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REPRESENTATIONS OF COMPONENT IMPORTANCE FOR COHERENT SYSTEMS WITH EXCHANGEABLE COMPONENTS

S. PITZEN,* AND
M. BURKSCHAT,* ** *RWTH Aachen University*

Abstract

Two definitions of Birnbaum's importance measure for coherent systems are studied in the case of exchangeable components. Representations of these measures in terms of distribution functions of the ordered component lifetimes are given. As an example, coherent systems with failure-dependent component lifetimes based on the notion of sequential order statistics are considered. Furthermore, it is shown that the two measures are ordered in the case of associated component lifetimes. Finally, the limiting behavior of the measures with respect to time is examined.

Keywords: coherent system; exchangeability; component importance; signature; order statistics; sequential order statistics; association

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Secondary 60E15; 60K10

1. Introduction

In reliability theory, importance measures are used to quantify the influence of components on the performance of coherent systems. For an overview of the topic we refer to the survey papers [36, 37, 53] and the monograph [35]. Recently, importance measures in settings with exchangeable components have been examined to a larger extent. In [22], explicit expressions for Birnbaum and Barlow–Proschan importance measures in the setting of linear consecutive k-out-of-n and m-consecutive-k-out-of-n systems are obtained. Moreover, [23] and [39] also study the joint reliability importance of components in k-out-of-n and linear m-consecutive-k-out-of-n systems with exchangeable components. In [25], general representations of the joint reliability importance are obtained for all coherent systems. An extension of the Barlow–Proschan importance index is studied in [42] and explicit expressions are obtained for component lifetimes without ties. References [40] and [41] consider the concept of subsignatures of systems, which includes the Barlow–Proschan importance index. Furthermore, for arbitrary coherent systems with exchangeable component lifetimes, time-dependent representations of the Birnbaum reliability importance are derived in [24]. A representation of the Birnbaum importance for coherent systems consisting of different types of exchangeable components has been obtained in [26]. Recent results on extensions of the Birnbaum importance to non-coherent systems can be found in [2, 51].

In this paper we primarily aim to contribute to the analysis of coherent systems with exchangeable component lifetimes. We derive representations of two reliability importance measures, the first based on Birnbaum's original definition and the second using an alternative

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* Postal address: Institute of Statistics, RWTH Aachen University, D-52056 Aachen, Germany.
** Email address: marco.burkschat@rwth-aachen.de

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definition for dependent components according to Barlow and Proschan [6, pp. 26–27]. The latter measure has also been considered recently in [44] and [52]. Building on the results in [24], we obtain expressions of the importance measures in terms of the distribution functions of order statistics of the underlying exchangeable component lifetimes. Our results show that these measures can be conveniently evaluated by employing the well-known distribution theory of order statistics (cf., e.g., [3, 18]). Furthermore, we highlight the different behavior of the measures by conducting a comparative analysis with regard to ordering and limiting properties.

In order to illustrate the results, we examine in detail failure-dependent lifetimes based on the model of sequential order statistics (see [31]; see also [14, 15, 32]). Typically, the model is used to describe load-sharing scenarios in engineering applications (for this and further areas of application, see the additional references at the beginning of Section 3 and at the end of Section 4).

The paper is organized as follows: In Section 2, the representations of the two importance measures described above based on order statistics are given for systems with exchangeable components. Additionally, we briefly touch upon the reliability importance of systems with several types of exchangeable dependent components which has been considered in [26], and present the corresponding representation of the reliability importance defined for dependent components by Barlow and Proschan [6, pp. 26–27]. We also apply our results in an elementary setting of component lifetimes distributed according to a Marshall–Olkin exponential distribution. Furthermore, in Section 3 an example of a system with failure-dependent component lifetimes based on sequential order statistics is presented in detail. In Section 4, observations from the previous section are generalized. In particular, it is shown that both measures are ordered for associated lifetimes. Moreover, the limiting behavior with respect to time is studied. In particular, explicit limits in the case of failure-dependent lifetimes are given and two illustrative examples are provided.

2. Component importance for exchangeable lifetimes

Let φ denote the structure function of a coherent system. Moreover, let $T_1, \ldots, T_n$ denote the random variables which describe the component lifetimes in the considered system. In the following, we assume that the support of the distribution of each lifetime T_i is given by the interval $[0, \infty)$. The corresponding component states at time $t \geq 0$ are given by

$$X_i(t) = 1_{(t,\infty)}(T_i), \qquad i = 1, \ldots, n,$$

with 1_A being the indicator function of the set A. This implies that the component state 1 indicates a working component (at time t) and 0 indicates a failed component (at time t). The vector of component states is denoted by $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$.

Definition 1. *cf. [9]* The time-dependent Birnbaum reliability importance of component i at a fixed time t is defined as

$$I_{\mathrm{B}}(i; t) = \mathbb{P}(\varphi(\boldsymbol{X}(t)) = 1 \mid X_i(t) = 1) - \mathbb{P}(\varphi(\boldsymbol{X}(t)) = 1 \mid X_i(t) = 0). \tag{1}$$

For arbitrary coherent systems, [24] obtained a representation of the preceding reliability importance for exchangeable lifetimes $T_1, \ldots, T_n$. Recall that the random vector $(T_1, T_2, \ldots, T_n)$ is called exchangeable if its joint distribution coincides with that of any permutation of the random variables $T_1, T_2, \ldots, T_n$, i.e.

$$\mathbb{P}(T_1 \leq t_1, T_2 \leq t_2, \ldots, T_n \leq t_n) = \mathbb{P}(T_{\pi(1)} \leq t_1, T_{\pi(2)} \leq t_2, \ldots, T_{\pi(n)} \leq t_n)$$

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for every $t_1, \dots, t_n \in \mathbb{R}$ and any permutation π of the numbers $1, \dots, n$. Then, the reliability importance in (1) can be expressed as (see [24])

$$I_{\mathrm{B}}(i;t) = \sum_{k=1}^{n} \left[s_k^{+i} \mathbb{P}(T_{k:n} > t \mid T_i > t) - s_k^{-i} \mathbb{P}(T_{k:n} > t \mid T_i \le t) \right], \tag{2}$$

where $T_{1:n} \le \cdots \le T_{n:n}$ are the order statistics of the random variables $T_1, \dots, T_n$, and

$$s_k^{+i} := \frac{r_{n-k+1}^{+i}}{\binom{n-1}{n-k}} - \frac{r_{n-k}^{+i}}{\binom{n-1}{n-k-1}}, \quad k \in \{1, \dots, n-1\}, \quad s_n^{+i} := \frac{r_1^{+i}}{\binom{n-1}{0}} = r_1^{+i}; \tag{3}$$

$$s_k^{-i} := \frac{r_{n-k+1}^{-i}}{\binom{n-1}{n-k+1}} - \frac{r_{n-k}^{-i}}{\binom{n-1}{n-k}}, \quad k \in \{2, \dots, n\}, \quad s_1^{-i} := 0 \tag{4}$$

for $i = 1, \dots, n$. The parameters r_k^{+i} and r_k^{-i} can be defined as the sums

$$r_k^{+i} := \sum_{|A|=k, i \in A} \varphi(1_A(1), \dots, 1_A(n)), \qquad r_k^{-i} := \sum_{|A|=k, i \notin A} \varphi(1_A(1), \dots, 1_A(n)),$$

ranging over sets $A \subset \{1, \dots, n\}$, i.e. they denote the number of path sets of size k of the considered system which contain and do not contain the component i, respectively. In particular, $r_n^{-i} = 0$, $r_n^{+i} = 1$, and $r_0^{-i} = r_0^{+i} = 0$. Observe that the representation in (2) involves conditional probabilities of the order statistics of the component lifetimes given the state of a component lifetime at time t. These conditional probabilities are in most cases not directly given and require further investigation.

In the next theorem we give another representation of the Birnbaum reliability importance. The joint distribution of the component lifetimes enters into this representation solely via the marginal distributions of the corresponding order statistics. The distribution (survival) function of an order statistic $T_{i:n}$ of the $T_1, \dots, T_n$ is denoted by $F_{i:n}$ ($\overline{F}_{i:n} = 1 - F_{i:n}$). Note that $F_{i:n}$ (and $\overline{F}_{i:n}$) can usually be easily evaluated (cf. Remarks 1 and 2).

Theorem 1. *Let $T_1, \dots, T_n$ be the component lifetimes of a coherent system of order n. If $T_1, \dots, T_n$ are exchangeable, then*

$$\begin{aligned} I_{\mathrm{B}}(i;t) = {} & \left(\sum_{j=1}^{n} \overline{F}_{j:n}(t) \right)^{-1} \sum_{k=1}^{n} s_k^{+i} \left(\sum_{m=1}^{k-1} \overline{F}_{m:n}(t) + (n-k+1) \overline{F}_{k:n}(t) \right) \\ & - \left(\sum_{j=1}^{n} F_{j:n}(t) \right)^{-1} \sum_{k=2}^{n} s_k^{-i} \left((k-1) \overline{F}_{k:n}(t) - \sum_{m=1}^{k-1} \overline{F}_{m:n}(t) \right) \end{aligned} \tag{5}$$

for $i = 1, \dots, n$ and $t > 0$.

Proof. Let $\phi_{k:n}$ denote the coherent life function (see, e.g., [27] or [6, p. 12]) of a k-out-of-n:F system, i.e. a coherent system which fails if and only if k or more of its components fail. Then $\phi_{k:n}(x_1, \dots, x_n)$ is invariant with respect to permutations of its arguments $x_1, \dots, x_n$. Furthermore, by definition $T_{k:n} = \phi_{k:n}(T_1, \dots, T_n)$ holds. Now, for $i, k \in \{1, \dots, n\}$, consider

$$\begin{aligned} \mathbb{P}(T_{k:n} > t, T_i > t) &= \mathbb{E}(1_{(t,\infty)}(T_{k:n}) \cdot 1_{(t,\infty)}(T_i)) \\ &= \mathbb{E}(1_{(t,\infty)}(\phi_{k:n}(T_1, \dots, T_n)) \cdot 1_{(t,\infty)}(T_i)). \end{aligned}$$

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Since $T_1, \ldots, T_n$ are exchangeable, we get

$$\mathbb{P}(T_{k:n} > t, T_i > t) = \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}(1_{(t,\infty)}(\phi_{k:n}(T_1, \ldots, T_n)) \cdot 1_{(t,\infty)}(T_m)).$$

Thus, because

$$\sum_{m=1}^{n} 1_{(t,\infty)}(T_m) = \sum_{m=1}^{n} 1_{(t,\infty)}(T_{m:n}),$$

we can further conclude that

$$\mathbb{P}(T_{k:n} > t, T_i > t) = \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}(1_{(t,\infty)}(\phi_{k:n}(T_1, \ldots, T_n)) \cdot 1_{(t,\infty)}(T_{m:n})).$$

Consequently, we obtain

$$\mathbb{P}(T_{k:n} > t, T_i > t) = \frac{1}{n} \sum_{m=1}^{n} \mathbb{P}(T_{k:n} > t, T_{m:n} > t)$$

for $i, k \in \{1, \ldots, n\}$. Analogously, it can be shown that

$$\mathbb{P}(T_{k:n} > t, T_i \leq t) = \frac{1}{n} \sum_{m=1}^{n} \mathbb{P}(T_{k:n} > t, T_{m:n} \leq t), \qquad \mathbb{P}(T_i > t) = \frac{1}{n} \sum_{m=1}^{n} \overline{F}_{m:n}(t).$$

Moreover, we have in general

$$\mathbb{P}(T_{k:n} > t, T_{m:n} > t) = \begin{cases} \overline{F}_{m:n}(t), & \text{if } 1 \leq m < k, \\ \overline{F}_{k:n}(t), & \text{if } k \leq m \leq n, \end{cases}$$

and

$$\begin{aligned} \mathbb{P}(T_{k:n} > t, T_{m:n} \leq t) &= \overline{F}_{k:n}(t) - \mathbb{P}(T_{k:n} > t, T_{m:n} > t) \\ &= \begin{cases} \overline{F}_{k:n}(t) - \overline{F}_{m:n}(t), & \text{if } 1 \leq m < k, \\ 0, & \text{if } k \leq m \leq n. \end{cases} \end{aligned}$$

By plugging the obtained expressions into (2), we get the stated representation. □

Remark 1. Considering independent and identically distributed (i.i.d.) lifetimes $T_1, \ldots, T_n$ with marginal distribution function F, the distribution of a single order statistic $T_{k:n}$, $k \in \{1, \ldots, n\}$, is given by

$$F_{k:n}(t) = \sum_{i=k}^{n} \binom{n}{i} \big(F(t)\big)^i \big(1 - F(t)\big)^{n-i}$$

as developed, e.g., in [18, p. 9]. In this case, the representation in (5) solely depends on the distribution function of a single component.

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Remark 2. The representation in Theorem 1 is primarily useful in situations where the distributions of the order statistics of the lifetimes $T_1, \ldots, T_n$ are given in an accessible form. However, for other joint distributions it might be easier to find the marginal distributions of random variables $T_{1:i} = \min(T_1, \ldots, T_i)$, or alternatively $T_{i:i} = \max(T_1, \ldots, T_i)$, $i = 1, \ldots, n$. In that case, Theorem 1 can still be used by applying the following relations for distributions of exchangeable random variables:

$$F_{r:n}(t) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \binom{n}{j} F_{1:j}(t) = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \binom{n}{j} F_{j:j}(t)$$

(see, e.g., [18, p. 46], or [46, Proposition 4.1]; see also [24, Lemma 1]).

Originally, [9] applied the importance measure to systems with independent components. Reference [6] used another representation of Birnbaum's measure to define an importance measure for the case of dependent components. Both measures coincide when the components in the system are independent. For fixed t, the time-dependent version of the reliability importance of component i considered in Barlow and Proschan [6, p. 26, Eq. (1.10c), and Remark (c) on p. 27] (see also [37, pp. 346–347]) is defined by

$$I_{\mathrm{B,\,BP}}(i; t) = \mathbb{E}\big(\varphi(1_i, \boldsymbol{X}(t)) - \varphi(0_i, \boldsymbol{X}(t))\big), \tag{6}$$

where for an arbitrary vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the usual notation $(\cdot_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ is employed (cf. [6, p. 4]). In particular,

$$(0_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n), \qquad (1_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n).$$

Clearly, if the component lifetimes or the component states are independent, then (1) and (6) are identical. Otherwise, both representations will usually lead to different values.

Remark 3. Note that the measure $I_{\mathrm{B,\,BP}}(i; t)$ should not be confused with the *Barlow-Proschan importance index*

$$I_{\mathrm{BP}}(i) = \mathbb{P}(T = T_i)$$

from [5], where T denotes the system lifetime, which was also considered earlier as the Shapley–Shubik index in cooperative game theory (see, e.g., [42] or [47, p. 75]). Moreover, if the component states are independent and, additionally,

$$\mathbb{P}(X_i(t) = 1) = \frac{1}{2}, \qquad i = 1, \ldots, n,$$

holds, then $I_{\mathrm{B}}(i; t) = I_{\mathrm{B,\,BP}}(i; t)$ and this value coincides with the *structural importance of component i* (see [6, p. 13, Eq. (3.10)]). In particular, the value has also been studied under the name absolute Banzhaf index in game theory (see, e.g., [47, p. 75]).

In the following theorem, we also give a representation of the importance measure (6) in terms of distribution functions of the ordered component lifetimes.

Theorem 2. *Let $T_1, \ldots, T_n$ be the component lifetimes of a coherent system of order n. If $T_1, \ldots, T_n$ are exchangeable, then*

$$I_{\mathrm{B,\,BP}}(i; t) = \frac{1}{n} \sum_{k=1}^{n} \big(s_k^{+i} - s_{k+1}^{-i}\big)\big((n-k)\overline{F}_{k:n}(t) + k\overline{F}_{k+1:n}(t)\big) \tag{7}$$

for $i = 1, \ldots, n$, $t > 0$ with $s_{n+1}^{-i} := 0$, $\overline{F}_{n+1:n} \equiv 1$.

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Proof. For $i = 1, \ldots, n$ and $t > 0$, the considered importance measure can be expressed as

$$\begin{aligned} I_{\mathrm{B,BP}}(i;t) &= \mathbb{P}(\varphi(1_i, \boldsymbol{X}(t)) = 1) - \mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1) \\ &= [\mathbb{P}(T > t \mid T_i > t) - \mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1)]\mathbb{P}(T_i > t) \\ &\quad + [\mathbb{P}(\varphi(1_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 0) - \mathbb{P}(T > t \mid T_i \le t)]\mathbb{P}(T_i \le t). \end{aligned} \tag{8}$$

In order to derive representations for the terms in (8), we will use arguments similar to those in [24]. According to equations (5) and (6) in that paper, we already know that

$$\mathbb{P}(T > t \mid T_i > t) = \sum_{k=1}^{n} s_k^{+i} \mathbb{P}(T_{k:n} > t \mid T_i > t), \tag{9}$$

$$\mathbb{P}(T > t \mid T_i \le t) = \sum_{k=2}^{n} s_k^{-i} \mathbb{P}(T_{k:n} > t \mid T_i \le t). \tag{10}$$

To obtain the remaining terms in (8), consider at first

$$\begin{aligned} \mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1) = \sum_{k=1}^{n} &\mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1, S_n(t) = k) \\ &\times \mathbb{P}(S_n(t) = k \mid T_i > t), \end{aligned} \tag{11}$$

where $S_n(t) = \sum_{j=1}^{n} X_j(t)$ denotes the number of active components at time t. By introducing the set

$$\begin{aligned} M_{\varphi}^{\mathcal{P},-i} := \big\{ &\big(1_{P'}(1), \ldots, 1_{P'}(n)\big) \in \{0,1\}^n \mid \\ &P' \subseteq \{1, \ldots, n\},\ P' \ni i,\ P' \setminus \{i\} \text{ is path set of } \varphi \big\}, \end{aligned}$$

we obtain

$$\mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1, S_n(t) = k) = \sum_{\substack{\boldsymbol{x} \in M_{\varphi}^{\mathcal{P},-i} \\ \sum_{j=1}^{n} x_j = k}} \mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{x} \mid X_i(t) = 1,\ S_n(t) = k). \tag{12}$$

Since exchangeability of the component lifetimes $T_1, \ldots, T_n$ transfers to the states $X_1(t), \ldots, X_n(t)$, we get

$$\mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{x}_1 \mid X_i(t) = 1,\ S_n(t) = k) = \mathbb{P}(\boldsymbol{X}(t) = \boldsymbol{x}_2 \mid X_i(t) = 1,\ S_n(t) = k)$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B_n^{k,+i} := \big\{\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0,1\}^n \mid x_i = 1, \sum_{j=1}^{n} x_j = k\big\}$, and hence

$$\mathbb{P}\big(\boldsymbol{X}(t) = \boldsymbol{x} \mid X_i(t) = 1,\ S_n(t) = k\big) = \frac{1}{\binom{n-1}{k-1}}$$

for every $\boldsymbol{x} \in B_n^{k,+i}$, because the number of elements in this set is given by $|B_n^{k,+i}| = \binom{n-1}{k-1}$. Therefore, it follows from (12) by using $|\{\boldsymbol{x} \in M_{\varphi}^{\mathcal{P},-i} \mid x_i = 1, \sum_{j=1}^{n} x_j = k\}| = r_{k-1}^{-i}$ that

$$\mathbb{P}(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1, S_n(t) = k) = \frac{r_{k-1}^{-i}}{\binom{n-1}{k-1}}. \tag{13}$$

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Furthermore, for $k \in \{1, \ldots, n-1\}$,

$$\mathbb{P}(S_n(t) = k \mid T_i > t) = \mathbb{P}(T_{n-k+1:n} > t \mid T_i > t) - \mathbb{P}(T_{n-k:n} > t \mid T_i > t),$$
$$\mathbb{P}(S_n(t) = n \mid T_i > t) = \mathbb{P}(T_{1:n} > t \mid T_i > t).$$

By plugging these expressions and (13) into (11), it can be shown after some calculations that

$$\mathbb{P}\big(\varphi(0_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 1\big) = \sum_{k=1}^{n-1} s_{k+1}^{-i} \mathbb{P}(T_{k:n} > t \mid T_i > t). \tag{14}$$

Applying a similar argument but now using the set

$$M_\varphi^{\mathcal{P},+i} := \big\{\big(1_{P'}(1), \ldots, 1_{P'}(n)\big) \in \{0,1\}^n \mid P' \subseteq \{1, \ldots, n\} \setminus \{i\},\ P' \cup \{i\} \text{ is path set of } \varphi\big\},$$

it can be proved that

$$\mathbb{P}(\varphi(1_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 0, S_n(t) = k) = \frac{r_{k+1}^{+i}}{\binom{n-1}{k}}. \tag{15}$$

Then, we obtain

$$\mathbb{P}(\varphi(1_i, \boldsymbol{X}(t)) = 1 \mid X_i(t) = 0) = \sum_{k=2}^{n+1} s_{k-1}^{+i} \mathbb{P}(T_{k:n} > t \mid T_i \le t), \tag{16}$$

where $T_{n+1:n} := \infty$. Utilizing (9), (10), (14), and (16) in (8) (with $s_{n+1}^{-i} := 0$) and arguing analogously to the proof of Theorem 1, we arrive at the representation in the theorem after some rearrangements. □

Remark 4. By definitions (1) and (6) of the importance measures, the representations (5) and (7) coincide for i.i.d. component lifetimes $T_1, \ldots, T_n$.

Remark 5. In a recent paper by Eryilmaz et al. [26], a representation of the Birnbaum importance I_{B} for systems with exchangeable components of $K \ge 2$ types is given. In the following, we add a corresponding representation for the measure $I_{\mathrm{B,BP}}$. Since no specific dependence between components of different types is assumed, the resulting expressions are clearly less tractable than in the case of only one type of exchangeable component. However, arguments similar to those used in [26] and for showing Theorem 2 above can be applied. Consider a coherent system with n components of $K \ge 2$ types. The number of components of type v is denoted by n_v, $1 \le v \le K$, with $n = n_1 + \cdots + n_K$. Moreover, the lifetimes of the n_v components of type v are denoted by random variables $T_1^{(v)}, \ldots, T_{n_v}^{(v)}$ and the corresponding component states at time $t \ge 0$ by $X_1^{(v)}(t), \ldots, X_{n_v}^{(v)}(t)$. Assume that in the random vector of all component lifetimes the random lifetimes belonging to components of the same type are exchangeable dependent. Let $S_{n_v}^{(v)}(t)$ denote the number of active components of type v at time $t \ge 0$. By arguing analogously to equations (9) and (10) of [26], and deriving the coefficients similarly to equations (13) and (15) above, the following representation of the importance measure $I_{\mathrm{B,BP}}(i;t)$ can be shown: If component i is of type a, then

$$I_{\mathrm{B,BP}}(i;t) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} c_{n_1,\ldots,n_K}(l_1, \ldots, l_K) \mathbb{P}\big(S_{n_1}^{(1)}(t) = l_1, \ldots, S_{n_K}^{(K)}(t) = l_K\big)$$

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with coefficients satisfying

$$c_{n_1,\ldots,n_K}(l_1,\ldots,l_K)\cdot\binom{n_1}{l_1}\cdots\binom{n_K}{l_K}$$
$$=\left[r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)+r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_a+1,\ldots,l_K)\right.$$
$$\left.-r^{-i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)-r^{-i}_{n_1,\ldots,n_K}(l_1,\ldots,l_a-1,\ldots,l_K)\right],$$

where we put

$$r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_{a-1},0,l_{a+1},\ldots,l_K)=r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_{a-1},n_a+1,l_{a+1},\ldots,l_K)=0$$

and $r^{-i}_{n_1,\ldots,n_K}(l_1,\ldots,l_{a-1},-1,l_{a+1},\ldots,l_K)=0$. Here, $r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)$ denotes the number of path sets of the system including component i of type a, l_1 components of type $1,\ldots,l_a-1$ additional components of type $a,\ldots,l_K$ components of type K, and $r^{-i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)$ denotes the number of path sets of the system that do not include component i of type a, and include l_1 components of type $1,\ldots,l_a$ components of type $a,\ldots,l_K$ components of type K (see [26]). In particular,

$$r_{n_1,\ldots,n_K}(l_1,\ldots,l_K)=r^{+i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)+r^{-i}_{n_1,\ldots,n_K}(l_1,\ldots,l_K)$$

holds, where $r_{n_1,\ldots,n_K}(l_1,\ldots,l_K)$ denotes the number of path sets of the system including exactly l_1 components of type $1,\ldots,$ exactly l_K components of type K.

Moreover, the probabilities $\mathbb{P}(S^{(1)}_{n_1}(t)=l_1,\ldots,S^{(K)}_{n_K}(t)=l_K)$ can be evaluated in terms of joint survival functions of the system lifetimes as follows (see [26], Eq. (3)):

$$\mathbb{P}(S^{(1)}_{n_1}(t)=l_1,\ldots,S^{(K)}_{n_K}(t)=l_K)=\binom{n_1}{l_1}\cdots\binom{n_K}{l_K}\cdot S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K),$$

where

$$S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K)$$
$$=\sum_{i_1=0}^{n_1-l_1}\cdots\sum_{i_K=0}^{n_K-l_K}(-1)^{i_1+\cdots+i_K}\binom{n_1-l_1}{i_1}\cdots\binom{n_K-l_K}{i_K}$$
$$\times\,\mathbb{P}\left(T^{(1)}_1>t,\ldots,T^{(1)}_{l_1+i_1}>t,\ldots,T^{(K)}_1>t,\ldots,T^{(K)}_{l_K+i_K}>t\right).$$

For the case of i.i.d. component lifetimes $T_1,\ldots,T_n$, [24] establishes a representation of the importance measure (1) based on survival functions of order statistics with sample size $n-1$. The corollary below shows that this representation continues to hold for exchangeable lifetimes when definition (6) is applied.

Corollary 1. *Let $T_1,\ldots,T_n$ be the component lifetimes of a coherent system of order $n\geq 2$. If $T_1,\ldots,T_n$ are exchangeable, then*

$$I_{\mathrm{B,BP}}(i;t)=\sum_{k=1}^{n}\left(s^{+i}_k\overline{F}_{k:n-1}(t)-s^{-i}_k\overline{F}_{k-1:n-1}(t)\right)$$

for $i=1,\ldots,n$ and $t>0$, where $\overline{F}_{0:n-1}\equiv 0$, $\overline{F}_{n:n-1}\equiv 1$.

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Proof. The result follows from Theorem 2 and the well-known triangle rule for distribution functions $F_{k:n-1} = \frac{n-k}{n} F_{k:n} + \frac{k}{n} F_{k+1:n}$, $1 \le k < n$, of order statistics based on exchangeable random variables $T_1, \ldots, T_n$ (cf., e.g., [17]). □

The coefficients s_k^{+i} and s_k^{-i} have properties similar to the entries of the well-known signature vector (see, e.g., [48]) of a coherent system. For example, in [24] it is shown that

$$\sum_{k=1}^{n} s_k^{+i} = 1, \qquad \sum_{k=1}^{n} s_k^{-i} = \begin{cases} 1, & \text{if there exists } \; k : s_k^{-i} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Moreover, the following lemma yields that the coefficients are also non-negative. Therefore, they can be interpreted as probability mass functions of a discrete distribution (at least if there is some positive coefficient).

Lemma 1. *The coefficients s_k^{+i} and s_k^{-i} given in (3) and (4) satisfy*

$$s_k^{+i} \ge 0, \quad s_k^{-i} \ge 0, \qquad k = 1, \ldots, n.$$

Proof. We will show the claim for s_k^{+i} at first. By definition, $r_k^{+i} \ge 0$ for all $i, k \in \{1, \ldots, n\}$. Hence, we get immediately $s_n^{+i} = r_1^{+i} \ge 0$. Furthermore,

$$\begin{aligned} & s_k^{+i} \ge 0 \quad \text{for all } k \in \{1, \ldots, n-1\} \\ \Leftrightarrow \quad & \frac{r_{k+1}^{+i}}{\binom{n-1}{k}} - \frac{r_k^{+i}}{\binom{n-1}{k-1}} \ge 0 \quad \text{for all } k \in \{1, \ldots, n-1\} \\ \Leftrightarrow \quad & k \cdot r_{k+1}^{+i} \ge (n-k) \cdot r_k^{+i} \quad \text{for all } k \in \{1, \ldots, n-1\}. \end{aligned}$$

In the following, we will show the latter inequality. As in [37], let $\mathcal{P}_i(d)$ denote the set of path sets of size d containing component i. Note that

$$|\{P \in \mathcal{P}_i(k+1) \mid P' \subset P\}| = n - k$$

for all $P' \in \mathcal{P}_i(k)$, because every path set of size k which contains component i can be extended to a path set of size $k+1$ by adding one of the remaining $n-k$ components. Then, we obtain

$$\begin{aligned} (n-k) \cdot r_k^{+i} &= \sum_{P' \in \mathcal{P}_i(k)} |\{P \in \mathcal{P}_i(k+1) \mid P' \subset P\}| = \sum_{P \in \mathcal{P}_i(k+1)} |\{P' \in \mathcal{P}_i(k) \mid P' \subset P\}| \\ &\le \sum_{P \in \mathcal{P}_i(k+1)} |\{P' \subset \{1, \ldots, n\} \mid i \in P', |P'| = k, P' \subset P\}| \\ &= k \cdot r_{k+1}^{+i}, \end{aligned}$$

since the number of subsets of size k, which contain the element i, of a set with $k+1$ elements including i is given by $\binom{k}{k-1} = k$. For proving the assertion for s_k^{-i}, just note that

$$s_n^{-i} = \frac{r_1^{-i}}{\binom{n-1}{1}} \ge 0, \qquad s_1^{-i} = 0,$$

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TABLE 1: Exemplary (rounded) values for the importance measures I_{B} and $I_{\mathrm{B,BP}}$ calculated in the setting of Example 1 with $\lambda_1 = 2 \cdot 10^{-3}$ and $\lambda_{12345} = 4 \cdot 10^{-3}$.

t	i	$I_{\mathrm{B}}(i;t)$	$I_{\mathrm{B,BP}}(i;t)$
2000	1, 2, 4, 5	0.7510	0.1432
	3	0.5909	0.0439
4000	1, 2, 4, 5	0.6467	0.0747
	3	0.4411	0.0247

and, furthermore, for all $k \in \{1, \ldots, n-2\}$,

$$s_k^{-i} \geq 0 \quad \text{for all } k \in \{2, \ldots, n-1\} \quad \Leftrightarrow \quad (k+1) \cdot r_{k+1}^{-i} \geq (n-k-1) \cdot r_k^{-i}.$$

Therefore, the result can be proved by arguing similarly to the first case. □

The above representations can be effectively employed due to the well-developed distribution theory of order statistics (see Remarks 1 and 2). Let us consider the following elementary example.

Example 1. Consider a classic bridge system of order 5 (cf., e.g., [48, p. 9]). Then, as developed in [24],

$$\boldsymbol{s}^{+3} = \left(0, \tfrac{1}{3}, \tfrac{2}{3}, 0, 0\right), \qquad \boldsymbol{s}^{-3} = \left(0, 0, \tfrac{2}{3}, \tfrac{1}{3}, 0\right),$$
$$\boldsymbol{s}^{+i} = \left(0, \tfrac{1}{6}, \tfrac{7}{12}, \tfrac{1}{4}, 0\right), \quad \boldsymbol{s}^{-i} = \left(0, \tfrac{1}{4}, \tfrac{7}{12}, \tfrac{1}{6}, 0\right),$$

for $i = 1, 2, 4, 5$. (Note that $\boldsymbol{s}^{-3}$ is actually affected by an arithmetic error in [24]).

Let $T_1, \ldots, T_5$ be the component lifetimes of the system, which we assume follow a particular Marshall–Olkin exponential distribution (cf. [6, 43]), i.e. the joint survival function of $(T_1, \ldots, T_5)$ is given by

$$\overline{F}(t_1, \ldots, t_5) = \exp\left\{-\lambda_1 \sum_{i=1}^{5} t_i - \lambda_{12345} \max\{t_1, \ldots, t_5\}\right\},$$

where $\lambda_1, \lambda_{12345} > 0$ and $t_i \geq 0$, $i = 1, \ldots, 5$. Note that then $T_1, \ldots, T_5$ are not independent, but exchangeable. Moreover, they are known to be associated. Using the representations from Theorems 1 and 2 and Remark 2, and

$$\begin{aligned}\overline{F}_{1:j}(t) &= \mathbb{P}(\min\{T_1, \ldots, T_j\} > t) = \mathbb{P}(T_1 > t, \ldots, T_j > t) \\ &= \overline{F}(\underbrace{t, \ldots, t}_{j \text{ times}}, \underbrace{0, \ldots, 0}_{5-j \text{ times}}) = \exp\{-(j\lambda_1 + \lambda_{12345})t\}\end{aligned}$$

for $j = 1, \ldots 5$, we are able to calculate $I_{\mathrm{B}}(i;t)$ and $I_{\mathrm{B,BP}}(i;t)$ for all components of the bridge system. Some exemplary values are given in Table 1.

3. Component importance for failure-dependent lifetimes

As an extended example, we will study so-called failure-dependent lifetimes based on the notion of the sequential order statistics defined in [31]. In this model, after the failure of a component, the underlying lifetime distribution of the remaining components may change in order

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to take into account effects of previous failures. Failure-dependent component lifetimes are modeled with exchangeable random variables obtained via a random permutation of sequential order statistics. In this way, the model represents a framework for describing systems, where the performance of active components may be influenced by previous component failures (see [1, 10, 45]; see also [29]). By construction, the order statistics of the failure-dependent component lifetimes coincide with sequential order statistics.

Note that the model of sequential order statistics can be applied to describe load-sharing scenarios in engineering applications, where the load of a failed component is distributed among the remaining active components. For modeling with sequential order statistics in this area, see, e.g, [4, 8, 15] and the references therein; see also [21] and [38] for general fields of applications.

In the following, let $T^*_{1:n}, \ldots, T^*_{n:n}$ denote the sequential order statistics based on absolutely continuous distribution functions $F_1, F_2, \ldots, F_n$ with corresponding density functions $f_1, \ldots, f_n$. Moreover, we denote the corresponding survival functions by $\overline{F}_i = 1 - F_i$, $i = 1, \ldots, n$.

Sequential order statistics can be introduced as follows (cf. [31]; see also [12]): Assume that the n components in a system start to work at time $t = 0$. The lifetimes of the components are described by independent random variables $T^{(1)}_1, T^{(1)}_2, \ldots, T^{(1)}_n$ which are identically distributed according to the distribution function F_1. The first failure time of a component is then given by

$$T^*_{1:n} = \min\left\{T^{(1)}_1, T^{(1)}_2, \ldots, T^{(1)}_n\right\}.$$

Given that the first failure occurs at time $t_1 = T^*_{1:n}$, we assume that the residual lifetimes of the $n-1$ remaining components have the same distribution as the residual lifetimes of i.i.d. components with age t_1 and distribution function F_2 (instead of F_1). The distribution function F_2 is used to model the influence of the first component failure on the behavior of the other components. The lifetimes of these components will be represented by random variables $T^{(2)}_1, T^{(2)}_2, \ldots, T^{(2)}_{n-1}$. Given that $T^*_{1:n} = t_1$, the random variables $T^{(2)}_1, T^{(2)}_2, \ldots, T^{(2)}_{n-1}$ are i.i.d. according to the truncated distribution function $F_2(\cdot \mid t_1)$ with $\overline{F}_2(x \mid t_1) = \overline{F}_2(x)/\overline{F}_2(t_1)$ for $x \geq t_1$. In particular, given that $T^*_{1:n} = t_1$, these random variables satisfy $T^{(2)}_i \geq t_1$ for $i = 1, 2, \ldots, n-1$. Then, the second failure time in the system is given by

$$T^*_{2:n} = \min\left\{T^{(2)}_1, T^{(2)}_2, \ldots, T^{(2)}_{n-1}\right\}.$$

Proceeding in this way, we assume that the ith component failure occurs at time $t_i = T^*_{i:n}$. Then we assume that the residual lifetimes of the $n-i$ remaining components have the same distribution as the residual lifetimes of i.i.d. components with age t_i and distribution function F_{i+1}. The distribution function F_{i+1} describes the behavior of the remaining components after the occurrence of the ith failure. The lifetimes of the components will be represented by $T^{(i+1)}_1, T^{(i+1)}_2, \ldots, T^{(i+1)}_{n-i}$. Given that $T^*_{i:n} = t_i$, the random variables $T^{(i+1)}_1, T^{(i+1)}_2, \ldots, T^{(i+1)}_{n-i}$ are i.i.d. according to the truncated distribution function $F_{i+1}(\cdot \mid t_i)$ with $\overline{F}_{i+1}(x \mid t_i) = \overline{F}_{i+1}(x)/\overline{F}_{i+1}(t_i)$ for $x \geq t_i$. Then, the $(i+1)$th component failure time is

$$T^*_{i+1:n} = \min\left(T^{(i+1)}_1, T^{(i+1)}_2, \ldots, T^{(i+1)}_{n-i}\right).$$

Finally, if the $(n-1)$th component failure occurs at time $t_{n-1} = T^*_{n-1:n}$, then the last failure time is given by $T^*_{n:n}$ with the reliability function $\overline{F}_n(x \mid t_{n-1}) = \overline{F}_n(x)/\overline{F}_n(t_{n-1})$ for $x \geq t_{n-1}$.

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The formal definition of sequential order statistics can be found in [31] (see also [14, 16]). In particular, the resulting joint density of the sequential order statistics $T^*_{1:n}, \ldots, T^*_{n:n}$ is given by

$$f^{T^*_{1:n},\ldots,T^*_{n:n}}(t_1, \ldots, t_n) = n! \left(\prod_{i=1}^{n-1} \left(\frac{1 - F_i(t_i)}{1 - F_{i+1}(t_i)} \right)^{n-i} f_i(t_i) \right) f_n(t_n)$$

for $t_1 \leq \cdots \leq t_n$, and is zero elsewhere. We will assume that the distribution functions satisfy

$$F_i = 1 - (1 - F)^{\alpha_i}, \qquad i = 1, \ldots, n, \tag{18}$$

based on an absolutely continuous distribution function F with density function f and parameters $\alpha_1, \ldots, \alpha_n > 0$. This relation of the distribution functions often considered in the model of sequential order statistics is known as the proportional hazard rates (PHR) assumption, because then the corresponding hazard rates are given by

$$\frac{f_i}{1 - F_i} = \alpha_i \frac{f}{1 - F}, \qquad i = 1, \ldots, n.$$

Under this assumption, increasingly ordered $\alpha_1 \leq \cdots \leq \alpha_n$ correspond to a growing load on each individual component after every failure. In particular, the case of no additional stress on the components, i.e. $\alpha_1 = \cdots = \alpha_n$, leads to usual order statistics from an i.i.d. sample based on the distribution function F_1. For convenience, in the remainder we will assume again that the support of the underlying F is given by $[0, \infty)$.

Under the PHR assumption (18), well-known results on the marginal distributions (see [16, 33]; see also [11]) of generalized order statistics can be utilized in order to evaluate the importance measures of systems with failure-dependent components. For example, if the parameters $\gamma_i = \alpha_i(n - i + 1)$, $i = 1, \ldots, n$, are pairwise different, i.e. $\gamma_k \neq \gamma_l$ for $k \neq l$, then the distribution function of the rth sequential order statistic is given by

$$F^{T^*_{r:n}}(t) = 1 - \left(\prod_{j=1}^{r} \gamma_j \right) \sum_{i=1}^{r} \frac{a_{i,r}}{\gamma_i} (1 - F(t))^{\gamma_i}, \tag{19}$$

$t \geq 0$, where

$$a_{i,r} := \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \qquad 1 \leq i \leq r \leq n,$$

with $a_{1,1} = 1$. Moreover, for $\gamma_1 = \cdots = \gamma_n$, the corresponding distribution function is given by

$$F^{T^*_{r:n}}(t) = 1 - (1 - F(t))^{\gamma_1} \sum_{j=0}^{r-1} \frac{1}{j!} (-\gamma_1 \ln(1 - F(t)))^j, \tag{20}$$

for $t \geq 0$.

Example 2. Consider the system with $n = 4$ components given by the structure function $\varphi(x_1, \ldots, x_4) = \min(x_1, \max(x_2, x_3, x_4))$, where $x_1, \ldots, x_4 \in \{0, 1\}$. We assume that the components have failure-dependent lifetimes based on sequential order statistics with underlying distribution functions that satisfy the PHR assumption (18) with the baseline F following

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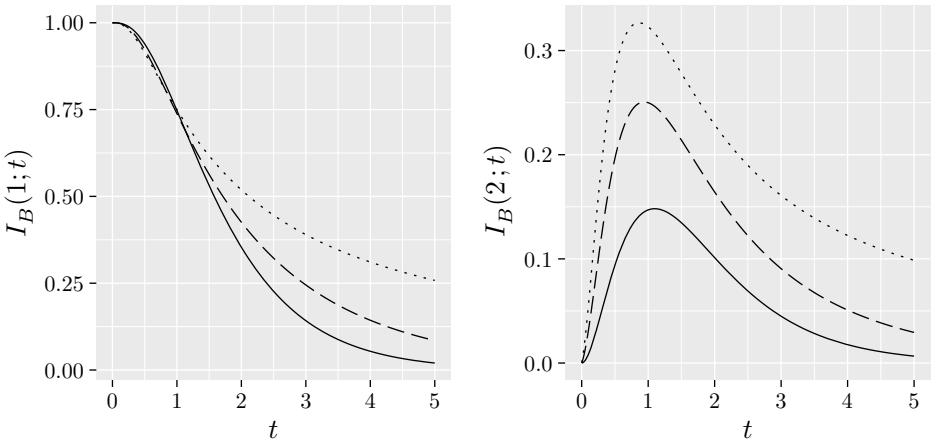


FIGURE 1: Plots of the importance measure I_{B} given in (5) for the components with failure-dependent lifetimes based on the standard exponential distribution from Example 2 (component 1 is displayed on the left and component 2 on the right). The cases (a), (b), and (c) are depicted with solid, dotted, and dashed lines, respectively.

the standard exponential distribution. We consider three choices for the parameters α_i, $i = 1, \ldots, 4$:

(a) $\alpha_1 = \cdots = \alpha_4 = 1$, i.e. $\gamma_1 = 4$, $\gamma_2 = 3$, $\gamma_3 = 2$, $\gamma_4 = 1$.

(b) $\alpha_1 = 1$, $\alpha_2 = \frac{4}{3}$, $\alpha_3 = 2$, $\alpha_4 = 4$, i.e. $\gamma_1 = \cdots = \gamma_4 = 4$.

(c) $\alpha_i = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{4}{5-i}$, i.e. $\gamma_i = \frac{9-i}{2}$, $i = 1, \ldots, 4$.

In case (a) the individual load on each component does not change with the number of failed units in the system. This case corresponds to the usual situation of i.i.d. component lifetimes. In case (b) the complete load of failed components is distributed evenly among the surviving ones. In this scenario, the total load on the system itself remains unchanged and the entirety of intact components is always stressed according to this initial load (cf., e.g., [4], Example 1). In particular, the first two cases can be considered as the possible extremal situations (cf., e.g., [13], Remark 2.3). Consequently, the third case (c) can then be interpreted as an intermediate scenario between the previous two.

By applying representations (19) and (20) and exploiting that for this system the vectors of coefficients (3) and (4) belonging to components 1 and 2 are given by

$$\boldsymbol{s}^{+1} = (0, 0, 1, 0), \quad \boldsymbol{s}^{-1} = (0, 0, 0, 0), \quad \boldsymbol{s}^{+2} = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0\right), \quad \boldsymbol{s}^{-2} = \left(0, \tfrac{1}{3}, \tfrac{2}{3}, 0\right),$$

we can evaluate the expressions for the importance measures given in this section. Note that, due to the structure of the system, the importance measures of components 3 and 4 coincide with those of component 2.

In Fig. 1 the importance measure I_{B} given in (5) is displayed for components 1 and 2 in all three cases. Figure 2 shows the respective graphs for $I_{\mathrm{B,BP}}$ given in (7).

We point out that in case (a) both importance measures, I_{B} and $I_{\mathrm{B,BP}}$, are identical (see Remark 4). It can be seen from the figures that the general shapes of the curves are also similar for both importance measures in the other considered situations. Moreover, the curves of

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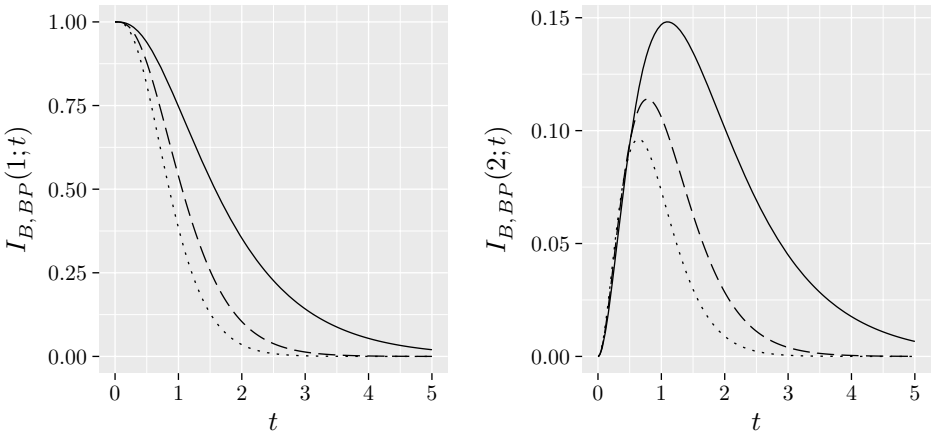


FIGURE 2: Plots of the importance measure $I_{\mathrm{B,BP}}$ given in (7) for the components with failure-dependent lifetimes based on the standard exponential distribution from Example 2 (component 1 is displayed on the left and component 2 on the right). The cases (a), (b), and (c) are depicted with solid, dotted, and dashed lines, respectively.

I_{B} always lie above the corresponding curves of $I_{\mathrm{B,BP}}$. This behavior will be explained in the next section (see Corollary 2). Furthermore, by studying the left graph in Fig. 1 in the interval (0, 1.5) and the right graph in Fig. 2 in the interval (0,1), it turns out that the curves cross each other in these areas. Therefore, although the parameter vectors $(\alpha_1, \ldots, \alpha_4)$ are component-wise ordered in the three cases, these orderings are not transferred to the importance measures. As one may expect, it can be observed for larger t that the values of the importance measure in case (c) lie between the values of cases (a) and (b). It is also seen for such t that the importance measure I_{B} assigns larger (smaller) values to the components in case (b) (case (a)), whereas the measure $I_{\mathrm{B,BP}}$ shows the opposite behavior with smaller (larger) values in the respective cases.

4. Comparison of component importance

In this section we will show that the importance measures I_{B} and $I_{\mathrm{B,BP}}$ from the preceding section are ordered when the component lifetimes satisfy a suitable notion of dependence. For this reason, let us recall the following concepts of positive and negative dependence (see [6, 28, 30]). Random variables $X_1, \ldots, X_n$ are called associated if

$$\mathrm{Cov}(g(X_1, \ldots, X_n), h(X_1, \ldots, X_n)) \geq 0$$

for all increasing functions $g, h\colon \mathbb{R}^n \to \mathbb{R}$ for which the covariance exists; $X_1, \ldots, X_n$ are called negatively associated if for all pairs of disjoint subsets $A_1, A_2 \subset \{1, \ldots, n\}$,

$$\mathrm{Cov}(g(X_i, i \in A_1), h(X_j, j \in A_2)) \leq 0$$

for all increasing functions $g\colon \mathbb{R}^{|A_1|} \to \mathbb{R}$, $h\colon \mathbb{R}^{|A_2|} \to \mathbb{R}$ for which the covariance exists. By applying these dependence concepts, we can establish the following ordering of the importance measures.

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Theorem 3. *Let $T_1, \ldots, T_n$ be associated (negatively associated) component lifetimes. Then the importance measures I_B and $I_{B,BP}$ satisfy*

$$I_B(i; t) \geq (\leq)\, I_{B,BP}(i; t)$$

for $i = 1, \ldots, n$ and every $t > 0$. In particular, $I_B(i; t) \geq 0$ holds for associated $T_1, \ldots, T_n$.

Proof. If the component lifetimes $T_1, \ldots, T_n$ are associated (negatively associated), then the component states $X_i(t) = 1_{(t,\infty)}(T_i)$ are associated (negatively associated), too, as increasing functions of the component lifetimes. Thus, the representation

$$I_B(i; t) - I_{B,BP}(i; t) = \frac{\mathrm{Cov}(\varphi(1_i, \boldsymbol{X}(t)), X_i(t))}{\mathbb{E}(X_i(t))} - \frac{\mathrm{Cov}(\varphi(0_i, \boldsymbol{X}(t)), 1 - X_i(t))}{\mathbb{E}(1 - X_i(t))}$$

yields the assertion, since the two included covariances $\mathrm{Cov}(\varphi(1_i, \boldsymbol{X}(t)), X_i(t))$ and $\mathrm{Cov}(\varphi(0_i, \boldsymbol{X}(t)), X_i(t))$ are non-negative (non-positive) due to the associated (negatively associated) component states. □

From the definitions (1) and (6), it follows immediately that $I_B(i; t) \leq 1$ and $0 \leq I_{B,BP}(i; t) \leq 1$. The preceding theorem provides a sufficient condition, namely association of the component lifetimes, which guarantees that I_B is non-negative. The following example illustrates that I_B can attain negative values in other situations.

Example 3. We examine the three-component system with structure function

$$\varphi(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3)),$$

where $x_1, x_2, x_3 \in \{0, 1\}$. The coefficients (3) and (4) for this system are given by (cf. [24])

$$\boldsymbol{s}^{+1} = (0, 0, 1), \quad \boldsymbol{s}^{-1} = (0, 1, 0), \quad \boldsymbol{s}^{+2} = (0, 1, 0), \quad \boldsymbol{s}^{-2} = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

The components are assumed to be failure dependent based on distribution functions from the PHR model (18) with underlying standard exponential distribution. The occurring parameters are chosen as

$$\alpha_1 = 1,\ \alpha_2 = \frac{2}{3},\ \alpha_3 = \frac{1}{3}, \quad \text{i.e. } \gamma_1 = 3,\ \gamma_2 = \frac{4}{3},\ \gamma_3 = \frac{1}{3}.$$

Therefore, in this scenario the individual load on each active unit is decreasing with the number of failed components. In Fig. 3 the resulting importance measures I_B and $I_{B,BP}$ of component 2 are displayed. Clearly, the graphs for component 3 are the same.

It is apparent from Fig. 3 that $I_B(2; t)$ attains negative values for larger t. Since the lifetimes are negatively correlated in the situation considered (see [8]), it is possible that the failure of component 2 has a beneficial effect on the survival probability of the system. If the failure of component 2 occurs earlier (later) than t, then the lifetimes of the remaining components may tend to be larger (smaller) than t. In this way, conditioning with the event of a component failure before time t may lead to an increase of the probability that the remaining components, and consequently also the system, survive time t. While this phenomenon can become visible from I_B (cf. definition (1)), it cannot be captured by $I_{B,BP}$, which measures the gain in probability by switching from a defective component 2 to a flawless one, but does not take into account the influence of component 2 on the other components (cf. definition (6)). However,

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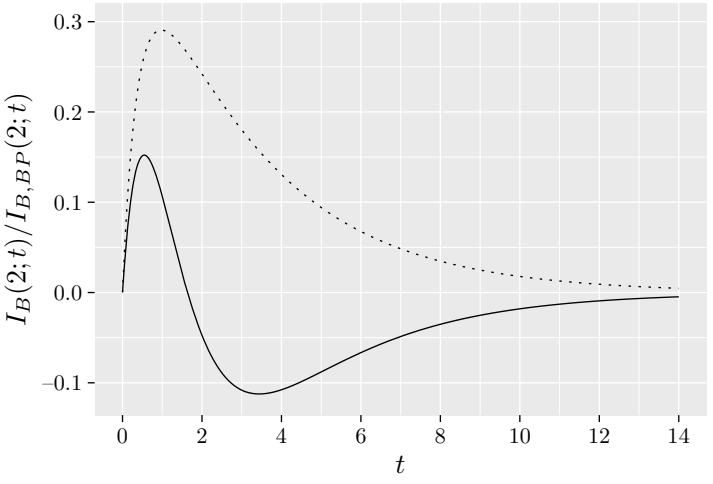


FIGURE 3: Plots of importance measures $I_{\mathrm{B}}(2;t)$ (solid) and $I_{\mathrm{B,BP}}(2;t)$ (dotted) of component 2 for the system with failure-dependent lifetimes based on the standard exponential distribution from Example 3.

as a consequence, the usual interpretation of the Birnbaum importance as (positive) reliability growth does not necessarily hold anymore for the measure I_{B} when facing dependence between the component lifetimes, whereas the measure $I_{\mathrm{B,BP}}$ is not affected by that due to its construction.

Remark 6. The preceding example illustrates the effect of the different definitions. The measure $I_{\mathrm{B,BP}}$ represents the difference of the probability that the system works at time t when component i cannot fail and the corresponding probability when component i cannot work. The measure I_{B} represents the difference of the probability that the system works at time t given component i works at time t and the corresponding probability given component i has failed at time t. In the former case only the dependence structure of $(T_1,\ldots,T_{i-1},T_{i+1},\ldots,T_n)$ is relevant; in the latter, additionally the influence of T_i on $(T_1,\ldots,T_{i-1},T_{i+1},\ldots,T_n)$ is crucial.

In [8], dependence properties of failure-dependent lifetimes are examined. For example, from results on association of failure-dependent lifetimes, we can draw a conclusion for the important situation of an increasing load on surviving components (see also [49], Section 6.1).

Corollary 2. *Let $T_1^*,\ldots,T_n^*$ be failure-dependent lifetimes and let the underlying distribution functions $F_1,\ldots,F_n$ satisfy the PHR model (18) with an absolutely continuous F with continuous positive density f on $(0,\infty)$. If $\alpha_1\le\cdots\le\alpha_n$, then*

$$0\le I_{\mathrm{B,BP}}(i;t)\le I_{\mathrm{B}}(i;t)\le 1,\qquad i=1,\ldots,n,\ t>0,$$

holds for every coherent system of order n with component lifetimes $T_1^*,\ldots,T_n^*$.

Finally, we will compare the asymptotic behavior of both importance measures. The limiting behavior of $I_{\mathrm{B,BP}}$ is directly seen as follows: As long as the lifetimes $T_1,\ldots,T_n$ satisfy $T_i>0$ almost surely for $i=1,\ldots,n$, we obtain for the component states

$$\lim_{t\to 0+}X_i(t)=1,\qquad \lim_{t\to\infty}X_i(t)=0,\qquad i=1,\ldots,n,$$

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almost surely. Hence, since the structure function φ can be chosen as a continuous function on $[0, 1]^n$, we conclude from (6) by using dominated convergence that

$$\lim_{t\to 0+} I_{\mathrm{B,BP}}(i;t) = 1 - \varphi(0_i, \mathbf{1}), \qquad \lim_{t\to\infty} I_{\mathrm{B,BP}}(i;t) = \varphi(1_i, \mathbf{0}),$$

where $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. In particular, both limits are either equal to zero or one. More precisely,

$$\lim_{t\to 0+} I_{\mathrm{B,BP}}(i;t) = 1 \;\Leftrightarrow\; \{i\} \text{ is a (minimal) cut set,}$$
$$\lim_{t\to\infty} I_{B,BP}(i;t) = 1 \;\Leftrightarrow\; \{i\} \text{ is a (minimal) path set.}$$

In order to study the limiting behavior of the measure I_{B}, it will be helpful to state the connection between the above limits given in terms of the structure function φ and the coefficients s_k^{+i} and s_k^{-i} from (3) and (4).

Lemma 2. *The structure function φ of a coherent system satisfies*

(i) $\varphi(0_i, \mathbf{1}) = \sum_{k=1}^{n} s_k^{-i} = \begin{cases} 1, & \text{if there exists } k : s_k^{-i} \neq 0, \\ 0, & \text{otherwise;} \end{cases}$

(ii) $\varphi(1_i, \mathbf{0}) = r_1^{+i} = s_n^{+i}$.

Proof. The second equality in (i) is given in (17). In order to obtain the first equality, note that it can be shown with (4) that

$$s_k^{-i} = 0 \text{ for all } k \in \{1, \ldots, n\} \;\Leftrightarrow\; r_k^{-i} = 0 \text{ for all } k \in \{1, \ldots, n\},$$

and with Lemma 1 it follows that

$$s_k^{-i} = 0 \text{ for all } k \in \{1, \ldots, n\} \;\Leftrightarrow\; \sum_{k=1}^{n} s_k^{-i} = 0.$$

Consequently, the claimed equality follows by observing that

$$\varphi(0_i, \mathbf{1}) = 0 \;\Leftrightarrow\; r_k^{-i} = 0 \text{ for all } k \in \{1, \ldots, n\}.$$

(ii) is obvious, because $\varphi(1_i, \mathbf{0}) = 1$ if and only if $r_1^{+i} = 1$. □

The limits of the importance measure I_{B} depend on the tail behavior of the ordered component lifetimes. Here, we give explicit results for the failure-dependent lifetimes considered in Section 3.

Theorem 4. *Let $T_1^*, \ldots, T_n^*$ be failure-dependent lifetimes and let the underlying distribution functions $F_1, \ldots, F_n$ satisfy the PHR model (18). The importance measure I_{B} in (5) satisfies*

$$\lim_{t\to 0+} I_{\mathrm{B}}(i;t) = 1 - \varphi(0_i, \mathbf{1}) = \begin{cases} 0, & \text{if there exists } k : s_k^{-i} \neq 0, \\ 1, & \text{otherwise;} \end{cases}$$

$$\lim_{t\to\infty} I_{\mathrm{B}}(i;t) = \varphi(1_i, \mathbf{0}) + \left(\sum_{j=1}^{n-1} \gamma^{(j,n)} + 1 \right)^{-1} \times \left(\sum_{k=1}^{n-1} s_k^{+i} \left(\sum_{m=1}^{k-1} \gamma^{(m,n)} + (n-k+1)\gamma^{(k,n)} \right) \right)$$

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with $\gamma_{1:n} = \min(\gamma_1, \ldots \gamma_n)$ *and* $\gamma^{(k,n)} := \prod_{l=k+1}^{n} \frac{\gamma_l - \gamma_{1:n}}{\gamma_l}$. *In particular, if* $\gamma_1 \geq \cdots \geq \gamma_n$, *then* $\lim_{t\to\infty} I_{\mathrm{B}}(i;t) = \varphi(1_i, \mathbf{0}) = s_n^{+i}$.

Proof. Denote the distribution (survival) function of a sequential order statistic $T_{i:n}^*$ by $F_{i:n}^*$ ($\overline{F}_{i:n}^*$). We decompose I_{B} into $I_{\mathrm{B}}(i;t) = I_{\mathrm{B}}^{(1)}(i;t) - I_{\mathrm{B}}^{(2)}(i;t)$, where

$$I_{\mathrm{B}}^{(1)}(i;t) := \left(\sum_{j=1}^{n} \overline{F}_{j:n}^*(t)\right)^{-1} \left(\sum_{k=1}^{n} s_k^{+i} \left(\sum_{m=1}^{k-1} \overline{F}_{m:n}^*(t) + (n-k+1)\overline{F}_{k:n}^*(t)\right)\right),$$

$$I_{\mathrm{B}}^{(2)}(i;t) := \left(\sum_{j=1}^{n} F_{j:n}^*(t)\right)^{-1} \left(\sum_{k=2}^{n} s_k^{-i} \left(\sum_{m=1}^{k-1} F_{m:n}^*(t) - (k-1)F_{k:n}^*(t)\right)\right).$$

It is easily seen that

$$\lim_{t\to 0+} I_{\mathrm{B}}^{(1)}(i;t) = \sum_{k=1}^{n} s_k^{+i}, \qquad \lim_{t\to\infty} I_{\mathrm{B}}^{(2)}(i;t) = 0.$$

For deriving the remaining limits, we recall the following limiting result (see [13, Lemma 2.2]): Under assumption (18),

$$\lim_{t\to 0+} \frac{F_{k:n}^*(t)}{F_{r:n}^*(t)} = 0, \qquad \lim_{t\to\infty} \frac{\overline{F}_{r:n}^*(t)}{\overline{F}_{k:n}^*(t)} = \gamma^{(r,k)}$$

holds for $1 \leq r < k \leq n$. Therefore, expanding the fraction in $I_{\mathrm{B}}^{(2)}(i;t)$ by $F_{1:n}^*(t)$ and letting t tend to zero, we obtain

$$\lim_{t\to 0+} I_{\mathrm{B}}^{(2)}(i;t) = \sum_{k=2}^{n} s_k^{-i}.$$

Thus, Lemma 2(i) yields the first limit, because $s_1^{-i} = 0$ and $\sum_{k=1}^{n} s_k^{+i} = 1$ (see (17)). By expanding $I_{\mathrm{B}}^{(1)}(i;t)$ by $\overline{F}_{n:n}^*(t)$, we get

$$\lim_{t\to\infty} I_{\mathrm{B}}^{(1)}(i;t) = \left(\sum_{j=1}^{n-1} \gamma^{(j,n)} + 1\right)^{-1} \left[s_n^{+i} \left(\sum_{m=1}^{n-1} \gamma^{(m,n)} + 1\right) + \sum_{k=1}^{n-1} s_k^{+i} \left(\sum_{m=1}^{k-1} \gamma^{(m,n)} + (n-k+1)\gamma^{(k,n)}\right)\right],$$

which leads to the second limit using Lemma 2(ii). If $\gamma_1 \geq \cdots \geq \gamma_n$, then all the $\gamma^{(m,n)}$ on the right-hand side become zero and the limit is equal to $s_n^{+i} = \varphi(1_i, \mathbf{0})$. □

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Remark 7.

(i) Recall that by definition (1), $I_B(i;\, t) \leq 1$ holds. Since the limit for $t \to \infty$ in Theorem 4 is non-negative (cf. Lemma 1), it follows that $\lim_{t\to\infty} I_B(i;\, t) \in [0,\, 1]$ in that situation. Clearly, $I_B(i;\, t)$ can still be negative for particular $t \in (0,\, \infty)$, as Example 3 has illustrated.

(ii) For the considered failure-dependent component lifetimes, the limits of I_B and $I_{B,\,BP}$ are identical as t tends to zero. The same holds true for t approaching ∞ when $\gamma_1 \geq \cdots \geq \gamma_n$ is fulfilled. Otherwise, the limits can be different and values in (0,1) for the limit of I_B are possible, as the following example shows.

Example 4. Consider the situation from Example 2. This time the load on the remaining components in the system is assumed to change according to the following pattern:

$$\alpha_1 = 1,\ \alpha_2 = \frac{5}{3},\ \alpha_3 = 3,\ \alpha_4 = 7, \quad \text{i.e. } \gamma_1 = 4,\ \gamma_2 = 5,\ \gamma_3 = 6,\ \gamma_4 = 7.$$

In this scenario, after every failure the individual load on each component increases to such a substantial extent that even the total load on the system rises (for example, due to damage caused by the component failures). In this case, it can be shown that the limits of the importance measures for the first component are given by

$$\lim_{t\to 0+} I_B(1;\, t) = \lim_{t\to 0+} I_{B,\,BP}(1;\, t) = 1, \qquad \lim_{t\to\infty} I_B(1;\, t) = \frac{9}{14}, \qquad \lim_{t\to\infty} I_{B,\,BP}(1;\, t) = 0.$$

As stated in Section 3, sequential order statistics are used to model load-sharing systems, where the lifetime distribution of remaining components may change after a component has failed. Practical situations in which failures reduce the expected remaining lifetime (e.g. mechanical systems, computer hardware systems, water filters; cf. [34]) are known as well as situations with a positive effect on the remaining lifetime (e.g. biological survival experiments; cf. [19, 20]). An application of the presented importance measures to a real data set from the literature is discussed in the following example.

Example 5. In a ReliaSoft data set (see, e.g., [7]) examined recently by [7, 34, 50], the failure times of 18 identical parallel systems consisting of $n = 2$ motors are given. Bedbur et al. [7] propose a sequential order statistics model under the PHR assumption with a Weibull distribution $W(\beta,\, 1)$ with shape parameter $\beta > 0$ as the baseline distribution F (cf. (18)). They find the corresponding MLEs of the parameters involved as $\hat{\alpha}_1 = 2.84$, $\hat{\alpha}_2 = 6.49$, and $\hat{\beta} = 2.77$. Since the coefficients (3) and (4) of the underlying parallel system are given by

$$\boldsymbol{s}^{+1} = (0,\, 1), \quad \boldsymbol{s}^{-1} = (0,\, 1), \quad \boldsymbol{s}^{+2} = (0,\, 1), \quad \boldsymbol{s}^{-2} = (0,\, 1),$$

estimators of the the measures I_B and $I_{B,\,BP}$ can be derived using the MLEs and the representations in Theorems 1 and 2 (cf. Eq. (19)). In Fig. 4, both measures are displayed. We observe that, as mentioned before, I_B lies above $I_{B,\,BP}$, and that the limiting behavior is identical in this case.

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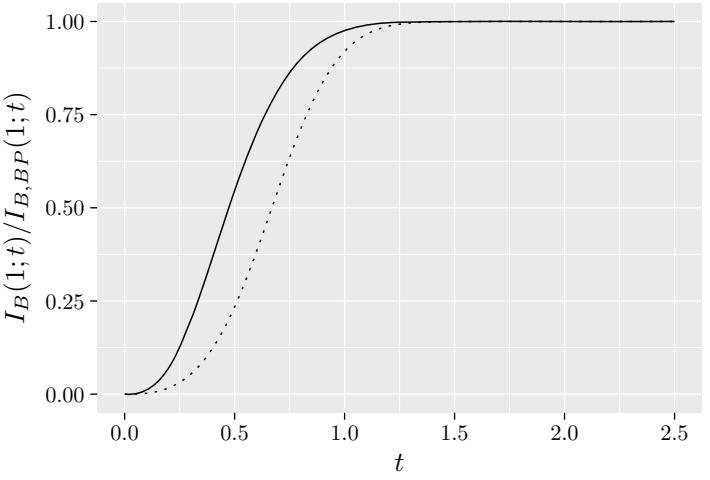


FIGURE 4: Plots of importance measures $I_B(1; t)$ (solid) and $I_{B,BP}(1; t)$ (dotted) of component 1 for the parallel system with failure-dependent lifetimes based on the Weibull distribution from Example 5.

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