\omega) + 2\sum_{n=1}^\infty \int_\Omega \int_0^1 (\varphi_{\omega,0} h_\omega) \cdot (\varphi_{\sigma^n\omega,0} \circ T_\omega^n)\, dm\, d\mathbb{P}(\omega),$$

and $\varphi_{\omega,0} := \varphi_\omega - \int_0^1 \varphi_\omega h_\omega\, dm$. Here, $\mathcal{N}(0, \Sigma^2)$ denotes the normal distribution with mean 0 and variance Σ^2 provided that $\Sigma^2 > 0$, and the unit mass in 0 in the case $\Sigma^2 = 0$.

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Proof. First, we assume that condition (i) holds. We will verify the assumptions of [25, Theorem 2.3.] with $Q = \Omega$ (that is, when there is no inducing involved) and with the fiberwise centered observable. Note that $\varphi_{\omega,0} \in C^1[0,1]$ for $\omega \in \Omega$. Since

$$\left|\int_0^1 \varphi_\omega \, d\mu_\omega\right| = \left|\int_0^1 \varphi_\omega h_\omega \, dm\right| \le \int_0^1 |\varphi_\omega| h_\omega \, dm \le \|\varphi_\omega\|_{C^1} \int_0^1 h_\omega \, dm = \|\varphi_\omega\|_{C^1},$$

we have that $\|\varphi_{\omega,0}\|_{C^1} \le 2\|\varphi_\omega\|_{C^1}$, and thus from (5), we get that

$$\omega \mapsto \|\varphi_{\omega,0}\|_{C^1} \in L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{7}$$

Observing that $(\mathbb{E}_{\mu_\omega}|\varphi_{\omega,0}|^2)^{1/2} \le \|\varphi_{\omega,0}\|_{C^1}$, we see that (7) implies [25, (2.7)].

Next, we aim to verify that [25, Theorem 2.3(i)] is true. Using (A.2), we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} |\mathbb{E}_{\mu_\omega}(\varphi_{\omega,0} \cdot (\varphi_{\sigma^n\omega,0} \circ T_\omega^n))| &= \sum_{n=1}^{\infty} \left|\int_0^1 \varphi_{\omega,0} h_\omega (\varphi_{\sigma^n\omega,0} \circ T_\omega^n) \, dm\right| \\
&= \sum_{n=1}^{\infty} \left|\int_0^1 \varphi_{\sigma^n\omega,0} \mathcal{L}_\omega^n(\varphi_{\omega,0} h_\omega) \, dm\right| \\
&\le \sum_{n=1}^{\infty} \|\mathcal{L}_\omega^n(\varphi_{\omega,0} h_\omega)\|_{L^1(m)} \cdot \|\varphi_{\sigma^n\omega,0}\|_{C^1} \\
&\le C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} \|\varphi_{\omega,0}\|_{C^1} \cdot \|h_\omega\|_{L^1(m)} \cdot \|\varphi_{\sigma^n\omega,0}\|_{C^1} \\
&= C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} \|\varphi_{\omega,0}\|_{C^1} \cdot \|\varphi_{\sigma^n\omega,0}\|_{C^1}.
\end{aligned} \tag{8}$$

Consequently, using Hölder's inequality and that σ preserves $\mathbb{P}$, we have that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}} \sum_{n=1}^{\infty} |\mathbb{E}_{\mu_\omega}(\varphi_{\omega,0} \cdot (\varphi_{\sigma^n\omega,0} \circ T_\omega^n))| \\
&\le C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} \mathbb{E}_{\mathbb{P}}(\|\varphi_{\omega,0}\|_{C^1} \cdot \|\varphi_{\sigma^n\omega,0}\|_{C^1}) \\
&\le C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} (\mathbb{E}_{\mathbb{P}}(\|\varphi_{\omega,0}\|_{C^1}^2))^{1/2} \cdot (\mathbb{E}_{\mathbb{P}}(\|\varphi_{\sigma^n\omega,0}\|_{C^1}^2))^{1/2} \\
&\le C_\alpha \mathbb{E}_{\mathbb{P}}(\|\varphi_\omega\|_{C^1}^2) \sum_{n=1}^{\infty} n^{-1/\alpha+1},
\end{aligned}$$

which is finite due to (7) and $\alpha < 1/2$.

It remains to verify [25, Theorem 2.3(ii')]. More precisely, we need to show that

$$\sum_{n=0}^{\infty} \mathbb{E}_{\mu_\omega} |L_{\sigma^{-n}\omega}^n \varphi_{\sigma^{-n}\omega,0}| \in L^2(\Omega, \mathcal{F}, \mathbb{P}), \tag{9}$$

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where

$$L_\omega(\psi) := \frac{\mathcal{L}_\omega(\psi h_\omega)}{h_{\sigma\omega}} \quad \text{and} \quad L_\omega^n := L_{\sigma^{n-1}\omega} \circ \cdots \circ L_{\sigma\omega} \circ L_\omega,$$

for $\omega \in \Omega$ and $n \in \mathbb{N}$. In addition, $L_\omega^0 := \mathrm{Id}$. Note that

$$L_\omega^n(\psi) = \frac{\mathcal{L}_\omega^n(\psi h_\omega)}{h_{\sigma^n\omega}}$$

for $\omega \in \Omega$ and $n \in \mathbb{N}$. Then, by relying on (A.2) again, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{E}_{\mu_\omega} |L_{\sigma^{-n}\omega}^n \varphi_{\sigma^{-n}\omega,0}| &= \sum_{n=1}^{\infty} \|L_{\sigma^{-n}\omega}^n \varphi_{\sigma^{-n}\omega}\|_{L^1(\mu_\omega)} \\ &= \sum_{n=1}^{\infty} \|\mathcal{L}_{\sigma^{-n}\omega}^n(\varphi_{\sigma^{-n}\omega,0} h_{\sigma^{-n}\omega})\|_{L^1(m)} \\ &\le C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} \|\varphi_{\sigma^{-n}\omega,0} h_{\sigma^{-n}\omega}\|_{L^1(m)} \\ &\le C_\alpha \sum_{n=1}^{\infty} n^{-1/\alpha+1} \|\varphi_{\sigma^{-n}\omega,0}\|_{C^1}. \end{aligned} \tag{10}$$

Therefore, using the fact that σ preserves $\mathbb{P}$, we have

$$\Big\| \sum_{n=1}^{\infty} \mathbb{E}_{\mu_\omega} |L_{\sigma^{-n}\omega}^n \varphi_{\sigma^{-n}\omega,0}| \Big\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} \le C_\alpha \| \|\varphi_\omega\|_{C^1} \|_{L^2(\Omega,\mathcal{F},\mathbb{P})} \sum_{n=1}^{\infty} n^{-1/\alpha+1},$$

which is finite due to (7) and $\alpha < \frac{1}{2}$. The proof of condition (i) is complete.

The proof in the case of condition (ii) is similar to that of condition (i) and proceeds via an application of [**25**, Theorem 2.3] relying on (4). To avoid repetition, we only give an outline of the proof. Note that $\varphi_{\omega,0} = \varphi_\omega$. Assuming condition (ii), we obtain [**25**, (2.7)] using (5) since $(\mathbb{E}_{\mu_\omega} |\varphi_\omega|^2)^{1/2} \le \|\varphi_\omega\|_{C^1}$. Arguing as in (8), using (4) and (6), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} |\mathbb{E}_{\mu_\omega}(\varphi_\omega \cdot (\varphi_{\sigma^n\omega} \circ T_\omega^n))| &\le \sum_{n=1}^{\infty} \|\mathcal{L}_\omega^n(\varphi_\omega h_\omega)\|_{L^1(m)} \cdot \|\varphi_{\sigma^n\omega}\|_{C^1} \\ &\le C \sum_{n=1}^{\infty} (C_\omega + \|\varphi_\omega\|_{C^1}) n^{1-(1+\min\{\alpha,\gamma\})/\alpha} \cdot C_{\sigma^n\omega}\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, where $C > 0$ depends only on a, α, γ. Consequently,

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}} \sum_{n=1}^{\infty} |\mathbb{E}_{\mu_\omega}(\varphi_\omega \cdot (\varphi_{\sigma^n\omega} \circ T_\omega^n))| \\ &\le C \sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{P}}[(C_\omega + \|\varphi_\omega\|_{C^1}) C_{\sigma^n\omega}] n^{1-(1+\min\{\alpha,\gamma\})/\alpha} \\ &\le C\{((\mathbb{E}_{\mathbb{P}}(C_\omega^2))^{1/2} + (\mathbb{E}_{\mathbb{P}}(\|\varphi_\omega\|_{C^1}^2))^{1/2})(\mathbb{E}_{\mathbb{P}}(C_\omega^2))^{1/2}\} \sum_{n=1}^{\infty} n^{1-(1+\min\{\alpha,\gamma\})/\alpha} < \infty,\end{aligned}$$

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since $\alpha < (1+\gamma)/2$. Hence, [25, Theorem 2.3(i)] holds. Finally, [25, Theorem 2.3(ii')] follows by a computation similar to (10), but using (4) instead of (A.2) to obtain

$$\sum_{n=1}^{\infty} \mathbb{E}_{\mu_\omega} |L^n_{\sigma^{-n}\omega} \varphi_{\sigma^{-n}\omega}| \le C \sum_{n=1}^{\infty} (C_{\sigma^{-n}\omega} + \|\varphi_{\sigma^{-n}\omega}\|_{C^1}) n^{1-(1+\min\{\alpha,\gamma\})/\alpha}. \qquad \square$$

Remark 4.

(1) In the case where (Ω, σ) is a full shift, $\mathbb{P}$ is an ergodic invariant measure for σ, and φ_ω are Lipschitz continuous with Lipschitz constants uniformly bounded in ω, Theorem 3(i) was established in [36, Corollary 3.8] (extending the previous results [32, 33]).

(2) The proof of Theorem 3 yields a stronger conclusion, namely that the quenched law of iterated logarithm holds, provided that $\Sigma^2 > 0$ (see the statement of [25, Theorem 2.3]).

(3) To the best of our knowledge, Theorem 3(ii) is new in the setting of random dynamics. For a single LSV map T_α with parameter $1/2 < \alpha < 1$ and absolutely continuous invariant probability measure μ, Gouëzel [22] proved that $n^{-1/2} \sum_{j=0}^{n-1} \varphi \circ T_\alpha^n$ converges weakly to a normal distribution on $([0,1], \mu)$, provided that φ is Hölder continuous with $\int \varphi \, d\mu = 0$ and $|\varphi(x)| \le C x^\gamma$ holds for $\gamma > \alpha - 1/2$. A corresponding weak invariance principle was proved by Dedecker and Merlevède [13].

(4) We present a simple class of examples of observables Φ satisfying (6). To this end, take C^1 functions $u, g\colon [0,1] \to \mathbb{R}$ such that

$$0 \le u(x) \le K x^\gamma, \quad x \in [0,1],$$

where $\gamma > 2\alpha - 1$ and u is not identically zero. Define now $\Phi\colon \Omega \times [0,1] \to \mathbb{R}$ by

$$\varphi_\omega = \Phi(\omega, \cdot) := u(g - c_\omega), \quad \omega \in \Omega,$$

where

$$c_\omega := \frac{\int_0^1 gu \, d\mu_\omega}{\int_0^1 u \, d\mu_\omega}, \quad \omega \in \Omega.$$

We note that c_ω is well defined for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, we recall that it was established in the proof of [17, Proposition 9] that there is $\rho > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $h_\omega \ge \rho$ m-almost everywhere. Hence,

$$\int_0^1 u \, d\mu_\omega \ge \rho \int_0^1 u \, dm > 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

In addition,

$$|c_\omega| \le \frac{|\int_0^1 gu \, d\mu_\omega|}{\int_0^1 u \, d\mu_\omega} \le \frac{\|g\|_\infty \int_0^1 u \, d\mu_\omega}{\int_0^1 u \, d\mu_\omega} = \|g\|_\infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Therefore,

$$|\varphi_\omega(x)| = |u(x)| \cdot |g(x) - c_\omega| \le 2K x^\gamma \|g\|_\infty \quad \text{for } x \in [0,1] \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

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Thus, the first requirement in (6) holds with $C_\omega := 2K\|g\|_\infty$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Clearly, the second requirement in (6) is also fulfilled.

3. *Continuity of the variance*

We are interested in studying the regularity of the variance Σ^2 in Theorem 3 under suitable perturbations of random dynamics.

Let us start by introducing the appropriate setup. We take $\varepsilon_0 > 0$, and measurable maps $\beta\colon \Omega \to (0, 1)$ and $\delta\colon \Omega \to (0, 1)$ such that

$$\alpha := \operatorname{esssup}_\omega \beta(\omega) + \varepsilon_0 < \tfrac{1}{2} \quad \text{and} \quad \operatorname{essinf}_\omega \beta(\omega) - \varepsilon_0 > 0.$$

For $\omega \in \Omega$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, let $T_{\omega,\varepsilon}$ be an LSV map with parameter $\beta(\omega) + \varepsilon\delta(\omega)$. We will write T_ω instead of $T_{\omega,0}$. By $\mathcal{L}_{\omega,\varepsilon}$, we will denote the transfer operator associated with $T_{\omega,\varepsilon}$, again writing $\mathcal{L}_\omega$ instead of $\mathcal{L}_{\omega,0}$. By $(\mu_{\omega,\varepsilon})_{\omega\in\Omega}$, we will denote the random a.c.i.m. associated with the cocycle $(T_{\omega,\varepsilon})_{\omega\in\Omega}$. We will write μ_ω instead of $\mu_{\omega,0}$. By $h_{\omega,\varepsilon} \in \mathcal{C}_* \cap \mathcal{C}_2$, we denote the density of $\mu_{\omega,\varepsilon}$ (again writing h_ω instead of $h_{\omega,0}$).

Let $F\colon \Omega \times [0, 1] \to \mathbb{R}$ be a measurable map such that $F(\omega, \cdot) \in C^1[0, 1]$ for $\omega \in \Omega$ and

$$\operatorname{esssup}_{\omega\in\Omega} \|F(\omega, \cdot)\|_{C^1} < +\infty. \tag{11}$$

In the following, we will denote $F_\omega := F(\omega, \cdot)$. Set

$$f_{\omega,\varepsilon} := F_\omega - \int_0^1 F_\omega \, d\mu_{\omega,\varepsilon}, \quad \omega \in \Omega,\ \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Moreover, let

$$\Sigma_\varepsilon^2 := \int_\Omega \int_0^1 f_{\omega,\varepsilon}^2 \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) + 2\sum_{n=1}^{\infty} \int_\Omega \int_0^1 f_{\omega,\varepsilon} \cdot (f_{\sigma^n\omega,\varepsilon} \circ T_{\omega,\varepsilon}^n) \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega). \tag{12}$$

By Theorem 3, Σ_ε^2 is finite for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then, we have the following theorem.

THEOREM 5. *The map $\varepsilon \mapsto \Sigma_\varepsilon^2$ is continuous in* 0.

Proof. We will first show that

$$\lim_{\varepsilon\to 0} \int_\Omega \int_0^1 f_{\omega,\varepsilon}^2 \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) = \int_\Omega \int_0^1 f_{\omega,0}^2 \, d\mu_\omega \, d\mathbb{P}(\omega). \tag{13}$$

To this end, observe that

$$\begin{aligned}
&\int_\Omega \int_0^1 f_{\omega,\varepsilon}^2 \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) - \int_\Omega \int_0^1 f_{\omega,0}^2 \, d\mu_\omega \, d\mathbb{P}(\omega) \\
&\quad = \int_\Omega \int_0^1 (f_{\omega,\varepsilon}^2 - f_{\omega,0}^2) \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) \\
&\qquad + \int_\Omega \int_0^1 f_{\omega,0}^2 (h_{\omega,\varepsilon} - h_\omega) \, dm \, d\mathbb{P}(\omega).
\end{aligned}$$

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Observe that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$f_{\omega,\varepsilon} - f_{\omega,0} = \int_0^1 F_\omega(h_\omega - h_{\omega,\varepsilon})\, dm, \tag{14}$$

and consequently,

$$|f_{\omega,\varepsilon} - f_{\omega,0}| \le \|F_\omega\|_{C^0} \|h_{\omega,\varepsilon} - h_{\omega,0}\|_{L^1(m)} \le C_{\alpha,F}|\varepsilon|,$$

where in the second inequality, we used Proposition A.3 and (11). However,

$$\|f_{\omega,\varepsilon} + f_{\omega,0}\|_{C^0} \le 4\|F_\omega\|_{C^0}.$$

Consequently,

$$\|f_{\omega,\varepsilon}^2 - f_{\omega,0}^2\|_{C^0} \le C_{\alpha,F}|\varepsilon|$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. This implies that

$$\left| \int_\Omega \int_0^1 (f_{\omega,\varepsilon}^2 - f_{\omega,0}^2)\, d\mu_{\omega,\varepsilon}\, d\mathbb{P}(\omega) \right| \le \int_\Omega \|f_{\omega,\varepsilon}^2 - f_{\omega,0}^2\|_{C^0}\, d\mathbb{P}(\omega) \le C_{\alpha,F}|\varepsilon|,$$

yielding

$$\lim_{\varepsilon \to 0} \int_\Omega \int_0^1 (f_{\omega,\varepsilon}^2 - f_{\omega,0}^2)\, d\mu_{\omega,\varepsilon}\, d\mathbb{P}(\omega) = 0.$$

Similarly, $|f_{\omega,0}^2| \le 4\|F_\omega\|_{C^0}^2$ and thus, using Proposition A.3 and (11), we have

$$\lim_{\varepsilon \to 0} \int_\Omega \int_0^1 f_{\omega,0}^2 (h_{\omega,\varepsilon} - h_\omega)\, dm\, d\mathbb{P}(\omega) = 0,$$

since

$$\begin{aligned} &\left| \int_\Omega \int_0^1 f_{\omega,0}^2 (h_{\omega,\varepsilon} - h_\omega)\, dm\, d\mathbb{P}(\omega) \right| \\ &\quad \le \int_\Omega \|f_{\omega,0}^2\|_{C^0} \cdot \|h_{\omega,\varepsilon} - h_\omega\|_{L^1(m)}\, d\mathbb{P}(\omega) \le C_{\alpha,F}|\varepsilon|. \end{aligned}$$

Therefore, we conclude that (13) holds.

Next, we observe that

$$\begin{aligned} &\left| \int_\Omega \int_0^1 f_{\omega,\varepsilon} \cdot (f_{\sigma^n\omega,\varepsilon} \circ T_{\omega,\varepsilon}^n)\, d\mu_{\omega,\varepsilon}\, d\mathbb{P}(\omega) \right| \\ &\quad = \left| \int_\Omega \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n (f_{\omega,\varepsilon} h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) \right| \\ &\quad \le \int_\Omega \|\mathcal{L}_{\omega,\varepsilon}^n (f_{\omega,\varepsilon} h_{\omega,\varepsilon})\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega) \\ &\quad \le C_\alpha n^{-1/\alpha+1} (\operatorname{esssup} \|F_\omega\|_{C^1})^2, \end{aligned}$$

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where we used (A.2) and (11). Since the series $\sum_{n=1}^{\infty} n^{-1/\alpha+1}$ is summable, to prove our theorem, it remains to show that for each $n \in \mathbb{N}$, the map

$$\begin{aligned}\varepsilon \mapsto &\int_\Omega \int_0^1 f_{\omega,\varepsilon} \cdot (f_{\sigma^n\omega,\varepsilon} \circ T^n_{\omega,\varepsilon})\, d\mu_{\omega,\varepsilon}\, d\mathbb{P}(\omega)\\ &= \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\end{aligned}$$

is continuous in 0. To this end, we first observe that

$$\begin{aligned}&\int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) - \int_\Omega \int_0^1 \mathcal{L}^n_{\omega}(f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega)\\ &\quad= \int_\Omega \int_0^1 \mathcal{L}^n_{\omega}(f_{\omega,0} h_{\omega})(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0})\, dm\, d\mathbb{P}(\omega)\\ &\qquad+ \int_\Omega \int_0^1 (\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega})(f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\\ &\qquad+ \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\\ &\quad= \int_\Omega \int_0^1 (\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_{\omega})(f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\\ &\qquad+ \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega),\end{aligned}$$

since $f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0}$ depends only on ε and ω (see (14)) and

$$\int_0^1 \mathcal{L}^n_{\omega}(f_{\omega,0} h_{\omega})\, dm = \int_0^1 f_{\omega,0} h_{\omega}\, dm = 0.$$

Next,

$$\begin{aligned}&\left|\int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\right|\\ &\quad\le \int_\Omega \|\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega})\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega)\\ &\quad\le \int_\Omega \|f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega)\\ &\quad\le 2\, \operatorname{esssup}_{\omega\in\Omega} \|F_\omega\|_{C^0} \int_\Omega \|f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}\|_{L^1(m)}\, d\mathbb{P}(\omega).\end{aligned}$$

Moreover,

$$\begin{aligned}\|f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}\|_{L^1(m)} &\le \|f_{\omega,\varepsilon}\|_{C^0} \|h_{\omega,\varepsilon} - h_{\omega}\|_{L^1(m)} + |f_{\omega,\varepsilon} - f_{\omega,0}|\\ &\le C_{\alpha,F}|\varepsilon|,\end{aligned}$$

where we used Proposition A.3. This together with (11) implies that

$$\lim_{\varepsilon\to 0} \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_{\omega}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) = 0.$$

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Finally, we have that

$$\int_\Omega \int_0^1 (\mathcal{L}_{\omega,\varepsilon}^n - \mathcal{L}_\omega^n)(f_{\omega,0}h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)$$
$$= \sum_{j=1}^n \int_\Omega \int_0^1 \mathcal{L}_{\sigma^j\omega,\varepsilon}^{n-j}(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega).$$

Therefore,

$$\left|\int_\Omega \int_0^1 (\mathcal{L}_{\omega,\varepsilon}^n - \mathcal{L}_\omega^n)(f_{\omega,0}h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)\right|$$
$$\le \sum_{j=1}^n \int_\Omega \|\mathcal{L}_{\sigma^j\omega,\varepsilon}^{n-j}(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega)$$
$$\le \sum_{j=1}^n \int_\Omega \|(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega)$$
$$\le 2(\operatorname{esssup}_{\omega\in\Omega} \|F_\omega\|_{C^0}) \sum_{j=1}^n \int_\Omega \|(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)\|_{L^1(m)}\, d\mathbb{P}(\omega).$$

In addition, we have

$$(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)$$
$$= -\int_{\beta(\sigma^{j-1}\omega)}^{\beta(\sigma^{j-1}\omega)+\varepsilon\delta(\sigma^{j-1}\omega)} (X_\gamma N_\gamma(\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)))'\, d\gamma,$$

since $\partial_\gamma \mathcal{L}_\gamma(\phi) = -(X_\gamma N_\gamma(\phi))'$ (see [9, p. 865]). It follows from $h_\omega \in \mathcal{C}_* \cap \mathcal{C}_2$ and (1) that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $f_{\omega,0}h_\omega = \psi_1 - \psi_2$ for some $\psi_i = \psi_i(\omega) \in \mathcal{C}_* \cap C^1(0, 1]$. Moreover, $\|\psi_i\|_{L^1} \le K$ for $i = 1, 2$, where K does not depend on ω. Hence,

$$\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega) = \mathcal{L}_\omega^{j-1}(\psi_1) - \mathcal{L}_\omega^{j-1}(\psi_2), \quad \mathcal{L}_\omega^{j-1}(\psi_i) \in \mathcal{C}_* \cap C^1(0, 1],\ i = 1, 2.$$

Consequently,

$$\|(X_\gamma N_\gamma(\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)))'\|_{L^1(m)} \le \sum_{i=1,2} \|(X_\gamma N_\gamma(\mathcal{L}_\omega^{j-1}(\psi_i)))'\|_{L^1(m)} \le \tilde{K},$$

where the last inequality is a consequence of bounds on X_γ and X'_γ in [9, (2.3) and (2.4)], the invariance of $\mathcal{C}_* \cap C^1(0, 1]$ with respect to N_γ and $\mathcal{L}_\gamma$ (see [17, Proposition 5]), and the observation (see [17, Remark 1]) that

$$\phi \in \mathcal{C}_* \cap C^1(0, 1] \implies |\phi'(x)| \le (\alpha + 1)x^{-1}\phi(x) \quad \text{for all } x \in (0, 1]. \tag{15}$$

This gives that

$$\int_\Omega \|(\mathcal{L}_{\sigma^{j-1}\omega,\varepsilon} - \mathcal{L}_{\sigma^{j-1}\omega})\mathcal{L}_\omega^{j-1}(f_{\omega,0}h_\omega)\|_{L^1(m)}\, d\mathbb{P}(\omega) \le C|\varepsilon|$$

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for some $C > 0$ independent on ε. Thus,

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^1 (\mathcal{L}^n_{\omega,\varepsilon}-\mathcal{L}^n_\omega)(f_{\omega,0}h_\omega)f_{\sigma^n\omega,\varepsilon}\,dm\,d\mathbb{P}(\omega)=0.$$

This completes the proof of the theorem. □

4. *Differentiability of the variance*

Take $\beta\colon \Omega\to(0,1)$, $\delta\colon \Omega\to[0,1)$ to be measurable maps and $\underline{\alpha}>0$ such that $0<\underline{\alpha}<\alpha<\frac{1}{2}$,

$$0<\underline{\alpha}\le \operatorname{essinf}_{\omega\in\Omega}\beta(\omega)-\varepsilon_0 \quad\text{and}\quad \operatorname{esssup}_{\omega\in\Omega}\beta(\omega)+\varepsilon_0\le\alpha.$$

Let $T_{\omega,\varepsilon}$ be the LSV map with parameter $\beta(\omega)+\varepsilon\delta(\omega)$ for every $\omega\in\Omega$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. By $(\mu_{\omega,\varepsilon})_{\omega\in\Omega}$, we will denote the random a.c.i.m. that corresponds to the cocycle $(T_{\omega,\varepsilon})_{\omega\in\Omega}$, where $h_{\omega,\varepsilon}$ is the density of $\mu_{\omega,\varepsilon}$. As in the previous section, we will drop the subscript ε when $\varepsilon=0$.

We will use the following result, which is a sharper version of [17, Theorem 14]. Its proof is given in Appendix C.

THEOREM 6. *There exists $C=C_{\alpha,\underline{\alpha}}>0$ such that for ε sufficiently close to* 0,

$$\|\varepsilon^{-1}(h_{\omega,\varepsilon}-h_\omega)-\hat{h}_\omega\|_{L^1(m)}\le C|\varepsilon|^{1-2\alpha} \tag{16}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$, where

$$\hat{h}_\omega:=-\sum_{i=0}^{\infty}\delta(\sigma^{-(i+1)}\omega)\mathcal{L}^i_{\sigma^{-i}\omega}(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))'\in L^1(m). \tag{17}$$

Moreover,

$$\operatorname{esssup}_{\omega\in\Omega}\|\hat{h}_\omega\|_{L^1(m)}<+\infty. \tag{18}$$

Remark 7. In Theorem 6, we show for $\mathbb{P}$-a.e. $\omega\in\Omega$, the map $(-\varepsilon_0,\varepsilon_0)\ni\varepsilon\mapsto h_{\omega,\varepsilon}\in L^1(m)$ is differentiable in $\varepsilon=0$. However, in [17, Theorem 14], a weaker statement is proved: for $\mathbb{P}$-a.e. $\omega\in\Omega$ and $\psi\in L^\infty(m)$, the map $\varepsilon\mapsto\int_0^1\psi h_{\omega,\varepsilon}\,dm$ is differentiable in $\varepsilon=0$. For other quenched linear response results for different classes of random dynamics, we refer to [12, 16, 19, 20, 34].

Remark 8. An anonymous referee raised a question about the possibility of obtaining higher order derivatives of the map $\varepsilon\mapsto h_{\omega,\varepsilon}$ in $\varepsilon=0$. We emphasize that this problem has not been addressed in the case of autonomous dynamics yet. However, it is briefly mentioned in [9, p. 859] that the cone approach developed in [9] (used in [17] and the present paper) is likely to yield such higher order regularity. However, it would require the construction of new families of invariant cones. In addition, we mention that if such a result holds, then it would have a somewhat different form than [12, Theorem 3.6]. More precisely, unlike in [12], where the quadratic response holds in the space weaker than that in which we have a linear response, in our setting, we should have both in $L^1(m)$.

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Let $F\colon \Omega \times [0,1] \to \mathbb{R}$ be a measurable map such that $F(\omega, \cdot) \in C^2[0,1]$ for $\omega \in \Omega$ and that

$$\operatorname{esssup}_{\omega\in\Omega} \|F(\omega,\cdot)\|_{C^2} < +\infty. \tag{19}$$

Let $f_{\omega,\varepsilon}$ and Σ_ε^2 be as in the previous section. Then, we have the following theorem.

THEOREM 9. *Let $\alpha < \frac{1}{5}$. Then, the map $\varepsilon \mapsto \Sigma_\varepsilon^2$ is differentiable in* 0.

Remark 10. We note that the proof of Theorem 9 yields

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Sigma_\varepsilon^2|_{\varepsilon=0} \\
&\quad= \int_\Omega \int_0^1 f_{\omega,0}^2 \hat{h}_\omega \, dm \, d\mathbb{P}(\omega) - 2\int_\Omega \left(\int_0^1 F_\omega \hat{h}_\omega \, dm\right) \int_0^1 f_{\omega,0} \, d\mu_\omega \, d\mathbb{P}(\omega) \\
&\qquad+ \sum_{n=1}^\infty \int_\Omega \int_0^1 \mathcal{L}_\omega^n(-L(\omega,F_\omega)h_\omega + f_{\omega,0}\hat{h}_\omega) f_{\sigma^n\omega,0} \, dm \, d\mathbb{P}(\omega) \\
&\qquad+ \sum_{n=1}^\infty \sum_{j=0}^{n-1} \int_\Omega \delta(\sigma^j\omega) \int_0^1 \mathcal{L}_{\sigma^{j+1}\omega}^{n-j-1}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}_\omega^j(f_{\omega,0}h_\omega))) f_{\sigma^n\omega,0} \, dm \, d\mathbb{P}(\omega),
\end{aligned}$$

where $L(\omega, F_\omega) := \int_0^1 F_\omega \hat{h}_\omega \, dm$. The proof of Theorem 9 shows that this formula for the derivative of Σ_ε^2 is well defined under the assumption $\alpha < 1/5$. In particular, Lemma 18 ensures the summability

$$\sum_{n=1}^\infty \sum_{j=0}^{n-1} \int_\Omega \int_0^1 |\mathcal{L}_{\sigma^{j+1}\omega}^{n-j-1}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}_\omega^j(f_{\omega,0}h_\omega)))| \, dm \, d\mathbb{P}(\omega) < \infty. \tag{20}$$

We do not know whether (20) remains valid beyond the range $\alpha < 1/5$.

Remark 11. For a special class of observables as in Theorem 3(ii), we can weaken the constraint on α to $\alpha < 1/4 + \delta$ for suitable $\delta > 0$. See Theorem 21 for a precise statement.

The proof of Theorem 9 will be established in a series of auxiliary results.

4.1. *Differentiability of the first term in (12).*

LEMMA 12. *The map*

$$\varepsilon \to \int_\Omega \int_0^1 f_{\omega,\varepsilon}^2 \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) \tag{21}$$

is differentiable in $\varepsilon = 0$.

Proof. Writing

$$\begin{aligned}
&\varepsilon^{-1}\left(\int_\Omega \int_0^1 f_{\omega,\varepsilon}^2 \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) - \int_\Omega \int_0^1 f_{\omega,0}^2 \, d\mu_\omega \, d\mathbb{P}(\omega)\right) \\
&\quad= \varepsilon^{-1} \int_\Omega \int_0^1 (f_{\omega,\varepsilon}^2 - f_{\omega,0}^2) \, d\mu_{\omega,\varepsilon} \, d\mathbb{P}(\omega) + \varepsilon^{-1} \int_\Omega \int_0^1 f_{\omega,0}^2 (h_{\omega,\varepsilon} - h_\omega) \, dm \, d\mathbb{P}(\omega),
\end{aligned}$$

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it follows from (16) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega\int_0^1 f_{\omega,0}^2(h_{\omega,\varepsilon}-h_{\omega,0})\,dm\,d\mathbb{P}(\omega)=\int_\Omega\int_0^1 f_{\omega,0}^2\hat{h}_\omega\,dm\,d\mathbb{P}(\omega),$$

since

$$\begin{aligned}&\left|\int_\Omega\int_0^1 f_{\omega,0}^2(\varepsilon^{-1}(h_{\omega,\varepsilon}-h_{\omega,0}))\,dm\,d\mathbb{P}(\omega)-\int_\Omega\int_0^1 f_{\omega,0}^2\hat{h}_\omega\,dm\,d\mathbb{P}(\omega)\right|\\&\quad\le\int_\Omega\|f_{\omega,0}^2\|_{C^0}\cdot\|\varepsilon^{-1}(h_{\omega,\varepsilon}-h_{\omega,0})-\hat{h}_\omega\|_{L^1(m)}\,d\mathbb{P}(\omega)\le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha}\end{aligned}$$

for ε sufficiently close to 0. Next, we note that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_\Omega\int_0^1(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)\,d\mu_{\omega,\varepsilon}\,d\mathbb{P}(\omega)-\int_\Omega\int_0^1(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)\,d\mu_\omega\,d\mathbb{P}(\omega)\right)=0,$$

as

$$\begin{aligned}&\left|\int_\Omega\int_0^1(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)\,d\mu_{\omega,\varepsilon}\,d\mathbb{P}(\omega)-\int_\Omega\int_0^1(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)\,d\mu_\omega\,d\mathbb{P}(\omega)\right|\\&\quad\le\int_\Omega\|f_{\omega,\varepsilon}^2-f_{\omega,0}^2\|_{C^0}\|h_{\omega,\varepsilon}-h_\omega\|_{L^1(m)}\,d\mathbb{P}(\omega)\\&\quad\le C_{\alpha,F}|\varepsilon|^2\end{aligned}$$

for some $C_{\alpha,F}>0$, where we used (14) and Proposition A.3. Finally,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega\int_0^1(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)\,d\mu_\omega\,d\mathbb{P}(\omega)=-2\int_\Omega\left(\int_0^1 F_\omega\hat{h}_\omega\,dm\right)\int_0^1 f_{\omega,0}\,d\mu_\omega\,d\mathbb{P}(\omega),$$

since

$$\begin{aligned}&\left\|\varepsilon^{-1}(f_{\omega,\varepsilon}^2-f_{\omega,0}^2)+2\left(\int_0^1 F_\omega\hat{h}_\omega\,dm\right)f_{\omega,0}\right\|_{C^0}\\&\quad\le\left|\varepsilon^{-1}(f_{\omega,\varepsilon}-f_{\omega,0})+\int_0^1 F_\omega\hat{h}_\omega\,dm\right|\cdot\|f_{\omega,\varepsilon}+f_{\omega,0}\|_{C^0}+\left|\int_0^1 F_\omega\hat{h}_\omega\,dm\right|\cdot|f_{\omega,\varepsilon}-f_{\omega,0}|\\&\quad\le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha}+C_{\alpha,F}|\varepsilon|\end{aligned}$$

for ε sufficiently close to 0, where we used (14), Proposition A.3, and Theorem 6. Thus, the map in (21) is differentiable in $\varepsilon=0$. □

4.2. *Differentiability of the second term in* (12). Next, we consider the second term in (12):

$$\begin{aligned}&\sum_{n=1}^\infty\int_\Omega\int_0^1(f_{\sigma^n\omega,\varepsilon}\circ T_{\omega,\varepsilon}^n)f_{\omega,\varepsilon}\,d\mu_{\omega,\varepsilon}\,d\mathbb{P}(\omega)\\&\quad=\sum_{n=1}^\infty\int_\Omega\int_0^1\mathcal{L}_{\omega,\varepsilon}^n(f_{\omega,\varepsilon}h_{\omega,\varepsilon})f_{\sigma^n\omega,\varepsilon}\,dm\,d\mathbb{P}(\omega).\end{aligned}$$

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Let

$$C_n(\varepsilon) := \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega)$$

and

$$D_n(\varepsilon) := \frac{C_n(\varepsilon) - C_n(0)}{\varepsilon}.$$

Observe that

$$|C_n(\varepsilon)| \le \int_\Omega \|\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,\varepsilon} h_{\omega,\varepsilon})\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0}\, d\mathbb{P}(\omega) \le C_{\alpha,F} n^{-1/\alpha+1}.$$

Consequently,

$$|D_n(\varepsilon)| \le C_{\alpha,F} |\varepsilon|^{-1} n^{-1/\alpha+1}.$$

Let us now take $\gamma \in (0, \frac{1}{2})$ such that

$$\gamma(1/\alpha - 2) > 1. \tag{22}$$

We note that this is possible since $\alpha < \frac{1}{4}$. Then, we have

$$\left| \sum_{n \ge |\varepsilon|^{-\gamma}} D_n(\varepsilon) \right| \le C|\varepsilon|^{\gamma(1/\alpha-2)-1},$$

where $C > 0$ is not dependent on ε. Hence, due to (22), we have

$$\lim_{\varepsilon \to 0} \sum_{n \ge |\varepsilon|^{-\gamma}} D_n(\varepsilon) = 0. \tag{23}$$

Next, note that

$$\begin{aligned} D_n(\varepsilon) &= \int_\Omega \int_0^1 \mathcal{L}^n_\omega(f_{\omega,0} h_\omega)(\varepsilon^{-1}(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0}))\, dm\, d\mathbb{P}(\omega) \\ &\quad + \frac{1}{\varepsilon} \int_\Omega \int_0^1 (\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_\omega)(f_{\omega,0} h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) \\ &\quad + \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(\varepsilon^{-1}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_\omega)) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) \\ &= \frac{1}{\varepsilon} \int_\Omega \int_0^1 (\mathcal{L}^n_{\omega,\varepsilon} - \mathcal{L}^n_\omega)(f_{\omega,0} h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) \\ &\quad + \int_\Omega \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(\varepsilon^{-1}(f_{\omega,\varepsilon} h_{\omega,\varepsilon} - f_{\omega,0} h_\omega)) f_{\sigma^n\omega,\varepsilon}\, dm\, d\mathbb{P}(\omega) \\ &=: D^1_n(\varepsilon) + D^2_n(\varepsilon), \end{aligned} \tag{24}$$

since as in the proof of Theorem 5, we have

$$\int_\Omega \int_0^1 \mathcal{L}^n_\omega(f_{\omega,0} h_\omega)(\varepsilon^{-1}(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0}))\, dm\, d\mathbb{P}(\omega) = 0.$$

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We first focus on $D_n^2(\varepsilon)$. Taking into account (14) and

$$\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(f_{\omega,\varepsilon}h_{\omega,\varepsilon} - f_{\omega,0}h_\omega)\,dm = \int_0^1 (f_{\omega,\varepsilon}h_{\omega,\varepsilon} - f_{\omega,0}h_\omega)\,dm = 0,$$

we have

$$D_n^2(\varepsilon) = \int_\Omega \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(\varepsilon^{-1}(f_{\omega,\varepsilon}h_{\omega,\varepsilon} - f_{\omega,0}h_\omega))f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega).$$

Let

$$\bar{D}_n^2(\varepsilon) := \int_\Omega \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(-L(\omega, F_\omega)h_\omega + f_{\omega,0}\hat{h}_\omega)f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega),$$

where $L(\omega, F_\omega) := \int_0^1 F_\omega \hat{h}_\omega\,dm$.

LEMMA 13. *We have*

$$|D_n^2(\varepsilon) - \bar{D}_n^2(\varepsilon)| \le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha}$$

for ε sufficiently close to 0.

Proof. Note that

$$\begin{aligned}
&|D_n^2(\varepsilon) - \bar{D}_n^2(\varepsilon)| \\
&\quad\le \int_\Omega \|\varepsilon^{-1}(f_{\omega,\varepsilon} - f_{\omega,0})h_{\omega,\varepsilon} + L(\omega, F_\omega)h_\omega \\
&\qquad + f_{\omega,0}(\varepsilon^{-1}(h_{\omega,\varepsilon} - h_\omega) - \hat{h}_\omega)\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,0}\|_{C^0}\,d\mathbb{P}(\omega) \\
&\quad\le C_F \int_\Omega \|\varepsilon^{-1}(f_{\omega,\varepsilon} - f_{\omega,0})h_{\omega,\varepsilon} \\
&\qquad + L(\omega, F_\omega)h_\omega + f_{\omega,0}(\varepsilon^{-1}(h_{\omega,\varepsilon} - h_\omega) - \hat{h}_\omega)\|_{L^1(m)}\,d\mathbb{P}(\omega).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|\varepsilon^{-1}(f_{\omega,\varepsilon} - f_{\omega,0})h_{\omega,\varepsilon} + L(\omega, F_\omega)h_\omega\|_{L^1(m)} \\
&\quad\le \|\varepsilon^{-1}(f_{\omega,\varepsilon} - f_{\omega,0})h_{\omega,\varepsilon} + L(\omega, F_\omega)h_{\omega,\varepsilon}\|_{L^1(m)} \\
&\qquad + |L(\omega, F_\omega)|\|h_{\omega,\varepsilon} - h_\omega\|_{L^1(m)} \\
&\quad= |\varepsilon^{-1}(f_{\omega,\varepsilon} - f_{\omega,0}) + L(\omega, F_\omega)| \\
&\qquad + |L(\omega, F_\omega)|\|h_{\omega,\varepsilon} - h_\omega\|_{L^1(m)} \\
&\quad\le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha} + C_{\alpha,F}|\varepsilon| \\
&\quad\le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha}
\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and ε sufficiently close to 0, where we used Proposition A.3 and Theorem 6. Similarly,

$$\begin{aligned}
\|f_{\omega,0}(\varepsilon^{-1}(h_{\omega,\varepsilon} - h_\omega) - \hat{h}_\omega)\|_{L^1(m)} &\le \|f_{\omega,0}\|_{C^0} \cdot \|\varepsilon^{-1}(h_{\omega,\varepsilon} - h_\omega) - \hat{h}_\omega\|_{L^1(m)} \\
&\le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha}
\end{aligned}$$

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for $\mathbb{P}$-a.e. $\omega \in \Omega$ and ε sufficiently close to 0. The conclusion of the lemma follows readily from the above estimates. □

By Lemma 13, we have

$$\sum_{n<|\varepsilon|^{-\gamma}} |D_n^2(\varepsilon) - \bar{D}_n^2(\varepsilon)| \le C_{\alpha,\underline{\alpha},F}|\varepsilon|^{1-2\alpha-\gamma}.$$

Since $1 - 2\alpha > \gamma$, we have

$$\lim_{\varepsilon\to 0} \sum_{n<|\varepsilon|^{-\gamma}} |D_n^2(\varepsilon) - \bar{D}_n^2(\varepsilon)| = 0. \tag{25}$$

Taking into account (23), (24), and (25), to conclude the differentiability of $\varepsilon \mapsto \Sigma_\varepsilon^2$, it remains to establish the following.

- The limits $\lim_{\varepsilon\to 0} D_n^1(\varepsilon)$ and $\lim_{\varepsilon\to 0} \bar{D}_n^2(\varepsilon)$ exist for each $n \ge 1$.
- There exist $\bar{\varepsilon}_0 > 0$ and u_n such that for all $n < |\varepsilon|^{-\gamma}$ and all $\varepsilon \in [-\bar{\varepsilon}_0, \bar{\varepsilon}_0] \setminus \{0\}$,

$$|D_n^1(\varepsilon)| \le u_n, \quad |\bar{D}_n^2(\varepsilon)| \le u_n \quad \text{and} \quad \sum_{n=1}^{\infty} u_n < \infty.$$

The first point above is established in Lemma 17 for $\bar{D}_n^2$ and in Lemma 19 for D_n^1. The second point is proved in Lemma 16 for $\bar{D}_n^2$ and in Lemma 18 for D_n^1.

4.3. *A series of auxiliary lemmas.* We write

$$\begin{aligned}
\bar{D}_n^2(\varepsilon) &= \int_\Omega \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(-L(\omega, F_\omega)h_\omega + f_{\omega,0}\hat{h}_\omega) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\
&= \int_\Omega \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(f_{\omega,0}\hat{h}_\omega) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\
&\quad - \int_\Omega L(\omega, F_\omega) \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(h_\omega) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\
&=: \bar{D}_{n,1}^2(\varepsilon) - \bar{D}_{n,2}^2(\varepsilon).
\end{aligned} \tag{26}$$

LEMMA 14. *We have*

$$|\bar{D}_{n,2}^2(\varepsilon)| \le C_{\alpha,F}(n^{-1/\gamma} + n^{-1/\alpha+1}), \quad n < |\varepsilon|^{-\gamma}.$$

Proof. Note that

$$\|\mathcal{L}_{\omega,\varepsilon}^n h_\omega - h_{\sigma^n\omega,\varepsilon}\|_{L^1(m)} = \|\mathcal{L}_{\omega,\varepsilon}^n(h_\omega - h_{\omega,\varepsilon})\|_{L^1(m)} \le C_\alpha n^{-1/\alpha+1}.$$

Then, using that

$$\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n(h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm = \int_0^1 h_{\sigma^n\omega,\varepsilon} f_{\sigma^n\omega,\varepsilon}\, dm = 0,$$

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we have

$$\begin{aligned}
&\left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(h_\omega) f_{\sigma^n\omega,0}\, dm\right| \\
&\le \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(h_\omega)(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0})\, dm\right| + \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(h_\omega) f_{\sigma^n\omega,\varepsilon}\, dm\right| \\
&= \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(h_\omega)(f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0})\, dm\right| + \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(h_\omega - h_{\omega,\varepsilon}) f_{\sigma^n\omega,\varepsilon}\, dm\right| \\
&\le \|\mathcal{L}^n_{\omega,\varepsilon} h_\omega\|_{L^1(m)} \cdot |f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0}| + \|\mathcal{L}^n_{\omega,\varepsilon}(h_\omega - h_{\omega,\varepsilon})\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0} \\
&\le |f_{\sigma^n\omega,\varepsilon} - f_{\sigma^n\omega,0}| + \|\mathcal{L}^n_{\omega,\varepsilon}(h_\omega - h_{\omega,\varepsilon})\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,\varepsilon}\|_{C^0} \\
&\le C_{\alpha,F}|\varepsilon| + C_{\alpha,F} n^{-1/\alpha+1},
\end{aligned}$$

which yields the desired conclusion. □

LEMMA 15. *We have*

$$|\bar{D}^2_{n,1}(\varepsilon)| \le C_{\alpha,\underline{\alpha},F}(n^{1-1/2\alpha} + n^{-1/\gamma}), \quad n < |\varepsilon|^{-\gamma}.$$

Proof. By using Theorem 6, we have

$$\begin{aligned}
&\left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}\hat{h}_\omega) f_{\sigma^n\omega,0}\, dm\right| \\
&\quad\le \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}(\hat{h}_\omega - r^{-1}(h_{\omega,r} - h_\omega))) f_{\sigma^n\omega,0}\, dm\right| \\
&\qquad + \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0} r^{-1}(h_{\omega,r} - h_\omega)) f_{\sigma^n\omega,0}\, dm\right| \\
&\quad\le \|\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}(\hat{h}_\omega - r^{-1}(h_{\omega,r} - h_\omega)))\|_{L^1(m)} \cdot \|f_{\sigma^n\omega,0}\|_{C^0} \\
&\qquad + \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0} r^{-1}(h_{\omega,r} - h_\omega)) f_{\sigma^n\omega,0}\, dm\right| \\
&\quad\le C_{\alpha,\underline{\alpha},F} r^{1-2\alpha} + \left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0} r^{-1}(h_{\omega,r} - h_\omega)) f_{\sigma^n\omega,0}\, dm\right| \qquad (27)
\end{aligned}$$

for $r \in (0, 1)$ that will be fixed later on. Next,

$$\begin{aligned}
&\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0} r^{-1}(h_{\omega,r} - h_\omega)) f_{\sigma^n\omega,0}\, dm \\
&\quad= \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}\left(r^{-1}\left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\right) f_{\sigma^n\omega,0}\, dm \\
&\qquad + \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}\left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\right) f_{\sigma^n\omega,0}\, dm \\
&\qquad - \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}\left(r^{-1}\left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_\omega\, dm\right) h_\omega\right) f_{\sigma^n\omega,0}\, dm \\
&\qquad - \int_0^1 \mathcal{L}^n_{\omega,\varepsilon}\left(r^{-1}\left(\int_0^1 f_{\omega,0} h_\omega\, dm\right) h_\omega\right) f_{\sigma^n\omega,0}\, dm.
\end{aligned}$$

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Since

$$\int_0^1 \left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\, dm = 0,$$

we have using (A.2) that

$$\begin{aligned}
&\left|\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\right) f_{\sigma^n\omega,0}\, dm\right| \\
&\quad \le C_F r^{-1} \left\| \mathcal{L}_{\omega,\varepsilon}^n \left(\left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\right)\right\|_{L^1(m)} \\
&\quad \le C_{\alpha,F} r^{-1} n^{-1/\alpha+1} \left\| f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right\|_{C^1} \\
&\quad \le C_{\alpha,F} r^{-1} n^{-1/\alpha+1}. \qquad (28)
\end{aligned}$$

In the same manner,

$$\left|\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(f_{\omega,0} - \int_0^1 f_{\omega,0} h_{\omega}\, dm\right) h_{\omega}\right) f_{\sigma^n\omega,0}\, dm\right| \le C_{\alpha,F} r^{-1} n^{-1/\alpha+1}. \qquad (29)$$

Moreover,

$$\begin{aligned}
&\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) h_{\omega,r}\right) f_{\sigma^n\omega,0}\, dm \\
&\quad - \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega}\, dm\right) h_{\omega}\right) f_{\sigma^n\omega,0}\, dm \\
&= \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) (h_{\omega,r} - h_{\omega})\right) f_{\sigma^n\omega,0}\, dm \\
&\quad + \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm - \int_0^1 f_{\omega,0} h_{\omega}\, dm\right) h_{\omega}\right) f_{\sigma^n\omega,0}\, dm.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left|\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right) (h_{\omega,r} - h_{\omega})\right) f_{\sigma^n\omega,0}\, dm\right| \\
&\quad \le C_F r^{-1} \left|\int_0^1 f_{\omega,0} h_{\omega,r}\, dm\right| \cdot \|\mathcal{L}_{\omega,\varepsilon}^n (h_{\omega,r} - h_{\omega})\|_{L^1(m)} \\
&\quad \le C_{\alpha,F} r^{-1} n^{-1/\alpha+1}. \qquad (30)
\end{aligned}$$

In addition

$$\begin{aligned}
&\int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm - \int_0^1 f_{\omega,0} h_{\omega}\, dm\right) h_{\omega}\right) f_{\sigma^n\omega,0}\, dm \\
&= \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n \left(\left(r^{-1}\left(\int_0^1 f_{\omega,0} h_{\omega,r}\, dm - \int_0^1 f_{\omega,0} h_{\omega}\, dm\right) - \int_0^1 f_{\omega,0} \hat{h}_{\omega}\, dm\right) h_{\omega}\right) f_{\sigma^n\omega,0}\, dm \\
&\quad + \int_0^1 f_{\omega,0} \hat{h}_{\omega}\, dm \int_0^1 \mathcal{L}_{\omega,\varepsilon}^n (h_{\omega}) f_{\sigma^n\omega,0}\, dm.
\end{aligned}$$

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By Theorem 6,

$$\begin{aligned}&\left|\int_0^1 \mathcal{L}^n_{\omega,\varepsilon}\left(\left(r^{-1}\left(\int_0^1 f_{\omega,0}h_{\omega,r}\,dm-\int_0^1 f_{\omega,0}h_\omega\,dm\right)-\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\right)h_\omega\right)f_{\sigma^n\omega,0}\,dm\right|\\&\quad\le C_F\left|r^{-1}\left(\int_0^1 f_{\omega,0}h_{\omega,r}\,dm-\int_0^1 f_{\omega,0}h_\omega\,dm\right)-\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\right|\\&\quad\le C_{\alpha,\underline{\alpha},F}r^{1-2\alpha}.\end{aligned}\tag{31}$$

Furthermore,

$$\begin{aligned}&\left|\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1\mathcal{L}^n_{\omega,\varepsilon}(h_\omega)f_{\sigma^n\omega,0}\,dm-\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1 h_{\sigma^n\omega,\varepsilon}f_{\sigma^n\omega,0}\,dm\right|\\&\quad\le C_F\|\mathcal{L}^n_{\omega,\varepsilon}(h_\omega-h_{\omega,\varepsilon})\|_{L^1(m)}\\&\quad\le C_{\alpha,F}n^{-1/\alpha+1},\end{aligned}\tag{32}$$

and thus (taking into account (27)–(32)),

$$\begin{aligned}&\left|\bar{D}^2_{n,1}(\varepsilon)-\int_\Omega\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1 h_{\sigma^n\omega,\varepsilon}f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega)\right|\\&\quad\le C_{\alpha,\underline{\alpha},F}(r^{1-2\alpha}+r^{-1}n^{-1/\alpha+1})\end{aligned}$$

for r close to 0. In addition, since $\int_0^1 f_{\omega,\varepsilon}h_{\omega,\varepsilon}\,dm=0$,

$$\begin{aligned}&\int_\Omega\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1 h_{\sigma^n\omega,\varepsilon}f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega)\\&\quad=\int_\Omega\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1 h_{\sigma^n\omega,\varepsilon}(f_{\sigma^n\omega,0}-f_{\sigma^n\omega,\varepsilon})\,dm\,d\mathbb{P}(\omega),\end{aligned}$$

which gives that

$$\left|\int_\Omega\int_0^1 f_{\omega,0}\hat{h}_\omega\,dm\int_0^1 h_{\sigma^n\omega,\varepsilon}f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega)\right|\le C_{\alpha,F}|\varepsilon|\le C_{\alpha,F}n^{-1/\gamma}$$

for $n<|\varepsilon|^{-\gamma}$. By choosing $r=n^{-(1-\alpha/\alpha(2-2\alpha))}$ (so that $r^{1-2\alpha}=r^{-1}n^{-1/\alpha+1}$), we obtain the desired conclusion. □

As a direct consequence of the previous two lemmas and (26), we obtain the following lemmas.

LEMMA 16. *We have*

$$|\bar{D}^2_n(\varepsilon)|\le C_{\alpha,\underline{\alpha},F}(n^{1-1/2\alpha}+n^{-1/\gamma}+n^{-1/\alpha+1}),\quad n<|\varepsilon|^{-\gamma}.$$

LEMMA 17. *For each $n\in\mathbb{N}$,*

$$\lim_{\varepsilon\to 0}\bar{D}^2_n(\varepsilon)=\bar{D}^2_n(0).$$

Proof. We first show that

$$\lim_{\varepsilon\to 0}\bar{D}^2_{n,1}(\varepsilon)=\bar{D}^2_{n,1}(0).\tag{33}$$

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We start by decomposing

$$\begin{aligned}\hat{h}_\omega &= -\sum_{i=0}^{\infty} \delta(\sigma^{-(i+1)}\omega)\mathcal{L}^i_{\sigma^{-i}\omega}(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))' \\ &= -\sum_{i=0}^{k} \delta(\sigma^{-(i+1)}\omega)\mathcal{L}^i_{\sigma^{-i}\omega}(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))' \\ &\quad -\sum_{i=k+1}^{\infty} \delta(\sigma^{-(i+1)}\omega)\mathcal{L}^i_{\sigma^{-i}\omega}(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))' \\ &=: \hat{h}_{\omega,k} + R_{\omega,k},\end{aligned}$$

where $k \in \mathbb{N}$. It follows from Theorem A.1 and Lemma A.2 that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|R_{\omega,k}\|_{L^1(m)} \le C_\alpha \sum_{i=k+1}^{\infty} i^{-1/\alpha+1} \le C_\alpha k^{-1/\alpha+2} \quad \text{for } k \in \mathbb{N}.$$

Thus, for $k \in \mathbb{N}$,

$$\begin{aligned}|\bar{D}^2_{n,1}(\varepsilon) - \bar{D}^2_{n,1}(0)| \le |\int_\Omega \int_0^1 [\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}\hat{h}_{\omega,k}) - \mathcal{L}^n_{\omega,0}(f_{\omega,0}\hat{h}_{\omega,k})] f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega)| \\ + C_{\alpha,F} k^{-1/\alpha+2}.\end{aligned}$$

By $h_\omega \in \mathcal{C}_* \cap \mathcal{C}_2$ and Lemma A.2,

$$(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))' = \psi_1 - \psi_2,$$

where $\psi_1, \psi_2 \in \mathcal{C}_* \cap C^1(0,1]$. Hence, [**30**, Lemma 3.3] implies that

$$f_{\omega,0}\mathcal{L}^i_{\sigma^{-i}\omega}(X_{\beta(\sigma^{-(i+1)}\omega)}N_{\beta(\sigma^{-(i+1)}\omega)}(h_{\sigma^{-(i+1)}\omega}))' = \tilde{\psi}_{i,1} - \tilde{\psi}_{i,2},$$

where $\tilde{\psi}_{i,j} \in C^1(0,1] \cap \mathcal{C}_*$. Consequently,

$$f_{\omega,0}\hat{h}_{\omega,k} = -\sum_{i=0}^{k}\sum_{j=1,2} \delta(\sigma^{-(i+1)}\omega)(-1)^{j-1}\tilde{\psi}_{i,j}. \tag{34}$$

Substituting (34) into the remaining expression and using $\varphi(x+\varepsilon) - \varphi(x) = \varepsilon \int_0^1 \varphi'(x + t\varepsilon)\, dt$, we obtain

$$\begin{aligned}&\int_\Omega \int_0^1 [\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}\hat{h}_{\omega,k}) - \mathcal{L}^n_{\omega,0}(f_{\omega,0}\hat{h}_{\omega,k})] f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\ &= -\sum_{i=0}^{k}\sum_{j=1,2} \delta(\sigma^{-(i+1)}\omega)(-1)^{j-1} \int_\Omega \int_0^1 [\mathcal{L}^n_{\omega,\varepsilon}(\tilde{\psi}_{i,j}) - \mathcal{L}^n_{\omega,0}(\tilde{\psi}_{i,j})] f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\ &= -\sum_{i=0}^{k}\sum_{j=1,2}\sum_{\ell=0}^{n-1} \delta(\sigma^{-(i+1)}\omega)(-1)^{j-1} \int_\Omega \int_0^1 \mathcal{L}^\ell_{\sigma^{n-\ell}\omega,\varepsilon} \\ &\quad \times [\mathcal{L}_{\beta(\sigma^{n-1-\ell}\omega)+\varepsilon\delta(\sigma^{n-1-\ell}\omega)} - \mathcal{L}_{\beta(\sigma^{n-1-\ell}\omega)}] \\ &\quad \times \mathcal{L}^{n-1-\ell}_\omega(\tilde{\psi}_{i,j}) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega)\end{aligned}$$

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$$= -\sum_{i=0}^{k}\sum_{j=1,2}\sum_{\ell=0}^{n-1}\delta(\sigma^{-(i+1)}\omega)(-1)^{j-1}\varepsilon\delta(\sigma^{n-1-\ell}\omega)$$
$$\times\int_\Omega\int_0^1\int_0^1 \mathcal{L}^{\ell}_{\sigma^{n-\ell}\omega,\varepsilon}\partial_\gamma\mathcal{L}_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}\mathcal{L}^{n-1-\ell}_{\omega}(\tilde{\psi}_{i,j})f_{\sigma^n\omega,0}\,dt\,dm\,d\mathbb{P}(\omega).$$

Moreover,

$$\partial_\gamma\mathcal{L}_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}\mathcal{L}^{n-1-\ell}_{\omega}(\tilde{\psi}_{i,j})$$
$$= (X_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}N_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}(\mathcal{L}^{n-1-\ell}_{\omega}(\tilde{\psi}_{i,j})))'.$$

It follows from (15), bounds on X_γ and X'_γ in [9, (2.3) and (2.4)], and invariance properties of the cones in (A.1) that

$$\|(X_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}N_{\beta(\sigma^{n-1-\ell}\omega)+t\varepsilon\delta(\sigma^{n-1-\ell}\omega)}(\mathcal{L}^{n-1-\ell}_{\omega}(\tilde{\psi}_{i,j})))'\|_{L^1(m)} \le C_{\alpha,\underline{\alpha},F}.$$

Consequently,

$$\left|\int_\Omega\int_0^1[\mathcal{L}^n_{\omega,\varepsilon}(f_{\omega,0}\hat{h}_{\omega,k}) - \mathcal{L}^n_{\omega,0}(f_{\omega,0}\hat{h}_{\omega,k})]f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega)\right| \le C_{\alpha,\underline{\alpha},F}|\varepsilon|kn.$$

We choose $k = \lfloor|\varepsilon|^{-\alpha/(1-\alpha)}\rfloor$ so that $|\varepsilon|k$ and $k^{-1/\alpha+2}$ are approximately equal. Since $\alpha < 1/2$,

$$|\bar{D}^2_{n,1}(\varepsilon) - \bar{D}^2_{n,1}(0)| \le C_{\alpha,\underline{\alpha},F}(|\varepsilon|kn + k^{-1/\alpha+2}) \le C_{\alpha,\underline{\alpha},F,n}|\varepsilon|^{1-\alpha/(1-\alpha)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, (33) holds.

We now claim that

$$\lim_{\varepsilon\to 0}\bar{D}^2_{n,2}(\varepsilon) = \bar{D}_{n,2}(0). \tag{35}$$

Note that

$$\mathcal{L}^n_{\omega,\varepsilon}(h_\omega) - \mathcal{L}^n_{\omega}(h_\omega) = \sum_{j=0}^{n-1}\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon}(\mathcal{L}_{\sigma^j\omega,\varepsilon} - \mathcal{L}_{\sigma^j\omega})\mathcal{L}^j_{\omega}(h_\omega)$$
$$= \sum_{j=0}^{n-1}\int_{\beta(\sigma^j\omega)}^{\beta(\sigma^j\omega)+\varepsilon\delta(\sigma^j\omega)}\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon}[\partial_\gamma\mathcal{L}_\gamma(h_{\sigma^j\omega})]\,d\gamma$$
$$= -\sum_{j=0}^{n-1}\int_{\beta(\sigma^j\omega)}^{\beta(\sigma^j\omega)+\varepsilon\delta(\sigma^j\omega)}\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon}[(X_\gamma N_\gamma(h_{\sigma^j\omega}))']\,d\gamma.$$

By applying Lemma A.2, we conclude (using the convention $0^{-1/\alpha+1} := 1$) that

$$\|\mathcal{L}^n_{\omega,\varepsilon}(h_\omega) - \mathcal{L}^n_{\omega}(h_\omega)\|_{L^1(m)} \le C_\alpha|\varepsilon|\sum_{j=0}^{n-1}(n-j-1)^{-1/\alpha+1} \le C_\alpha|\varepsilon|,$$

which easily implies (35). Finally, it remains to note that the conclusion of the lemma follows readily from (33) and (35). □

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We now consider $D_n^1(\varepsilon)$ and note that

$$D_n^1(\varepsilon) = \frac{1}{\varepsilon}\int_\Omega \int_0^1 (\mathcal{L}_{\omega,\varepsilon}^n - \mathcal{L}_\omega^n)(f_{\omega,0}h_\omega) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega).$$

LEMMA 18. *We have*

$$|D_n^1(\varepsilon)| \le C_{\alpha,\underline{\alpha},F}(n^{1-1/\gamma} + n^{1-(1-\alpha)/(2\alpha)}), \quad n < |\varepsilon|^{-\gamma}. \tag{36}$$

Proof. Observe that

$$\begin{aligned} D_n^1(\varepsilon) &= \int_\Omega \int_0^1 \sum_{j=0}^{n-1} \mathcal{L}_{\sigma^{j+1}\omega,\varepsilon}^{n-j-1}(\varepsilon^{-1}(\mathcal{L}_{\sigma^j\omega,\varepsilon} - \mathcal{L}_{\sigma^j\omega})\mathcal{L}_\omega^j(f_{\omega,0}h_\omega)) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\ &= \int_\Omega \int_0^1 \sum_{j=0}^{n-1} \delta(\sigma^j\omega)\mathcal{L}_{\sigma^{j+1}\omega,\varepsilon}^{n-j-1}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}_\omega^j(f_{\omega,0}h_\omega))) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\ &\quad + \varepsilon \int_\Omega \sum_{j=0}^{n-1} (\delta(\sigma^j\omega))^2 \int_0^1 \int_0^1 (1-t) f_{\sigma^n\omega,0}\mathcal{L}_{\sigma^{j+1}\omega,\varepsilon}^{n-j-1} \\ &\quad \times (\partial_\gamma^2 \mathcal{L}_{\beta(\sigma^j\omega)+t\varepsilon\delta(\sigma^j\omega)}(\mathcal{L}_\omega^j(f_{\omega,0}h_\omega)))\, dm\, dt\, d\mathbb{P}(\omega) \\ &=: D_{n,1}^1(\varepsilon) + D_{n,2}^1(\varepsilon). \end{aligned} \tag{37}$$

It follows from Lemmas B.2 and B.3 that

$$|D_{n,2}^1(\varepsilon)| \le C_{\alpha,\underline{\alpha},F} n|\varepsilon| \le C_{\alpha,\underline{\alpha},F} n^{1-1/\gamma}. \tag{38}$$

We now turn to $D_{n,1}^1(\varepsilon)$. By definition,

$$D_{n,1}^1(\varepsilon) = \sum_{j=0}^{n-1} I_j,$$

where

$$I_j := \int_\Omega \delta(\sigma^j\omega) \int_0^1 \mathcal{L}_{\sigma^{j+1}\omega,\varepsilon}^{n-j-1}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}_\omega^j(f_{\omega,0}h_\omega))) f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega).$$

For j such that $n - j - 1 \ge \lfloor n/2 \rfloor - 1$, we have (using Lemma B.2 and Theorem A.1) that

$$|I_j| \le C_{\alpha,F}(n-j-1)^{-1/\alpha+1} \le C_{\alpha,F} n^{-1/\alpha+1}. \tag{39}$$

We now consider j so that $j \ge \lfloor n/2 \rfloor$. For $r > 0$ to be fixed later, we have by writing

$$r^{-1}(\mathcal{L}_{\beta(\sigma^j\omega)+r} - \mathcal{L}_{\beta(\sigma^j\omega)})(\varphi) = \partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\varphi) + r\int_0^1 (1-t)\partial_\gamma^2 \mathcal{L}_{\beta(\sigma^j\omega)+tr}(\varphi)\, dt,$$

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and using Lemmas B.2 and B.3 that

$$\|\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)) - r^{-1}(\mathcal{L}_{\beta(\sigma^j\omega)+r} - \mathcal{L}_{\beta(\sigma^j\omega)})(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega))\|_{L^1(m)} \le C_{\alpha,\underline{\alpha},F} r.$$

Writing I_j as

$$\begin{aligned} I_j = &\\ &= \int_\Omega \delta(\sigma^j\omega) \int_0^1 \mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon}((\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)} \\ &\quad - r^{-1}(\mathcal{L}_{\beta(\sigma^j\omega)+r} - \mathcal{L}_{\beta(\sigma^j\omega)}))(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega) \\ &\quad + r^{-1}\int_\Omega \delta(\sigma^j\omega)\int_0^1 \mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon} \\ &\quad \times ((\mathcal{L}_{\beta(\sigma^j\omega)+r} - \mathcal{L}_{\beta(\sigma^j\omega)})(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega), \end{aligned}$$

we have that

$$|I_j| \le C_{\alpha,\underline{\alpha},F} r + C_{\alpha,F} r^{-1} j^{-1/\alpha+1} \le C_{\alpha,\underline{\alpha},F}(r + r^{-1}n^{-1/\alpha+1}), \tag{40}$$

where we used that

$$\begin{aligned} &\|\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon}((\mathcal{L}_{\beta(\sigma^j\omega)+r} - \mathcal{L}_{\beta(\sigma^j\omega)})(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))\\ &\quad\|_{L^1(m)} \le 2\|\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)\|_{L^1(m)} \le 2C_{\alpha,F} j^{-1/\alpha+1}. \end{aligned}$$

Setting $r = n^{-(1-\alpha)/2\alpha}$, it follows that

$$|I_j| \le C_{\alpha,\underline{\alpha},F} n^{-(1-\alpha)/2\alpha}, \quad j \ge \lfloor n/2 \rfloor. \tag{41}$$

From (39) and (41), we conclude that

$$|D^1_{n,1}(\varepsilon)| \le \sum_{j=0}^{n-1} |I_j| \le C_{\alpha,\underline{\alpha},F} n^{1-(1-\alpha)/2\alpha}. \tag{42}$$

Estimate (36) now follows readily from (38) and (42). □

Observe that the upper bound in Lemma 16 is summable if $\alpha < 1/4$, and the upper bound in Lemma 18 is summable if $\alpha < 1/5$. To complete the proof of Theorem 9, it remains to prove the following lemma.

LEMMA 19. *We have*

$$\lim_{\varepsilon\to 0} D^1_n(\varepsilon) = \sum_{j=0}^{n-1} \int_\Omega \delta(\sigma^j\omega) \int_0^1 \mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))f_{\sigma^n\omega,0}\, dm\, d\mathbb{P}(\omega).$$

Proof. We decompose as in (37):

$$D^1_n(\varepsilon) = D^1_{n,1}(\varepsilon) + D^1_{n,2}(\varepsilon),$$

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where $\lim_{\varepsilon\to 0} D^1_{n,2}(\varepsilon) = 0$ (see (38)). Next,

$$\begin{aligned} D^1_{n,1}(\varepsilon) &= \int_\Omega \int_0^1 \sum_{j=0}^{n-1} \delta(\sigma^j\omega)\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega) \\ &\quad + \int_\Omega \int_0^1 \sum_{j=0}^{n-1} \delta(\sigma^j\omega)(\mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega,\varepsilon} - \mathcal{L}^{n-j-1}_{\sigma^{j+1}\omega}) \\ &\quad \times (\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))f_{\sigma^n\omega,0}\,dm\,d\mathbb{P}(\omega) \\ &=: \mathcal{D}^1_{n,1} + \mathcal{D}^1_{n,2}(\varepsilon). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathcal{D}^1_{n,2}(\varepsilon) = \sum_{j=0}^{n-1}\sum_{k=0}^{n-j-2} \int_\Omega \delta(\sigma^j\omega) \int_0^1 \mathcal{L}^{n-j-k-2}_{\sigma^{j+k+2}\omega,\varepsilon} \int_{\beta(\sigma^{j+1}\omega)}^{\beta(\sigma^{j+1}\omega)+\varepsilon\delta(\sigma^{(j+1)}\omega)} \\ \times \partial_\gamma \mathcal{L}_\gamma(\mathcal{L}^k_{\sigma^{j+1}\omega}(\partial_\gamma \mathcal{L}_{\beta(\sigma^j\omega)}(\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)))) \\ \times f_{\sigma^n\omega,0}\,d\gamma\,dm\,d\mathbb{P}(\omega). \end{aligned}$$

Using Lemma A.2 together with Lemmas B.2 and B.4, we have

$$|\mathcal{D}^1_{n,2}(\varepsilon)| \le C_{\alpha,\underline{\alpha},F} n^2 |\varepsilon|.$$

Hence, $\lim_{\varepsilon\to 0} \mathcal{D}^1_{n,2}(\varepsilon) = 0$. This immediately implies the desired conclusion. □

4.4. *An improvement of Theorem 9 for a special class of observables.* Here, we indicate how we can relax the assumption $\alpha < 1/5$ in the statement of Theorem 9 provided that we impose additional assumptions on F. We first note that the restriction $\alpha < 1/5$ is used only in Lemma 18; the rest of the proof remains valid under the weaker condition $\alpha < 1/4$.

We continue to assume that $F\colon \Omega \times [0,1] \to \mathbb{R}$ is measurable with $F(\omega,\cdot) \in C^2[0,1]$ for $\omega \in \Omega$ and that (19) is valid. We need the following version of Lemma 18.

LEMMA 20. *Suppose that there exist* $0 \le \eta \le \alpha$ *and* $C > 0$ *such that, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and* $x \in [0,1]$,

$$|F_\omega(x)| \le Cx^\eta \quad \text{and} \quad \int_0^1 F_\omega h_\omega\,dm = 0. \tag{43}$$

Then,

$$|D^1_n(\varepsilon)| \le C_{\alpha,\underline{\alpha},F}(n^{1-1/\gamma} + n^{1-(1+\eta-\alpha)/2\alpha}), \quad n < |\varepsilon|^{-\gamma}. \tag{44}$$

Proof. The proof proceeds as the proof of Lemma 18 up to (40). It follows from (4) that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}^j_\omega(f_{\omega,0}h_\omega)\|_{L^1(m)} \le C_{\alpha,\underline{\alpha},F} n^{1-(1+\eta)/\alpha}, \quad j \ge \lfloor n/2 \rfloor.$$

Thus, choosing $r = n^{\frac{1}{2}(1-(1+\eta)/\alpha)}$, we obtain

$$|I_j| \le C_{\alpha,\underline{\alpha},F}(r + r^{-1}n^{1-(1+\eta)/\alpha}) \le C_{\alpha,\underline{\alpha},F} n^{-(1+\eta-\alpha)/2\alpha}, \quad j \ge \lfloor n/2 \rfloor,$$

which yields (44). □

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THEOREM 21. *Suppose that (43) holds with $0 \le \eta \le \alpha$ and $C > 0$. Provided that $\alpha < (1+\eta)/5$, we have that $\varepsilon \mapsto \Sigma_\varepsilon^2$ is differentiable in $\varepsilon = 0$.*

Proof. We proceed exactly as in the proof of Theorem 9 with the only difference being that we use Lemma 20 instead of Lemma 18. The conclusion follows since the upper bound (44) is summable if $\alpha < (1+\eta)/5$. □

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A. *Appendix*

For the reader's convenience, we gather in this section preliminary results from [17] that are instrumental in the proofs of Theorems 5 and 9 concerning the regularity of the variance in the CLT.

Recall the definitions of the cones $\mathcal{C}_* = \mathcal{C}_*(a)$ and $\mathcal{C}_2 = \mathcal{C}_2(b_1, b_2)$ from §2. By [17, Lemma 2 and Proposition 5], there exist $a = a(\alpha) > 1$ and $b_1, b_2 > 0$ such that for $0 < \beta \le \alpha$, we have

$$\mathcal{L}_\beta(\mathcal{C}_*) \subset \mathcal{C}_*, \quad \mathcal{L}_\beta(\mathcal{C}_2) \subset \mathcal{C}_2, \quad N_\beta(\mathcal{C}_2) \subset \mathcal{C}_2. \tag{A.1}$$

The following theorem follows from [17, Theorem 7].

THEOREM A.1. *There exists $C_\alpha > 0$ with the property that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \in \mathbb{N}$, and $\phi, \psi \in \mathcal{C}_*$ such that $\int_0^1 \phi \, dm = \int_0^1 \psi \, dm$, we have*

$$\int_0^1 |\mathcal{L}_\omega^n(\phi - \psi)| \, dm \le C_\alpha(\|\phi\|_{L^1(m)} + \|\psi\|_{L^1(m)})n^{-1/\alpha+1}. \tag{A.2}$$

Furthermore, by [17, Lemma 12], we have the following result.

LEMMA A.2. *For $0 < \gamma_0 \le \alpha < 1$, there exist $0 < \delta \le \alpha$ and $C_1, C_2 > 0$ such that*

$$|(X_\gamma N_\gamma(\phi))'(x)| \le C_1 a x^{-\delta} \quad \text{and} \quad |(X_\gamma N_\gamma(\phi))''(x)| \le C_2 a x^{-\delta-1} \tag{A.3}$$

for $x \in (0,1]$, $\gamma \in [\gamma_0, \alpha]$, and $\phi \in \mathcal{C}_*(a) \cap \mathcal{C}_2(b_1, b_2)$ with $m(\phi) = 1$. Moreover, there exist $\psi_i \in \mathcal{C}_*(a) \cap C^1(0,1]$, $i \in \{1,2\}$ such that

$$(X_\gamma N_\gamma(\phi))' = \psi_1 - \psi_2, \tag{A.4}$$

with $\int_0^1 \psi_1 \, dm = \int_0^1 \psi_2 \, dm$ and $\|\psi_i\|_{L^1(m)} \le D$, $i \in \{1,2\}$ for some $D = D_{\gamma_0,\alpha} > 0$.

The following result on statistical stability is a consequence of [17, Theorem 13].

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PROPOSITION A.3. *Let $h_{\omega,\varepsilon}$ be defined as in §3 with $h_{\omega,0} = h_\omega$. There exists $C_\alpha > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$\|h_{\omega,\varepsilon} - h_\omega\|_{L^1(m)} \le C_\alpha|\varepsilon|, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

B. *Appendix*

LEMMA B.2. *Let $\varphi \in C^2$ and $h \in \mathcal{C}_* \cap \mathcal{C}_2$ with $\int_0^1 h\, dm = 1$. Then, there exist $\psi_i \in \mathcal{C}_* \cap \mathcal{C}_2$, $i = 1, 2$ such that*

$$\varphi h = \psi_1 - \psi_2. \tag{B.1}$$

Moreover, $\|\psi_i\|_{L^1(m)} \le D\|\varphi\|_{C^2}$ for $i = 1, 2$, where D depends only on a, α, b_1, and b_2 (parameters of cones).

Proof. We recall the construction from [32]. Namely, take $\psi_1 = (\varphi + \lambda x + A)h + B$ and $\psi_2 = (\lambda x + A)h + B$. Observe that (B.1) holds. Moreover, it is proved in [32] that $\psi_1 \in \mathcal{C}_*$ provided that:

- $\lambda = -2\|\varphi\|_{C^1}$ (so that $|\lambda| > \|\varphi'\|_\infty$ and $\lambda < 0$);
- $A = 3\|\varphi\|_{C^1}$ so that $\varphi + \lambda x + A \ge 0$;
- $B \ge (a/(\alpha+1))\|\varphi\|_{C^1}$ so that

$$(\alpha+1)x^\alpha B \ge (\alpha+2)\lambda x^{\alpha+1}h(x) + x^{\alpha+1}h(x)\varphi'(x),$$

  where we also recall that $\lambda < 0$;
- $B \ge (4a/(a-1))\|\varphi\|_{C^1}$, so that

$$B \ge \frac{a}{a-1}[\sup(\varphi + \lambda x + A)] - \inf(\varphi + \lambda x + A)].$$

Note that

$$\psi_1'(x) = (\varphi'(x) + \lambda)h(x) + (\varphi(x) + \lambda x + A)h'(x).$$

Consequently,

$$\begin{aligned}
|\psi_1'(x)| &\le (\|\varphi'\|_\infty + |\lambda|)h(x) + (\varphi(x) + \lambda x + A)|h'(x)| \\
&\le a(\|\varphi'\|_\infty + |\lambda|)x^{-\alpha} + \frac{b_1}{x}(\varphi(x) + \lambda x + A)h(x) \\
&\le a(\|\varphi'\|_\infty + |\lambda|)x^{-1} + \frac{b_1}{x}(\varphi(x) + \lambda x + A)h(x) \\
&\le B\frac{b_1}{x} + \frac{b_1}{x}(\varphi(x) + \lambda x + A)h(x) \\
&= \frac{b_1}{x}\psi_1(x),
\end{aligned}$$

provided that $B \ge (3a/b_1)\|\varphi\|_{C^1}$.

Next, note that

$$\psi_1''(x) = \varphi''(x)h(x) + 2(\varphi'(x) + \lambda)h'(x) + (\varphi(x) + \lambda x + A)h''(x).$$

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Thus,

$$\begin{aligned}|\psi_1''(x)| &\le \|\varphi''\|_\infty h(x) + 2\frac{b_1}{x}(\|\varphi'\|_\infty + |\lambda|)h(x) + \frac{b_2}{x^2}(\varphi(x) + \lambda x + A)h(x)\\ &\le a\|\varphi''\|_\infty x^{-2} + 2ab_1(\|\varphi'\|_\infty + |\lambda|)x^{-2} + \frac{b_2}{x^2}(\varphi(x) + \lambda x + A)h(x)\\ &\le B\frac{b_2}{x^2} + \frac{b_2}{x^2}(\varphi(x) + \lambda x + A)h(x)\\ &= \frac{b_2}{x^2}\psi_1(x),\end{aligned}$$

provided that $B \ge (a/b_2)\|\varphi''\|_\infty + (6ab_1/b_2)\|\varphi\|_{C^1}$. Taking into account various conditions for B, we observe that we can choose B of the form $B = d\|\varphi\|_{C^2}$, where constant $d > 0$ depends only on α, a, b_1, and b_2 to fulfill these. With this choice, we have that $\psi_1 \in \mathcal{C}_2$. Finally, we observe that

$$\|\psi_1\|_{L^1(m)} \le \|\varphi\|_\infty + |\lambda| + A + B,$$

which readily implies the desired conclusion. □

LEMMA B.3. *Let $0 < \underline{\alpha} \le \alpha < 1$. Then, there exist $\delta = \delta(\underline{\alpha}, \alpha) \in (0, \alpha]$ such that*

$$|\partial_\gamma^2 \mathcal{L}_\gamma(\phi)(x)| \le C_{\alpha,\underline{\alpha}} x^{-\delta}\|\phi\|_{L^1(m)}$$

for any $x \in (0, 1]$, $\gamma \in [\underline{\alpha}, \alpha]$, and $\phi \in \mathcal{C}_* \cap \mathcal{C}_2$. In particular,

$$\|\partial_\gamma^2 \mathcal{L}_\gamma(\phi)\|_{L^1(m)} \le C_{\alpha,\underline{\alpha}}\|\phi\|_{L^1(m)}.$$

Proof. The desired conclusion is obtained in the proof of [17, Lemma 16], that is, in the arguments that yield the first estimate in [17, (27)]. We note that these arguments do not use the assumption that ϕ belongs to the cone $\mathcal{C}_3$ made in the statement of [17, Lemma 16]. □

LEMMA B.4. *Let $0 < \underline{\alpha} \le \alpha < 1$. Then, there exist $\delta = \delta(\underline{\alpha}, \alpha) \in (0, \alpha]$ such that*

$$|\partial_\gamma \mathcal{L}_\gamma(\phi)(x)| \le C_{\alpha,\underline{\alpha}} x^{-\delta}\|\phi\|_{L^1(m)}$$

for $x \in (0, 1]$, $\phi \in C^1(0, 1] \cap \mathcal{C}_*$, and $\gamma \in [\underline{\alpha}, \alpha]$. In particular,

$$\|\partial_\gamma \mathcal{L}_\gamma(\phi)\|_{L^1(m)} \le C_{\alpha,\underline{\alpha}}\|\phi\|_{L^1(m)}.$$

Proof. The proof proceeds as the proof of [17, Lemma 12] for verification of the first estimate in [17, (15)]. The only difference concerns the term $(N_\gamma(\phi))'$, which in [17] was handled by relying on the assumption that $\phi \in \mathcal{C}_2$.

In the present context, assuming without loss of generality that $\int_0^1 \phi\, dm = 1$, we have $N_\gamma(\phi) \in \mathcal{C}_*$ (since $\phi \in \mathcal{C}_*$). Moreover, $N_\gamma(\phi) \in C^1(0, 1]$ as $\phi \in C^1(0, 1]$. Consequently, [17, Remark 1] implies that

$$|(N_\gamma(\phi))'(x)| \le \frac{\alpha + 1}{x} N_\gamma(\phi)(x) \le C_\alpha x^{-1-\alpha},$$

which is analogous to [17, (18)]. □

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C. *Appendix*

Here, we provide the proof of Theorem 6. We stress that we closely follow the proof of [17, Theorem 14].

Proof of Theorem 6. Using Theorem A.1 and Lemma A.2, it can easily be shown that the series in (17) converges in $L^1(m)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and that (18) holds.

For $\mathbb{P}$-a.e. $\omega \in \Omega$ and each ε, we have

$$\begin{aligned} h_{\omega,\varepsilon} - h_\omega &= \mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} h_{\sigma^{-l}\omega,\varepsilon} - \mathcal{L}^l_{\sigma^{-l}\omega} h_{\sigma^{-l}\omega} \\ &= \mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} 1 - \mathcal{L}^l_{\sigma^{-l}\omega} 1 + \mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon}(h_{\sigma^{-l}\omega,\varepsilon} - 1) + \mathcal{L}^l_{\sigma^{-l}\omega}(1 - h_{\sigma^{-l}\omega}) \end{aligned}$$

for $l \in \mathbb{N}$. Hence, Theorem A.1 implies that

$$\|h_{\omega,\varepsilon} - h_\omega - (\mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} 1 - \mathcal{L}^l_{\sigma^{-l}\omega} 1)\|_{L^1(m)} \le C_\alpha l^{1-1/\alpha}.$$

Therefore, it follows that there exists $l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that for $l \ge l_0$,

$$\left\| \frac{h_{\omega,\varepsilon} - h_\omega - \varepsilon \hat{h}_\omega}{\varepsilon} - \frac{\mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} 1 - \mathcal{L}^l_{\sigma^{-l}\omega} 1 - \varepsilon \hat{h}_\omega}{\varepsilon} \right\|_{L^1(m)} \le C_\alpha |\varepsilon|. \tag{C.1}$$

However,

$$\mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} 1 - \mathcal{L}^l_{\sigma^{-l}\omega} 1 = \sum_{j=0}^{l-1} \mathcal{L}^j_{\sigma^{-j}\omega,\varepsilon} (\mathcal{L}_{\sigma^{-(j+1)}\omega,\varepsilon} - \mathcal{L}_{\sigma^{-(j+1)}\omega}) \mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1).$$

Writing

$$\begin{aligned} &(\mathcal{L}_{\sigma^{-(j+1)}\omega,\varepsilon} - \mathcal{L}_{\sigma^{-(j+1)}\omega})(\varphi)(x) \\ &\quad = \varepsilon \delta(\sigma^{-(j+1)}\omega) \partial_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)}(\varphi)(x) \\ &\qquad + \varepsilon^2 (\delta(\sigma^{-(j+1)}\omega))^2 \int_0^1 (1-t) \partial^2_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)+t\epsilon\delta(\sigma^{-(j+1)}\omega)}(\varphi)(x)\, dt, \end{aligned}$$

we see that

$$\begin{aligned} &\frac{\mathcal{L}^l_{\sigma^{-l}\omega,\varepsilon} 1 - \mathcal{L}^l_{\sigma^{-l}\omega} 1 - \varepsilon \hat{h}_\omega}{\varepsilon} \\ &\quad = \sum_{j=0}^{l-1} \delta(\sigma^{-(j+1)}\omega) \mathcal{L}^j_{\sigma^{-j}\omega,\varepsilon} \partial_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)} (\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)) - \hat{h}_\omega \\ &\qquad + \varepsilon \sum_{j=0}^{l-1} (\delta(\sigma^{-(j+1)}\omega))^2 \int_0^1 (1-t) \mathcal{L}^j_{\sigma^{-j}\omega,\varepsilon} \partial^2_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)+t\epsilon\delta(\sigma^{-(j+1)}\omega)} (\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1))\, dt. \end{aligned}$$

By Theorem A.1 and [17, Lemma 16], we have that

$$\begin{aligned} &\left\| \varepsilon \sum_{j=0}^{l-1} (\delta(\sigma^{-(j+1)}\omega))^2 \int_0^1 (1-t) \mathcal{L}^j_{\sigma^{-j}\omega,\varepsilon} \partial^2_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)+t\epsilon\delta(\sigma^{-(j+1)}\omega)} (\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1))\, dt \right\|_{L^1(m)} \\ &\quad \le C_{\alpha,\underline{\alpha}} |\varepsilon|. \end{aligned} \tag{C.2}$$

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However, similarly to [17, Remark 21] for $n = \lfloor |\varepsilon|^{-\alpha} \rfloor$ and $l > n$,

$$\begin{aligned}
&\Big\| \sum_{j=0}^{l-1} \delta(\sigma^{-(j+1)}\omega) \mathcal{L}^{j}_{\sigma^{-j}\omega,\varepsilon} \partial_\gamma \mathcal{L}_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)) - \hat{h}_\omega \Big\|_{L^1(m)} \\
&\quad \le \Big\| \sum_{j=0}^{n} \delta(\sigma^{-(j+1)}\omega)(\mathcal{L}^{j}_{\sigma^{-j}\omega,\varepsilon} - \mathcal{L}^{j}_{\sigma^{-j}\omega})[(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))'] \Big\|_{L^1(m)} \\
&\qquad + \Big\| \sum_{j=0}^{n} \delta(\sigma^{-(j+1)}\omega) \mathcal{L}^{j}_{\sigma^{-j}\omega}((X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))' - \\
&\qquad - (X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(h(\sigma^{-(j+1)}\omega)))') \Big\|_{L^1(m)} + C_\alpha |\varepsilon|^{1-2\alpha}. \qquad \text{(C.3)}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Big\| \sum_{j=0}^{n} \delta(\sigma^{-(j+1)}\omega)(\mathcal{L}^{j}_{\sigma^{-j}\omega,\varepsilon} - \mathcal{L}^{j}_{\sigma^{-j}\omega})[(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))'] \Big\|_{L^1(m)} \\
&\quad \le \sum_{j=1}^{n} \|(\mathcal{L}^{j}_{\sigma^{-j}\omega,\varepsilon} - \mathcal{L}^{j}_{\sigma^{-j}\omega})[(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))']\|_{L^1(m)} \\
&\quad \le C_{\alpha,\underline{\alpha}} n^2 |\varepsilon| \le C_{\alpha,\underline{\alpha}} |\varepsilon|^{1-2\alpha}, \qquad \text{(C.4)}
\end{aligned}$$

where we used that in the proof of [17, Theorem 14], it is showed that

$$\|(\mathcal{L}^{j}_{\sigma^{-j}\omega,\varepsilon} - \mathcal{L}^{j}_{\sigma^{-j}\omega})[(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))']\|_{L^1(m)} \le C_{\alpha,\underline{\alpha}} n |\varepsilon|$$

for each $1 \le j \le n$. Next,

$$\begin{aligned}
&\Big\| \sum_{j=0}^{n} \delta(\sigma^{-(j+1)}\omega) \mathcal{L}^{j}_{\sigma^{-j}\omega}((X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))' \\
&\qquad - (X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(h(\sigma^{-(j+1)}\omega)))') \Big\|_{L^1(m)} \\
&\quad \le \sum_{j=0}^{n} \|(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))' \\
&\qquad - (X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(h(\sigma^{-(j+1)}\omega)))'\|_{L^1(m)} \\
&L \le n|\varepsilon| = O(|\varepsilon|^{1-\alpha}), \qquad \text{(C.5)}
\end{aligned}$$

where we used (see [17, Remark 21]) that (by making l sufficiently large)

$$\begin{aligned}
&\|(X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(\mathcal{L}^{l-1-j}_{\sigma^{-l}\omega}(1)))' \\
&\quad - (X_{\beta(\sigma^{-(j+1)}\omega)} N_{\beta(\sigma^{-(j+1)}\omega)}(h(\sigma^{-(j+1)}\omega)))'\|_{L^1(m)} \le |\varepsilon|
\end{aligned}$$

for $0 \le j \le n$. The conclusion of the theorem follows by putting (C.1)–(C.5) together. □

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