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ORIGINAL RESEARCH PAPER

On RVaR-based optimal partial hedging

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Abstract

The main aim of this paper is to develop an optimal partial hedging strategy that minimises an investor's shortfall subject to an initial wealth constraint. The risk criterion we employ is a robust tail risk measure called Range Value-at-Risk (RVaR) which belongs to a wider class of distortion risk measures and contains the well-known measures VaR and CVaR as important limiting cases. Explicit forms of such RVaR-based optimal hedging strategies are derived. In addition, we provide a numerical example to demonstrate how to apply this more comprehensive methodology of partial hedging in the area of mixed finance/insurance contracts in the market with long-range dependence.

Keywords: Partial hedging; Range Value-at-Risk; Mixed Fractional Brownian motion; Equity-linked life insurance contracts

1. Introduction

Control of risk is a main and permanent research topic of modern mathematical finance. It gives a perfect motivation for theoretical developments and is vitally important for practice. According to the option pricing theory, any contingent claim can be hedged perfectly if the initial wealth is no less than the fair price of the claim in complete markets or if it is no less than the minimal super-hedging price in incomplete markets. However, perfect hedging costs are usually too high to be of practical interest in most cases and that is why the partial hedging comes out. During such hedging processes, investors allocate initial capitals less than perfect hedging costs and would like to construct strategies that minimise their shortfall under certain risk measures. Another reason that makes the partial hedging interesting is that although it has some downside risk, it brings opportunities to gain benefits. Some financial institutions such as insurance companies indeed exploit risk to make profits. Föllmer and Leukert are pioneers in the field of optimal partial hedging. They studied quantile hedging (see Föllmer & Leukert, 1999) and efficient hedging (see Föllmer & Leukert, 2000) for semimartingale financial market models. In their papers, explicit solutions in complete markets were provided with the help of the classical Neyman-Pearson lemma while theoretical solutions in incomplete markets were given according to the convex duality approach. Moreover, partial hedging problems have been discussed in more sophisticated markets. For example, Spivak & Cvitanic (1999) investigated quantile hedging in markets with partial information and in markets with large investors. Nakano (2011) solved problems of optimal quantile hedging and efficient hedging with linear loss functions for claims with a single default time. The book of Melnikov & Nosrati (2017) discussed several partial hedging methods and their applications in pricing and hedging of insurance contracts.

One important component that investors need to consider when they construct a partial hedging is the risk criterion employed. The theory of risk measures was discussed in Artzner *et al.* (1999) and was applied to pricing and hedging of contingent claims by Xu (2006), where the

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associated optimal portfolio is determined by minimising a convex measure of risk. We also note some other works: Bernard *et al*. (2015) for Law-Invariance risk measures and Madan & Schoutens (2016) for risk measures generated by distortion functions. In financial institutions, Value-at-risk (VaR) and Conditional Value-at-Risk (CVaR) are most commonly used risk measures and there is a long list of references regarding them. For instance, Melnikov & Smirnov (2012) studied partial hedging with the measure CVaR where the semi-explicit solution of the optimal CVaR hedging problem in complete markets was given. Cong *et al*. (2013) discussed VaR-based optimal hedging, while in Cong *et al*. (2014), the CVaR-based optimal hedging problem was solved without the restriction regarding to the completeness of markets. In this paper, we would like to implement a more general tail risk measure named Range Value-at-Risk (RVaR) which was applied to hedging problems by Cont *et al*. (2010) and Embrechts *et al*. (2018). Such a measure includes VaR and CVaR as two limiting cases, and hence, it provides insights into connections between these two other measures and it is more customised since investors can set their risk appetite by choosing two risk level parameters α as well as β. Another risk measure called GlueVaR that combines both VaR and CVaR using the technique of distortion functions developing in mathematical finance and statistics can be found in Jaume *et al*. (2014). Meanwhile, Cont *et al*. (2010) have indicated that a risk measure cannot be both robust and coherent, but RVaR constitutes a trade-off between the sensitivity of CVaR and the robustness of VaR nonetheless.

The main objective of this paper is to derive an optimal hedging strategy that minimises the RVaR of a hedger's risk exposure subject to an initial wealth constraint and compare it with VaR and CVaR-based optimal hedging strategies to investigate relationships among them. Similar to the procedures in Cong *et al*. (2013) and (2014), the optimisation problem is solved in two steps. First, we search for an optimal partition between the hedged loss which is the part to be hedged with the initial capital and the retained loss that cannot be hedged. Then, a hedging strategy of the hedged loss should be constructed. It is worth to emphasize that the method in this paper can be applied in any arbitrage free market even if it is incomplete, and hence, our methodology has some advantages since as was mentioned in Föllmer & Leukert (2000) that deriving explicit solutions of optimal partial hedging in incomplete markets was extremely hard. In addition, the structure of our optimal partial hedging is independent of the dynamic of the underlying asset and such a strategy can be easily derived even though the financial model is complex. Furthermore, as an application, we construct optimal RVaR hedging strategies of life insurance contracts in Mixed Fractional Brownian motion (MFBM) markets which have the property of a long-range dependence. Bratyk & Mishura (2008) investigated the quantile hedging problem in such markets and pointed out that explicit solutions of partial hedging problems were rather difficult to be derived because of the complicated structure of the martingale measure. Such a difficulty is overcome by our method. In our opinion, MFBM markets are not used yet in equity-linked life insurance, but they may bring better modelling and pricing properties in actuarial calculations.

The rest of the paper is organised as follows. In section 2, we provide the definition of RVaR and introduce some properties of it. In section 3, we begin with the formalisation of our RVaR-based hedging problem, and then, we derive two explicit solutions of it depending on the size of the initial wealth. Most importantly, in section 3.2, we show that VaR-based hedging and CVaR-based hedging can be seen as two special cases of RVaR hedging. In section 4, a numerical example is provided to explain our method and to describe how it can be implemented to hedge life insurance contracts in sophisticated markets. Section 5 concludes the paper.

2. Range Value-at-Risk

Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space and $\mathcal{S}$ be the set of real-valued random variables denoting loss amounts (a negative value represents gains). A risk measure ρ is a mapping from $\mathcal{S}$ to $\mathbb{R}$. For instance, VaR and CVaR are commonly used risk measures in the financial industry

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which are defined as

$$VaR_\alpha(L) = \inf\{v \in \mathbb{R} : P(L > v) \leq 1 - \alpha\} \tag{1}$$

and

$$CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_s(L)ds \tag{2}$$

where $\alpha \in (0, 1)$ is the risk level and the random variable $L \in \mathcal{S}$ with finite mean.

In this paper, we would like to investigate a measure called Range Value-at-Risk (*RVaR*) that can be seen as a bridge between VaR and CVaR.

The RVaR at the level (α, β) such that $0 < \alpha \leq \alpha + \beta < 1$ is

$$RVaR_{\alpha,\beta}(L) = \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s(L)ds, & if\ \ \beta > 0, \\ VaR_\alpha(L), & if\ \ \beta = 0 \end{cases} \tag{3}$$

which is the average value of VaR among specific risk levels $[\alpha, \alpha + \beta]$.

A risk measure is called coherent if it satisfies (see Artzner *et al.* 1999 for details):

(1) Monotonicity: If $L_1, L_2 \in \mathcal{S}$ such that $L_1 \leq L_2$, then $\rho(L_1) \leq \rho(L_2)$;
(2) Positive homogeneity: If $L \in \mathcal{S}$ and $s \in \mathbb{R}^+$, then $\rho(sL) = s\rho(L)$;
(3) Translation invariance: If $L \in \mathcal{S}$ and $s \in \mathbb{R}$, then $\rho(L + s) = \rho(L) + s$;
(4) Subadditivity: If $L_1, L_2 \in \mathcal{S}$, then $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.

CVaR is a coherent risk measure. However, Embrechts *et al.* (2018) have proved that both VaR and RVaR only satisfy properties (1)–(3) listed above while they do not satisfy subadditivity. Instead, for $L_1, L_2 \in \mathcal{S}$, the following inequalities hold true:

$$VaR_{\alpha_1+\alpha_2}(L_1 + L_2) \leq VaR_{\alpha_1}(L_1) + VaR_{\alpha_2}(L_2), \tag{4}$$

$$RVaR_{\tilde{\alpha}+\tilde{\beta}}(L_1 + L_2) \leq RVaR_{\alpha_1,\beta_1}(L_1) + RVaR_{\alpha_2,\beta_2}(L_2) \tag{5}$$

where $\tilde{\alpha} = \alpha_1 + \alpha_2$ and $\tilde{\beta} = \max\{\beta_1, \beta_2\}$. (4) and (5) are called the special form of subadditivity.

Desired properties of RVaR are that it is robust, and it is a general tail risk measure which includes VaR as well as CVaR as two limiting cases, that is,

$$RVaR_{\alpha,0}(L) = VaR_\alpha(L) = \lim_{\beta \to 0^+} RVaR_{\alpha,\beta}(L), \tag{6}$$

$$RVaR_{\alpha,1-\alpha}(L) = CVaR_\alpha(L) \tag{7}$$

Additionally, all three measures belong to a wide class of distortion risk measures, such that

$$\rho(L) = \int_0^1 VaR_s(L)dh(s) \tag{8}$$

where $h(s)$ called a distortion function is a non-decreasing and left-continuous function from [0,1] to [0,1] satisfying $h(0) = 0$ and $h(1) = 1$. Specifically, for $\alpha \in (0, 1)$, $\beta \in [0, 1)$, the distortion function of $RVaR_{\alpha,\beta}$ is

$$h_{\alpha,\beta}(s) = \begin{cases} \min\left\{I_{\{s>\alpha\}}\frac{s-\alpha}{\beta}, 1\right\}, & if\ \ \beta > 0, \\ I_{\{s>\alpha\}}, & if\ \ \beta = 0 \end{cases} \tag{9}$$

As we know that while VaR and CVaR have some common properties, they are still quite different and consequently, investigating RVaR which is a bridge between them helps us to understand their connections and to gain a more integrated picture regarding to risk criteria.

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3. Optimal RVaR-based Hedging and Connection with CVaR and VaR Hedging

3.1 Optimal RVaR-based hedging

A European contingent claim X with maturity T is a non-negative $\mathcal{F}$ measurable random variable which can be thought of as a contract or agreement that pays X at the maturity T. Assume that a financial market is arbitrage free but does not have to be complete. In this case, arbitrage free price of a claim may not be unique and a popular option pricing approach in incomplete markets is the utility-based indifference pricing (UBIP). The idea behind this pricing method is that all investors' risk appetite can be fully described by a utility function, and all investors are presumed to maximise their expected utility of wealth. Then, the utility-based indifference price of a claim is the amount that makes no difference to investors' expected utility no matter whether they include the claim into their portfolio or not. One important property of utility-based indifference price is that it does not depend on the completeness of the market, and in the special case that a market is complete, it coincides with the fair price of that claim (see Henderson & Hobson 2004 for more properties regarding UBIP).

Suppose a hedger is exposed to a future obligation X at time T. Option pricing theorem points out that a claim can be hedged perfectly if the initial hedging capital is no less than its minimal super-hedging price (or fair price in complete markets). However, if the initial hedging capital $\tilde{v}_0$ is not enough, the hedger is exposed to some downside risk. The main goal of this paper is to find an optimal hedging strategy that achieves the minimal risk under the measure RVaR subject to an initial wealth constraint.

We start with a partition of the contingent claim X such that $X = f(X) + R_f(X)$. $f(X)$ named the hedged loss function represents the part that would be hedged with the initial capital $\tilde{v}_0$ and $R_f(X)$ called the retained loss function is the payout the hedger retains. Thus, the problem of optimal hedging can be solved in two steps: first, find the optimal partition $f^*(X)$, $R_{f^*}(X)$ satisfying the initial capital constraint and some assumptions regarding to hedged loss functions; second, hedge the payout $f^*(X)$ perfectly. Cong *et al.* (2013, 2014) considered VaR and CVaR hedging problems with some restrictive assumptions regarding to the hedged loss functions. Assumptions they imposed are

(a) Not globally over-hedged: $f(x) \le x$ for all $x \ge 0$;
(b) Not locally over-hedged: $f(x_2) - f(x_1) \le x_2 - x_1$ for all $0 \le x_1 \le x_2$;
(c) Non-negativity of the hedged loss: $f(x) \ge 0$ for all $x \ge 0$.

Assumption (a) ensures that the hedged loss would be bounded from above by the original risk to be hedged. Assumption (b) indicates that the increment of the hedged part should be no more than the increment of the risk itself. Although adding assumption (b) makes hedged loss functions less general, it indeed brings advantages. For instance, as we will show later, the resulting optimal hedged loss function does not depend on the underlying market model if we have the assumption (b). Assumption (c) is commonly imposed that ensures hedging strategies take positive values, and Cong *et al.* (2013) mentioned that without the assumption (c), the optimal partial hedging problem might be ill-posed.

Hence, the admissible set $\mathcal{D}$ of the hedged loss function is represented as

$$\mathcal{D} = \big\{0 \le f(x) \le x : \ R_f(x) = x - f(x) \text{ is a non-decreasing and left-continuous function}\big\} \tag{10}$$

Based on the above considerations, the RVaR hedging problem in our setting is

$$\begin{cases} \min\limits_{f \in \mathcal{D}} \ RVaR_{\alpha,\beta}(R_f(X)), \\ s.t \ \Pi\big(f(X)\big) \le \tilde{v}_0 < \Pi\big(X\big) \end{cases} \tag{11}$$

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where $\Pi(X)$ denotes the utility indifference price of X at time 0. Conclusions in this paper can be applied for all forms of utility indifference price.

Depending on the size of the initial capital $\tilde{v}_0$, we represent optimal hedged loss functions of the problem (11) in Theorems 1 and 4 correspondingly.

Theorem 1. *If* $\Pi\left(XI_{\{X\leq VaR_{\alpha+\beta}(X)\}}\right) \leq \tilde{v}_0$, *the optimal hedged loss function satisfies*

$$f^*(x) = xI_{\{x\leq v_{\alpha+\beta}\}} \tag{12}$$

where $v_{\alpha+\beta} = VaR_{\alpha+\beta}(X)$.

Moreover, the minimal value of RVaR is zero, that is,

$$RVaR_{\alpha,\beta}(R_{f^*}(X)) = 0$$

Proof. For $f^*(x)$ defined as (12), the retained loss function is

$$R_{f^*}(x) = x - f^*(x) = x - xI_{\{x\leq v_{\alpha+\beta}\}} = xI_{\{x>v_{\alpha+\beta}\}}$$

which is non-decreasing and left continuous regarding to x, and hence, the hedged loss function $f^*(x) \in \mathcal{D}$ and $f^*(X)$ satisfies the wealth constraint.

Notice that, for any $s \leq \alpha + \beta$, we have

$$\begin{aligned} P\Big(XI_{\{X>v_{\alpha+\beta}\}} > 0\Big) &\leq P\Big(X > v_{\alpha+\beta}\Big) \\ &\leq \big(1 - (\alpha+\beta)\big) \leq 1 - s \end{aligned}$$

and consequently we get

$$VaR_s\Big(XI_{\{X>v_{\alpha+\beta}\}}\Big) = 0, \quad \forall s \leq \alpha + \beta$$

which implies

$$\begin{aligned} RVaR_{\alpha,\beta}\big(R_{f^*}(X)\big) &= RVaR_{\alpha,\beta}\big(XI_{\{X>v_{\alpha+\beta}\}}\big) \\ &= \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} VaR_s\big(XI_{\{X>v_{\alpha+\beta}\}}\big)ds \\ &= 0 \end{aligned}$$

Furthermore, for any $f \in \mathcal{D}$, we have $RVaR_{\alpha+\beta}\big(R_f(X)\big) \geq 0 = RVaR_{\alpha+\beta}\big(R_{f^*}(X)\big)$, and hence, we conclude that f^* is the optimal hedged loss function. □

Let us now consider the second case: $\tilde{v}_0 < \Pi\left(XI_{\{X\leq VaR_{\alpha+\beta}(X)\}}\right)$. To begin with, we need to recall the definition of stop-loss ordering between two random variables X_1 and X_2:

Suppose X_1 and X_2 are two random variables with finite means under a probability measure P. We say X_1 is smaller than X_2 in stop-loss order under P, if

$$E\big((X_1 - m)^+\big) \leq E\big((X_2 - m)^+\big), \quad \forall m \in \mathbb{R} \tag{13}$$

Such a relationship is denoted as $X_1 \leq_{sl}^{P} X_2$.

Lemma 2. *For a given random variable* X *and any function* $f \in \mathcal{D}$, *let*

$$g_f(x) = \min\left\{(x-d)^+, \bar{u}\right\} I_{\{x\leq v_{\alpha+\beta}\}} \tag{14}$$

where $v_\alpha = VaR_\alpha(X)$, $d = v_\alpha - f(v_\alpha)$ *and* $\bar{u}$ *is chosen such that*

$$RVaR_{\alpha,\beta}(R_f(X)) = RVaR_{\alpha,\beta}(R_{g_f}(X)) \tag{15}$$

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Then, we have

(1) g_f is well defined and $g_f \in \mathcal{D}$;
(2) $g_f(X)$ is smaller than $f(X)$ in stop-loss order under the measure P, that is, $g_f(X) \leq_{sl}^{P} f(X)$.

Remark 3. Let us denote $u = d + \bar{u}$. Since $\bar{u} \in [0, v_{\alpha+\beta} - d]$, and $d = v_\alpha - f(v_\alpha) \geq 0$, we have $d \leq u \leq v_{\alpha+\beta}$. Consequently, $g_f(x)$ can be rewritten as

$$\begin{aligned} g_f(x) &= \min\left\{(x-d)^+, u-d\right\} I_{\{x \leq v_{\alpha+\beta}\}} \\ &= \left[(x-d)^+ - (x-u)^+\right] I_{\{x \leq v_{\alpha+\beta}\}} \end{aligned} \tag{16}$$

where $0 \leq d \leq v_\alpha$ and $d \leq u \leq v_{\alpha+\beta}$.

According to Cong *et al.* (2014), the utility-based indifference price preserves the stop-loss order, that is, $\Pi(X_1) \leq \Pi(X_2)$, if $X_1 \leq_{sl}^{P} X_2$, and therefore, Lemma 2 indicates that for any $f \in \mathcal{D}$, there is a $g_f \in \mathcal{D}$ such that

$$RVaR_{\alpha,\beta}\big(R_f(X)\big) = RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big);$$

$$\Pi\big(g_f(X)\big) \leq \Pi\big(f(X)\big)$$

Thereby, we can focus on hedged loss functions with the form (16). Let us formulate another theorem about the problem (11).

Theorem 4. *If* $\tilde{v}_0 < \Pi\big(XI_{\{X \leq v_{\alpha+\beta}\}}\big)$, *the optimal hedged loss function that solves the RVaR minimisation problem (11) is*

$$f^*(x) = \left[(x-d^*)^+ - (x-u^*)^+\right] I_{\{x \leq v_{\alpha+\beta}\}} \tag{17}$$

where (d^*, u^*) *is the solution to the following 2-dimensional optimisation problem*

$$\begin{cases} \min\limits_{0 \leq d \leq v_\alpha, d \leq u \leq v_{\alpha+\beta}} d + \frac{1}{\beta}\int_\alpha^{\alpha+\beta} (VaR_s(X) - u)^+ ds, \\ s.t.\ \Pi\Big(\big[(X-d)^+ - (X-u)^+\big] I_{\{X \leq v_{\alpha+\beta}\}}\Big) \leq \tilde{v}_0 \end{cases} \tag{18}$$

Proof. By Remark 3, we know that the optimal hedged loss function admits the form (16), and hence, the objective function in (11) becomes

$$\begin{aligned} RVaR_{\alpha+\beta}\big(R_{g_f}(X)\big) &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s\big(R_{g_f}(X)\big) ds \\ &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} R_{g_f}\big(VaR_s(X)\big) ds \\ &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s(X) - g_f(VaR_s(X)) ds \\ &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s(X) - (VaR_s(X) - d) + (VaR_s(X) - u)^+ ds \\ &= d + \frac{1}{\beta}\int_\alpha^{\alpha+\beta} (VaR_s(X) - u)^+ ds \end{aligned}$$

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such that $d \in [0, v_\alpha]$ and $u \in [d, v_{\alpha+\beta}]$. Hence, (11) is rewritten as (18) and Theorem 4 is proved. □

Theorem 4 shows that the optimal strategy is to long a knock-out call option on the payout X with a strike price d^* and a barrier $v_{\alpha+\beta}$ while short another knock-out call option with a higher strike price u^* but with the same barrier. If those knock-out call options are available in the market, our optimal partial hedging is to buy (sell) and hold those options. Otherwise, it is to dynamically hedge those knock-out call options with the initial capital $\tilde{v}_0$. It is worth to mention that both Theorem 1 and Theorem 4 indicate that the optimal strategy does not cover the extreme loss that exceeds $v_{\alpha+\beta}$. This is because we only focus on the risk between levels $[\alpha, \alpha+\beta]$.

3.2 Connection among RVaR, CVaR, and VaR optimal hedging

As discussed in section 2, VaR and CVaR can be seen as two limiting cases of RVaR. In this subsection, we want to discuss connections among optimal RVaR hedging, optimal VaR hedging, and optimal CVaR hedging.

By the definition of CVaR, we have

$$CVaR_\alpha(X) = RVaR_{\alpha,1-\alpha}(X) \tag{19}$$

that is, $\beta = 1-\alpha$, and hence, we get

$$I_{\{X \le v_{\alpha+\beta}\}} = 1; \;\; P-a.s$$

According to (17), the optimal hedged loss function satisfies the form

$$\begin{aligned} f^*(x) &= \big((x-d)^+ - (x-u)^+\big) I_{\{x \le v_{\alpha+\beta}\}} \\ &= (x-d)^+ - (x-u)^+ \end{aligned} \tag{20}$$

Cong *et al*. (2014) considered the optimal CVaR hedging and showed the optimal hedged loss function f^*_{CVaR} admitted the form

$$f^*_{CVaR}(x) = (x-d)^+ - (x-u)^+ \tag{21}$$

which is consistent with (20). Therefore, the optimal CVaR hedging can be seen as a special case of the RVaR-based hedging.

As for VaR, we have

$$VaR_\alpha(X) = RVaR_{\alpha,0}(X)$$

that is, $\beta = 0$. In this case, (16) becomes

$$g_f(x) = \min\left\{(x-d)^+, \bar{u}\right\} I_{\{x \le v_\alpha\}}$$

On the one hand, we know that

$$\begin{aligned} RVaR_{\alpha,0}\big(R_f(X)\big) &= VaR_\alpha\big(R_f(X)\big) \\ &= VaR_\alpha(X) - f\big(VaR_\alpha(X)\big) \end{aligned} \tag{22}$$

On the other hand, we have

$$\begin{aligned} RVaR_{\alpha,0}\big(R_{g_f}(X)\big) &= VaR_\alpha\big(R_{g_f}(X)\big) \\ &= VaR_\alpha(X) - g_f\big(VaR_\alpha(X)\big) \\ &= VaR_\alpha(X) - \min\left\{(VaR_\alpha(X)-d)^+, \bar{u}\right\} \\ &= VaR_\alpha(X) - \min\left\{f\big(VaR_\alpha(X)\big), \bar{u}\right\} \end{aligned} \tag{23}$$

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Since $\bar{u}$ is derived from the equation

$$RVaR_{\alpha,0}\big(R_{g_f}(X)\big) = RVaR_{\alpha,0}\big(R_f(X)\big)$$

comparing (22) with (23), we have $\bar{u} \geq f\big(VaR_\alpha(X)\big)$, while from the previous discussions, we know $\bar{u} \leq VaR_{\alpha+\beta}(X) + f\big(VaR_\alpha(X)\big) - VaR_\alpha(X)$. The right hand side is $f\big(VaR_\alpha(X)\big)$ when $\beta = 0$, and hence, we arrive to $\bar{u} = f\big(VaR_\alpha(X)\big)$.

Meanwhile, the following relationship

$$\Big(x + f\big(VaR_\alpha(X)\big) - VaR_\alpha(X)\Big)^+ I_{\{x \leq VaR_\alpha(X)\}} \leq f\big(VaR_\alpha(X)\big) = \bar{u}$$

implies that the optimal hedged loss function has the form

$$\begin{aligned} f^*(x) &= \min\big\{(x-d)^+, \bar{u}\big\} I_{\{x \leq v_\alpha\}} \\ &= (x-d)^+ I_{\{x \leq v_\alpha\}} \end{aligned} \tag{24}$$

This result is consistent with the conclusion in Cong *et al.* (2013) that the optimal hedged loss function f^*_{VaR} of the VaR-based hedging problem has the form

$$f^*_{VaR}(x) = (x-d)^+ I_{\{x \leq v_\alpha\}} \tag{25}$$

and thereby, the optimal VaR hedging can be seen as a special case of the RVaR hedging.

4. Application in Equity-Linked Life Insurance Contracts

One important application of partial hedging is to deal with pricing and hedging of equity-linked life insurance contracts. Well-known papers are Melnikov & Skornyakova (2005), Kirch & Melnikov (2005), where authors discussed the implementation of quantile hedging and efficient hedging in this area. In this section, we focus on the RVaR-based optimal hedging of life insurance contracts and would compare RVaR, CVaR, and VaR hedging results.

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. Consider a financial market with a terminal time $T \in (0, \infty)$ consisting two assets. One is the riskless asset

$$S^0_t = e^{rt}, \quad t \in [0, T] \tag{26}$$

where r is the constant risk-free interest rate.

Another one is the risky asset, $(S_t)_{t\in[0,T]}$, described by a Mixed Fractional Brownian motion (MFBM) model:

$$S_t = S_0 \exp\left(\mu t + \sigma_1 B_t + \sigma_2 B^H_t\right), \quad S_0 > 0 \tag{27}$$

where B is a Brownian motion (BM) independent of a Fraction Brownian motion B^H (FBM). The constant μ is the drift, $\sigma_1 > 0$ is the volatility of the Brownian motion B, and $\sigma_2 > 0$ is the volatility of B^H.

The reason why we assume the risky asset follows MFBM model is that, for $H > \frac{1}{2}$, this model has the property of long-range dependence. Equity-linked life insurance contracts usually have long-term maturities, and hence, the factor of long-term dependence should be included in the list of key factors having a certain influence on pricing and hedging.

Let us denote the mixed process $M^\sigma_t = B_t + \sigma B^H_t$, $t \in [0, T]$, where $\sigma = \frac{\sigma_2}{\sigma_1}$. It is well known that except for $H = \frac{1}{2}$, B^H is neither a Markov process nor a semimartingale, and thus, the mixed process is also not a semimartingale with respect to the filtration generated by the BM and the FBM. However, Cheridito (2001) has proved that in the case $H \in (\frac{3}{4}, 1)$, the process M^σ is a semimartingale with respect to its natural filtration: $\mathbb{F}^M = \{\mathcal{F}^M_t \quad 0 < t < T\}$, $\mathcal{F}^M_t = \sigma\{M^\sigma_u, \quad 0 \leq u \leq t\}$. Moreover, it is equivalent in measure to a Brownian motion. In this paper, we consider the filtration $\mathbb{F}^M$ and assume $H \in (\frac{3}{4}, 1)$, $\mathcal{F} = \mathcal{F}^M_T$, and hence, there is a Brownian motion $\{W_t\}_{t\in[0,T]}$ and

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a unique real-valued Volterra kernal $r_\sigma \in L_2([0,T]^2)$ such that the following relationship holds:

$$M_t^\sigma = W_t + \int_0^t \int_0^s r_\sigma(s,u) dW_u ds, \quad t \in [0,T] \tag{28}$$

where r_σ is the unique solution of the equation:

$$\sigma^2 H(2H-1)(t-s)^{2H-2} = r_\sigma(t,s) + \int_0^s r_\sigma(t,x) r_\sigma(s,x) dx, \quad 0 \le s < t \le T$$

The MFBM market is complete, and the unique martingale measure P^* is defined by the relation (see, Melnikov & Mishura 2011):

$$Z_t^* = \frac{dP^*}{dP}|_{\mathcal{F}_t^M} = \exp\left\{ -\int_0^t \left(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u) dW_u \right) dW_s \right. \tag{29}$$

$$\left. - \frac{1}{2} \int_0^t \left(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u) dW_u \right)^2 ds \right\}$$

Note that, in this complete market, the utility indifference price of a claim is equivalent to its fair price, that $\Pi(X) = e^{-rT} E^*(X)$, where $E^*()$ is the expectation under the martingale measure P^*.

According to the Girsanov theorem, the process

$$\tilde{W}_t = W_t + \left(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2}\right) t + \int_0^t \int_0^s r_\sigma(s,u) dW_u ds \tag{30}$$

is a Wiener process under the measure P^*.

With the help of (28) and (30), the stock price process can be rewritten as

$$S_t = S_0 \exp\left\{ \mu t + \sigma_1 \left(W_t + \int_0^t \int_0^s r_\sigma(s,u) dW_u ds \right) \right\} \tag{31}$$

$$= S_0 \exp\left\{ \sigma_1 \tilde{W}_t + \left(r - \frac{\sigma_1^2}{2}\right) t \right\}$$

For equity-linked life insurance contracts, there are two sources of risk: market risk associated with the underlying asset price and insurance risk reflected by the insureds' mortality, and hence, besides characteristics of the financial market, we also need to describe mortality properties of insureds. Following actuarial traditions, let a random variable $T(x)$ on an "actuarial" probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$ denote the remaining life time of a person of current age x and ${}_Tp_x = \tilde{P}(T(x) > T)$ be the survival probability for the next T years of the insured. Since usually the insurance risk and the financial market risk have no effect on each other, we would take a natural assumption that $(\Omega, \mathcal{F}, P)$ and $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$ can be treated as independent.

Let us consider a pure endowment life insurance contract with the payoff $X = \max(S_T, K)$ provided that an insured is alive at T, where K is a constant guarantee amount. Since the mortality risk is essentially independent of the financial market, according to Brennan & Schwartz (1976), the premium of such a contract is defined as

$${}_TU_x = E^*(e^{-rT} X) E^{\tilde{P}}(I_{\{T(x) > T\}}) = {}_Tp_x V_0 \tag{32}$$

where x is the insured's age, T is the maturity time of the contract, and V_0 is perfect hedging costs of the claim X. According to the most recently published United States 2015 Life Table (National Vital Statistics Reports volume 67, Number 7), the survival probability ${}_Tp_x$ of a given insured can be found. Obviously, ${}_Tp_x V_0 < V_0$, and thus, a perfect hedging of the option is impossible and a

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partial hedging strategy should be constructed. With the help of methodologies in section 3, we derive the minimal value of RVaR that can be achieved with the initial wealth $\tilde{v}_0 = {}_T p_x V_0$.

Proposition 5. *The fair price of the claim* $X = \max\{S_T, K\}$ *is*

$$\begin{aligned} V_0 &= e^{-rT} E^*(X) \\ &= S_0 \Phi\big(\Lambda_+(K)\big) + K e^{-rT} \Phi\big(-\Lambda_-(K)\big) \end{aligned} \tag{33}$$

where Φ *is the distribution function of a standard normal random variable and*

$$\Lambda_\pm(x) = \frac{\ln \frac{S_0}{x} + (r \pm \frac{\sigma_1^2}{2})T}{\sigma_1 \sqrt{T}}$$

Proof. See Appendix A. □

In addition, applying Theorems 1 and 4 to a claim $X = \max\{S_T, K\}$, we arrive to the following results.

Theorem 6.

(a) If $\tilde{v}_0 \geq V^{\alpha+\beta} = e^{-rT} E^*(X I_{\{X \leq v_{\alpha+\beta}\}})$*, the optimal hedging strategy is a perfect hedge of the claim* $X I_{\{X \leq v_{\alpha+\beta}\}}$ *and the minimal value of risk under the measure RVaR*$_{\alpha,\beta}$ *is 0.*

(b) If $\tilde{v}_0 < V^{\alpha+\beta}$*, the optimal hedging strategy is a perfect hedge of the claim*

$$\big[(X - d^*)^+ - (X - u^*)^+\big] I_{\{X \leq v_{\alpha+\beta}\}}$$

such that (d^*, u^*) *are points of minimum of the system*

$$\begin{cases} \min\limits_{0 \leq d \leq v_\alpha, d \leq u \leq v_{\alpha+\beta}} d + \frac{1}{\beta} \int_\alpha^{\alpha+\beta} (v_s - u)^+ ds, \\ s.t.\ C(d) - C(u) \leq \tilde{v}_0 \end{cases} \tag{34}$$

where

$$\begin{aligned} V^{\alpha+\beta} &= e^{-rT} E^*(X I_{\{X \leq v_{\alpha+\beta}\}}) \\ &= S_0 \Phi^{(2)}\big(\Lambda_+(K), -\Lambda_+(v_{\alpha+\beta}), -1\big) + e^{-rT} K \Phi\big(-\Lambda_-(K)\big), \end{aligned}$$

$$\sigma_M^2 = \sigma_1^2 T + \sigma_2^2 T^{2H},$$

$$v_s = \begin{cases} K, & \text{if } \Phi\big(\frac{\ln \frac{S_0}{K} + \mu T}{\sigma_M}\big) \leq 1 - s, \\ S_0 e^{\mu T - \sigma_M z_{1-s}}, & \text{otherwise;} \end{cases}$$

$$\Lambda_{d,K} = \min\{\Lambda_-(d), \Lambda_-(K)\}, \quad \Lambda_{u,K} = \min\{\Lambda_-(u), \Lambda_-(K)\},$$

$$\begin{aligned} C(x) = {} & S_0 \Phi^{(2)}\big(\Lambda_{x,K} + \sigma_1 \sqrt{T}, -\Lambda_+(v_{\alpha+\beta}), -1\big) - x e^{-rT} \Phi^{(2)}\big(\Lambda_{x,K}, -\Lambda_-(v_{\alpha+\beta}), -1\big) \\ & + e^{-rT} (K - x)^+ \Phi\big(-\Lambda_-(K)\big) \end{aligned}$$

Here, z_{1-s} is the $(1-s)$ quantile of a standard normal random variable. $\Phi^{(2)}(z_1, z_2, \rho)$ denotes the cumulative distribution function of two jointly normally distributed random variables (Z_1, Z_2) with zero means, unit variances, and the correlation ρ.

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Table 1. Optimal hedged loss and minimal RVaR for different levels of β.

	$f^*(X)$	$RVaR_{0.95,\beta}$
$\beta = 10^{-6}$	$(X - 4.3584)^+ I_{\{X \le 1641.8\}}$	4.3584
$\beta = 0.025$	$(X - 4.35847)^+ I_{\{X \le 2105.5\}}$	4.358
β=0.04	$(X - 4.3588)^+ I_{\{X \le 2811.6\}}$	4.3588
β=0.05	$(X - 4.3588)^+$	4.3625

Proof. See Appendix B. □

We illustrate our results with the following numerical example.

Example: Assume a MFBM market with following parameters:

$$\mu = 0.1,\ \ \sigma_1 = 0.15,\ \ \sigma_2 = 0.1,\ \ S_0 = 100,\ \ H = 0.8,\ \ r = 0.05$$

An insured at age 40 buys a 10 years contract $X = \max\{S_T, K\}$ with the guarantee amount $K = 110$. By the United States 2015 Life Table, the survival probability of the insured is ${}_{10}p_{40} = 0.97462$, and hence according to equations (32) and (33), the premium that the insurance company can receive is ${}_T U_x = 0.97462 * 104.16 = 101.521$. For a fixed $\alpha = 0.95$, Table 1 shows optimal hedged loss functions and corresponding minimal values of RVaR for different levels of β.

Meanwhile, subject to the same initial capital constraint, we provide optimal hedged loss functions of VaR-based hedging and CVaR-based hedging correspondingly.

For VaR-based hedging at the level 95%, with the help of the method in Cong *et al.* (2013), we derive the optimal hedged loss function which is

$$f^*_{VaR}(X) = (X - 4.3584)^+ I_{\{X \le 1641.8\}} \tag{35}$$

For CVaR-based hedging at the level 95%, according to the method in Cong *et al.* (2014), we arrive to the conclusion that the optimal hedged loss function is

$$f^*_{CVaR}(X) = (X - 4.3588)^+ \tag{36}$$

Comparing (35) and (36) with results in Table 1, it is not hard to find that, as $\beta \to 0$, the optimal hedged loss function of RVaR hedging coincides with $f^*_{VaR}(X)$, while in another extreme case $\beta = 1 - \alpha$, it coincides with $f^*_{CVaR}(X)$. Such results are consistent with conclusions in section 3.2. Moreover, for a small value of β, the optimal RVaR hedging is close to the optimal VaR-based hedging while for a big value of β, it is closer to the optimal CVaR hedging, which indicates that the measure RVaR is a bridge between VaR and CVaR.

5. Conclusion

In this paper, explicit forms of optimal hedging strategies that minimise RVaR of a hedger's risk exposure subject to an initial wealth constraint are derived. We show that for sufficiently large hedging budget $\left(\Pi(X) > \tilde{\nu}_0 \ge \Pi(X I_{\{X < \nu_{\alpha+\beta}\}})\right)$, the optimal strategy is to hedge the entire risk up to the level $\nu_{\alpha+\beta}$. On the other hand, if $\tilde{\nu}_0 < \Pi(X I_{\{X < \nu_{\alpha+\beta}\}})$, the optimal strategy is to hedge a bull call spread on the claim itself with a knock-out barrier $\nu_{\alpha+\beta}$. In both cases, it is optimal not to hedge at all on the set $\{X > \nu_{\alpha+\beta}\}$. Our RVaR-based partial hedging method has some advantages. First, it can be applied in incomplete markets and the solution can be easily derived. Furthermore, the optimal strategy is model independent. Most importantly, we demonstrate that CVaR hedging and VaR hedging can be seen as two limiting cases of RVaR hedging, and hence, our RVaR hedging method is more general, and it is more customised since investors can set their risk appetite by choosing two risk level parameters α and β. Finally, a numerical example is provided to explain

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how such a method can be implemented to the area of life insurance even if the financial model is sophisticated.

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Appendix A: Proof for Lemma 2

First of all, we note that $RVaR_{\alpha,\beta}(R_{g_f}(X))$ is continuous and non-increasing as a function of $\bar{u}$. For $\bar{u} = 0$, we have

$$R_{g_f}(X) = X \geq R_f(X)$$

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and thereby the following inequality holds:

$$RVaR_{\alpha,\beta}\left(R_{g_f}(X)\right) \geq RVaR_{\alpha,\beta}\left(R_f(X)\right)$$

Moreover, for $\bar{u} = v_{\alpha+\beta} + f(v_\alpha) - v_\alpha$, we get

$$(x-d)^+ I_{\{x \leq v_{\alpha+\beta}\}} \leq (v_{\alpha+\beta} - d)^+ = \bar{u} \tag{A.1}$$

which implies

$$g_f(x) = (x-d)^+ I_{\{x \leq v_{\alpha+\beta}\}}$$

In this case, we have following inequalities

$$\begin{aligned} RVaR_{\alpha,\beta}\left(R_{g_f}(X)\right) &= RVaR_{\alpha,\beta}\left(X - (X-d)^+ I_{\{x \leq v_{\alpha+\beta}\}}\right) \\ &= RVaR_{\alpha,\beta}\left((X-(X-d)^+) I_{\{X \leq v_{\alpha+\beta}\}} + X I_{\{X > v_{\alpha+\beta}\}}\right) \\ &\leq RVaR_{0,\beta}\left((X-(X-d)^+) I_{\{X \leq v_{\alpha+\beta}\}}\right) + RVaR_{\alpha,\beta}\left(X I_{\{X > v_{\alpha+\beta}\}}\right) \\ &= RVaR_{0,\beta}\left(\min\{X, d\} I_{\{X < v_{\alpha+\beta}\}}\right) \\ &\leq d \end{aligned} \tag{A.2}$$

where the first inequality is due to (5) and the last inequality is because of the monotonicity of RVaR.

On the other hand, since $R_f(x)$ is non-decreasing and left continuous regarding to x, by Theorem 1 in Dhaene et al. (2002), we have $R_f(VaR_\alpha(X)) = VaR_\alpha(R_f(X))$ and hence

$$\begin{aligned} RVaR_{\alpha,\beta}\left(R_f(H)\right) &\geq VaR_\alpha(R_f(X)) = R_f(VaR_\alpha(X)) \\ &= VaR_\alpha(X) - f(VaR_\alpha(X)) \\ &= d \end{aligned} \tag{A.3}$$

Combining (A.2) and (A.3), we arrive to $RVaR_{\alpha,\beta}\left(R_f(X)\right) \geq RVaR_{\alpha,\beta}\left(R_{g_f}(X)\right)$ when $\bar{u} = v_{\alpha+\beta} - d$, and hence, there is a $\bar{u} \in [0, v_{\alpha+\beta} - d]$ that solves the equation $RVaR_{\alpha,\beta}(R_f(X)) = RVaR_{\alpha,\beta}(R_{g_f}(X))$.

Meanwhile, by the definition of $g_f(x)$, $R_{g_f}(x)$ is non-decreasing and left continuous as a function of x and therefore, we have proved $g_f(x) \in \mathcal{D}$.

In order to show that $g_f(X) \leq_{sl}^P f(X)$, let us introduce a new random variable u_α^β that is uniformly distributed on $[\alpha, \alpha+\beta]$ and is independent of all other random variables involved in the paper.

If

$$g_f\left(VaR_{u_\alpha^\beta}(X)\right) \leq_{sl}^P f\left(VaR_{u_\alpha^\beta}(X)\right) \tag{A.4}$$

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holds true, we have, for any $m \in \mathbb{R}$, that

$$\begin{aligned}
E\Big[\big(g_f(X)-m\big)^+\Big] &= \int_0^1 \big(g_f(VaR_s(X))-m\big)^+ ds \\
&= \int_0^\alpha \big(g_f(VaR_s(X))-m\big)^+ ds + \int_\alpha^{\alpha+\beta} \big(g_f(VaR_s(X))-m\big)^+ ds \\
&\quad + \int_{\alpha+\beta}^1 \big(g_f(VaR_s(X))-m\big)^+ ds \\
&= \int_0^\alpha \big(g_f(VaR_s(X))-m\big)^+ ds + \beta E\Big[\big(g_f(VaR_{u_\alpha^\beta}(X))-m\big)^+\Big] + \int_{\alpha+\beta}^1 (-m)^+ ds \\
&\le \int_0^\alpha \big(f(VaR_s(X))-m\big)^+ ds + \beta E\Big[\big(f(VaR_{u_\alpha^\beta}(X))-m\big)^+\Big] \\
&\quad + \int_{\alpha+\beta}^1 \big(f(VaR_s(X))-m\big)^+ ds \\
&= E\Big[\big(f(X)-m\big)^+\Big]
\end{aligned}$$

where the inequality is due to the non-decreasing property of $R_f(x)$ which leads to

$$VaR_s(X) - f(VaR_s(X)) \le VaR_\alpha(X) - f(VaR_\alpha(X))$$

for $s \le \alpha$, and hence,

$$g_f\big(VaR_s(X)\big) \le \big(VaR_s(X) - VaR_\alpha(X) + f(VaR_\alpha(X))\big)^+ \le f\big(VaR_s(X)\big)$$

Thus, in order to prove $g_f(X) \le_{sl}^P f(X)$, it is sufficient to show (A.4).

Rolski et al. (1999) indicated that for two random variables X_1 and X_2 with finite means, a sufficient condition for the stop-loss order $X_1 \le_{sl}^P X_2$ is as follows:

i. $E(X_1) \le E(X_2)$, and
ii. there exists $t_0 \in \mathbb{R}$ such that $P(X_1 \le t) \le P(X_2 \le t)$ for $t < t_0$ while $P(X_1 \le t) \ge P(X_2 \le t)$ for $t > t_0$.

Let us now put that $X_1 = g_f(VaR_{u_\alpha^\beta}(X))$, $X_2 = f(VaR_{u_\alpha^\beta}(X))$ and prove that these random variables satisfy the above two conditions.

According to Theorem 1 in Dhaene et al. (2002), we have

$$\begin{aligned}
RVaR_{\alpha,\beta}\big(R_f(X)\big) &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s\big(R_f(X)\big) ds \\
&= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} R_f\big(VaR_s(X)\big) ds \\
&= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} VaR_s(X) - f\big(VaR_s(X)\big) ds \\
&= RVaR_{\alpha,\beta}(X) - E\Big[f\big(VaR_{u_\alpha^\beta}(X)\big)\Big]
\end{aligned}$$

Similarly, it is true that

$$RVaR_{\alpha,\beta}\big(R_{g_f}(X)\big) = RVaR_{\alpha,\beta}(X) - E\Big[g_f\big(VaR_{u_\alpha^\beta}(X)\big)\Big]$$

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Since $RVaR_{\alpha,\beta}\left(R_{g_f}(X)\right) = RVaR_{\alpha,\beta}\left(R_f(X)\right)$, we conclude that

$$E\left[g_f\left(VaR_{u_\alpha^\beta}(X)\right)\right] = E\left[f\left(VaR_{u_\alpha^\beta}(X)\right)\right] \tag{A.5}$$

which is the desired property (i).

Notice that $VaR_{u_\alpha^\beta}(X) \geq VaR_\alpha(X)$, and therefore, we have $VaR_{u_\alpha^\beta}(X) - f(VaR_{u_\alpha^\beta}(X)) \geq VaR_\alpha(X) - f(VaR_\alpha(X))$, which implies that, for any $t < \bar{u}$,

$$\begin{aligned} P\left(g_f(VaR_{u_{\alpha^\beta}}(X)) \leq t\right) &= P\left(VaR_{u_\alpha^\beta}(X) - VaR_\alpha(X) + f\left(VaR_\alpha(X)\right) \leq t\right) \\ &\leq P\left(f(VaR_{u_\alpha^\beta}(X)) \leq t\right) \end{aligned} \tag{A.6}$$

On the other hand, for $t > \bar{u}$, we have

$$P\left(g_f(VaR_{u_\alpha^\beta}(X)) \leq t\right) = 1 \geq P\left(f(VaR_{u_\alpha^\beta}(X)) \leq t\right) \tag{A.7}$$

Combining (A.6) and (A.7), we arrive to the property (ii).

As a consequence, we have proved that $g_f(VaR_{u_\alpha^\beta}(X)) \leq_{sl}^P f(VaR_{u_\alpha^\beta}(X))$ which implies $g_f(X) \leq_{sl}^P f(X)$. □

Appendix B: Proof of Proposition 5

In the complete MFBM market, the utility indifference price of the claim $X = \max\{S_T, K\}$ is equal to its fair price, that is,

$$\Pi(X) = e^{-rT}E^*(X) = e^{-rT}\left[E^*(S_T I_{\{S_T > K\}}) + E^*(K I_{\{S_T \leq K\}})\right]$$

Note that

$$\begin{aligned} \{S_T > K\} &= \{S_0 e^{\sigma_1 \tilde{W}_T + (r - \frac{\sigma_1^2}{2})T} > K\} \\ &= \{\tilde{W}_T > \frac{\ln \frac{K}{S_0} - (r - \frac{\sigma_1^2}{2})T}{\sigma_1}\} \\ &= \{Z_1 < \Lambda_-(K)\} \end{aligned} \tag{B.1}$$

where $Z_1 = -\frac{\tilde{W}_T}{\sqrt{T}} \sim N(0,1)$ under the measure P^* and $\Lambda_-(K) = \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

To simplify the calculation, let us introduce the "Multi-asset Theorem" (see Melnikov & Romanyuk 2008):

Let $(Z_1, Z_2, \ldots, Z_J, Y)$ be $J+1$ jointly normally distributed random variables with means $(\mu_1, \mu_2, \ldots, \mu_J, \mu_y)$, variance $(\sigma_1, \sigma_2, \ldots, \sigma_J, \sigma_y)$, and the correlation matrix of $(Z_1, Z_2, \ldots, Z_J)$ is Σ. Then, for given constants z_i,

$$E(e^{-Y}\prod_{i=1}^{J} I_{\{Z_i < z_i\}}) = e^{-\mu_y + \frac{\sigma_y^2}{2}}\Phi^J(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_J; \Sigma),$$

$$\hat{z}_i = \frac{z_i - \mu_i + cov(Z_i, Y)}{\sigma_i}$$

where $\Phi^J(z_1, z_2, \ldots, z_J, \Sigma)$ denotes the cumulative distribution of J jointly normally distributed random variables $(Z_1, Z_2, \ldots, Z_n)$ with zero means, unit variance, and the correlation matrix Σ.

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By such a theorem, the price of the claim can be rewritten as:

$$\begin{aligned}\Pi(X) &= e^{-rT}\left[E^*(S_0 e^{\sigma_1 \tilde{W}_T + (r-\frac{\sigma_1^2}{2})T} I_{\{Z_1 < \Lambda_-(K)\}}) + KP^*\big(Z_1 \geq \Lambda_-(K)\big)\right] \\ &= S_0 e^{-\frac{\sigma_1^2}{2}T} E^*(e^{\sigma_1 \tilde{W}_T} I_{\{Z_1 < \Lambda_-(K)\}}) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big) \\ &= S_0 \Phi\big(\Lambda_+(K)\big) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big)\end{aligned}$$

where $\Lambda_+(K) = \Lambda_-(K) + \sigma_1\sqrt{T} = \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

Appendix C: Proof of Theorem 6

By the definition of VaR, we have

$$v_s = VaR_s(X) = \inf\left\{v : P(\max\{S_T, K\} > v) \leq 1 - s\right\}, \quad s \in (0,1)$$

Note that, if $v < K$, we get

$$P(\max\{S_T, K\} > v) = 1$$

and hence v_s satisfies $v_s \geq K$.

For $v \geq K$, the above probability can be calculated as

$$\begin{aligned}P(\max\{S_T, K\} > v) &= P\big(S_T > v\big) \\ &= P\big(S_0 e^{\mu T + \sigma_1 M_T^\sigma} > v\big) \\ &= P\big(\sigma_1 M_T^\sigma > \ln\frac{v}{S_0} - \mu T\big) \\ &= \Phi\Big(\frac{\ln\frac{S_0}{v} + \mu T}{\sigma_M}\Big) \qquad \text{(C.1)}\end{aligned}$$

where in the last equation we use the fact $\sigma_1 M_T^\sigma \sim N(0, \sigma_M^2)$ with $\sigma_M^2 = \sigma_1^2 T + \sigma_2^2 T^{2H}$, under the measure P.

If

$$\Phi\Big(\frac{\ln\frac{S_0}{K} + \mu T}{\sigma_M}\Big) \leq 1 - s$$

we conclude $v_s = K$. Otherwise, since equation (B.1) is a decreasing function regarding v, v_s is the unique solution of the equation

$$\Phi\Big(\frac{\ln\frac{S_0}{v} + \mu T}{\sigma_M}\Big) = 1 - s$$

at the interval $[K, +\infty)$, which can be rewritten as

$$v_s = S_0 e^{\mu T - \sigma_M z_{1-s}}$$

where z_{1-s} is the $1 - s$ quantile of a standard normal random variable.

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Let us start with the calculation of $V^{\alpha+\beta}$:

$$\begin{aligned} V^{\alpha+\beta} &= \Pi\big(XI_{\{X\le v_{\alpha+\beta}\}}\big) = e^{-rT}E^*(XI_{\{X\le v_{\alpha+\beta}\}}) \\ &= e^{-rT}\big[E^*(S_TI_{\{S_T\ge K\}}I_{\{S_T\le v_{\alpha+\beta}\}}) + E^*(KI_{\{S_T<K\}}I_{\{K\le v_{\alpha+\beta}\}})\big] \\ &= e^{-rT}\big[E^*(S_TI_{\{S_T\ge K\}}I_{\{S_T\le v_{\alpha+\beta}\}}) + E^*(KI_{\{S_T<K\}})\big] \end{aligned}$$

where in the last equation we utilise the fact that $K \le v_{\alpha+\beta}$.

As for $\{S_T \le v_{\alpha+\beta}\}$, we have

$$\begin{aligned} \{S_T \le v_{\alpha+\beta}\} &= \{S_0 e^{\sigma_1\tilde{W}_T + (r-\frac{\sigma_1^2}{2})T} \le v_{\alpha+\beta}\} \\ &= \{\tilde{W}_T \le \frac{\ln\frac{v_{\alpha+\beta}}{S_0} - (r-\frac{\sigma_1^2}{2})T}{\sigma_1}\} \\ &= \{Z_2 \le -\Lambda_-(v_{\alpha+\beta})\} \end{aligned}$$

where $Z_2 = \frac{\tilde{W}_T}{\sqrt{T}} \sim N(0,1)$ under the measure P^* and $\Lambda_-(v_{\alpha+\beta}) = \frac{\ln\frac{S_0}{v_{\alpha+\beta}} + (r-\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

Hence, with the help of multi-asset theorem, we arrive to

$$\begin{aligned} V^{\alpha+\beta} &= e^{-rT}\big[S_0 e^{(r-\frac{\sigma_1^2}{2})T}E^*(e^{\sigma_1\tilde{W}_T}I_{\{Z_1\le\Lambda_-(K)\}}I_{\{Z_2\le-\Lambda_-(v_{\alpha+\beta})\}}) \\ &\quad + KP^*\big(Z_1 \ge \Lambda_-(K)\big)\big] \\ &= S_0\Phi^{(2)}\big(\Lambda_+(K), -\Lambda_+(v_{\alpha+\beta}), -1\big) + Ke^{-rT}\Phi\big(-\Lambda_-(K)\big) \end{aligned}$$

where $\Lambda_+(v_{\alpha+\beta}) = \frac{\ln\frac{S_0}{v_{\alpha+\beta}} + (r+\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

Now, let us move to $\Pi\Big(\big[(X-d)^+ - (X-u)^+\big]I_{\{X\le v_{\alpha+\beta}\}}\Big) = E^*\Big(e^{-rT}\big[(X-d)^+ - (X-u)^+\big]I_{\{X\le v_{\alpha+\beta}\}}\Big)$.

Similar to the derivation of (A.1), we have

$$\{S_T > d\} = \{Z_1 < \Lambda_-(d)\}$$

where $\Lambda_-(d) = \frac{\ln\frac{S_0}{d} + (r-\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

And therefore,

$$\begin{aligned} &E^*\big((\max\{S_T,K\}-d)^+I_{\{X\le v_{\alpha+\beta}\}}\big) \\ &= E^*\big((S_T-d)^+I_{\{S_T>K\}}I_{\{S_T\le v_{\alpha+\beta}\}}\big) + E^*\big((K-d)^+I_{\{S_T\le K\}}I_{\{K\le v_{\alpha+\beta}\}}\big) \\ &= E^*\big((S_T-d)I_{\{S_T>d\}}I_{\{S_T>K\}}I_{\{S_T\le v_{\alpha+\beta}\}}\big) + E^*\big((K-d)^+I_{\{S_T\le K\}}\big) \\ &= E^*\big(S_TI_{\{Z_1<\Lambda_{d,K}\}}I_{\{Z_2\le-\Lambda_-(v_{\alpha+\beta})\}}\big) - dP^*\big(Z_1<\Lambda_{d,K}, Z_2\le-\Lambda_-(v_{\alpha+\beta})\big) \\ &\quad + (K-d)^+P^*\big(S_T\le K\big) \end{aligned}$$

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$$= S_0 e^{(r-\frac{\sigma_1^2}{2})T} E^*\big(e^{\sigma_1\sqrt{T}Z_2} I_{\{Z_1<\Lambda_{d,K}\}} I_{\{Z_2\le -\Lambda_-(\nu_{\alpha+\beta})\}}\big) - d\Phi^{(2)}\big(\Lambda_{d,K}, -\Lambda_-(\nu_{\alpha+\beta}), -1\big)$$
$$+ (K-d)^+\Big(1 - \Phi\big(\Lambda_-(K)\big)\Big)$$
$$= S_0 e^{rT}\Phi^{(2)}\big(\Lambda_{d,K} + \sigma_1\sqrt{T}, -\Lambda_+(\nu_{\alpha+\beta}), -1\big) - d\Phi^{(2)}\big(\Lambda_{d,K}, -\Lambda_-(\nu_{\alpha+\beta}), -1\big)$$
$$+ (K-d)^+\Phi\big(-\Lambda_-(K)\big)$$

where $\Lambda_{d,K} = \min\{\Lambda_-(d), \Lambda_-(K)\}$, and $\Lambda_+(\nu_{\alpha+\beta}) = \frac{\ln\frac{S_0}{\nu_{\alpha+\beta}} + (r+\frac{\sigma_1^2}{2})T}{\sigma_1\sqrt{T}}$.

With similar calculations, it is clear that

$$E^*\big((\max\{S_T, K\} - u)^+ I_{\{X\le\nu_{\alpha+\beta}\}}\big)$$
$$= S_0 e^{rT}\Phi^{(2)}\big(\Lambda_{u,K} + \sigma_1\sqrt{T}, -\Lambda_+(\nu_{\alpha+\beta}), -1\big) - u\Phi^{(2)}\big(\Lambda_{u,K}, -\Lambda_-(\nu_{\alpha+\beta}), -1\big)$$
$$+ (K-u)^+\Phi\big(-\Lambda_-(K)\big)$$

where $\Lambda_{u,K} = \min\{\Lambda_-(u), \Lambda_-(K)\}$.

Consequently, we arrive to

$$E^*\Big(e^{-rT}\big[(X-d)^+ - (X-u)^+\big] I_{\{X\le\nu_{\alpha+\beta}\}}\Big)$$
$$= e^{-rT}\Big[E^*\big((\max\{S_T, K\} - d)^+ I_{\{X\le\nu_{\alpha+\beta}\}}\big) - E^*\big((\max\{S_T, K\} - u)^+ I_{\{X\le\nu_{\alpha+\beta}\}}\big)\Big]$$
$$= C(d) - C(u)$$

where

$$C(x) = S_0\Phi^{(2)}\big(\Lambda_{x,K} + \sigma_1\sqrt{T}, -\Lambda_+(\nu_{\alpha+\beta}), -1\big) - xe^{-rT}\Phi^{(2)}\big(\Lambda_{x,K}, -\Lambda_-(\nu_{\alpha+\beta}), -1\big)$$
$$+ e^{-rT}(K-x)^+\Phi\big(-\Lambda_-(K)\big)$$

The proof of Theorem 6 is completed.

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