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Partially hyperbolic and transitive dynamics generated by heteroclinic cycles

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Abstract. We study $\mathcal{C}^k$-diffeomorphisms, $k \geq 1$, $f : M \to M$, exhibiting heterodimensional cycles (i.e. cycles containing periodic points of different stable indices). We prove that if f can not be $\mathcal{C}^k$-approximated by diffeomorphisms with homoclinic tangencies, then f is in the closure of an open set $\mathcal{U} \subset \mathrm{Diff}^k(M)$ consisting of diffeomorphisms g with a non-hyperbolic transitive set Λ_g which is locally maximal and strongly partially hyperbolic (the partially hyperbolic splitting at Λ_g has three non-trivial directions). As a consequence, in the case of 3-manifolds, we give new examples of open sets of $\mathcal{C}^1$-diffeomorphisms for which residually infinitely many sinks or sources coexist ($\mathcal{C}^1$-Newhouse's phenomenon). We also prove that the occurrence of non-hyperbolic dynamics has persistent character in the unfolding of heterodimensional cycles.

1. *Introduction*

The concept of a homoclinic orbit (an orbit of intersection of the stable and unstable manifolds of a hyperbolic saddle) plays a key role in dynamics: transverse homoclinic orbits imply the existence of horseshoes (hyperbolic, transitive, and invariant sets where the periodic orbits are dense) which in their turn are a characteristic ingredient of complex dynamical behavior. Thus, an important problem in dynamics is to understand the main mechanisms leading to the appearance of new homoclinic orbits. This question leads us to study the phenomena of *creation and unfolding of cycles.* These phenomena yield infinitely many changes in the dynamics, called *bifurcations*. If the cycle is associated to hyperbolic periodic points of the same *index* (dimension of the stable manifold), then its creation involves homoclinic tangencies. These homoclinic bifurcations, at least in the setting of dissipative surface diffeomorphisms, give rise to relevant dynamical phenomena such as cascades of period doubling bifurcations [**YA**], Hénon-like attractors [**MV**], and

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the coexistence of infinitely many sinks and Hénon-like attractors [**N2, PT3, C**], among others. See also [**PV**] and [**R**] for the generalizations of these results to higher dimensions.

In this article we focus our attention on cycles associated to periodic points having different (stable) indices, the so-called *heterodimensional cycles*. These cycles were considered in the 1970s in [**NP**] and developed in a series of recent papers (see [**D1, D2, DU, DR1, DR2**]). Besides the intrinsic interest of these cycles, motivated mainly by bifurcation theory, the importance of the heterodimensional cycles has been renewed by recent results showing that such cycles appear in a natural way in the study of transitive partially hyperbolic dynamics, see [**BD1, DPU, H**].

Heterodimensional cycles do not lead necessarily to homoclinic tangencies, so thus the features of the dynamics arising from these bifurcations are rather different from the ones described in the homoclinic setting. For instance, bifurcations across heterodimensional cycles typically lead to the coexistence of horseshoes of different indices. In some cases one gets hyperbolic dynamics and these horseshoes are disjoint (see [**DR2**]). However, in some relevant cases these horseshoes are *intermingled:* they are contained in the same transitive set. Such a topological configuration implies that the dynamics is non-hyperbolic. But this non-hyperbolic dynamics may keep some memory of the hyperbolicity before the unfolding of the cycle so that the dynamics satisfies a weak form of hyperbolicity, called *partial hyperbolicity*.

Here, for a large class of diffeomorphisms with heterodimensional cycles, including those which can not be approximated by homoclinic tangencies, we prove that the unfolding of the cycle leads to diffeomorphisms g for which the periodic points in the cycle, say P and Q where $\mathrm{index}(P) > \mathrm{index}(Q)$, are such that $W^s(P)$ is contained in the closure of $W^s(Q)$. In rough terms, this means that the *topological dimension* of $W^s(Q)$ is larger than the dimension of $W^s(Q)$ (equal to the index of Q). This property is used as the main step towards proving that P and Q are in the same transitive set of g, and thus g is non-hyperbolic. We point out that our arguments work for any $k \geq 1$.

In this paper we study heterodimensional cycles from two points of view: the structure of the non-wandering set and the measure of the bifurcation set of arcs of diffeomorphisms. In fact, these two approaches are closely related.

1.1. *Structure of the non-wandering set: transitivity, creation of cycles, and partial hyperbolicity.* Our first results are motivated by the following theorem of Mañé [**M3**] for surface diffeomorphisms: generically (for a residual subset) a $\mathcal{C}^1$-surface diffeomorphism f satisfies one of the following two possibilities; either f satisfies the Axiom A and the no-cycles properties (in particular, the non-wandering set $\Omega(f)$ of f is the union of finitely many pairwise disjoint basic sets), or f has infinitely many sinks or sources.

Let us first observe that it is an open question whether the second possibility may be eliminated. Moreover, Mañé's result does not hold in higher dimensions, see for instance [**M1**] and [**Sh**]. After [**BD2**], we also know that the coexistence of infinitely many sinks/sources is locally generic for $\mathcal{C}^1$-diffeomorphisms of three manifolds. Thus a natural aim is to obtain a suitable version of Mañé's result in higher dimensions. Some progress in this direction has been obtained by replacing the usual uniform hyperbolicity assumption

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by a weaker one (*partial hyperbolicity* in dimension three and *dominated splittings* in higher dimensions, see [**DPU, BDP, BV**]).

A second motivation is the study of partially hyperbolic diffeomorphisms, for previous results in this subject see, chronologically, [**BP, PeSi, Sh, M1, GPS, BD1, DPU, BV, ABV**]. Here we study non-hyperbolic transitive sets having some kind of stability or persistence. We consider diffeomorphisms with heterodimensional cycles and state conditions implying that after small perturbations one gets partially hyperbolic and robustly transitive sets containing the continuations of the points in the cycle, see Theorem A below. Observe that in the case of *equidimensional cycles* (all the points in the cycle have the same index) this result is trivial: after a perturbation all points in the cycle are in the same *horseshoe*.

Let us be more precise. Let M denote a compact, connected, and boundaryless n-dimensional Riemannian manifold. A diffeomorphism f defined on M exhibits a *heterodimensional cycle* Γ *associated to the hyperbolic periodic points* P *and* Q if $W^s(P)$ and $W^u(Q)$ have a non-trivial transverse intersection, and $W^u(P)$ and $W^s(Q)$ have a quasi-transverse intersection of codimension one. In this case one has $\text{index}(P) = \text{index}(Q) + 1$. We write $\Gamma = \Gamma(f, P, Q)$. Observe that if f has a heterodimensional cycle then the dimension of M is at least three. Denote by $\text{Diff}^k(M)$ the space of $\mathcal{C}^k$-diffeomorphisms defined on M endowed with the usual topology.

THEOREM A. *Suppose that* $f \in \text{Diff}^k(M)$, $k \geq 1$, *has a heterodimensional cycle* $\Gamma(f, P, Q)$ *and that* f *can not be approximated by diffeomorphisms* g *with homoclinic tangencies associated to the continuations* P_g *and* Q_g *of* P *and* Q. *Then for every neighborhood* $\mathcal{U}_f$ *of* f *in* $\text{Diff}^k(M)$ *there is an open subset* $\mathcal{V} \subset \mathcal{U}_f$ *such that every* $g \in \mathcal{V}$ *has a strong partially hyperbolic transitive set containing* P_g *and* Q_g.

Here by *strong partially hyperbolic set* we mean a partially hyperbolic set having three non-trivial directions, see the precise definition in §2.2. Let us make some comments on this result. First we observe that our proof gives that the strong partially hyperbolic set in Theorem A, say Λ_g, is contained in the closure of the transverse homoclinic points of Q_g. Moreover, there is an open set V containing the orbits of the periodic points and the points of quasi-transverse intersections between $W^s(P_g)$ and $W^u(Q_g)$ such that Λ_g is the maximal invariant set of g in V. Actually, Λ_g is the closure of the homoclinic points of Q_g in such a locally maximal set (i.e. the set Λ_g is the *relative homoclinic class of* Q_g *in* V).

For a second remark let us recall that Hayashi's connecting lemma [**H**] asserts (in rough terms) that by a $\mathcal{C}^1$-perturbation of a *robustly non-hyperbolic transitive set* one can create heterodimensional cycles. Thus Theorem A is a converse (which holds in any $\mathcal{C}^k$-topology, $k \geq 1$) of the connecting lemma in the partially hyperbolic context.

Finally, let us recall the notion of *persistent connection*. Two hyperbolic periodic points P and Q with different indices of a diffeomorphism f are $\mathcal{C}^k$*-persistently connected* in a $\mathcal{C}^k$-neighborhood $\mathcal{U}$ of f if there is a dense subset $\mathcal{D}$ of $\mathcal{U}$ such that every $g \in \mathcal{D}$ has a transitive set Λ_g containing P_g and Q_g. From [**BD2**, Proposition 1.1], P and Q are $\mathcal{C}^1$-persistently connected in $\mathcal{U}$ if and only if the diffeomorphisms g exhibiting cycles $\Gamma(g, P_g, Q_g)$ are dense in $\mathcal{U}$. Theorem A gives conditions on a cycle $\Gamma(f, P, Q)$ (satisfied

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for every cycle *far from homoclinic tangencies*) implying that P and Q are $\mathcal{C}^k$-persistently connected ($k \geq 1$) after a perturbation of f.

As in [**DPU**, Theorems C and E], the phenomena of approximation by tangencies and the existence of *central* (i.e. weak expanding or contracting) non-real eigenvalues are closely related. Given a periodic point R of f let $|\lambda_1(R)| \leq \cdots \leq |\lambda_n(R)|$ be the eigenvalues of $D_R f^m$, m being the period of R. Suppose that the points P and Q in $\Gamma(f, P, Q)$ have indices r and $r-1$. Then the hypothesis on the approximation by tangencies in Theorem A may be replaced by:

- Let g be close to f and R be a periodic point of g homoclinically related either to P_g or Q_g. Then the *central eigenvalue* $\lambda_r(R, g)$ is a real number.

1.2. *Coexistence of infinitely many sinks/sources.* A diffeomorphism f satisfies the $\mathcal{C}^k$-*Newhouse's phenomenon* if for every $\mathcal{C}^k$-neighborhood $\mathcal{U}_f$ of f there are an open subset $\mathcal{V}$ of $\mathcal{U}_f$ and a residual subset $\mathcal{R}$ of $\mathcal{V}$ consisting of diffeomorphisms with infinitely many sinks or sources. For $k \geq 2$, Newhouse's phenomenon is associated to the existence of homoclinic tangencies and the creation of thick horseshoes, see [**N1, N2, PT3**] for surface diffeomorphisms and [**PV**] for higher dimensions. On the other hand, for $k = 1$ the situation is rather different and it is an open question whether the $\mathcal{C}^1$-Newhouse's phenomenon holds for surface diffeomorphisms (see also [**U, PuSa**]). Recently, in the case of three-dimensional diffeomorphisms, [**BD2**, Theorem C] gives the following mechanism leading to the $\mathcal{C}^1$-Newhouse's phenomenon: let f be a $\mathcal{C}^1$-diffeomorphism having two hyperbolic periodic points A and B (of indices 2 and 1), which are $\mathcal{C}^1$-persistently connected in $\mathcal{U}$, and such that $D_A f^n$ and $D_B f^m$ (n and m are the periods of A and B) both have two complex (non-real) eigenvalues, then f satisfies the $\mathcal{C}^1$-Newhouse's phenomenon.

In [**BD2**] to obtain points A and B as before (different indices, derivatives with non-real eigenvalues, and persistently connected) two types of dynamical configurations are analyzed: perturbations of the diffeomorphisms in the proof of the non-genericity of Ω-stability by Abraham–Smale (see [**AS, Si**]) and bifurcations of *non-critical connected heterodimensional cycles*. Let us now discuss this second type of examples and give some definitions.

Let $\Gamma(f, P, Q)$ be a heterodimensional cycle, where $\text{index}(P) = r$. We say that $\Gamma(f, P, Q)$ has *real central eigenvalues* if $|\lambda_{r-1}(P)| < |\lambda_r(P)| < 1 < |\lambda_r(Q)| < |\lambda_{r+1}(Q)|$. If $\Gamma(f, P, Q)$ has real central eigenvalues then we can define the *strong stable foliation* $\mathcal{F}^{ss}$ of $W^s(P)$ and the *strong unstable foliation* $\mathcal{F}^{uu}$ of $W^u(Q)$ (i.e. the unique f-invariant foliations of codimension one in $W^s(P)$ and $W^u(Q)$, respectively). A component γ of the transverse intersection between $W^s(P)$ and $W^u(Q)$ is *non-critical* if it is transverse to $\mathcal{F}^{ss}$ (in $W^s(P)$) and to $\mathcal{F}^{uu}$ (in $W^u(Q)$). Similarly, γ is *s-critical* (respectively *u-critical*) if it is tangent to $\mathcal{F}^{ss}$ (respectively $\mathcal{F}^{uu}$) at some point. Finally, γ is *bicritical* if it is simultaneously s- and u-critical. We say that the cycle $\Gamma(f, P, Q)$ is *connected* if the transverse intersection between $W^s(P)$ and $W^u(Q)$ has a connected component γ such that $f^h(\gamma) = \gamma$ for some $h \neq 0$. Let us observe that the main source of examples of cycles satisfying Theorem A are the connected ones.

For the next two results we focus our attention on $\mathcal{C}^1$-diffeomorphisms of 3-manifolds.

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Let $\Gamma(f, P, Q)$ be a connected cycle with real central eigenvalues and γ a periodic component of $W^s(P) \pitchfork W^u(Q)$. Since the dimension of M is three, we have that the indices of P and Q are 2 and 1, respectively. Suppose that P (respectively Q) is homoclinically related to some periodic point A (respectively B) with a complex (non-real) eigenvalue. In [**BD2**], assuming that γ is non-critical as well as some conditions on the restriction of f to the intersection γ, it is proved that after a perturbation of f one gets g such that the continuations A_g and B_g of A and B are $\mathcal{C}^1$-persistently connected. Thus, g satisfies the $\mathcal{C}^1$-Newhouse's phenomenon. The next result asserts that these two hypotheses are not necessary.

THEOREM B. *Let f be a $\mathcal{C}^1$-diffeomorphism defined on a three-dimensional manifold such that:*

(i) *f has a connected heterodimensional cycle $\Gamma(f, P, Q)$ with real central eigenvalues;*

(ii) *there are hyperbolic periodic points A and B homoclinically related to P and Q, respectively, such that the derivative of f at A (respectively B) has a complex (non-real) eigenvalue.*

Then f can be $\mathcal{C}^1$-approximated by a diffeomorphism g such that A_g and B_g are $\mathcal{C}^1$-persistently connected. In particular, f satisfies the $\mathcal{C}^1$-Newhouse's phenomenon.

Finally, let us state a corollary of the previous theorem.

COROLLARY C. *Let f be a three-dimensional $\mathcal{C}^1$-diffeomorphism exhibiting a bicritical connected heterodimensional cycle. Then f satisfies the $\mathcal{C}^1$-Newhouse's phenomenon.*

1.3. *Parametrized families of diffeomorphisms: measure of the bifurcation set.* As the creation of cycles is a fundamental mechanism to produce a string of complicated changes in the dynamics, it seems relevant to consider arcs of diffeomorphisms $(f_t)_{t\in I}$ exhibiting a cycle (for instance at $t = 0$) and describe the dynamical features more frequently displayed by the diffeomorphisms f_t (in the sense of the Lebesgue measure of parameters). A dynamical feature $\mathcal{P}$ is *persistent* (at the bifurcation) if for every $t_0 > 0$ there is a set $E \subset [-t_0, t_0]$ having positive measure such that f_t satisfies $\mathcal{P}$ for all t in E. One also aims to determine these phenomena $\mathcal{P}$ which are *prevalent* at the bifurcation: there is $c > 0$ such that for all small $t_0 > 0$ the set E of parameters t such that f_t satisfies $\mathcal{P}$ has density larger than c in $[-t_0, t_0]$ (i.e. $|E \cap [-t_0, t_0]| > ct_0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}$).

For bifurcations associated to tangencies, especially in two dimensions, the results in [**N1, PT1, PT2, PY1, PY2**] have brought up the importance of fractal dimensions (thickness, Hausdorff dimension, and capacity) of the hyperbolic sets in the cycle for measuring the frequency of hyperbolic dynamics after the bifurcation, see also [**P**]. Until now, and as far as we know, in the case of heterodimensional cycles these fractal dimensions did not play such a key role. In the results obtained previously the frequency of hyperbolic dynamics has been mainly determined by the geometry of $W^s(P) \cap W^u(Q)$ (number and relative size of the connected components of $W^s(P) \cap W^u(Q)$, and the existence of criticalities) as well as by the dynamics of f in the *central direction*, see

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[**D1, D2, DU, DR1, DR2**]. So we analyze the dynamical behavior of heterodimensional cycles according to these features. Therefore, let us introduce some notation and definitions.

Denote by $\mathcal{P}(M)$ the space of $\mathcal{C}^\infty$-arcs of diffeomorphisms (endowed with the usual $\mathcal{C}^\infty$-topology), by $\mathcal{H}(M)$ the subset of $\mathcal{P}(M)$ of arcs unfolding a heterodimensional cycle generically, and by $\mathcal{C}(M)$ the subset of $\mathcal{H}(M)$ of arcs unfolding connected cycles.

The *bifurcation set* B *of* $(f_t)_{t\in I}$ is the set of parameters t such that the limit set of f_t, $L(f_t)$, is not hyperbolic. From now on, we assume that $I = [-1, 1]$ and that the arcs unfold the cycle at $t = 0$. The arc has a *first bifurcation at* $t = 0$ if f_t satisfies Axiom A and the no-cycles condition for every $t < 0$. Let $\mathcal{FbC}(M)$ denote the subset of $\mathcal{C}(M)$ of arcs having a first bifurcation at $t = 0$. Finally, let $\mathbb{RC}(M)$ be the subset of $\mathcal{C}(M)$ of arcs such that the cycle has at least one real central eigenvalue associated to P or Q. Let us observe that if the cycle is a first bifurcation then (generically) its central eigenvalues are both real and positive (for details see the arguments in [**DR2**]).

THEOREM D.

(i) Persistence of non-hyperbolicity. *There is an open and dense subset* $\mathcal{A}_0(M)$ *of* $\mathcal{H}(M)$ *such that the bifurcation set* B *of every* $(f_t)_{t\in I}$ *in* $\mathcal{A}_0(M)$ *satisfies*

$$|B \cap [-t, t]| > 0$$

for all $t > 0$.

(ii) *There is an open and dense subset* $\mathcal{A}_1(M)$ *of* $\mathbb{RC}(M)$ *such that for every* $(f_t)_{t\in I}$ *in* $\mathcal{A}_1(M)$ *there is a sequence of non-trivial intervals* $B_n = [b_n^-, b_n^+]$ *contained in the bifurcation set of* $(f_t)_{t\in I}$ *such that* $b_n^\pm \to 0$.

(iii) Prevalence of non-hyperbolicity. *There is an open and dense subset* $\mathcal{A}_2(M)$ *of* $\mathcal{FbC}(M)$ *such that for every* $(f_t)_{t\in I}$ *in* $\mathcal{A}_2(M)$ *the sequence of intervals* B_n *in* (ii) *can be taken satisfying*

$$\liminf_{t\to 0^+} \frac{|B' \cap [-t, t]|}{2t} > 0, \quad B' = \bigcup_n B_n.$$

Finally, we describe some topological features of the non-wandering set arising from the unfolding of heterodimensional cycles. Consider an arc $(f_t)_{t\in I}$ unfolding a cycle $\Gamma(f_0, P_0, Q_0)$. The *resulting non-wandering set of* f_t, denoted by $\Omega'(f_t)$, is the restriction of the non-wandering set of f_t to a fixed *neighborhood of the cycle* (an open set containing the intersections of the invariant manifolds of P_0 and Q_0). In some cases (including first bifurcations), for a prevalent set of parameters t we prove that after the bifurcation the *homoclinic class* $H(Q_t)$ of Q_t (i.e. the closure of the transverse intersections between $W^s(Q_t)$ and $W^u(Q_t)$) is strong partially hyperbolic and contains P_t. Observe that $H(Q_t)$ is always transitive. In fact, we obtain somewhat more: $W^s(P_t) \cap W^u(Q_t) \subset H(Q_t) \subset \Omega'(f_t)$. Thus, to get an appropriate description of the resulting non-wandering set one needs to impose some conditions on $W^s(P_0) \cap W^u(Q_0)$. Let us now state such conditions.

A cycle $\Gamma(f, P, Q)$ is *elementary* if $W^s(P)$ is transverse to $W^u(Q)$ and $W^s(P) \cap W^u(Q)$ has finitely many connected components. In particular, every elementary cycle is connected. Moreover, for an open and dense subset of arcs unfolding elementary cycles one

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has real central eigenvalues (see [**DR2**]) and one can define the strong stable and unstable foliations and speak of non-critical cycles. Denote by $\mathcal{E}(M)$ the subset of $\mathcal{C}(M)$ of arcs unfolding an elementary cycle and by $\mathcal{NIE}(M)$ the subset of $\mathcal{E}(M)$ of arcs unfolding a non-critical cycle associated to isolated points of the limit set.

Given $(f_t)_{t\in I}$ with a cycle $\Gamma(f_0, P_0, Q_0)$ let S denote the set of parameters t such that the resulting limit set $\Omega'(f_t)$ is strongly partially hyperbolic and equal to $H(Q_t)$. Since P_t and Q_t belong to $\Omega'(f_t)$, the set S is contained in the bifurcation set of $(f_t)_{t\in I}$.

THEOREM E. *There is an open and dense subset $\mathcal{A}_3(M)$ of $\mathcal{NIE}(M)$ such that*

$$\liminf_{t\to 0} \frac{|S \cap [-t, t]|}{2t} > 0$$

for every $(f_t)_{t\in I}$ in $\mathcal{A}_3$.

2. *Basic definitions and auxiliary results*

2.1. *Heterodimensional cycles.* Let $f : M \to M$ be a diffeomorphism and P and Q hyperbolic periodic points of f. We say that f exhibits a *heterodimensional cycle $\Gamma(f, P, Q)$ associated to P and Q* if $W^s(P)$ and $W^u(Q)$ have non-empty transverse intersection, and $W^u(P)$ and $W^s(Q)$ have a quasi-transverse intersection (i.e. there is an $x \in W^u(P) \cap W^s(Q)$ such that $\dim(T_x W^u(P) + T_x W^s(Q)) = n - 1$, where n is the dimension of M). Recall that the *index* of a hyperbolic periodic point is the number of eigenvalues, counted with multiplicity, of modulus less than one of its derivative. Then the points P and Q in $\Gamma(f, P, Q)$ satisfy

$$\text{index}(P) = \text{index}(Q) + 1. \tag{2.1}$$

2.1.1. *Connected, elementary, and non-critical heterodimensional cycles.* The cycle $\Gamma(f, P, Q)$ is *connected* if there is a connected component γ (called the *connection*) of $W^s(P) \pitchfork W^u(Q)$ that is periodic (i.e. $f^k(\gamma) = \gamma$ for some $k \geq 1$). The cycle is *elementary* if there is a connection γ such that

$$W^s(P) \cap W^u(Q) = W^s(P) \pitchfork W^u(Q) = \gamma \cup \cdots \cup f^k(\gamma).$$

Let λ_c and β_c, $|\lambda_c| < 1 < |\beta_c|$, be eigenvalues of $D_P f^k$ and $D_Q f^m$, with k and m the periods of P and Q. We say that λ_c and β_c are *central eigenvalues of* $\Gamma(f, P, Q)$ if $|\lambda| \leq |\lambda_c|$ (respectively $|\beta| \geq |\beta_c|$) for every contracting (respectively expanding) eigenvalue of $D_P f^k$ (respectively $D_Q f^m$). Observe that, *a priori*, the central eigenvalues are not uniquely defined (for instance, if λ_c is non-real). The cycle has *real central eigenvalues* if the central eigenvalues are uniquely defined, in particular $\lambda_c, \beta_c \in \mathbb{R}$, and have multiplicity one. Let us observe that if $\Gamma(f, P, Q)$ is elementary then (for an open and dense subset of diffeomorphisms exhibiting such cycles) it has real central eigenvalues λ_c and β_c which are positive, see the arguments in [**DR2**, §3] for details. In this case, $f^p(\gamma) = \gamma$, where $p = \text{lcm}(k, m)$, and γ is the connection of $\Gamma(f, P, Q)$.

If $\Gamma(f, P, Q)$ has real central eigenvalues then there is a unique *strong stable foliation* $\mathcal{F}^{ss}$ defined on $W^s(P)$ with f-invariant leaves of dimension $\text{index}(P) - 1$ (respectively

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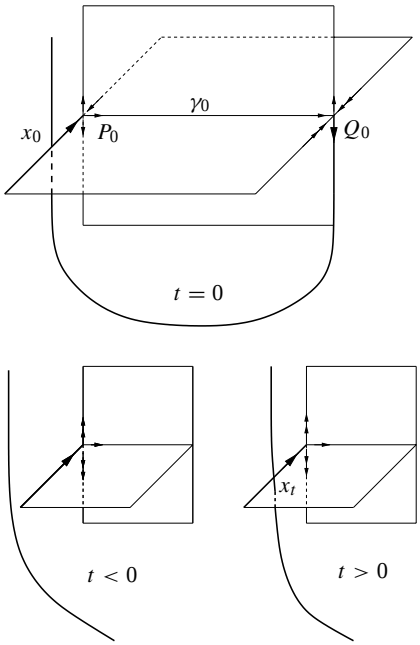


FIGURE 1. The unfolding of a heterodimensional cycle.

strong unstable foliation $\mathcal{F}^{uu}$ defined on $W^u(Q)$ with leaves of dimension ($n - \text{index}(Q) - 1$)). We say that $\Gamma(f, P, Q)$ is *non-critical* if every connected component of $W^s(P) \pitchfork W^u(Q)$ is transverse to $\mathcal{F}^{ss}$ (in $W^s(P)$) and to $\mathcal{F}^{uu}$ (in $W^u(Q)$).

2.1.2. *Generic unfolding of cycles.* Consider a cycle $\Gamma(f, P, Q)$. The following conditions hold generically (for an open and dense subset of diffeomorphisms exhibiting heterodimensional cycles):

(HC1) $W^u(P) \cap W^s(Q) = \{f^i(x)\}_{i\in\mathbb{Z}}$;

(HC2) $T_x W^u(P) + T_x W^s(Q) = T_x W^u(P) \oplus T_x W^s(Q)$; and

(HC3) the transverse intersection between $W^s(P)$ and $W^u(Q)$ is a countable or finite union of pairwise disjoint curves (i.e. it is one-dimensional).

Consider now an arc $(f_t)_{t\in I=[-1,1]}$ with a *generic* cycle $\Gamma(f_0, P_0, Q_0)$ at $t = 0$ (i.e. satisfying (HC1)–(HC3)). Let $\{f_0^i(x_0)\}_{i\in\mathbb{Z}}$ be the orbit of intersection between $W^u(P_0)$ and $W^s(Q_0)$. We can assume that

$$x_0 \in f_0^{h_0}(W^u_{\text{loc}}(P_0)) \cap W^s_{\text{loc}}(Q_0), \quad \text{for some } h_0 \geq 0,$$

where $W^{s,u}_{\text{loc}}(R)$ denotes the local stable/unstable manifold of R. We say that the arc $(f_t)_{t\in I}$ *unfolds the cycle at* $t = 0$ *generically* if there are $\mathcal{C}^1$-maps $x : I \to M$, $x(0) = x_0$, and $d : I \to R^+$, $d(0) \neq 0$, such that:

(GU1) $x(t) = x_t \in f_t^{h_0}(W^u_{\text{loc}}(P_t))$ for all t;

(GU2) $\text{dist}(x_t, W^s_{\text{loc}}(Q_t)) = |t|\, d(t)$;

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(GU3) $T_{x_0}M = T_{x_0}W^u(P_0) \oplus T_{x_0}W^s(Q_0) \oplus L$, where L is the one-dimensional space spanned by $x'(0)$, the vector tangent to the curve $(x_t)_{t\in I}$ at $t = 0$.

2.2. *Basic definitions, standing notation, and generic hypotheses.*

2.2.1. *Df-invariant splittings.* Let us now recall some definitions and introduce some notation. An f-invariant set Σ is *transitive* if it is the closure of the orbit of some point x in Σ. We consider three types of Df-invariant splittings for $T_\Sigma M$: uniformly dominated, hyperbolic, and strongly partially hyperbolic. We proceed now to describe them. A Df-invariant splitting $T_\Sigma M = E \oplus F$ over Σ is *uniformly dominated* if there are $m > 0$ and $K < 1$ such that

$$\|D_x f^m/_E\| \|D_{f^m(x)} f^{-m}/_F\| < K, \quad \text{for every } x \in \Sigma,$$

where $\|\cdot\|$ denotes the norm.

A Df-invariant splitting $T_\Sigma M = E^s \oplus E^u$ is *hyperbolic* if there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left.\begin{aligned} \|D_x f^n(v^s)\| &\le C\lambda^n \|v^s\| \\ \|D_x f^{-n}(v^u)\| &\le C\lambda^n \|v^u\| \end{aligned}\right\} \quad \text{for all } x \in \Sigma,\ v^i \in E^i_x,\ i = s, u, \text{ and } n \ge 0.$$

If $T_\Sigma M$ admits a hyperbolic splitting we say that Σ is *hyperbolic*. If Σ is transitive and hyperbolic then $\dim(E^s_x) = \dim(E^s_y)$ for all $x, y \in \Sigma$. This number is the *index of* Σ.

Finally, Σ is *strongly partially hyperbolic* if there is a Df-invariant splitting of $T_\Sigma M$ into three non-trivial directions E^s, E^c, and E^u (the stable, center, and unstable directions), $T_\Sigma M = E^s \oplus E^c \oplus E^u$, and constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n(v^s)\| \le C\lambda^n \|v^s\| \quad \text{and} \quad \|D_x f^{-n}(v^u)\| \le C\lambda^n \|v^u\|,$$

for all $x \in \Sigma$ and $v^i \in E^i_x$, $i = s, u$, and $n \ge 0$. Moreover, if we let $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$, then the splittings $E^s \oplus E^{cu}$ and $E^{cs} \oplus E^u$ are both uniformly dominated.

2.2.2. *The resulting non-wandering set.* Consider an arc $(f_t)_{t\in I}$ with a cycle $\Gamma(f_0, P_0, Q_0)$ at $t = 0$. An open set V is a *neighborhood of the cycle* if it contains $H(P_0)$, $H(Q_0)$, $W^u(P_0) \cap W^s(Q_0)$, and $W^s(P_0) \cap W^u(Q_0)$. The *maximal invariant set of f_t in V*, denoted by $\Lambda_t(V)$, is the set of points whose f_t-orbit is contained in V, i.e. $\Lambda_t(V) = \cap_{i\in\mathbb{Z}} f_t^i(V)$. We define the *forward (respectively backward) invariant set of f_t in V*, $\Lambda_t^+(V)$ (respectively $\Lambda_t^-(V)$), by $\Lambda_t^+(V) = \cap_{i\le 0} f_t^i(V)$ (respectively $\Lambda_t^-(V) = \cap_{i\ge 0} f_t^i(V)$). The *resulting non-wandering set $\Omega'(f_t)$ in V* is $\Omega(f_t) \cap V$. Finally, the *bifurcation set associated to V of $(f_t)_{t\in I}$* is the set of parameters t such that $L(f_t) \cap \Lambda_t(V)$ is not hyperbolic.

To obtain a topological description of the resulting non-wandering set, we focus our attention on the set $\mathcal{NIE}(M)$ of arcs unfolding non-critical elementary cycles associated to isolated points of $\Omega(f_0)$. In this case, if V is small, $\Omega'(f_t)$ only depends on the level of the filtration containing the cycle (see [**DR2**, §3.3]). Thus, let us omit the neighborhood V and write Λ_t instead of $\Lambda_t(V)$. Let

$$S = \{t \in I : \Omega'(f_t) = L(f_t) \cap \Lambda_t = \Lambda_t = H(Q_t) \text{ is strongly partially hyperbolic}\}.$$

From the transitivity of $H(Q_t)$, and since $\Omega'(f_t)$ contains the points P_t and Q_t of different indices, $L(f_t)$ is not hyperbolic for every t in S, i.e. S consists of bifurcation parameters.

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2.3. *Auxiliary results.* For the sake of clearness we first focus our attention on arcs $(f_t)_{t\in I} \in \mathcal{H}(M)$ with two central real eigenvalues. Denote such a subset by $\mathbb{R}_2\mathcal{H}(M)$. Given $(f_t)_{t\in I} \in \mathbb{R}_2\mathcal{H}(M)$ with a cycle $\Gamma(f_0, P_0, Q_0)$, let $\lambda_c(0)$ and $\beta_c(0)$, $|\lambda_c(0)| < 1 < |\beta_c(0)|$, be its central eigenvalues. We also let $\lambda_c(t)$ and $\beta_c(t)$ be the continuations of $\lambda_c(0)$ and $\beta_c(0)$ for small $|t|$. For simplicity, let us assume that P_0 and Q_0 are fixed points. Next consider the dense subset $\mathcal{J}(M)$ of $\mathbb{R}_2\mathcal{H}(M)$ (see [**St**]) of arcs such that:

(J1) for every small $|t|$ the diffeomorphism f_t is $\mathcal{C}^1$-linearizable at P_t and Q_t, and the linearizations can be taken to depend continuously on t;
(J2) $\log(\lambda_c(0))/\log(\beta_c(0))$ is irrational;
(J3) $\lambda_c'(0) \neq 0 \neq \beta_c'(0)$.

Writing $\mathcal{JE}(M) = \mathcal{J}(M) \cap \mathcal{E}(M)$, we prove the following result.

THEOREM 2.1. *Let $(f_t)_{t\in I} \in \mathcal{JE}(M)$ be an arc with a cycle $\Gamma(f_0, P_0, Q_0)$. Then there is a neighborhood $\mathcal{W}$ of $(f_t)_{t\in I}$ in $\mathcal{H}(M)$ such that for every $(g_t)_{t\in I} \in \mathcal{W}$ there is a sequence $(B_n)_n$ of pairwise disjoint closed intervals converging to zero such that:*

(i) $\liminf_{t\to 0} \dfrac{|B \cap [-t, t]|}{2t} > \varsigma > 0, \quad B = \bigcup_n B_n;$
(ii) $W^s(P_{g_t}) \pitchfork W^u(Q_{g_t}) \subset H(Q_{g_t}) \subset L(g_t)$ *for all* $t \in B_n$.
(iii) *Suppose that P_0 and Q_0 are isolated in $L(f_0)$ and that the cycle is non-critical, i.e. $(f_t)_{t\in I} \in \mathcal{NIE}(M)$. Then, for every small neighborhood V of the cycle and every large n, there is $S_n' \subset (S \cap B_n)$ with full measure in B_n.*

Observe that the tangent space of any curve δ_t in $W^s(P_{g_t}) \pitchfork W^u(Q_{g_t})$ does not admit a hyperbolic splitting: every vector tangent to δ_t is asymptotically contracted by both Dg_t and Dg_t^{-1}. Thus, the intervals B_n are in the bifurcation set associated to any neighborhood of the cycle.

Consider now a cycle $\Gamma(f, P, Q)$ with real central eigenvalues, say $|\lambda_c| < 1 < |\beta_c|$. Then the strong stable and unstable foliations, $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$, in $W^s(P)$ and $W^u(Q)$ are defined. Consider a one-dimensional *centre stable manifold of P*, $W^{cs}(P)$, tangent to the eigenspace associated to λ_c, and the projection π^{ss} from $W^s(P)$ to $W^{cs}(P)$ along the leaves of $\mathcal{F}^{ss}$. Analogously, consider a *centre unstable manifold of Q*, $W^{cu}(Q)$, and the projection π^{uu} from $W^u(Q)$ to $W^{cu}(Q)$ along the leaves of $\mathcal{F}^{uu}$. The cycle $\Gamma(f, P, Q)$ *contains fundamental domains* if there are curves $\gamma_1, \ldots, \gamma_k$ in $W^s(P) \pitchfork W^u(Q)$ such that either the union of the interiors of $\pi^{ss}(\gamma_1), \ldots, \pi^{ss}(\gamma_k)$ contains a fundamental domain of $W^{cs}(P)$, or the union of the interiors of $\pi^{uu}(\gamma_1), \ldots, \pi^{uu}(\gamma_k)$ contains a fundamental domain of $W^{cu}(Q)$. This definition does not depend on the choice of the centre manifolds. Moreover, every connected cycle contains fundamental domains.

THEOREM 2.2. *Let $(f_t)_{t\in I}$ be an arc in $\mathcal{J}(M)$ with a cycle $\Gamma(f_0, P_0, Q_0)$ containing fundamental domains. Then there is a neighborhood $\mathcal{W}$ of $(f_t)_{t\in I}$ such that for every $(g_t)_{t\in I} \in \mathcal{W}$ there is a sequence $(B_n)_n$ of pairwise disjoint closed intervals converging to zero such that:*

(i) $W^s(P_{g_t}) \cap W^u(Q_{g_t}) \subset L(g_t)$ *for all* $t \in B_n$, *thus B_n is contained in the bifurcation set;*

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(ii) *for every $t \in B_n$ there is a g_t-invariant set Λ_{g_t} containing P_{g_t} and Q_{g_t} that is transitive, locally maximal, and strongly partially hyperbolic.*

Finally, we have the following version of Theorem 2.2 for $\mathcal{C}^k$-diffeomorphisms, $k \geq 1$.

COROLLARY 2.3. *Let f be a $\mathcal{C}^k$-diffeomorphism, $k \geq 1$, with a cycle $\Gamma(f, P, Q)$ with real central eigenvalues and containing fundamental domains. Then for every $\mathcal{C}^k$-neighborhood $\mathcal{U}_f$ of f there is a $\mathcal{C}^k$-open subset $\mathcal{V}$ of $\mathcal{U}_f$ such that for every $g \in \mathcal{V}$ one has:*

(i) *$W^s(P_g) \cap W^u(Q_g) \subset L(g)$;*

(ii) *there is a g-invariant set Λ_g containing P_g and Q_g that is transitive, locally maximal, and strongly partially hyperbolic.*

Here, for simplicity, we first prove Theorem 2.1 in the non-critical case. This proof allows us to introduce the main tools of our constructions: families of transitions and returns (§3.2), secondary cycles (§3.4), and blenders (§3.5). In §6 we explain how the general version of Theorem 2.1 and the proofs of Theorem 2.2 and Corollary 2.3 follow from the arguments in the proof of Theorem 2.1 in the non-critical case.

3. *Main tools: transitions, itineraries, secondary cycles, and blenders*

To prove Theorem 2.1 we begin by considering arcs $(f_t)_{t\in I} \in \mathcal{JE}(M)$ with a non-critical cycle, that is, satisfying:

(NC) the connection γ_0 of $\Gamma(f_0, P_0, Q_0)$ satisfies $\gamma_0 = W^s(P_0) \cap W^u(Q_0) = W^s(P_0) \pitchfork W^u(Q_0)$ and it is transverse to the strong stable and unstable foliations.

From now on consider a fixed arc $(f_t)_{t\in I} \in \mathcal{JE}(M) \cap \mathcal{NE}(M)$.

3.1. *Coordinates, foliations, and projections.*

3.1.1. *Linearizing coordinates.* Let $n \geq 3$ be the dimension of M and let $\lambda_1(t), \ldots, \lambda_n(t)$ (respectively $\beta_1(t), \ldots \beta_n(t)$) be the eigenvalues of $D_{P_t} f_t$ (respectively $D_{Q_t} f_t$), where $|\sigma_i(t)| \leq |\sigma_{i+1}(t)|$, $\sigma = \lambda, \beta$. From (2.1) the indices of P_t and Q_t are r and $r - 1 \geq 1$, respectively. Since $(f_t)_{t\in I} \in \mathcal{E}(M)$ one has

(CE) $|\lambda_{r-1}(t)| < \lambda_r(t) < 1 < \beta_r(t) < |\beta_{r+1}(t)|$ for every $|t|$ close to 0.

For simplicity from now on we write $\lambda_r(t) = \lambda_t$ and $\beta_r(t) = \beta_t$. For small t denote by γ_t the continuation of the connection γ_0 of the cycle, i.e. the connected component of $W^s(P_t) \pitchfork W^u(Q_t)$ close to γ_0. Generically the closure of γ_t is tangent at P_t (respectively Q_t) to the eigenspace associated to λ_t (respectively to β_t), see [**D1**, §3] for details. In other words, γ_t is a separatrix of a centre stable manifold of P_t and of a centre unstable manifold of Q_t. Consider the neighborhoods $\mathcal{U}_P$ and $\mathcal{U}_Q$ of P_t and Q_t (independent of t) where f_t is linearizable. From the previous comments we can take the linearizing coordinates such that

$$\gamma_t \cap \mathcal{U}_P = \{x = (x_1, \ldots, x_n) \text{ such that } x_i = 0 \text{ for all } i \neq r \text{ and } x_r < 0\},$$
$$\gamma_t \cap \mathcal{U}_Q = \{x = (x_1, \ldots, x_n) \text{ such that } x_i = 0 \text{ for all } i \neq r \text{ and } x_r > 0\}.$$

Given $x \in \mathcal{U}_P \cup \mathcal{U}_Q$ we write $x = (x_s, x_c, x_u)$, where $x_s = (x_1, \ldots, x_{r-1})$, $x_c = x_r$, and $x_u = (x_{r+1}, \ldots, x_n)$. In $\mathcal{U}_Q$ we define the subsets

$$S(Q_t) = \{(x_s, x_c, 0_u)\}, \quad S(Q_t)^+ = S(Q_t) \cap \{x_c > 0\}.$$

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Due to the geometry of the cycle and the non-critical assumption (NC), since every $z = (0_s, z_c, 0_u) \in \mathcal{U}_Q$ with $z_c > 0$ belongs to γ_t, we can take the linearizing coordinates in $\mathcal{U}_Q$ such that

$$S(Q_t)^+ \subset W^s(P_t).$$

Analogously, using the coordinates in $\mathcal{U}_P$, we can assume that

$$S(P_t) \subset W^s(P_t), \quad \text{where } S(P_t) = \{(x_s, x_c, 0_u)\} \subset \mathcal{U}_P.$$

3.1.2. *Foliations.* In $\mathcal{U}_P$ and $\mathcal{U}_Q$ consider the extensions of the strong stable and unstable foliations with leaves $F_t^{ss} = \{(x_s, 0, 0_u)\}$ and $F_t^{uu} = \{(0_s, 0, x_u)\}$, respectively. Using the non-criticality condition, for small $|t|$ we can extend these foliations to a small neighborhood U (independent of t) of the connection γ_t. Such extensions, also denoted by $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$, *a priori*, are not f_t-invariant. Recalling that $|\lambda_t| > |\lambda_i(t)|$ for all $1 \le i \le (r-1)$ and $|\beta_t| < |\beta_i(t)|$ for all $(r+1) \le i \le n$, and using the non-criticality condition one obtains

$$\lim_{n\to\infty} C(f_t^{-n}(x), F_t^{ss}(f_t^{-n}(x)) \cap \mathcal{U}_Q) \to W^s_{\text{loc}}(Q_t),$$
$$\lim_{n\to\infty} C(f_t^{n}(x), F_t^{uu}(f_t^{n}(x)) \cap \mathcal{U}_P) \to W^u_{\text{loc}}(P_t),$$

for any $x \in \gamma_t$, where the convergence is understood to be in the $\mathcal{C}^1$-topology and $C(x, A)$ denotes the connected component of the set A containing x.

Observe that we can assume that $x_0 \in \mathcal{U}_Q$ and $f_0^{-h_0}(x_0) \in \mathcal{U}_P$, h_0 as in (GU1) in the definition of generic unfolding of the cycle. Let $\mathcal{B} = f_0^{h_0}(\mathcal{U}_P)$ and take a small neighborhood $U_0 \subset \mathcal{U}_Q$ of x_0 with $U_0 \subset \mathcal{B}$. Consider the neighborhood V of the cycle given by

$$V = \mathcal{U}_P \cup \mathcal{U}_Q \cup U \cup (\cup_{i=0}^{h_0} f_0^{-i}(U_0)). \tag{3.1}$$

Our next objective is to extend the foliations to the whole V by iterations of $f_t^{\pm}$. For instance, in $\mathcal{B}$ we consider the extension $\tilde{\mathcal{F}}_t^{uu}$ with leaves

$$\tilde{F}_t^{uu}(x) = f_t^{h_0}(\varphi_{P_t}^{-1}(\{(a_s, b_c, x_u), x_u \in \mathbb{R}^u\})),$$

where $\varphi_{P_t}(x) = (a_s, b_c, d_u)$ and φ_{P_t} is the linearizing coordinate at P_t. As before, if $x \in \mathcal{B} \cap \mathcal{U}_Q$, $\tilde{F}_t^{uu}(x)$ may be different from $F_t^{uu}(x)$. But, one has that these leaves are arbitrarily $\mathcal{C}^1$-close (the proximity is of the order of Ct^α, for some $\alpha > 1$ and $C > 0$ independent of t, see [**D1**] and §6). For notational simplicity we also denote such extensions by $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ and suppose that:

(IF) the foliations $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ defined in V are f_t-invariant.

The general case is handled as in [**D1**] or [**DR2**]. For completeness, in §6 we explain how one removes this hypothesis in the general case.

3.1.3. *Projections.* Define projections $\pi_t^{u,s}$ along the leaves of the foliations $\mathcal{F}_t^{ss,uu}$ in V,

$$\begin{aligned} &\pi_t^u : S(P_t) \to S(Q_t), \quad x \mapsto F_t^{uu}(x) \cap S(Q_t), \\ &\pi_t^s : S(R_t) \to \{0_s, x_c, 0_u)\}, \quad (x_s, x_c, 0_u) \mapsto (0_s, x_c, 0_u), \quad R = P, Q. \end{aligned} \tag{3.2}$$

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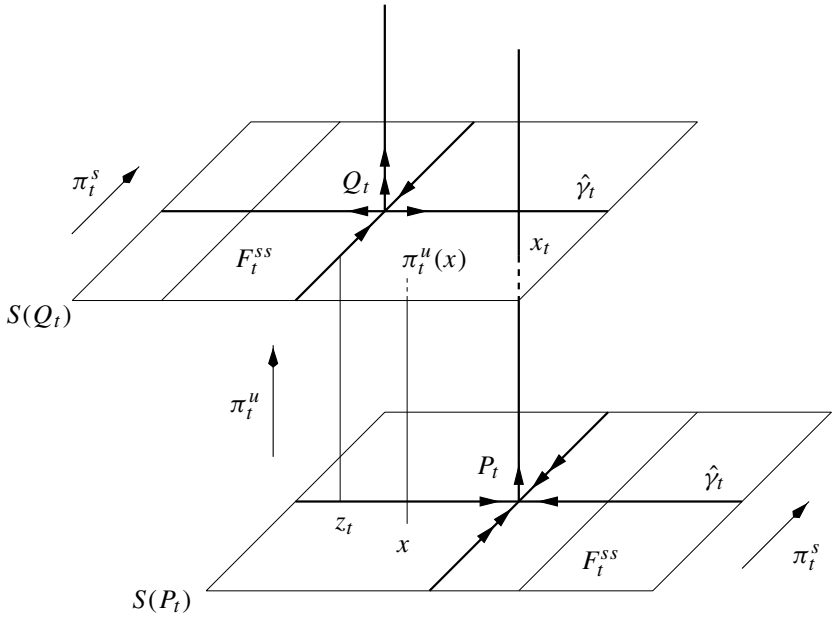


FIGURE 2. Projections.

Consider the extension $\hat{\gamma}_t$ of γ_t given by

$$\hat{\gamma}_t = \gamma_t \cup \varphi_{P_t}^{-1}(\{(0_s, x_c, 0_u)\}) \cup \varphi_{Q_t}^{-1}(\{(0_s, x_c, 0_u)\})$$

and the natural ordering $<$ in $\hat{\gamma}_t$ with $Q_t < P_t$. Given $A \subset \hat{\gamma}_t$ we say that a map $\rho : A \to \hat{\gamma}_t$ *is order preserving* if $\rho(x) < \rho(y)$ for all $x < y$. We have the following result (for the proof see [**DR2**, Lemma 3.2]).

LEMMA 3.1. *Suppose that $(f_t) \in \mathcal{E}(M)$ unfolds a non-critical cycle. Then $\pi_t^s \circ \pi_t^u : \hat{\gamma}_t \cap \mathcal{U}_P \to \hat{\gamma}_t \cap \mathcal{U}_Q$ is order preserving for every small $|t|$.*

The geometry of the cycle implies that the points x_t in (GU1) and (GU2) belong to $S(Q_t)$. Assume that $x_t \in S(Q_t)^+$ for $t > 0$, otherwise consider the unfolding of the cycle for negative t. In particular, since $S(Q_t)^+ \subset W^s(P_t)$, using the non-criticality condition, we have that $x_t \in H(P_t)$.

To emphasize the main ideas in our proof and to avoid technicalities in our constructions, let us assume that in our coordinates $\pi_t^s \circ \pi_t^u$ has the simplest form
(AP) $\pi_t^s \circ \pi_t^u(x) = (a_s(x), x_c + t, 0_u)$, $x = (x_s, x_c, 0_u) \in S(P_t)$.
In particular,

$$\pi_t^s \circ \pi_t^u(P_t) = x_t = (a_s(t), t, 0_u),$$

where $a_s(x)$ and $a_s(t)$ are $\mathcal{C}^1$ maps. Finally, for small $t > 0$ define z_t as the (unique) point such that

$$z_t \in \gamma_t \cap \mathcal{U}_P \quad \text{and} \quad \pi_t^u(z_t) = (x_s, 0_c, 0_u), \quad \text{i.e. } \pi_t^u(z_t) \in W^s_{\text{loc}}(Q_t). \tag{3.3}$$

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From (AP), in our local coordinates in $\mathcal{U}_P$,

$$z_t = (0_s, -t, 0_u).$$

In §6 we explain how to proceed in the general case. In very rough terms, one can think that $\pi_t^s \circ \pi_t^u$ is an affine map. Essentially, this is the same as supposing that the distortion of the *returns* is included in the *transition* T_t from Q_t to P_t which we are going to introduce. In the following sections we also explain what we mean exactly by returns and transitions.

3.2. *Transitions and returns.* From now on we focus our attention on the dynamics of points x in $\Lambda_t(V) = \cap_{i\in\mathbb{Z}} f_t^i(V)$, where V is the neighborhood of the cycle defined in (3.1). In this section V only plays an auxiliary role and thus our results do not depend on V. Consider x in $\Lambda_t(V)$, small $t > 0$. If $x \notin W^s(P_t) \cup W^s(Q_t)$ then its forward orbit returns infinitely many times to $\mathcal{U}_Q$. To analyze such returns we fix the fundamental domain $\tilde{D}_t^+ \subset \mathcal{U}_Q$ of the restriction of f_t to γ_t and the cube $\tilde{C}_t^+$ defined by

$$\tilde{D}_t^+ = \{0_s\} \times [\beta_t^{-1}, 1] \times \{0_u\},$$
$$\tilde{C}_t^+ = \{x = (x_s, x_c, x_u) \in \mathcal{U}_Q \text{ such that } (0_s, x_c, 0_u) \in \tilde{D}_t^+\}.$$

The *forward returns* $\{n_i(x)\}_{i\geq 0}$ *of* x *to* $\tilde{C}_t^+$ are given by

$$x_i = f_t^{n_i(x)}(x) \in \tilde{C}_t^+ \quad \text{and} \quad f_t^n(x) \notin \tilde{C}_t^+, \quad \text{for all } n_i(x) + 1 < n < n_{i+1}(x).$$

Observe that every $x \in \Lambda_t(V)$, $x \neq P_t, Q_t$, has at least one iterate (backward or forward) in $\tilde{C}_t^+$. Thus, replacing x by some iterate, we can assume that $x \in \tilde{C}_t^+$ and $n_0(x) = 0$.

Remark 3.2. Let $m_i(x) = n_{i+1}(x) - n_i(x)$. The action of the derivative $D_{x_i} f_t^{m_i(x)}$ in the strong stable and unstable directions (tangent to $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$) is hyperbolic: there is $\lambda > 1$ such that $D_{x_i} f_t^{m_i(x)}$ is a uniform contraction of ratio $\lambda^{-m_i(x)}$ in the stable direction and a uniform expansion of ratio $\lambda^{m_i(x)}$ in the unstable direction.

The remark follows by using the standard cone field construction and (IF). Thus, in our proof we focus our attention on the action of $D_{x_i} f_t^{m_i(x)}$ in the *central direction* (almost parallel to γ_t, i.e. on vectors $v \simeq (0_s, v_c, 0_u)$). A key idea we borrow from [**D1**] and [**DR2**] is that such a derivative is *essentially* determined by the restriction of f_t to γ_t. This justifies the introduction of the family of transitions (T_t) below. To define such transitions we fix $h_1 > 0$ such that $f_t^{h_1}(\tilde{D}_t^+) = \tilde{D}_t^- \subset \mathcal{U}_P$. For simplicity, let us suppose that for every t

$$\tilde{D}_t^- = \{0_s\} \times [-1, -\lambda_t] \times \{0_u\}.$$

The *transition* T_t *from* $\tilde{D}_t^+$ *to* $\tilde{D}_t^-$ is defined by

$$T_t : \tilde{D}_t^+ \to \tilde{D}_t^-, \quad x \mapsto T_t(x) = f_t^{h_1}(x).$$

These transitions can be extended differentially, and we do so to a compact neighborhood $K(\tilde{D}_t^+)$ of $\tilde{D}_t^+$ in γ_t in the natural way. We also denote this extension by T_t. Since the derivative T_t' depends continuously on t there are $0 < m < M$ such that

$$m < |T_t'(x)| < M, \quad \text{for all } x \in K(\tilde{D}_t^+) \text{ and small } t. \tag{3.4}$$

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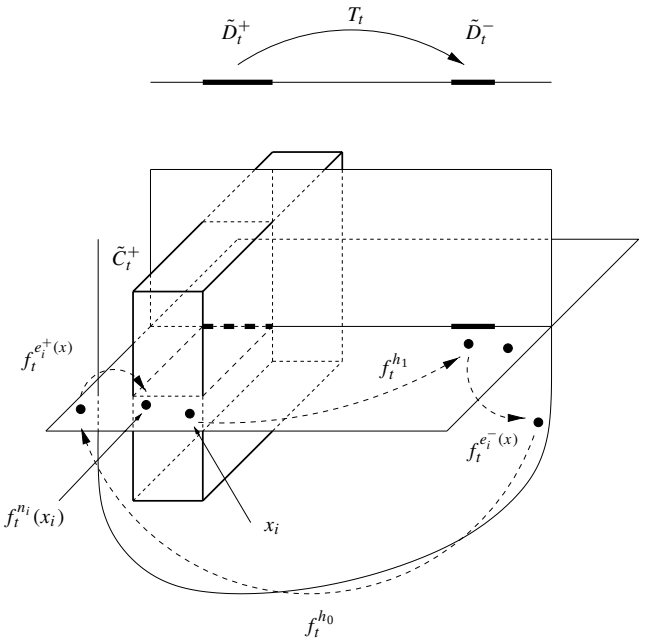


FIGURE 3. Returns.

A key ingredient of our construction is an appropriate control of the distortion of the derivatives of returns to $\tilde{C}_t^+$ in the central direction. For this we divide the segment of the orbit of x between two consecutive returns, say $f_t^{n_i(x)}(x)$, $f_t^{n_i(x)+1}(x), \ldots, f_t^{n_{i+1}(x)}(x)$, into four parts:

(i) the transition from $\mathcal{U}_Q$ to $\mathcal{U}_P$ (i.e. the h_1 iterates of f_t corresponding to the transition T_t);
(ii) the iterates in $\mathcal{U}_P$, say $e_i^-(x)$;
(iii) the transition from $\mathcal{U}_P$ to $\mathcal{U}_Q$ (i.e. the h_0 iterates in (GU1)); and
(iv) the iterates in $\mathcal{U}_Q$ before returning to $\tilde{C}_t^+$, say $e_i^+(x)$.

The estimates of the distortion in $\mathcal{U}_P$ and $\mathcal{U}_Q$ (parts (ii) and (iv)) are trivial (the diffeomorphisms are linearizable in these sets). In our coordinates the h_0 iterates in the transition from $\mathcal{U}_P$ to $\mathcal{U}_Q$ (part (iii)) correspond to an isometry: we are concerned with the central direction and assume that $\pi_t^s \circ \pi_t^u$ is a translation in the x_c-direction, see (AP). Thus the crucial point is to control the derivative of T_t. This is done in the following section.

3.3. *Choice of the itineraries of returns. Estimates on the distortion of T_t.* To estimate the distortion of T_t given small $\varepsilon_1 > 0$ (to be fixed in (3.6)) we divide $K(\tilde{D}_t^+)$ into intervals U_k such that $DT_t(x)/DT_t(y) \in (1-\varepsilon_1, 1+\varepsilon_1)$ for all $x, y \in U_k$; that is, in each U_k the distortion of T_t is fairly bounded. The next step is to obtain an accurate control of the central coordinate of the returns x_i of x. We will get such an appropriate control for parameters s close to the parameters corresponding to *secondary cycles*, see the definition

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in §3.4. Finally, we prove that such parameters s have positive density at the bifurcation, see Lemma 3.5 and Proposition 3.6. Let us now go into the details of our construction.

3.3.1. *Distortion constants.* In this section we fix some constants which allow us to control the distortion of T_t. The reasons for the choice of such constants will be clear throughout the proof of Theorem 2.1. We begin by taking constants $1 < \rho_1 < \rho_2$ close to 1 such that

$$(1-\rho_2^{-1}) < \frac{m}{2M\beta_0^{10}} \quad \text{and} \quad \frac{\beta_0^{-14}\lambda_0^{13}m^2}{M(1-\rho_2^{-1})} > 3, \tag{3.5}$$

with m and M as in (3.4). Clearly, (3.5) holds for ρ_2 close to 1. Fix $\varepsilon_1 > 0$ such that

$$0 < \varepsilon_1 < \sqrt{\frac{2\rho_2}{\rho_1+\rho_2}} - 1 < \sqrt{\frac{\rho_2}{\rho_1}} - 1. \tag{3.6}$$

If ρ_1 is close enough to ρ_2 then ε_1 is arbitrarily small. Associated with ε_1 we consider a covering $\mathcal{U} = \{U_1, U_2, \dots, U_\ell\}$ of $\tilde{D}_t^+$ (independent of small t) by open intervals contained in $K(\tilde{D}_t^+)$ such that for each i there are m_i and M_i, $0 < m \le m_i < M_i \le M$, satisfying

$$\begin{aligned} &m_i < |DT_t(x)| < M_i, \quad \text{for all } x \in U_i \text{ and small } t, \\ &\frac{3}{4} < (1-\varepsilon_1)^{1/2} < \frac{m_i}{M_i} < \frac{M_i}{m_i} < (1+\varepsilon_1)^{1/2}. \end{aligned} \tag{3.7}$$

Pick $\upsilon = \lambda_0^2\beta_0^{-2}$; this constant will appear in a natural way in (3.14). In $[\upsilon, \upsilon^{-1}]$ take the partition induced by $\upsilon = \upsilon_0 < \upsilon_1 < \cdots < \upsilon_j < \upsilon_{j+1} < \cdots < \upsilon_k = \upsilon^{-1}$, where

$$(1-\varepsilon_1)^{1/2} < \frac{\upsilon_{j-2}}{\upsilon_{j+2}} < \frac{\upsilon_{j+2}}{\upsilon_{j-2}} < (1+\varepsilon_1)^{1/2}, \quad \text{for every } j \in [2, k-2]. \tag{3.8}$$

Note that we can assume that $k = \ell$ (the number of intervals of the covering $\mathcal{U}$ of $\tilde{D}_t^+$).

3.3.2. *Itineraries of returns.* By hypothesis $\log(\beta_0)/\log(\lambda_0)$ is irrational, see (J2). From this it follows that

$$\{\lambda_0^i\beta_0^j\}_{i,j>0} \text{ is dense in } \mathbb{R}^+. \tag{3.9}$$

This fact will allow us to choose returns with fine distortion properties. As $\{\lambda_0^i\beta_0^j\}_{i,j>0}$ is dense in $\mathbb{R}^+$, for each $e, g \in \{1, \dots, \ell\}$, $g \ge 1$, there are integers $k_{e,g}$, $j_{e,g} > 0$ (that we can take arbitrarily large) such that

$$\lambda_0^{k_{e,g}}\beta_0^{j_{e,g}} m_e^2 \upsilon_{g-1}^2 \in \left(\rho_1, \rho_1 + \frac{\rho_2-\rho_1}{2}\right) = \left(\rho_1, \frac{\rho_2+\rho_1}{2}\right).$$

From (3.7), (3.8), and (3.6), one has

$$\lambda_0^{k_{e,g}}\beta_0^{j_{e,g}} M_e^2 \upsilon_{g+2}^2 = \lambda_0^{k_{e,g}}\beta_0^{j_{e,g}} m_e^2 \upsilon_{g-2}^2 \frac{M_e^2}{m_e^2}\frac{\upsilon_{g+2}^2}{\upsilon_{g-2}^2} < \frac{\rho_1+\rho_2}{2}(1+\varepsilon_1)^2 < \rho_2,$$

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where $2 \le g \le \ell - 2$. The two inequalities above are summarized as follows: given $e, g \in \{1, \dots, \ell\}$, $2 \le g \le \ell - 2$, there are $k_{e,g},\ j_{e,g} > 0$ (arbitrarily large) such that

$$\rho_1 < \lambda_0^{k_{e,g}} \beta_0^{j_{e,g}} m_e^2 \upsilon_{g-2}^2 < \lambda_0^{k_{e,g}} \beta_0^{j_{e,g}} M_e^2 \upsilon_{g+2}^2 < \rho_2, \quad \text{for all } e, g \in \{1, \dots, \ell\}. \tag{3.10}$$

Remark 3.3. We first choose the constants $1 < \rho_1 < \rho_2$ with ρ_1 and ρ_2 close to 1 and satisfying (3.5). Next we fix a small $\varepsilon_1 > 0$ (depending on ρ_1 and ρ_2, see (3.6)). In its turn, ε_1 determines the covering $\mathcal{U} = \{U_i\}_{i=1}^{\ell}$ of $\tilde{D}_t^+$ as well as its Lebesgue number τ (i.e. for every $x \in \tilde{D}_t^+$ the ball of radius τ centered at x is contained in some U_j). The constant ε_1 also determines the partition $\upsilon_0, \dots, \upsilon_\ell$ of $[\upsilon^{-1}, \upsilon]$, see (3.8). The covering $\mathcal{U}$ determines the constants $\{m_i, M_i\}_{i=1}^{\ell}$, in (3.7). Finally, we take $k_{e,g}$ and $j_{e,g}$, $e, g \in \{1, \dots, \ell\}$, satisfying (3.10). Since $k_{i,g}$ and $j_{i,g}$ can be taken arbitrarily large, we can assume that

$$\lambda_0^{k_{i,g}} + \beta_0^{-j_{i,g}} M_i^2 < \tau, \quad \text{for all } i, g \in \{1, \dots, \ell\}. \tag{3.11}$$

Let us end this section with the following remark we use in §6.

Remark 3.4. Let $(f_t)_{t\in I} \in \mathcal{JE}(M)$ and denote by $\lambda_c(f_0)$ and $\beta_c(f_0)$ the central eigenvalues of $(f_t)_{t\in I}$. The maps

$$(g_t)_{t\in I} \mapsto (\lambda_c(g_0), \beta_c(g_0)) \quad \text{and} \quad t \mapsto (\lambda_c(f_t), \beta_c(f_t)),$$

are both continuous at $(f_t)_{t\in I}$ and $t = 0$, respectively. In particular, (3.5), (3.7), (3.10), and (3.11) hold for all small $|t|$ and every $(g_t)_{t\in I}$ in a neighborhood $\mathcal{W}$ of a fixed arc $(f_t)_{t\in I} \in \mathcal{JE}(M)$.

3.4. *Secondary cycles.* Given $x = (x_s, x_c, x_u) \in \mathcal{U}_R$, $R = P, Q$, let $x^c = (0_s, x_c, 0_u)$. We use the same notation for vectors: given a vector $v = (v_s, v_c, v_u)$ then $v^c = (0_s, v_c, 0_u)$.

Let $t > 0$. The *time of exit of* x_t, N_t^+ (see (GU1) for the definition of x_t) is the minimum number of positive iterations that $x_t^c = (0_s, t, 0_u)$ needs to arrive at $\tilde{D}_t^+$. Similarly, the *time of entry of* z_t, N_t^- (see (3.3) for the definition of z_t) is the minimum number of negative iterations that $z_t^c = z_t = (0_s, -t, 0_u)$ needs to arrive at $\tilde{D}_t^-$. These times satisfy

$$\begin{aligned} \beta_t^{N_t^+} t \in \tilde{D}_t^+ = [\beta_t^{-1}, 1), \quad N_t^+ \simeq \frac{-\log(t)}{\log(\beta_t)}, \\ \lambda_t^{-N_t^-}(-t) \in \tilde{D}_t^- = (-1, -\lambda_t], \quad N_t^- \simeq \frac{\log(t)}{\log(\lambda_t)}. \end{aligned} \tag{3.12}$$

The *scaled fundamental domains* $D_t^\pm$ of $\tilde{D}_t^\pm$ are now defined by

$$D_t^+ = f_t^{-N_t^+}(\tilde{D}_t^+) \quad \text{and} \quad D_t^- = f_t^{N_t^-}(\tilde{D}_\mu^-). \tag{3.13}$$

Write

$$\sigma_t = \beta_t^{N_t^+} \lambda_t^{N_t^-}.$$

From (3.12) it follows that

$$0 < \upsilon = \beta_0^{-2} \lambda_0^2 < \sigma_t < \beta_0^2 \lambda_0^{-2} = \upsilon^{-1}, \quad \text{for small } |t|. \tag{3.14}$$

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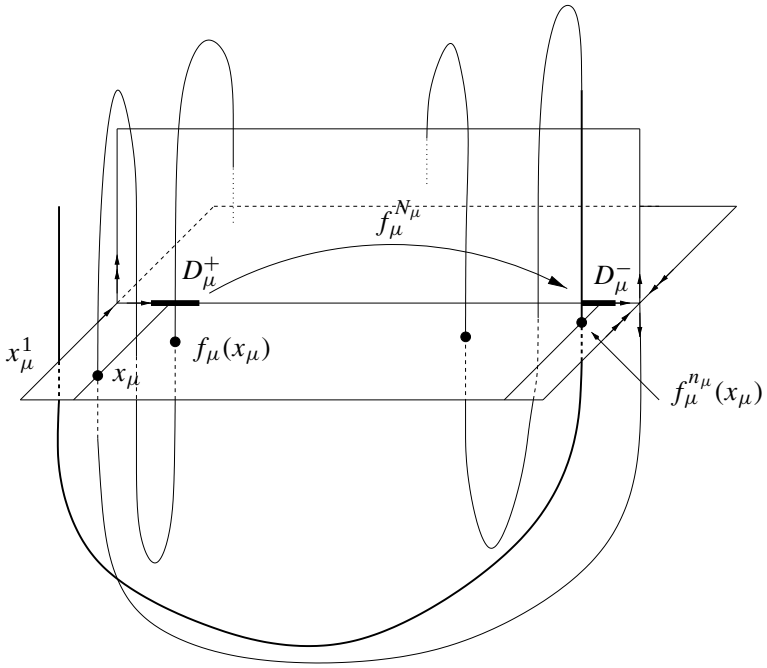


FIGURE 4. Secondary cycle.

We say that f_μ, $\mu > 0$, *has a secondary cycle* $\Gamma(f_\mu, P_\mu, Q_\mu)$ if

$$C(f_\mu^{N_\mu}(x_\mu), W^u(P_\mu) \cap \mathcal{B}) \cap W^s_{\mathrm{loc}}(Q_\mu) = \{x_\mu^1\}, \quad N_\mu = N_\mu^+ + h_1 + N_\mu^-, \tag{3.15}$$

and $(f_t)_{t\in I}$ unfolds the cycle at μ generically. From (IF) (f_t-invariance of $\mathcal{F}_t^{ss,uu}$) and (AP) ($\pi_s^u \circ \pi_t^s$ is a translation in the x_c-direction by a vector of norm $|t|$), (3.15) is equivalent to

$$\pi_\mu^s(\pi_\mu^u(f_\mu^{N_\mu}(x_\mu))) = \pi_\mu^s(x_\mu^1) = Q_\mu. \tag{3.16}$$

We have the following result on the frequency of secondary cycles.

LEMMA 3.5. *Let $(f_t)_{t\in I} \in \mathcal{E}(M)$ be an arc unfolding a non-critical cycle $\Gamma(f_0, P_0, Q_0)$. There are $K > 0$ and a sequence of parameters $(t_n)_n$, $t_n \to 0^+$, corresponding to secondary cycles $\Gamma(f_{t_n}, P_{t_n}, Q_{t_n})$ such that*

$$0 < K < \frac{t_{n+1}}{t_n} < 1.$$

We omit the proof of this lemma and refer the reader to the proofs of [**DR1**, Lemmas 3.3–3.4].

We now classify the parameters corresponding to secondary cycles according to the intervals of the covering $\mathcal{U}$ and of the partition $\{\upsilon_1, \ldots, \upsilon_\ell\}$ containing $f_t^{N_t^+}(x_t^c)$ and $\sigma_t = \beta_t^{N_t^+} \lambda_t^{N_t^-}$, respectively. Given t corresponding to a secondary cycle, consider the

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interval $I_\tau(f_t^{N_t^+}(x_t^c))$ of radius τ (the Lebesgue number of $\mathcal{U}$) centered at $f_t^{N_t}(x_t^c)$ in γ_t. By definition of τ, $I_\tau(f_t^{N_t^+}(x_t^c)) \subset U_e$ for some $e \in \{1, \dots, \ell\}$. On the other hand, by the definitions of σ_t and the partition (υ_i), see (3.14), $\upsilon_{g-1} < \sigma_t \le \upsilon_g$ for some $g \in \{1, \dots, \ell\}$. Then we say that the secondary cycle at t is of *type* (e, g). The control of these two positions will allow us to have an accurate control of the distortion of the derivative in the central direction of the returns we consider.

3.5. *Blenders of m symbols.* In R^n, $n \ge 3$, consider the n-cube $C^n = [-1, 1]^s \times [-1, 1] \times [-1, 1]^u$, $s, u \ge 1$, $s + u + 1 = n$, and a $\mathcal{C}^1$-embedding of it in the ambient manifold, $\phi : C^n \to M$. In $C = \phi(C^n)$ take the coordinates (x_s, x_c, x_u) induced by ϕ, identify $[-1, 1]$ with $\phi(\{0_s\} \times [-1, 1] \times \{0_u\})$, and write

$$D^s = \phi([-1, 1]^s \times \{(0, 0_u)\}), \quad D^u = \phi(\{(0_s, 0)\} \times [-1, 1]^u).$$

Then $C = D^s \times [-1, 1] \times D^u$. Consider the following subsets of the boundary ∂C of C:

$$\partial^{ss} C = \partial(D^s) \times [-1, 1] \times D^u, \quad \partial^{uu} C = D^s \times [-1, 1] \times \partial(D^u),$$
$$\partial^u C = D^s \times \partial([-1, 1] \times D^u).$$

Let $\mathcal{C}^{ss}$, $\mathcal{C}^{uu}$, and $\mathcal{C}^u$ be cone fields around the tangent spaces of $D^s \times \{(a_c, a_u)\}$, $\{(a_s, a_c)\} \times D^u$, and $\{a_s\} \times [-1, 1] \times D^u$, respectively. Note that $\mathcal{C}^{uu} \subset \mathcal{C}^u$. Let $\mathcal{C}^j(x)$ be the cone of $\mathcal{C}^j$ at x.

Definition 3.1.

- A *vertical disk* $V \subset C$ is a u-disk tangent to $\mathcal{C}^{uu}$ (i.e. $T_x V \subset \mathcal{C}^{uu}(x)$, for all $x \in V$) such that $\partial V \subset \partial^{uu} C$.
- A *vertical strip* S is the image of a $\mathcal{C}^1$-embedding $\Phi : [-1, 1] \times [-1, 1]^u \to C$ such that $T_x(S) \subset \mathcal{C}^u(x)$ and $S_t = \Phi(\{t\} \times [-1, 1]^u)$ is a vertical disk for all t.

The following definition for the *blender of m symbols* generalizes the concept of a center-stable blender in [**BD1**, pp. 365–366].

Definition 3.2. Given an n-cube C and a $\mathcal{C}^1$-diffeomorphism $f : M \to M$ the pair (C, f) is a *generalized blender of m symbols* if there are a constant $\rho > 1$, positive integers $\{q_i\}_1^m$, n-cubes $\{C_i\}_1^m$ contained in C, and cone fields $\mathcal{C}^{ss}$, $\mathcal{C}^{uu}$ and $\mathcal{C}^u$ as before satisfying (B1)–(B4):

(B1) $f^{q_i}(C_i) \subset C$ for every $i = 1, \dots, m$;

(B2) for every $i \in \{1, \dots, m\}$ the cone fields $\mathcal{C}^u$, $\mathcal{C}^{uu}$, and $\mathcal{C}^{ss}$ satisfy
- $D_x f^{q_i}(\mathcal{C}^u(x)) \subset \mathcal{C}^u(f^{q_i}(x))$ and $|D_x f^{q_i}(v)| > \rho|v|$ for every $x \in C_i$ and $v \in C^u(x)$,
- $D_x f^{q_i}(\mathcal{C}^{uu}(x)) \subset \mathcal{C}^{uu}(f^{q_i}(x))$ for every $x \in C_i$,
- $D_x f^{-q_i}(\mathcal{C}^{ss}(x)) \subset \mathcal{C}^{ss}(f^{-q_i}(x))$ and $|D_x f^{-q_i}(v)| > \rho|v|$ for every $x \in C_i$ and $v \in C^{ss}(x)$;

(B3) the cube C_m contains a unique hyperbolic periodic point A of period q_m. Moreover, A is in the interior of C.

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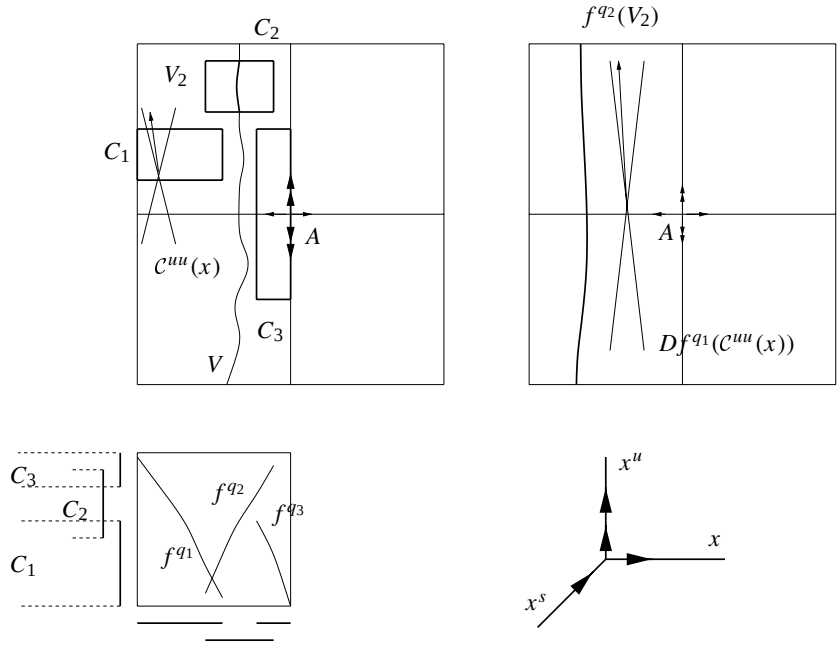


FIGURE 5. Projection of a blender of three symbols and its central dynamics.

We say that A is the *periodic point of the blender* (C, f). By (B2) the index of A is s. Condition (B2) also implies that the local stable manifold of A, $W^s_{\text{loc}}(A)$, intersects transversely the two connected components of $\partial^{ss}C$. Thus we can consider two types of vertical disks in C that do not intersect $C(A, W^s(A) \cap C) = W^s_{\text{loc}}(A)$: *vertical disks at the left of* A, briefly, ℓ*-disks* (the class of $\{(0_s, -1)\} \times D^u$), and *vertical disks at the right of* A, r*-disks* (the class of $\{(0_s, 1)\} \times D^u$). We define an ℓ*-strip* as a vertical strip consisting of ℓ-disks.

Given an ℓ-disk V write $V_i = C_i \cap V$, $i \in \{1, 2, \ldots, m\}$ (some of the V_i may be empty). We also assume that:

(B4) Given any ℓ-disk V there is $i \in \{1, 2, \ldots, m\}$ such that $f^{q_i}(V_i)$ is an ℓ-disk.

The main step towards proving Theorem 2.1 is the following result.

PROPOSITION 3.6. *Let* $(f_t) \in \mathcal{JE}(M)$ *be an arc unfolding a cycle* $\Gamma(f_0, P_0, Q_0)$ *generically. There exist a neighborhood* $\mathcal{W}$ *of* $(f_t)_{t\in I}$ *in* $\mathcal{H}(M)$ *and constants* $\delta, K > 0$ *such that for every* $(g_t)_{t\in I}$ *in* $\mathcal{W}$ *there is a decreasing sequence* $(s_n)_n$ *of parameters,* $s_n \to 0^+$, *such that:*

(a) $s_{n+1}/s_n > K$;

(b) *for every* $t \in [s_n, s_n(1+\delta)]$ *there is a generalized blender* (C_t, g_t) *such that*

(I1) $W^s(Q_t)$ *intersects transversely every* ℓ*-strip of* (C_t, g_t),

(I2) $W^u(P_t)$ *contains an* ℓ*-disk such that its interior is in the interior of* C_t.

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The existence of the blenders in Proposition 3.6 will be derived from the existence of secondary cycles and the control on the distortion of the transition maps. We postpone the proof of Proposition 3.6 to §5. We now prove Theorem 2.1 using this proposition.

4. *Proof of Theorem 2.1*

4.1. *Positive density of the bifurcation set.* To prove the theorem take $B_n = [s_n, s_n(1+\delta)]$, s_n and δ as in Proposition 3.6.

CLAIM 4.1. *Let $(g_t)_{t\in I}$ in $\mathcal{W}$ and $t \in B_n$, with n large enough. Then*

$$W^s(P_t) \pitchfork W^u(Q_t) \subset H(Q_t) \subset L(g_t).$$

In particular, $\gamma_t \subset L(g_t) \cap V$ for every neighborhood V of the cycle and $L(g_t) \cap \Lambda_t(V)$ is non-hyperbolic.

The claim immediately implies (ii) in Theorem 2.1 and the fact that B_n is contained in the bifurcation set for all large n. From the claim we also get (i) in Theorem 2.1: by shrinking δ we can assume that $s_{n+1}(1+\delta) < s_n$, where $(s_n)_n$ is the sequence of parameters in Proposition 3.6; thus

$$\liminf_{t\to 0} \frac{|\cup_n B_n \cap [-t,t]|}{2t} \geq \frac{K^2\delta}{2(1-K)} > 0.$$

We now prove the claim.

Proof. Pick a parameter t in B_n, a point x in $W^s(P_t) \pitchfork W^u(Q_t)$, and any small $(u+1)$-disk V_x in $W^u(Q_t)$ containing x in its interior. By the λ-lemma, the forward g_t-orbit of V_x accumulates on $W^u(P_t)$. By (b(I2)) in Proposition 3.6 and the cycle configuration, there is (large) $k > 0$ such that $g_t^k(V_x)$ contains an ℓ-strip $\tilde{V}_x$. From (b(I1)) we get $W^s(Q_t) \pitchfork \tilde{V}_x \neq \emptyset$. Therefore, $V_x \cap H(Q_t) \neq \emptyset$. Since this assertion holds for every $(u+1)$-disk V_x in $W^u(Q_t)$ containing x, one gets the claim. □

4.2. *Structure of the resulting non-wandering set.* To prove (iii) in Theorem 2.1 assume that P_0 and Q_0 are isolated in $L(f_0)$ and that the neighborhood V of the cycle is small. Let D denote the set of parameters t such that g_t has a cycle $\Gamma(g_t, P_t, Q_t)$, i.e. $W^u(P_t) \cap W^s(Q_t) \neq \emptyset$. Generically, D has measure zero. Write $S_n = B_n \setminus D$, then (generically) $|S_n| = |B_n|$. We claim that $S_n \subset S$ for all large n. In the following paragraphs we state the two key properties of $\Lambda_t = \Lambda_t(V)$ for $t \in S_n$: the density of $H(Q_t)$ in Λ_t and the existence of a strong partially hyperbolic splitting. These two properties of Λ_t imply that $S_n \subset S$.

4.2.1. *Density of $H(Q_t)$ in Λ_t.*

PROPOSITION 4.2. *Let $t \in S_n$ (large n) and let V be small (independent of n). Then $H(Q_t) = \Lambda_t(V) = \Lambda_t$.*

The main step for proving the proposition is the next lemma.

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LEMMA 4.3. *Let $t \in S_n$ (large n) and let V be small (independent of n). Then $\Lambda_t = \Lambda_t(V) \subset \overline{W^s(P_t) \pitchfork W^u(Q_t)}$.*

Proof. The lemma follows from arguments as in the proof of [**D1**, Lemma 5], so we just sketch it here. In the proof we use the fact that, for small V, the leaves of the strong unstable (respectively stable) foliation are expanded by the forward (respectively backward) iterations of g_t, see also Remark 3.2. More precisely, suppose first for simplicity that

$$x \in \Lambda_t \setminus (W^s(P_t) \cup W^u(P_t) \cup W^s(Q_t) \cup W^u(Q_t)).$$

Then the forward and backward orbits of x return infinitely many times to the cube $\tilde{C}_t^+$. So let us suppose that x itself is in $\tilde{C}_t^+$. Take the leaf $F_t^{uu}(x)$. We now get a sequence (z_i) such that $z_i \in F_t^{uu}(x) \pitchfork W^s(P_t)$ and $z_i \to x$ as follows. Let $\tilde{z}_i = F_t^{uu}(x_i) \cap W^s(P_t)$ and $z_i = g_t^{-n_i(x)}(\tilde{z}_i)$. Since $\text{dist}(z_i, x) < \lambda^{-n_i(x)}$, $\lambda > 1$ as in Remark 3.2, we have $z_i \to x$. Thus $W^s(P_t)$ accumulates on x. Analogously, using the fact that the backward orbit of x returns infinitely many times to $\tilde{C}_t^+$, we get that $W^u(Q_t)$ accumulates on x. Finally, using the non-criticalities assumption (NC) one has that $W^s(P_t)$ and $W^u(Q_t)$ have a point of transverse intersection close to x.

In the remaining cases we use the no-cycles condition ($t \notin D$ and thus $W^u(P_t) \cap W^s(Q_t) = \emptyset$). Suppose, for instance, that the orbit of x only returns finitely many times to $\tilde{C}_t^+$, then

$$x \in (W^s(P_t) \cup W^s(Q_t)) \cap (W^u(P_t) \cup W^u(Q_t)).$$

Since $W^s(Q_t) \cap W^u(P_t) = \emptyset$ there are three possibilities: (i) $x \in W^s(P_t) \cap W^u(Q_t)$ (in this case the lemma is obvious); (ii) $x \in W^s(Q_t) \cap W^u(Q_t)$ (in this case the lemma follows observing that $W^s(Q_t)$ is contained in the closure of $W^s(P_t)$); and (iii) $x \in W^s(P_t) \cap W^u(P_t)$. Arguing as above it is not difficult to see that case (iii) also satisfies the conclusion of the lemma. □

We are now ready to prove the proposition. Lemma 4.3 and Claim 4.1 imply

$$\Lambda_t = \Lambda_t(V) \subset \overline{W^s(P_t) \pitchfork W^u(Q_t)} \subset H(Q_t).$$

The converse, $H(Q_t) \subset \Lambda_t$, follows by using a filtration. The existence of such a filtration follows from the fact that P_0 and Q_0 are both isolated in $L(f_0)$. Since this is a standard construction we omit the details and refer the reader to [**DR2**, §3.3].

4.2.2. *Partial hyperbolicity of Λ_t.* Let $\text{Per}(\Lambda_t)$ denote the set of periodic points of g_t in Λ_t. From Remark 3.2 the actions of Dg_t in $\mathcal{F}_t^{uu}$ and in $\mathcal{F}_t^{ss}$ are hyperbolic (expanding and contracting, respectively). Bearing in mind the fact that the dimensions of the leaves of these foliations are $n - r$ and $r - 1$ (r is the index of P_0), it is not hard to see that for every R in $\text{Per}(\Lambda_t)$ the derivative $D_R g_t^m$ (m is the period of R) has at least $(r - 1)$ contracting eigenvalues, say $\varsigma_1, \ldots, \varsigma_{r-1}$, and $(n-r)$ expanding eigenvalues, say $\varsigma_{r+1}, \ldots, \varsigma_n$, where $|\varsigma_j| \leq |\varsigma_{j+1}|$. Moreover, there exists $\varsigma > 1$ (independent of t) such that

$$|\varsigma_{r-1}| < \varsigma^{-1} < 1 < \varsigma < |\varsigma_{r+1}|.$$

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Using the linearizing coordinates, the cycle configuration, and the non-critical property (also recall that the composition $\pi_t^s \circ \pi_t^t$ preserves $<$, see Lemma 3.1) it is not difficult to see that $D_R g_t^m$ has a positive central eigenvalue ς_r such that

$$\min\left\{\frac{|\varsigma_r|}{|\varsigma_{r-1}|}, \frac{|\varsigma_{r+1}|}{|\varsigma_r|}\right\} > (1+\delta)^m, \tag{4.1}$$

for some $\delta > 0$ independent of R and small $|t|$.

Let us give the main ideas of the proofs of the inequalities above. By replacing R by some iterate of it we can suppose that R belongs to $\tilde{C}_t^+$. Observe that between two consecutive returns of the orbit of R to $\tilde{C}_t^+$ the number of iterates of the orbit outside the linearizing neighborhoods $\mathcal{U}_P$ and $\mathcal{U}_Q$ is $h_1 + h_0 \pm 1$, thus it is uniformly bounded independently of t. On the other hand, the number of iterates of the orbit in the linearizing neighborhoods goes to infinity as t goes to zero: the number of iterates of the orbit in $\mathcal{U}_Q$ and $\mathcal{U}_P$, say q_t and p_t, is (at least) of order of N_t^+ and N_t^-, respectively, and $N_t^\pm \to \infty$ as $t \to 0$, see (3.12). Finally, in $\mathcal{U}_P \cup \mathcal{U}_Q$ the ratio between the expansion in the strong unstable direction and the central one, (respectively, between the contraction in the strong stable direction and the central one) after the $p_t + q_t$ iterations, goes to infinity (respectively zero) exponentially. Observe that to prove (4.1) one does not use the fact that t belong to S. Finally, the fact that the central eigenvalue ς_r is positive follows by observing that the projection $\pi_t^s \circ \pi_t^u$ preserves $<$, see Lemma 3.1.

Using (4.1), given $R \in \mathrm{Per}(\Lambda_t)$ define the $D_R g_t$-invariant linear spaces E_R^{ss}, E_R^{uu}, and E_R^c spanned by the eigenvectors associated to $\{\varsigma_1, \ldots, \varsigma_{r-1}\}$, $\{\varsigma_{r+1}, \ldots, \varsigma_n\}$, and ς_r. Write

$$E_R^{sc} = E_R^{ss} \oplus E_R^c, \quad E_R^{uc} = E_R^{uu} \oplus E_R^c.$$

By (4.1) the splittings $(E_R^{sc} \oplus E_R^{uu})_{R\in\mathrm{Per}(\Lambda_t)}$ and $(E_R^{uc} \oplus E_R^{ss})_{R\in\mathrm{Per}(\Lambda_t)}$ are both *dominated in the period*: there are $0 < \lambda < 1$ and $C > 0$ such that every $R \in \mathrm{Per}(\Lambda_t)$ of period m satisfies

$$\|Df^m/_{E_R^{ss}}\| \|Df^{-m}/_{E^{cu}_{f^m(R)}}\| < C\lambda^m \quad \text{and} \quad \|Df^{-m}/_{E_R^{uu}}\| \|Df^m/_{E^{cs}_{f^{-m}(R)}}\| < C\lambda^m.$$

Moreover, these splittings satisfy a stronger form of domination.

LEMMA 4.4. *The splittings $(E_R^{sc} \oplus E_R^{uu})_{R\in\mathrm{Per}(\Lambda_t)}$ and $(E_R^{uc} \oplus E_R^{ss})_{R\in\mathrm{Per}(\Lambda_t)}$ are uniformly dominated: there are $K > 0$, $0 < \mu < 1$, and $j > 0$ such that*

$$\|Df^j/_{E_R^{ss}}\| \|Df^{-j}/_{E^{cu}_{f^j(R)}}\| < C\mu^j \quad \text{and} \quad \|Df^{-j}/_{E_R^{uu}}\| \|Df^j/_{E^{cs}_{f^{-j}(R)}}\| < C\mu^j$$

for every R in $\mathrm{Per}(\Lambda_t)$.

The lemma follows from the arguments in the proof of (4.1) recalling that if j is large enough, for instance of the order of $N_t^+ + N_t^-$, the number of iterates of the orbit of any $R \in \mathrm{Per}(\Lambda_t)$ in the linearizing neighborhoods $\mathcal{U}_P$ and $\mathcal{U}_Q$ (at least of the order of $N_t^\pm$) is much larger than the number of iterations (of the order, at most, $2(h_0 + h_1)$) outside such vicinities.

Now, from Proposition 1.3 in [**M2**], Lemma 4.4 allows us to extend $(E_R^{sc} \oplus E_R^{uu})_{R\in\mathrm{Per}(\Lambda_t)}$ and $(E_R^{uc} \oplus E_R^{ss})_{R\in\mathrm{Per}(\Lambda_t)}$ to a uniformly dominated bundle defined on the

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closure of $\mathrm{Per}(\Lambda_t)$, which is the whole Λ_t (see Proposition 4.2). Denote such extensions also by E^{sc}, E^{uc}, E^{ss}, and E^{uu}. Let $E^c_L = E^{sc}_L \cap E^{uc}_L$, $L \in \Lambda_t$. By construction, $(E^{ss}_L \oplus E^c_L \oplus E^{uu}_L)_{L\in\Lambda_t}$ is a strong partially hyperbolic splitting. This completes the proof of (iii) in Theorem 2.1.

5. *Construction of blenders: proof of Proposition 3.6 (non-critical case)*

We first prove the proposition for the fixed arc $(f_t)_{t\in I}$ in $\mathcal{TE}(M)$ satisfying the non-critical assumption (NC). After that we extend our constructions to a neighborhood $\mathcal{W}$ of $(f_t)_{t\in I}$.

5.1. *Periodic points with central eigenvalue close to* 1^+.

The first step to construct the blenders in Proposition 3.6 is to find a decreasing sequence of parameters $(\tilde t_n)_n$ and a constant $K' > 0$ such that:

- $\tilde t_n \to 0^+$ and $\tilde t_{n+1}/\tilde t_n > K' > 0$;
- for every $\tilde t_n$ there is a hyperbolic point $A_n \in \mathrm{Per}(\Lambda_{\tilde t_n})$ of index $(r-1)$ and having central (weak expanding) eigenvalue of modulus close to 1^+.

We use the point A_n for constructing the cube of the blender, actually the continuations of A_n will be the periodic points of such blenders. To obtain the $\tilde t_n$ we first consider parameters t_n corresponding to secondary cycles in Lemma 3.5: each $\tilde t_n$ will be taken close to t_n, $|\tilde t_n - t_n| \simeq Ct_n$. Let us go into the details of our construction.

Fix a parameter t_n (large n) corresponding to a secondary cycle of type $(e(n), g(n))$. Then

$$I_\tau(f^{N^+_{t_n}}_{t_n}(x^c_{t_n})) \subset U_{e(n)} \quad \text{and} \quad \sigma_{t_n} = \beta^{N^+_{t_n}}_{t_n}\lambda^{N^-_{t_n}}_{t_n} \in (\upsilon_{g(n)-1}, \upsilon_{g(n)}].$$

For simplicity we write $j_n = j_{e(n),g(n)}$ and $k_n = k_{e(n),g(n)}$, where $j_{e(n),g(n)}$ and $k_{e(n),g(n)}$ are as in (3.10). Let us also write N_n and $N^\pm_n$ in place of N_{t_n} and $N^\pm_{t_n}$.

The goal of this subsection is to prove the following proposition.

PROPOSITION 5.1. *There is a decreasing sequence of parameters $(\tilde t_n)$, with $t_n < \tilde t_n < t_{n+1}$, such that every diffeomorphism $f_{\tilde t_n}$ has a hyperbolic periodic point $A_n \in \mathrm{Per}(\Lambda_{\tilde t_n}) \cap \mathcal{U}_Q$ of index $(r-1)$ and period $p_n = 2N_n + j_n + k_n$, such that:*

(1) $A^c_n = x^c_{\tilde t_n}$;

(2) $(D_{A_n} f^{p_n}_{\tilde t_n}(0_s, 1, 0_u))^c = v^c$, *where* $v^c \in (\rho_1, \rho_2)$, *and ρ_1 and ρ_2 are the constants in (3.5) and (3.6);*

(3) $W^s(A_n, f_{\tilde t_n}) \pitchfork W^u(Q_{\tilde t_n}, f_{\tilde t_n}) \neq \emptyset$;

(4) $W^u(A_n, f_{\tilde t_n}) \pitchfork W^s(P_{\tilde t_n}, f_{\tilde t_n}) \neq \emptyset$;

(5) $|t_n - \tilde t_n| \simeq (\lambda^{k_n}_{t_n} + \beta^{-j_n}_{t_n})t_n$.

For the sake of clearness we divide the proof of the proposition into two parts: parameter estimates and the existence of periodic (in the x_c-direction) unstable leaves (i.e. $z^c = w^c$ for every $z \in F^{uu}_t(x)$ and $w = F^{uu}_t(f^p_t(x))$ for some x and $p \geq 1$).

5.1.1. *Parameter estimates.*

Take t_n corresponding to a secondary cycle and the interval $U_n = U_{e(n)}$ of the covering $\mathcal{U}$ containing $I_\tau(f^{N^+_n}_{t_n}(x^c_{t_n}))$. For t close to t_n define the *scaled*

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interval $U_n(t)$ by

$$U_n(t) = f_t^{-N_n^+}(U_n), \quad \text{where diameter } (U_n(t)) \geq \tfrac{1}{2} t \beta_t^{-1} \text{ diameter } (U_n).$$

The estimate on the diameter of $U_n(t)$ follows from (3.12). Fix $\tau' = (\beta_c^{-1}\tau)/2$. Now, since τ is the Lebesgue number of $\mathcal{U}$, we have $I_{t_n\tau'}(x^c_{t_n}) \subset U_n(t_n)$, recall again (3.12). By continuity, $I_{t\tau'}(x^c_t) \subset U_n(t)$ for all t close to t_n. The next lemma gives $\hat\tau > 0$ and a lower bound for the size of the interval I_n centered at t_n of the parameters t such that $I_{t\hat\tau}(x^c_t) \subset U_n(t)$.

LEMMA 5.2. *There are constants* $\hat\tau$ *and* Δ, $0 < \Delta < \hat\tau$, *independent of* t_n *(hence independent of* k_n *and* j_n*) such that*

$$I_{t\hat\tau}(x^c_t) \subset U_n(t), \quad \textit{for every } t \in I_n = [t_n(1-\Delta), t_n(1+\Delta)].$$

Proof. First observe that since $f_{t_n}^{N_n^+}(I_{t_n\tau'}(x^c_{t_n})) \subset U_n$ we have

$$\operatorname{dist}(f_{t_n}^{N_n^+}(x^c_{t_n}), (\gamma_t \setminus U_n)) \geq K_0\tau \tag{5.1}$$

for some K_0 independent of large n. Now take $\hat\tau$ given by

$$3\hat\tau = \min\{K_0\tau, \tau\}. \tag{5.2}$$

Next, fixing an interval $J_n = [t_n(1-\kappa), t_n(1+\kappa)]$, for large n and small $\kappa > 0$, we estimate the distance from $f_t^{N_n^+}(x^c_t)$ to the complement of U_n in γ_t. For that we first obtain an upper bound for $|f_t^{N_n^+}(x^c_t) - f_{t_n}^{N_n^+}(x^c_{t_n})|$, $t \in J_n$. If $t \in J_n$ then

$$|f_t^{N_n^+}(x^c_t) - f_{t_n}^{N_n^+}(x^c_{t_n})| = |t\beta_t^{N_n^+} - t_n\beta_{t_n}^{N_n^+}|.$$

Write $g_n(t) = t\beta_t^{N_n^+}$, $t \in J_n$. By the mean value theorem,

$$|f_t^{N_n^+}(x^c_t) - f_{t_n}^{N_n^+}(x^c_{t_n})| = |g_n(t) - g_n(t_n)| \leq |\beta_s^{N_n^+} + N_n^+\beta_s^{N_n^+-1}\beta_s' s||t - t_n|,$$

for some $s \in J_n$. On the other hand, from (3.12), one has

$$|N_n^+ s| \simeq \frac{-\log(t_n)}{\log(\beta_{t_n})} t_n \frac{s}{t_n} \leq \frac{1+\kappa}{\log(\beta_{t_n})}(-\log(t_n)t_n) \to 0 \quad \text{as } n \to \infty.$$

Since $\beta_s^{N_n^+} \simeq 1/s$ and β_s' is uniformly upper bounded (recall (3.12) and (J3)), for small s, one has

$$|f_t^{N_n^+}(x^c_t) - f_{t_n}^{N_n^+}(x^c_{t_n})| \leq \text{constant}\,\frac{|t - t_n|}{s} \leq K_1\kappa.$$

Now pick small $\Delta > 0$ satisfying

$$3\hat\tau - K_1\Delta > 2\hat\tau.$$

From (5.1) and (5.2) (definition of $\hat\tau$) for every $t \in [t_n(1-\Delta), t_n(1+\Delta)]$ we have

$$\operatorname{dist}(f_t^{N_n^+}(x^c_t), (\gamma_t \setminus U_n)) > K_0\tau - K_1\Delta \geq 3\hat\tau - K_1\Delta \geq 2\hat\tau. \tag{5.3}$$

From (3.12) for every $y \in I_{t\hat\tau}(x^c_t)$ we have

$$|f_t^{N_n^+}(y^c) - f_t^{N_n^+}(x^c_t)| \leq \beta_t^{N_n^+}\hat\tau t \simeq \beta_t^{-\log(t_n)/\log(\beta_{t_n})}\hat\tau t = t_n^{-\log(\beta_t)/\log(\beta_{t_n})}\hat\tau t < \tfrac{3}{2}\hat\tau. \tag{5.4}$$

Inequalities (5.3) and (5.4) now imply that Δ and $\hat\tau$ satisfy the lemma.

Finally, note that the choices of Δ and $\hat\tau$ do not involve calculations with j_n and k_n, thus Δ and $\hat\tau$ are independent of t_n. □

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Since j_n and k_n can be taken arbitrarily large we can assume that

$$M\lambda_0^{k_n} + \beta_0^{-j_n} < \Delta, \quad \Delta \text{ as in Lemma 5.2.} \tag{5.5}$$

5.1.2. *Almost periodic leaves: periodicity in the central direction.* Let

$$\mathcal{U}_Q^+ = \{(x_s, x_c, x_u) \in \mathcal{U}_Q \text{ such that } x_c \geq 0\}.$$

Our next objective is to obtain a natural number p_n, a parameter s_n close to t_n ($s_n \in I_n$, see Lemma 5.2, and t_n a parameter corresponding to a secondary cycle of type (e_n, g_n)), and $w_{s_n} \in W^u(P_{s_n})$ close to x_{s_n} such that

$$z^c = y^c = x_{s_n}^c, \quad \begin{array}{l}\text{for all } z \in C(x_{s_n}, W^u(P_{s_n}) \cap \mathcal{U}_Q) \\ \text{and } y \in C(f_{s_n}^{p_n}(w_{s_n}), W^u(P_{s_n}) \cap \mathcal{U}_Q).\end{array}$$

This is done in the next proposition.

PROPOSITION 5.3. *For every n sufficiently large there are $s_n \in I_n$, I_n as in Lemma 5.2, and $w_{s_n} \in C(x_{s_n}, W^u(P_{s_n}) \cap \mathcal{U}_Q) = F_{s_n}^{uu}(x_{s_n})$ such that*

$$(f_{s_n}^{p_n}(w_{s_n}))^c = w_{s_n}^c = x_{s_n}^c, \quad p_n = 2N_n + j_n + k_n.$$

Moreover, the orbit of w_{s_n} satisfies:

- $f_{s_n}^i(w_{s_n}) \in V$ *for all* $0 \leq i \leq p_n$;
- $f_{s_n}^i(w_{s_n}) \in \mathcal{U}_P$ *for all* $N_n \leq i \leq N_n + k_n - h_0$;
- $f_{s_n}^{N_n+k_n}(w_{s_n}) \in \mathcal{U}_Q^+$;
- $f_{s_n}^{2N_n+k_n+i}(w_{s_n}) \in \mathcal{U}_Q^+$ *for all* $0 \leq i \leq j_n$,

where $j_n = j_{e(n),g(n)}$ and $k_n = k_{e(n),g(n)}$ are defined as in (3.10).

To prove the proposition, fixing $t \in I_n = [t_n(1-\Delta), t_n(1+\Delta)]$ write

$$S_n(t) = \{x \in S(Q_t) \text{ such that } x^c \in U_n(t)\}$$

and consider the projections π_t^1 and π_t^2 defined by

$$\pi_t^1, \pi_t^2 : S_n(t) \to S(Q_t), \quad \begin{cases} x^1 = \pi_t^1(x) = \pi_t^u(f_t^{N_n}(x)), \\ x^2 = \pi_t^2(x) = \pi_t^u(f_t^{N_n+k_n}(x)). \end{cases} \tag{5.6}$$

Recalling that we are assuming that $\pi_t^s \circ \pi_t^u$ is a *translation in the central direction* (see (AP)), from the estimates on the distortion of T_t in (3.7) and the definitions of N_n (3.15) and σ_t (3.14) for every $t \in I_n$ one has

$$\begin{aligned} \sigma_t m_n |x^c - y^c| < |(x^1)^c - (y^1)^c| < \sigma_t M_n |x^c - y^c|, \\ \sigma_t m_n \lambda_t^{k_n} |x^c - y^c| < |(x^2)^c - (y^2)^c| < \sigma_t M_n \lambda_t^{k_n} |x^c - y^c|. \end{aligned} \tag{5.7}$$

LEMMA 5.4. *The image of $S_n(t)$ by π_t^2, $\pi_t^2(S_n(t))$, is contained in $S_n(t)$ for all $t \in I_n$.*

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Proof. From the definition of N_n (see (3.15)) and (3.12) there are functions $d_n^{\pm}(t)$ such that

$$|f_t^{N_n+k_n}(x^c)| \in [\lambda_t^{k_n} d_n^-(t)t, \lambda_t^{k_n} d_n^+(t)t], \quad \text{where } \lambda_t^3 < d_n^{\pm}(t) < \lambda_t^{-2}, \tag{5.8}$$

for every $x = (x_s, x_c, 0_u)$ with $x^c \in D_t^+$. Since $\pi_t^u(P_t) = x_t$, using (AP) the last equation gives

$$\begin{aligned} \lambda_t^{k_n} d_n^-(t)t &\le |(\pi_t^u(f_t^{N_n+k_n}(x)))^c - (\pi_t^u(P_t))^c| = |(\pi_t^2(x))^c - (\pi_t^u(P_t))^c| \\ &= |(x^2)^c - (x_t)^c| \le \lambda_t^{k_n} d_n^+(t)t, \end{aligned} \tag{5.9}$$

for every $x = (x_s, x_c, 0_u)$ such that $x^c \in D_t^+$. On the other hand, since k_n is large enough (see (5.5)), we have $\lambda_t^{k_n-2} < \hat{\tau}$. Thus from (5.8)

$$|(x^2)^c - (x_t)^c| \le \lambda_t^{k_n} d_n^+(t)t \le \lambda_t^{k_n-2} t < t\hat{\tau}. \tag{5.10}$$

By Lemma 5.2, the last inequality gives $(x^2)^c \in U_n(t)$ for all $x \in S_n(t)$, that is $x^2 \in S_n(t)$. This ends the proof of the lemma. □

The previous lemma allows us to define the projection $\pi_t^3 = \pi_t^1 \circ \pi_t^2$,

$$\begin{gathered} \pi_t^3 : S_n(t) \to S(Q_t), \\ x^3 = \pi_t^3(x) = \pi_t^u(f_t^{N_n}(\pi_t^2(x))) = \pi_t^1(\pi_t^2(x)) = \pi_t^1(x^2) = (x^2)^1. \end{gathered} \tag{5.11}$$

Since we are considering points x in $S_n(t)$, from (5.7) and Lemma 5.4 we have

$$\sigma_t^2 m_n^2 \lambda_t^{k_n} |x^c - y^c| < |(x^3)^c - (y^3)^c| < \sigma_t^2 M_n^2 \lambda_t^{k_n} |x^c - y^c|. \tag{5.12}$$

Finally, to prove Proposition 5.3 we need the following lemma.

LEMMA 5.5. *Let t_n be a parameter corresponding to a secondary cycle. Then*

$$(f_{s_n}^{j_n}(x_{s_n}^3))^c = (x_{s_n})^c,$$

for some $s_n = t_n(1 + \hat{\Delta}_n)$, where

$$\hat{\Delta}_n^- < \hat{\Delta}_n < \hat{\Delta}_n^+,$$

and

$$\begin{aligned} \hat{\Delta}_n^+ &\simeq \frac{a_n^+}{\sigma_t K^- - a_n^+} \simeq a_n^+, \quad a_n^+ = \sigma_t M_n \lambda_{t_n}^{k_n} d(t_n)^+ + \beta_{t_n}^{-j_n}, \\ \hat{\Delta}_n^- &\simeq \frac{a_n^-}{\sigma_t K^+ - a_n^-} \simeq a_n^-, \quad a_n^- = \sigma_t m_n \lambda_{t_n}^{k_n} d(t_n)^- + \beta_{t_n}^{-j_n}, \end{aligned} \tag{5.13}$$

where K^- and K^+ are positive constants such that

$$\lambda_t^{k_n} + \beta_t^{-j_n} < K^- < K^+.$$

Before proving Lemma 5.5 let us prove Proposition 5.3 assuming it.

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Proof of Proposition 5.3. Take s_n as in Lemma 5.5. We claim that there is a u-disk δ^u contained in $F_t^{uu}(x_{s_n})$, $x_{s_n} \in \delta^u$, consisting of points z having the itinerary in the proposition and such that

$$(f_{s_n}^{p_n}(z))^c = (z)^c = x_{s_n}^c, \quad p_n = 2N_n + k_n + j_n.$$

Recalling the definitions of π_t^1, π_t^2, and π_t^3 (see (5.6) and (5.11)) and the f_t-invariance of the foliations $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ (IF), it is enough to take

$$\delta^u = f_{s_n}^{-p_n}(F(f_{s_n}^{j_n}(x_{s_n}^3))), \quad F(f_{s_n}^{j_n}(x_{s_n}^3)) = C(f_{s_n}^{j_n}(x_{s_n}^3), f_{s_n}^{p_n}(F(x_{s_n})) \cap \mathcal{U}_Q).$$

Since k_n and j_n can be taken arbitrarily large, $\hat{\Delta}_n$ is of the order of $\lambda_{s_n}^{k_n} + \beta_{s_n}^{-j_n}$. Thus

$$\hat{\Delta}_n \le \hat{\Delta}_n^+ < \Delta$$

and s_n belongs to I_n. This completes the proof of the proposition assuming the lemma. □

Before proving Lemma 5.5 let us state a remark we will use in §6. Observing that there are finitely many pairs (j_n, k_n) (at most ℓ^2, where ℓ is the number of intervals of the covering $\mathcal{U}$) and that $d_n^{\pm}(t)$ and σ_t are both uniformly bounded (see (5.8) and (3.14)), we get the following.

Remark 5.6. There is $\hat{\delta} > 0$ such that $0 < \hat{\delta} < \hat{\Delta}_n^- \le \hat{\Delta}_n < \hat{\Delta}$ for all n sufficiently large.

Proof of Lemma 5.5. We begin by observing that from Lemma 3.1 one can state the following.

FACT 5.7. *The projections π_t^1, π_t^2, and π_t^3 preserve the ordering $<$ in γ_t.*

Observe that to prove the lemma it is enough to find small $\mu > 0$ such that

$$(x_{t_n(1+\mu)}^3)^c = \beta_{t_n(1+\mu)}^{-j_n} t_n(1+\mu) = \beta_{t_n(1+\mu)}^{-j_n} x_{t_n(1+\mu)}^c. \tag{5.14}$$

We first get some preliminary estimates on the distortion of π_t^i. The definitions of t_n (secondary cycle), π_t^1, and π_t^3, and Fact 5.7 give

$$(x_{t_n}^1)^c = 0 \quad \text{and} \quad (x_{t_n}^3)^c < 0. \tag{5.15}$$

Arguing as in Lemma 5.2 one gets constants $K^{\pm}$, independent of t and n, such that

$$\sigma_t K^-(t - t_n) \le (x_t^1)^c \le \sigma_t K^+(t - t_n), \quad \text{for } t > t_n \text{ and close to } t_n. \tag{5.16}$$

Obviously, the $K^{\pm}$ do not depend on k_n and j_n. So, taking j_n and k_n large enough, we get

$$\lambda_t^{k_n} + \beta_t^{-j_n} < K^- < K^+.$$

On the other hand, by Lemma 5.2, $(x_t^2)^c \in U_n(t)$ (i.e. $x_t^2 \in S_n(t)$) and we can define x_t^3. From (5.7) and (5.9) we get

$$\begin{aligned} |(x_t^3)^c - (x_t^1)^c| = |((x_t^2)^1)^c - ((x_t)^1)^c| &> \sigma_t m_n |(x_t^2)^c - (x_t)^c| \\ &\ge m_n \sigma_t \lambda_t^{k_n} d_n^-(t) t. \end{aligned} \tag{5.17}$$

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Similarly,

$$|(x_t^3)^c - (x_t^1)^c| < \sigma_t M_n |(x_t^2)^c - (x_t)^c| \le \sigma_t M_n \lambda_t^{k_n} d_n^+(t)t. \tag{5.18}$$

Recalling that $(x_t^2)^c < (x_t)^c$ and π_t^1 is order preserving, from (5.18) and (5.16) one has

$$(x_t^3)^c > (x_t^1)^c - \sigma_t M_n \lambda_t^{k_n} d_n^+(t)t \ge \sigma_t (K^-(t - t_n) - M_n \lambda_t^{k_n} d_n^+(t)t). \tag{5.19}$$

Analogously, from (5.17) and (5.16), one gets

$$(x_t^3)^c < (x_t^1)^c - \sigma_t m_n \lambda_t^{k_n} d_n^-(t)t < \sigma_t (K^+(t - t_n) - m_n \lambda_t^{k_n} d_n^-(t)t).$$

Take $t = t_n$, from (5.15) we get

$$(x_{t_n}^3)^c < (x_{t_n}^1)^c - \sigma_t m_n \lambda_{t_n}^{k_n} d_n^-(t_n)t_n = 0 - \sigma_t m_n \lambda_{t_n}^{k_n} d_n^-(t_n)t_n < 0 < \beta_{t_n}^{-j_n} t_n. \tag{5.20}$$

We claim that to prove the lemma it is enough to get small $\nu > 0$ such that

$$\sigma_{t_n(1+\nu)}(K^- t_n \nu - M_n \lambda_{t_n(1+\nu)}^{k_n} d_n^+(t_n(1+\nu))t_n(1+\nu)) = \beta_{t_n(1+\nu)}^{-j_n} t_n(1+\nu). \tag{5.21}$$

In fact observe that (5.19) implies that

$$(x_{t_n(1+\nu)}^3)^c \ge \sigma_{t_n(1+\nu)}(K^- t_n \nu - M_n \lambda_{t_n(1+\nu)}^{k_n} d_n^+(t_n(1+\nu))t_n(1+\nu)).$$

The mean value theorem and (5.20) now give μ in $(0, \nu)$ satisfying (5.14). Thus it is enough to see that the ν in (5.21) is positive and small. Such a number is of the order of

$$\nu \simeq \frac{\beta_{t_n}^{-j_n} + \sigma_{t_n} M_n \lambda_{t_n}^{k_n} d_n^+(t_n)}{\sigma_{t_n} K^- - \sigma_{t_n} M_n \lambda_{t_n}^{k_n} d_n^+(t_n) - \beta_{t_n}^{-j_n}},$$

which is positive and small. Now it suffices to take $\hat{\Delta}_n^+$ equal to the ν satisfying (5.21). This implies that $0 < \mu < \hat{\Delta}_n^+$. Similarly, we have that $\mu > \hat{\Delta}_n^-$ for some $\hat{\Delta}_n^-$ satisfying

$$\hat{\Delta}_n^- \simeq \frac{\beta_{t_n}^{-j_n} + \sigma_{t_n} m_n \lambda_{t_n}^{k_n} d_n^-(t_n)}{\sigma_{t_n} K^+ - \sigma_{t_n} m_n \lambda_{t_n}^{k_n} d_n^-(t_n) - \beta_{t_n}^{-j_n}}.$$

This completes the proof of the lemma. □

5.1.3. *Proof of Proposition 5.1.* We divide the proof of the proposition into two parts: construction of the periodic point A_n satisfying (1)–(4) and parameter estimates (conclusion (5) in the proposition).

Construction of the periodic point A_n. Write $\mu = s_n$ and $w = (w_s, w_c, w_u) = w_{s_n} = w_\mu$, where s_n and w_{s_n} are as in Proposition 5.3. Using the linearizing coordinates and the non-criticality condition, from the choice of the itinerary of w ($w \in \tilde{C}_t$, $f_\mu^{N_n+k_n}(w) \in \mathcal{U}_Q^+$, and $f_\mu^i(w) \in V$ for all $0 \le i \le N_n + k_n$) we get

$$\|w_u\| \simeq \lambda_{r+1}(\mu)^{-N_n^- - k_n} \beta_{r+1}(\mu)^{-N_n^+}.$$

Given z in $\mathcal{U}_Q$, let $z_{sc} = (z_s, z_c, 0_u)$. Similarly, given a set H we let $H^{sc} = \{z_{sc}, z \in H\}$. In $\mathcal{U}_Q$, for some suitable constant K (independent of $\mu = s_n$ and n) to be fixed, consider the n-cube $C = C(w)$ around w given by

$$C = C(w_{sc}) = \{(x_s, x_c, x_u),\ |x_c - w_c| \le \hat{\tau}\mu,\ \|x_u\| \le K\lambda_{r+1}(\mu)^{-N_n^- - k_n} \beta_{r+1}(\mu)^{-N_n^+}\},$$

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where $\hat{\tau}$ is as in Lemma 5.2. Since $w^c = x_\mu^c$ and $I_{\hat{\tau}\mu}(x_\mu^c) \subset U_n(\mu)$, we have that $C^{sc} \subset S_n(\mu)$ and $\pi_\mu^3(x)$ is defined for all x in C^{sc}. Moreover, we control the distortion of $\pi_\mu^3(x)$ in terms of (5.12). Let

$$C_0 = C(f_\mu^{p_n}(w), f_\mu^{p_n}(C) \cap \mathcal{U}_Q), \quad p_n = 2N_n + j_n + k_n.$$

Having in mind (IF) ($\mathcal{F}_\mu^{uu}$ is f_μ-invariant), we can choose K such that C_0 is given by

$$C_0 = C_0^{sc} \times F_\mu^u = \{x \in \mathcal{U}_Q \text{ such that } x_{sc} \in C_0^{sc}\}, \quad C_0^{sc} = f_\mu^{j_n} \circ \pi_\mu^3(C^{sc}). \tag{5.22}$$

Let

$$\pi_\mu^0 = f_\mu^{j_n} \circ \pi_\mu^3 \tag{5.23}$$

and define the *width* of a subset H of $\mathcal{U}_Q$, denoted by $\varpi(H)$, as the diameter of $\pi_\mu^s(H^{sc})$.

CLAIM 5.8. *Let ρ_1 and ρ_2 be the constants in (3.5) and (3.6), then*

$$\rho_1 \varpi(C) < \varpi(C_0) < \rho_2 \varpi(C).$$

Proof. Suppose that the cycle at t_n is of type (e, g). From (5.12),

$$(\sigma_\mu m_e)^2 \lambda_\mu^{k_n} \varpi(C^{sc}) \le \varpi(\pi_\mu^3(C^{sc})) \le (\sigma_\mu M_e)^2 \lambda_\mu^{k_n} \varpi(C^{sc}).$$

Now, from (5.22),

$$(\sigma_\mu m_e)^2 \lambda_\mu^{k_n} \beta_\mu^{j_n} \varpi(C^{sc}) \le \varpi(C_0^{sc}) \le (\sigma_\mu M_e)^2 \lambda_\mu^{k_n} \beta_\mu^{j_n} \varpi(C^{sc}).$$

By (3.10) the last equation gives

$$\rho_1 \varpi(C^{sc}) < \varpi(C_0^{sc}) < \rho_2 \varpi(C^{sc})$$

ending the proof of the claim. □

Let us observe that the estimates in the proof of the claim also give

$$\rho_1 |x_c - y_c| \le |(\pi_\mu^0(x))_c - (\pi_\mu^0(y))_c| \le \rho_2 |x_c - y_c|, \quad \text{for all } x, y \in C^{sc}. \tag{5.24}$$

We are now ready to obtain the p_n-periodic point A_n. Claim 5.8 and Proposition 5.3 imply that C_0 is a cube around w of width strictly larger than the width of C. In fact our arguments give somewhat more: there are z_ℓ and z_r in C_0^{sc} such that

$$z_\ell^c < w^c < z_r^c, \quad \rho_1 \hat{\tau} < |(z_j)_c - w_c| < \rho_2 \hat{\tau}, \quad j = r, \ell.$$

This property and a standard argument (involving the expansion/contraction of Df_μ in the unstable/stable directions, recall Remark 3.2) imply that there is a unique hyperbolic periodic point $A_n \in C$ of period p_n, which is close to w. It is not hard to check that our construction also shows that:

- $(A_n)^c = (w)^c$;
- A_n has index $(r-1)$ and its central (expanding) eigenvalue is in between ρ_1 and ρ_2;
- $W_{\text{loc}}^s(A_n) = \{x = (x_s, x_c, x_u) \in C \text{ such that } x^c = A_n^c, x_u = (A_n)_u\}$;
- $W_{\text{loc}}^u(A_n) = \{x = (x_s, x_c, x_u) \in C \text{ such that } x^s = (A_n)_s\}$.

Finally, the expansion of Df_μ (respectively Df_μ^{-1}) along the leaves of the unstable foliation (respectively stable foliation) implies immediately that:

- $W^s(A_\mu)$ intersects $W^u(Q_\mu)$ and $W^u(A_\mu)$ intersects $W^s(P_\mu)$ (where both intersections are transverse).

All these comments imply (1)–(4) in Proposition 5.1.

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Parameter estimates. Lemma 5.5 implies that

$$s_n - t_n = \hat{\Delta}_n t_n, \quad \text{where } \hat{\Delta}_n \simeq \lambda_{t_n}^{k_n} + \beta_{t_n}^{-j_n}.$$

This implies (5) in the proposition. Now the proof of the proposition is complete. □

5.2. *Construction of the cubes of the blenders.* The first step is to construct the blender for the parameters s_n in Proposition 5.1.

5.2.1. *Projections and returns.* We begin by defining the auxiliary cube C_t^+

$$C_t^+ = \{x = (x_s, x_c, x_u) \in \mathcal{U}_Q \text{ such that } x^c \in D_t^+\}.$$

Condition (IF) (f_t-invariance of $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{ss}$) gives immediately the following relation between forward returns of points to C_t^+ and projections along leaves of the unstable foliation.

Remark 5.9. Let $x \in \Lambda_t^+ \cap C_t^+$ and $k = N_t + m$, $m \geq 0$, be the first forward iterate of x in C_t^+, i.e. $f_t^k(x) \in C_t^+$ and $f_t^i(x) \notin C_t^+$ for every $i \in \{2, \dots, k-1\}$. Then

$$(f_t^k(x))_{sc} = \pi_t^u(f_t^k(x_{sc})).$$

Bearing in mind this remark we introduce the family $(\varrho_t^m)_{m\geq 0}$ of projections defined by

$$\varrho_t^m : S(Q_t)^+ \to S(Q_t), \quad x \mapsto \varrho_t^m(x) = \pi_t^u(f_t^{N_t+m}(x)).$$

Using this notation one has $\pi_t^1 = \varrho_t^0$ (recall (5.6)).

5.2.2. *Determination of the basis of the cube.* Consider parameters s_n and periodic points A_n in Proposition 5.1. Given $0 < \nu \leq \hat{\tau}$, $\hat{\tau}$ as in Lemma 5.2, define the rectangle $R(\nu, s_n) \subset \mathcal{U}_Q$ by

$$R(\nu, s_n) = \{(x_s, x_c, 0_u) : |x_c - (A_n)_c| \leq s_n \nu\}.$$

The *points at the left of* A_n^c defines the subrectangle $R^-(\nu, s_n)$ of $R(\nu, s_n)$ as follows

$$R^-(\nu, s_n) = \{x \in R(\nu, s_n) : x^c \leq A_n^c\}.$$

Finally, the cube $C_{R(\nu,s_n)}$ of basis $R(\nu, s_n)$ is

$$C_{R(\nu,s_n)} = \{x \in C_{s_n}^+ : x_{sc} \in R(\nu, s_n)\}.$$

The cube of the blender will be of the form $C_{R(\nu_n,s_n)}$ for some ν_n we are going to determine. For simplicity, from now we omit the parameter s_n in the definitions above. The size of ν_n will be chosen to guarantee that the following two key properties hold:

(a) $\pi_{s_n}^1(R^-(\nu_n)) = \varrho_{s_n}^0(R^-(\nu_n)) \subset S(Q_{s_n})^+$;

(b) the distance from $\varrho_{s_n}^0(R^-(\nu_n))$ to $W_{\text{loc}}^s(Q_{s_n})$ is $\xi_n s_n$, for some small $\xi_n < K^- \Delta_n^-$ to be fixed precisely in (5.29).

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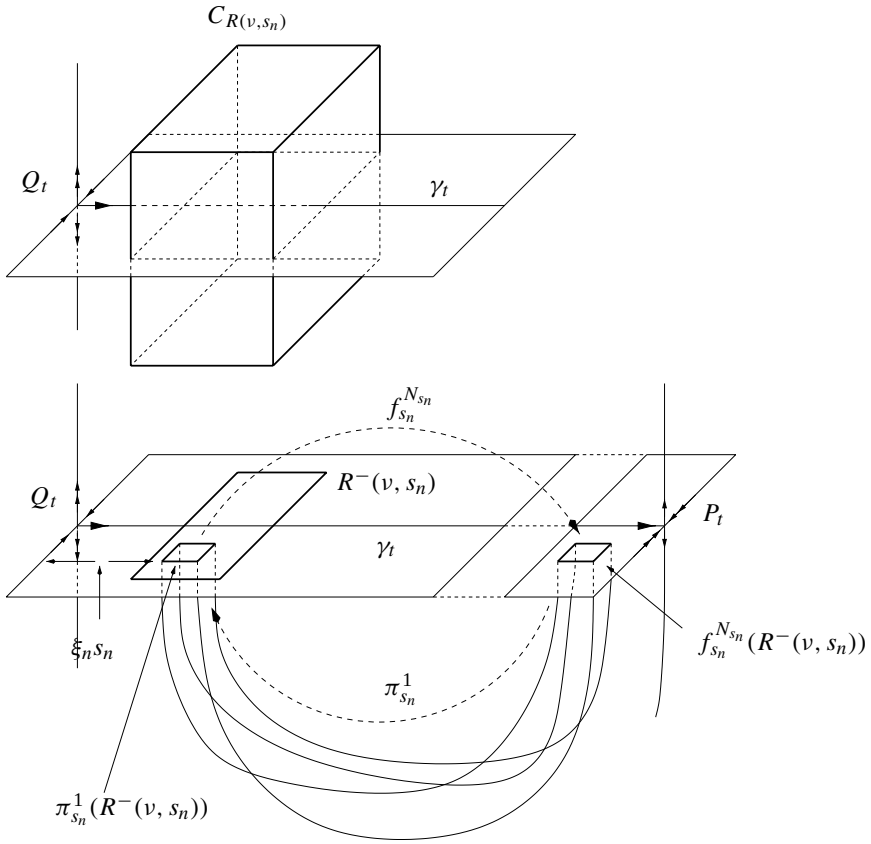


FIGURE 6. The cube $C_{R(\nu,s_n)}$: key properties.

These two conditions (especially (b)) will allow us to obtain expansive ℓ-returns to $C_{R(\nu_n)}$ for every ℓ-disk in $C_{R(\nu_n)}$. Let us now go into the details of our construction. Having in mind (AP) and Fact 5.7, to get (a) and (b) it is enough to see that

$$\begin{gathered}(\varrho_{s_n}^0(x_\ell))_c = (\pi_{s_n}^1(x_\ell))_c = \xi_n s_n,\\ x_\ell = (0_s, (A_n)_c - \nu_n s_n, 0_u) = (0_s, (x_{s_n})_c - \nu_n s_n, 0_u).\end{gathered} \tag{5.25}$$

From (5.16) and $s_n - t_n = \hat{\Delta}_n t_n$ (see Proposition 5.3 and Lemma 5.5) we have

$$\begin{gathered}(\pi_{s_n}^1(x_{s_n}))_c = (x_{s_n}^1)_c = \sigma_{s_n}\kappa_n(s_n - t_n) = \sigma_{s_n}\kappa_n\delta_n s_n,\\ K^- \le \kappa_n \le K^+, \quad \delta_n = \frac{\hat{\Delta}_n}{1+\hat{\Delta}_n}.\end{gathered} \tag{5.26}$$

Thus, from (5.7) and $(\pi_{s_n}^1(x_\ell))^c < (\pi_{s_n}^1(x_{s_n}))^c$,

$$\sigma_{s_n} s_n(\kappa_n\delta_n - M_n\nu_n) \le (\pi_{s_n}^1(x_\ell))_c \le \sigma_{s_n} s_n(\kappa_n\delta_n - m_n\nu_n).$$

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Hence, (5.25) holds for some ν_n with

$$\nu_n \in \left[\frac{\sigma_{s_n}\kappa_n\delta_n - \xi_n}{\sigma_{s_n}M_n}, \frac{\sigma_{s_n}\kappa_n\delta_n - \xi_n}{\sigma_{s_n}m_n} \right]. \tag{5.27}$$

Taking ξ_n close to zero one gets $\sigma_{s_n}\kappa_n\delta_n - \xi_n > 0$, thus ν_n is well defined (strictly positive). Note that ν_n increases as ξ_n decreases, that is ξ_n/ν_n can be taken arbitrarily small. To get expansive returns we need to choose ξ_n small, but uniformly bounded from below (for n sufficiently large) by some $\xi > 0$, satisfying the following two inequalities:

$$\frac{1}{4} > \frac{\xi_n}{\nu_n} \frac{\beta_0^8 - 1}{m\lambda_0^2\beta_0^{-2}}, \tag{5.28}$$

and

$$\frac{\beta_0^{-12}\lambda_0^{11}m^2}{(\xi_n/\nu_n) + \lambda_0^{-2}\beta_0^2 M(1-\rho_2^{-1})} \geq 2 > \rho_1, \tag{5.29}$$

where the last inequality follows from (3.5).

5.2.3. *Returns (at the left) of vertical ℓ-disks.* Fixing ν_n and ξ_n as before, consider the cube $C_{R(\nu_n)}$ containing A_n. Given a vertical disk Σ in $C_{R(\nu_n)}$, let $x_\Sigma = \Sigma \cap R(\nu_n)$. Having in mind (IF), we can focus our attention on vertical disks Σ contained in leaves of F_t^{uu} ($\Sigma = \{x \in C_t^+ : x_{sc} = x_\Sigma\}$). As in the definition of the blender, the disk Σ is at the left of A_n (Σ is an *ℓ-disk*) if $(x_\Sigma)^c < (A_n)^c$ (i.e. $x_\Sigma \in R^-(\nu_n)$).

Given a vertical disk Σ in $C_{R(\nu_n)}$ we say that $k = N_{s_n} + m$, $m \geq 0$, is an *ℓ-return of* Σ if $f_{s_n}^k(\Sigma) \cap C_{R(\nu_n)}$ contains an ℓ-disk Σ_k' such that $f_{s_n}^{-i}(\Sigma_k') \subset V$ for all $0 \leq i \leq k$. By Remark 5.9, this definition can be stated in terms of projections $\varrho_{s_m}^m$ and iterations of f_{s_n}.

Remark 5.10. Let k be an ℓ-return of Σ. Then there are $h \geq 1$, $j_1, \ldots, j_h$, and $r_1, \ldots, r_h$, $j_i, r_i \geq 0$, with $k = hN_{s_n} + \sum_{i=1}^h (j_i + r_i)$, such that

$$\Sigma_k' = \{x \in C(\nu_n) : x_{sc} = \hat{x}_\Sigma\}, \quad \text{where } \hat{x}_\Sigma = (f_{s_n}^{j_h} \circ \varrho_{s_n}^{r_h}) \circ \cdots \circ (f_{s_n}^{j_1} \circ \varrho_{s_n}^{r_1})(x_\Sigma) \in R^-(\nu_n).$$

Then we say that the ℓ-return k of Σ is of *type* $J = (j_h, r_h) \cdots (j_1, r_1)$.

In view of this remark we introduce projections $\varrho_{s_n}^J$ *associated to ℓ-returns k_J of type* $J = (j_h, r_h) \cdots (j_1, r_1)$, $k_J = hN_{s_n} + \sum_{i=0}^h (j_i + r_i)$, defined on subrectangles R_J of $S^+(Q_{s_n})$ by

$$\varrho_{s_n}^J : R_J \to S(Q_{s_n}), \quad x \mapsto \varrho_{s_n}^J(x) = (f_{s_n}^{j_h} \circ \varrho_{s_n}^{r_h}) \circ \cdots \circ (f_{s_n}^{j_1} \circ \varrho_{s_n}^{r_1})(x).$$

We say that the *projection $\varrho_{s_n}^J$ is ρ-expanding in the rectangle R_J* if

$$\varpi(\varrho_{s_n}^J(\alpha)) \geq \rho\varpi(\alpha), \quad \text{for every curve } \alpha \text{ in } R_J.$$

Similarly, an *ℓ-return of type J is ρ-expanding in R_J* if $\varrho_{s_n}^J$ is ρ-expanding in R_J.

Bearing in mind the comments above, in our context (f_t-invariance of $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ and hyperbolicity of Df_t in E^{ss} and E^{uu}) we can reformulate conditions (B1)–(B4) in the definition of blender in terms of projections $\varrho_{s_n}^J$ and expansive ℓ-returns (which give the expansion in the central direction). Then we prove that $(C_{R(\nu_n)}, f_{s_n})$ satisfies (P1)–(P3) and that (P1)–(P3) imply (B1)–(B4):

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(P1) there exists a covering $(R_i)_{i=0}^{\vartheta}$ of $R^-(\nu_n)$ by rectangles and families of pairs $(J_i)_{i=0}^{\vartheta}$ such that $\varrho_{s_n}^{J_i}(R_i) \subset R^-(\nu_n)$;
(P2) $\varrho_{s_n}^{J_i}$ is ρ_1-expansive in R_i for all i, $\rho_1 > 1$ as in (3.6);
(P3) $A_n^c \in R_0$ and the period of A_n is $k_{J_0} = p_n$ (A_n is the periodic point in Proposition 5.1).

LEMMA 5.11. *(P1)–(P3) imply (B1)–(B4).*

Proof. In the definition of the blender take $q_i = k_{J_i}$.

(P1) implies (B1). It suffices to take $C_i = f^{-q_i}(C_{\varrho_{s_n}^{J_i}(R_i)})$.

(P1) implies (B4). Given any ℓ-disk V in $C_{R(\nu_n)}$ let $\{x_V\} = V \cap R^-(\nu)$. If $x_V \in R_i$, $i \in \{0, \dots, \vartheta\}$, let $V_i = V \cap C_i$. By definition of C_i, $f_{s_n}^{q_i}(V_i) = F_{s_n}^{uu}(\varrho_{s_n}^{J_i}(x_V))$, which is an ℓ-disk; recall that by (P1), $\varrho_{s_n}^{J_i}(R_i) \subset R^-(\nu_n)$.

(P2) implies (B2). Using $\mathcal{F}_{s_n}^{uu}$ (respectively $\mathcal{F}_{s_n}^{ss}$) we construct the strong unstable (respectively stable) cone field $\mathcal{C}^{uu}$ around $E_{s_n}^{uu}$ (respectively $\mathcal{C}^{ss}$ around $E_{s_n}^{ss}$). The ρ_1-expansiveness of $\varrho_{s_n}^{J_i}$ implies that $D_x f_{s_n}^{q_i}$, $x \in C_i$, is ρ_1-expansive in the central direction. This allows us to construct the unstable bundle $\mathcal{C}^u$-around $T_x F_{s_n}^{uu}(x) \oplus (0_s, v_c, 0_u)$.

(P3) implies (B3). By construction, A_n is in the interior of $C_{R(\nu_n)}$. Moreover, by definitions of C_0 and q_0 and by construction, A_n belongs to C_0 and is the unique point of period q_0 in C_0. □

Construction of the covering (R_i). Expansive returns. Our target now is the construction of the covering (R_i) of $R^-(\nu_n)$ satisfying (P1)–(P3). For that we define expansive ℓ-returns of vertical disks Σ with $x_\Sigma \in R^-(\nu_n)$.

Definitions of ℓ-returns: returns of type $J_0 = (j_n, 0)(0, k_n)$. These returns are associated to $\varrho_{s_n}^{J_0}$. From (5.23) we have

$$\pi_{s_n}^0 = \varrho_{s_n}^{J_0}, \quad J_0 = (j_n, 0)(0, k_n).$$

Consider the rectangle

$$R_0 = \{x \in R^-(\nu_n) : (A_n)_c - \rho_2^{-1}\nu_n s_n \le x_c \le (A_n)_c\} \subset R^-(\nu_n) \subset S_n(s_n).$$

Since $R_0 \subset S_n(s_n)$, by (5.24), the definition of A_n, and Fact 5.7, for each $x \in R_0$ one has

$$(A_n)_c - \nu_n s_n \le (A_n)_c - \rho_2|x_c - (A_n)_c| \le (\pi_{s_n}^0(x))_c \le (\pi_{s_n}^0(A_n))_c = (A_n)_c.$$

Thus

$$\varrho_{s_n}^{J_0}(R_0) = \pi_{s_n}^0(R_0) \subset R^-(\nu_n).$$

This inclusion and the arguments in the proof of Claim 5.8 immediately imply the following.

LEMMA 5.12. *The projection $\varrho_{s_n}^{J_0} = \pi_{s_n}^0$ is ρ_1-expanding. In particular, every ℓ-disk Σ in $C_{R(\nu_n)}$, such that $x_\Sigma \in R_0$, has a ρ_1-expanding ℓ-return of type $J_0 = (j_n, 0)(0, k_n)$.*

By definition it is obvious that the rectangle R_0 satisfies (P3).

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Further expansive ℓ-returns. In view of Lemma 5.12 we are now left to get expansive ℓ-returns for ℓ-disks Σ with $x_\Sigma \in (R^-(\nu_n) \setminus R_0)$. The domains of definition of such returns will determine the rectangles $(R_i)_{i=1}^{\vartheta}$. For $i \geq 1$ define the subrectangle R_i of $R^-(\nu_n)$ by

$$R_i = \{x \in R^-(\nu_n) : (\pi^1_{s_n}(x))_c \in [s_n\beta_{s_n}^{i-1}\xi_n, s_n\beta_{s_n}^{i+2}\xi_n] \text{ and } (\pi^1_{s_n}(x))_c < (A_n)_c\}.$$

These rectangles are not disjoint, for instance R_i intersects R_{i+1}, R_{i+2}, and R_{i+3}, $i \geq 1$. Moreover, since $\pi^1_{s_n}$ preserves $<$, *R_{i+1} is at the right of R_i*, meaning that $x^c < y^c$ for every $x \in (R_i \setminus R_{i+1})$ and $y \in (R_{i+1} \setminus R_i)$. Finally, if i is large R_i may be empty.

We say that R_i *has full width* if there are $x, y \in R_i$ such that $(\pi^1_{s_n}(x))_c = \beta_{s_n}^{i-1}\xi_n s_n$ and $(\pi^1_{s_n}(y))_c = \beta_{s_n}^{i+2}\xi_n$, i.e. $(\pi^1_{s_n}(R_i))_c = [\beta_{s_n}^{i-1}\xi_n s_n, \beta_{s_n}^{i+2}\xi_n s_n]$. If ξ_n is small then $(\pi_{s_n}(R_1))_c = [s_n\xi_n, s_n\beta_{s_n}^3\xi_n]$ and R_1 has full width. Moreover, if R_i has full width then $R_1, R_2, \ldots, R_{i-1}$ also have full width and

$$(\pi^1_{s_n}(R_1 \cup R_2 \cup \cdots \cup R_i))_c = [\xi_n s_n, \beta_{s_n}^{i+2}\xi_n s_n].$$

From (5.7) it follows that the diameter $d(i)$ of $(R_1 \cup \cdots \cup R_i)_c$ satisfies

$$\frac{\xi_n s_n(\beta_{s_n}^{i+2} - 1)}{M_n\sigma_{s_n}} \leq d(i) \leq \frac{\xi_n s_n(\beta_{s_n}^{i+2} - 1)}{m_n\sigma_{s_n}}.$$

Let δ be the diameter of $(R^-(\nu_n) \setminus R_0)^c$, then

$$\delta = (1 - \rho_2^{-1})s_n\nu_n.$$

Let i_n be the first i such that R_i intersects the interior of R_0. We prove that in this case $R_1, \ldots, R_{i_n}$ have full width (see Claim 5.13 below). From $R_1 \cup \cdots \cup R_{i_n} \not\subset (R^-(\nu_n) \setminus R_0)$ and the position of the rectangles we get that $d(i_n) \geq \delta$. Similarly, $(R_1 \cup \cdots \cup R_{i_n-1}) \cap \operatorname{int}(R_0) = \emptyset$ implies $d(i_n - 1) \leq \delta$. The estimates on $d(i)$ before give

$$\frac{\xi_n(\beta_{s_n}^{i_n+1} - 1)}{M_n\sigma_{s_n}} \leq (1 - \rho_2^{-1})\nu_n \leq \frac{\xi_n(\beta_{s_n}^{i_n+2} - 1)}{m_n\sigma_{s_n}}.$$

In particular,

$$\xi_n\beta_{s_n}^{i_n+1} \leq M_n\sigma_{s_n}(1 - \rho_2^{-1})\nu_n + \xi_n. \tag{5.30}$$

CLAIM 5.13. *The rectangle R_{i_n+5} has full width, i.e. $[\xi_n s_n\beta_{s_n}^{i_n+4}, \xi_n s_n\beta_{s_n}^{i_n+7}] = (\pi^1_{s_n}(R_{i_n+5}))_c$. In particular, R_i has full width and intersects R_0 for every $i_n \leq i \leq i_n + 5$.*

Proof. By construction $\xi_n s_n$ is the left extreme of $(\pi^1_{s_n}(R^-(\nu_n)))_c$. Thus, to prove the claim it is enough to see that

$$[\xi_n s_n, \beta_{s_n}^{i_n+8}\xi_n s_n] \subset \operatorname{int}(\pi^1_{s_n}(R^-(\nu_n)))_c.$$

Having in mind the estimates on the expansion of $\pi^1_{s_n}$ in (5.7), to get this inclusion it is enough to check that

$$\beta_{s_n}^{i_n+8}\xi_n < \xi_n + m_n\sigma_{s_n}\nu_n.$$

Thus, from (5.30), it suffices to see that

$$\beta_{s_n}^{i_n+8}\xi_n \leq \beta_{s_n}^7\xi_n + \beta_{s_n}^7 M_n\sigma_{s_n}\nu_n(1 - \rho_2^{-1}) \leq \xi_n + m_n\sigma_{s_n}\nu_n.$$

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That is, it is enough to check that

$$(1-\rho_2^{-1}) \le \frac{m_n \nu_n \sigma_{s_n} + \xi_n - \beta_{s_n}^7 \xi_n}{\beta_{s_n}^7 M_n \sigma_{s_n} \nu_n}.$$

But for large n (thus small s_n) the choice of ρ_2 in (3.5) and $m \le M$ give

$$(1-\rho_2^{-1}) \le \frac{1}{2}\frac{m}{M\beta_{s_n}^7} \le \frac{1}{2\beta_{s_n}^7}.$$

Therefore, it suffices to prove that

$$\frac{1}{2\beta_{s_n}^7} \le \frac{m_n \nu_n \sigma_{s_n} + \xi_n(1-\beta_{s_n}^7)}{\beta_{s_n}^7 M_n \sigma_{s_n} \nu_n}.$$

Clearly, this inequality holds if

$$\frac{m_n}{M_n} \ge \frac{1}{2} + \frac{\xi_n(\beta_{s_n}^7 - 1)}{m\sigma_{s_n}\nu_n}.$$

Since $m_n/M_n > 3/4$ (see (3.7)) this follows from (5.28) and the estimates on σ_{s_n}, see (3.14). This completes the proof of the claim. □

The covering (R_i). Write $\vartheta_n = i_n + 5$ and consider the covering $(R_i)_0^{\vartheta_n}$ of $R^-(\nu_n)$. For simplicity let us write ϑ instead of ϑ_n. We claim that these rectangles satisfy (P1)–(P3). Recall that $A_n \in R_0$, so we are left to check properties (P1) and (P2). Recall also that for every $x \in R_0$ there is a ρ_1-expansive ℓ-return associated to $\pi_{s_n}^0$ (Lemma 5.12). Therefore, it is enough to define ρ_1-expansive ℓ-returns for $R_1, \dots, R_\vartheta$. For each $i \in \{1, \dots, \vartheta\}$ take g_i minimum such that

$$f_{s_n}^{g_i}(\pi_{s_n}^1(R_i)) \subset \{x \in S(P_t) : |x_c| \le \lambda_{s_n}\nu_n s_n\}$$

and write

$$g_i = a_i + N_{s_n} + b_i, \quad a_i, b_i \ge 0 \text{ and } a_i \text{ minimum such that } f_{s_n}^{a_i}(\pi_{s_n}^1(R_i)) \cap D_{s_n}^+ \ne \emptyset.$$

From the definitions of R_i and $D_{s_n}^+$, and $1 \le i \le \vartheta = i_n + 5$, we get

$$\beta_{s_n}^{a_i}\beta_{s_n}^{i+2}\xi_n s_n \in [\beta_{s_n}^{-1}, 1] \Longrightarrow \beta_{s_n}^{a_i} \ge \frac{\beta_{s_n}^{-8}}{\beta_{s_n}^{i_n}\xi_n s_n}. \tag{5.31}$$

Using the estimates in (5.30) and in (3.14) (the definition of σ_{s_n}), the last inequality gives

$$\begin{aligned}\beta_{s_n}^{a_i} \ge \frac{\beta_{s_n}^{-8}}{\beta_{s_n}^{i_n}\xi_n s_n} &\ge \frac{\beta_{s_n}^{-7}}{(\xi_n + M_n\sigma_{s_n}(1-\rho_2^{-1})\nu_n)s_n} \\ &> \frac{\beta_0^{-9}\lambda_0^2}{(\xi_n + M\beta_0^2\lambda_0^{-2}(1-\rho_2^{-1})\nu_n)s_n}.\end{aligned} \tag{5.32}$$

On the other hand, from the definitions of g_i, N_{s_n}, and b_i one immediately gets

$$\lambda_{s_n}^{b_i-3} \in [\lambda_{s_n}^2\nu_n s_n, \lambda_{s_n}\nu_n s_n) \Longrightarrow \lambda_{s_n}^{b_i} > \lambda_0^6 \nu_n s_n \quad \text{for every large } n. \tag{5.33}$$

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Finally, the choice of g_i gives

$$\pi^u_{s_n}(f^{g_i}_{s_n}(\pi^1_{s_n}(x))) \in R^-(\nu_n), \quad \text{for all } x \in R_i \text{ and } i = 1, \dots, \vartheta.$$

Thus we can define projections

$$\phi^i_{s_n} : R_i \to R^-(\nu_n), \quad x \mapsto \phi^i_{s_n}(x) = \varrho^{H_i}_{s_n}(x), \quad H_i = (0, b_i)(a_i, 0),\ i = 1, \dots, \vartheta.$$

For completeness, let $\phi^0_{s_n} : R_0 \to R^-(\nu_n)$, $\phi^0_{s_n}(x) = \pi^0_{s_n}(x)$. The definition of the $\phi^i_{s_n}$ implies that $(R_i)^{\vartheta}_{i=0}$ satisfies (P1). The fact that $(R_i)^{\vartheta}_{i=0}$ satisfies (P2) follows from the next lemma.

LEMMA 5.14. *Every projection $\phi^i_{s_n}$, $i \in \{0, 1, \dots, \vartheta\}$, is ρ_1-expansive.*

Proof. As we have mentioned the result holds for $\phi^0_{s_n} = \pi^0_{s_n}$. So we focus our attention on the projections $\phi^i_{s_n}$, $i = 1, \dots, \vartheta$. Let

$$E_A = \frac{\beta_0^{-12} m \lambda_0^4}{\xi_n + \beta_0^2 \lambda_0^{-2} M(1 - \rho_2^{-1})\nu_n} \frac{1}{s_n}, \quad E_B = \lambda_0^6 m \nu_n s_n.$$

We claim that the constant of expansion $E(\phi^i_{s_n})$ of $\phi^i_{s_n}$ satisfies

$$E(\phi^i_{s_n}) \geq E_A E_B = \frac{\beta_0^{-12} \lambda_0^{10} m^2}{(\xi_n/\nu_n) + \beta_0^2 \lambda_0^{-2} M(1 - \rho_2^{-1})} > 2 > \rho_1,$$

where the last inequality follows from (5.29). To prove the claim write $\phi^i_{s_n} = \varrho^{B_i}_{s_n} \circ \varrho^{A_i}_{s_n}$, $B_i = (0, b_i)$ and $A_i = (a_i, 0)$, and observe that in our case (hypotheses (IF) and (AP)) we have

$$E(\phi^i_{s_n}) = E(\varrho^{B_i}_{s_n}) E(\varrho^{A_i}_{s_n}). \tag{5.34}$$

From $R_i \subset R(\nu_n) \subset S_{s_n}(s_n)$, the definition of $\varrho^{A_i}_{s_n}$, and the estimates on $\beta^{a_i}_{s_n}$, and σ_{s_n}, see (5.32) and (3.14), we get

$$E(\varrho^{A_i}_{s_n}) \geq \sigma_{s_n} m_n \beta^{a_i}_{s_n} \geq \frac{m \beta_0^{-12} \lambda_0^5}{\xi_n + \lambda_0^{-2} \beta_0^2 M(1 - \rho_2^{-1})\nu_n} \frac{1}{s_n} = E_A. \tag{5.35}$$

Similarly, from the definition of $\varrho^{B_i}_{s_n}$ and the estimates for $\lambda^{b_i}_{s_n}$ in (5.33),

$$E(\varrho^{B_i}_{s_n}) \geq m \lambda^{b_i}_{s_n} s_n > m \lambda_0^6 \nu_n s_n = E_B. \tag{5.36}$$

The claim (and thus the lemma) now follows from (5.34), (5.35), and (5.36). □

This concludes the proof of the fact that the covering $(R_i)^{\vartheta_n}_{i=0}$ satisfies (P1)–(P3). Lemma 5.11 now implies that $(C_{R(\nu_n)}, f_{s_n})$ is a blender of $\vartheta_n + 1$ symbols.

5.2.4. *Geometric properties of the blender $(C_{R(\nu_n)}, f_{s_n})$.* In this section we prove that the blender $(C_{R(\nu_n)}, f_{s_n})$ satisfies the geometric properties (a) and (b) in Proposition 3.6. This follows from the next proposition.

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PROPOSITION 5.15. *The blender $(C_{R(\nu_n)}, f_{s_n})$ satisfies:*

- *$W^s(Q_{s_n})$ intersects transversely every ℓ-strip in $C_{R(\nu_n)}$;*
- *$W^u(P_{s_n})$ contains a vertical ℓ-disk in the interior of $C_{R(\nu_n)}$.*

Given a vertical strip S in $C_{R(\nu_n)}$ let $\alpha_S = S \cap R(\nu_n)$ be its *basis*. In the proof of the proposition we use the following property of the rectangles R_i, $i \in \{0, 1, \dots, \vartheta_n\}$.

LEMMA 5.16. *Let S be an ℓ-strip in $C_{R(\nu_n)}$ such that α_S is not contained in any R_i. Then $W^s(Q_{s_n})$ intersects S transversely.*

Proof. We claim that if α_S is not contained in any R_i then $(\pi^1_{s_n}(\alpha_S))^c$ contains at least two consecutive fundamental domains of γ_{s_n}. This claim implies that

$$C(\pi^1_{s_n}(\alpha_S), f^{N_n}_{s_n}(S) \cap \mathcal{U}_Q) \pitchfork W^s(Q_{s_n}) \neq \emptyset.$$

This assertion follows by observing that given any fundamental domain $D \subset \mathcal{U}_Q$ of f_{s_n} in γ_{s_n} there is a transverse homoclinic point $Z = (0_s, z_c, z_u)$ of Q_{s_n} such that $Z^c = (0_s, z_c, 0_u) \in D$ and $|z_u| < C|\lambda_{r+1}(s_n)|^{-N_t^-}$.

To prove the claim let ℓ_α and r_α, $(\ell_\alpha)^c < (r_\alpha)^c$, be the extremes of α_S. Define i^ℓ as the maximum i such that R_i contains ℓ_α. By hypothesis, r_α does not belong to R_{i^ℓ} (otherwise $\alpha \subset R_{i^\ell}$). There are two cases: either $i^\ell \leq \vartheta_n - 1$ or $i^\ell = \vartheta_n$. We show the claim in the first case; the second one follows analogously and we omit its proof. We split each R_i $(1 \leq i \leq \vartheta_n)$ into three parts,

$$R_i = R_i^0 \cup R_i^1 \cup R_i^2, \quad \text{where } (\pi^1_{s_n}(R_i^j))_c = [\beta^{i+j-1}_{s_n}\xi_n s_n, \beta^{i+j}_{s_n}\xi_n s_n], \ j = 0, 1, 2.$$

Remark that $R_i^0 = R_{i-1}^1 = R_{i-2}^2$ and that R_i^k is at the left of R_i^{k+1}.

Observe that $\ell_\alpha \in R^0_{i^\ell}$: otherwise, by the definitions of R_i^j, $\ell_\alpha \in R_{i^\ell+1}$, $i^\ell + 1 \leq \vartheta_n$, contradicting the definition of i^ℓ. On the other hand, since r_α does not belong to R_{i^ℓ}, $(R^1_{i^\ell} \cup R^2_{i^\ell})^c \subset (\alpha_S)^c$. Let $\hat{\alpha}_S = \alpha_S \cap (R^1_{i^\ell} \cup R^2_{i^\ell})$. The definition of R_i^j implies that $(\pi^1_{s_n}(\hat{\alpha}_S))_c = [\beta^{i_\ell}_{s_n}\xi_n s_n, \beta^{i_\ell+2}_{s_n}\xi_n s_n]$. That is, $(\pi^1_{s_n}(\hat{\alpha}_S))^c$ is the union of two consecutive fundamental domains of γ_{s_n}. This ends the proof of the claim, and thus of the lemma. □

Proof of Proposition 5.15. To state its first part consider any ℓ-strip S_0 and its basis $\alpha_0 = \alpha_{S_0} = S_0 \cap R(\nu_n)$. If α_0 is not contained in any R_i, from Lemma 5.16, $S_0 \pitchfork W^s(Q_{s_n}) \neq \emptyset$ and we are done. So assume that α_0 is contained in some R_{i_0}. By definition of g_{i_0}, $f^{g_{i_0}}_{s_n}(S_0)$ contains the ℓ-strip S_1 given by

$$S_1 = C(\alpha_1, f^{g_{i_0}}_{s_n}(S_0) \cap C_{s_n}), \quad \alpha_1 = \alpha_{S_1} = \phi^{i_0}_{s_n}(\alpha_0) \subset R^-(\nu_n),$$

and such that, see Lemma 5.14,

$$\varpi(\alpha_1) > \rho_1 \varpi(\alpha_0), \quad \rho_1 > 1.$$

As before, if α_1 is not contained in any R_i then S_1 intersects transversely $W^s(Q_{s_n})$ and we are done (recall that $S_1 \subset f^{g_{i_0}}_{s_n}(S_0)$). So assume that $\alpha_1 \subset R_{i_1}$ for some i_1. At this stage we argue inductively defining ℓ-curves $\alpha_0, \dots, \alpha_k$ and ℓ-strips $S_0, \dots, S_k$, such that:

- $\alpha_j = \alpha_{S_j} \subset R_{i_j}$;
- $S_j = C(\alpha_j, f^{g_{i_{j-1}}}_{s_n}(S_{j-1}) \cap C_{s_n}) \subset f^{g_{i_{j-1}}}_{s_n}(S_{j-1})$;

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- $\alpha_j = \phi_{s_n}^{i_{j-1}}(\alpha_{j-1})$ and $\varpi(\alpha_j) > \rho_1 \varpi(\alpha_{j-1}) > \rho_1^j \varpi(\alpha_0)$, $\rho_1 > 1$.

Since the width of the R_i is upper bounded, there is a first k such that α_k is not contained in any R_i. As before, $S_k \pitchfork W^s(Q_{s_n}) \neq \emptyset$. Since $S_k \subset f_{s_n}^q(S_0)$, $q = \sum_{j=0}^{k-1} g_j$, this ends the proof of the first part of the proposition.

The second part of the proposition follows immediately from the λ-lemma and the geometric configuration of the cycle: $W^u(P_{s_n})$ accumulates on the left of x_{s_n}. More precisely, there is a sequence $(y_k)_k$ of transverse homoclinic points of P_{s_n} contained in $S(Q_{s_n})^+$ such that

$$y_k \to x_{s_n}, \quad (y_k)^c < (x_{s_n})^c, \quad \text{and} \quad |(y_k)_c - (x_{s_n})_c| \simeq \lambda_{s_n}^k.$$

This ends the proof of Proposition 5.15. □

This completes the proof of Proposition 3.6 for the parameter s_n in the non-critical case. The next step is to see that there is a uniform δ, depending only on the choice of the constants in §§3 and 5, such that our constructions hold for every $t \in [s_n, s_n(1+\delta)]$.

5.3. *End of the proof of Proposition 3.6.* Fixing the arc $(f_t)_{t\in I} \in \mathcal{TE}(M)$ with a non-critical cycle, for every s_n as in Proposition 5.1 we have constructed a blender $(C_{R(\nu_n)}, f_{s_n})$ satisfying the intersection properties in (b) in Proposition 3.6:

(I1) $W^s(Q_{s_n})$ intersects transversely every ℓ-strip;

(I2) $W^u(P_{s_n})$ contains an ℓ-disk in the interior of C_{s_n}.

For every g $\mathcal{C}^1$-close to f_{s_n} we can define in the natural way a blender (C_g, g) which is a *continuation* of $(C_{R(\nu_n)}, f_{s_n})$, see [**BD1**, Lemma 1.11]. Moreover, if g is close to f then (C, g) also satisfies (I1) and (I2): just observe that to get such properties we only use compact parts of the invariant manifolds of P_{s_n} and Q_{s_n} (whose sizes depend only on N_n, k_n, and j_n) and that compact parts of invariant manifolds depend continuously on the diffeomorphism. Thus to end the proof of Proposition 3.6 we need to get some uniform $\delta > 0$ such that if $|t - s_n| \le \delta s_n$ then the continuation (C_t, f_t) of $(C_{R(\nu_n)}, f_{s_n})$ also satisfies (I1) and (I2).

LEMMA 5.17. *There is $\mu > 0$ such that for every $t \in I(s_n) = [s_n(1-\mu), s_n(1+\mu)]$ the continuation (C_t, f_t) of $(C_{R(\nu_n)}, f_{s_n})$ is well defined and satisfies (I1) and (I2).*

From Lemmas 3.5 and 5.5 the parameters s_n satisfy $s_{n+1}/s_n > K' > 0$ (large n). Thus Proposition 3.6 follows immediately from the lemma.

Proof of Lemma 5.17. We divide the proof of the lemma into two parts.

Existence of continuations. Suppose that the secondary cycle t_n (s_n is associated to t_n) is of type (e, g). Then the *central size* of $C_{R(\nu_n)}$ is $\nu_n s_n$, where ν_n depends only on the following constants:

- σ_{s_n}, defined in (3.14);
- κ_n, defined in (5.26);
- $\delta_n = \hat{\Delta}_n/(1+\hat{\Delta}_n)$, $\hat{\Delta}_n$ defined in (5.13) in Lemma 5.5;
- ξ_n, introduced in (b) in §5.2.2 and fixed in (5.28) and (5.29);
- M_n and m_n, defined in (3.7).

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These constants are uniformly bounded (the bounds depend on the type (e, g) of the secondary cycle considered). Thus there are constants $\nu^-_{e,g}$ and $\nu^+_{e,g}$ such that

$$0 < \nu^-_{e,g} < \nu_n < \nu^+_{e,g}. \tag{5.37}$$

That is, the scaled size of the cube of the blender is determined by the type (e, g) of the secondary cycle t_n used in its construction.

We also observe that the number of symbols of the blender, $\vartheta_n + 1$, depends exclusively on (e, g). As before, for large n, there are $\vartheta^-_{e,g}$ and $\vartheta^+_{e,g}$ such that

$$0 < \vartheta^-_{e,g} \le \vartheta_n \le \vartheta^+_{e,g}.$$

This follows from the fact that ϑ_n is determined by ε_n (recall that $\vartheta_n = i_n + 5$ and equation (5.30)) and that the ξ_n are uniformly bounded.

From the considerations above one gets $\mu_{e,g} > 0$ such that the continuation (C_s, f_s) of $(C_{R(\nu_n)}, f_{s_n})$ is defined for every $s \in I(s_n) = [s_n(1 - \mu_{e,g}), s_n(1 + \mu_{e,g})]$.

Finally, there are at most ℓ^2 types of secondary cycles (ℓ the number of sets of the covering $\mathcal{U}$), thus it is enough to take μ equal to the minimum of the $\mu_{e,g}$.

In rough terms, one can think (up to the type (e, g) of the cycle at t_n) that for each s_n there is an interval $I(s_n)$ (in the parameter space) centered at s_n and of size $2\mu_{e,g}s_n$ and maps (*renormalizations*) $\varphi_{s,n}$, $s \in I(s_n)$, $\varphi_{s,n} : \mathbb{R}^n \to M$, such that the cube C_s is the image by $\varphi_{s,n}$ of a universal cube $C_{e,g}$ in $\mathbb{R}^n$ and that in the x_c-direction the derivative of $\varphi_{t,n}$ is a contraction of ratio s_n.

Intersection properties. We prove that the $\mu_{e,g}$ can be taken such that (C_t, f_t) satisfies the intersection properties in Proposition 3.6 for all $t \in I(s_n) = [s_n(1 - \mu_{e,g}), s_n(1 + \mu_{e,g})]$.

(I1) $W^s(Q_t)$ intersects transversely every ℓ-strip. This fact follows from the choice of the cubes C_i (associated to rectangles R_i). Recall that to get (I1) for the parameter s_n we only use the fact that $\pi^s_{s_n}\pi^1_{s_n}(R_i)$, $i \ge 1$, contains three fundamental domains of γ_{s_n} (actually, in the proof it is enough to take such images containing two fundamental domains). Clearly, the continuations can be taken satisfying such a property.

(I2) $W^u(P_t)$ contains an ℓ-disk. The fact that (I2) holds for $W^u(P_{s_n})$ follows from the existence of a sequence of ℓ-disks $(W_k)_k \subset W^u(P_{s_n})$ accumulating on the left of A_n (the periodic point of the blender) and such that

$$\operatorname{dist}(W_k, A_n) \simeq \lambda^k_{s_n} s_n.$$

Thus there exists $r_0 > 0$ such that

$$2\lambda^{r_0}_{s_n} < \nu^-, \quad \nu^- = \min\{\nu^-_{e,g}\},$$

where $\nu^-_{e,g}$ is the lower bound of the scaled size of the cubes of the blenders in (5.37). If $\mu_{e,g}$ is small and $t \in [s_n(1-\mu_{e,g}), s_n(1+\mu_{e,g})]$, $W^u(P_t)$ contains a disk W_t which is $(\lambda^{r_0}_t s_n)$-close to W_{r_0}. By construction, W_t is an ℓ-disk. The proof of the lemma is now complete. □

This ends the proof of Proposition 3.6 for the arc $(f_t)_{t\in I}$.

To see that our constructions work for arcs in a small neighborhood of $(f_t)_{t\in I}$, just observe that we only use the fact that, fixing suitable ρ_1 and ρ_2 close to 1^+, and constants

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m_i and M_i in $[m, M]$ (see (3.4)), the central eigenvalues of the cycle satisfy (3.10) and the transition verifies (3.7). Remark 3.4 and the continuous dependence of the transitions guarantee that these inequalities hold in a neighborhood of $(f_t)_{t\in I}$. This completes the proof of Proposition 3.6 assuming the simplifying hypotheses (IF), (AP), and (NC). In the next section we explain how one argues in the general case. □

6. *Proof of Proposition 3.6: general case*

In this section we explain how one removes the simplifying assumptions (IF), (AP), and (NC) in §3.4.

6.1. *Almost-invariance and regularity of $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ (AF).*

6.1.1. *Almost-invariance of $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$.* In general, the foliations $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$ we consider (induced by the linearizing coordinates) are not f_t-invariant: if $x \in \Lambda_t(V) \cap \mathcal{U}_Q$ and $f_t^n(x)$ is a return of x to $\mathcal{U}_Q$ then $F_t^{uu}(f_t^n(x))$ and $f_t^n(F_t^{uu}(x))$ are (in general) different. But we claim that the compact parts of these leaves we consider in our constructions are $\mathcal{C}^1$-$t^{1+\alpha}$-close for some $\alpha > 0$ independent of small t. In the proof we use the component $C_1 = C(f_t^n(x), f_t^n(F_t^{uu}(x)) \cap \mathcal{U}_Q)$ of $f_t^n(F_t^{uu}(x)) \cap \mathcal{U}_Q$ containing $f_t^n(x)$, and assuming f_t-invariance of $\mathcal{F}_t^{uu}$ we replace it by $C_2 = C(f_t^n(x), F_t^{uu}(f_t^n(x)) \cap \mathcal{U}_Q)$. We claim that doing this replacement we introduce an error of order of $t^{1+\alpha}$, which is negligible in our context.

Take $x \in \Lambda_t(V)$ having a return $f_t^n(x)$ such that $f_t(x), \dots, f_t^r(x)$ do not belong to $\mathcal{U}_Q$, $f_t^{r+1}(x), \dots, f^n(x)$ are in $\mathcal{U}_Q$, and $f_t^{n+1}(x) \notin \mathcal{U}_Q$. It is not hard to check that $n - r$ is (at least) of order of $N_t^+ \simeq |\log(t)|/\log(\beta_t)$. Similarly, the number of consecutive iterates of this segment of orbit in $\mathcal{U}_P$ is (at least) of order of $N_t^- \simeq |\log(t)|/|\log(\lambda_t)|$. Using the linearizing coordinates and these bounds for the number of consecutive iterates of the orbit of x in $\mathcal{U}_P$ and $\mathcal{U}_Q$ we get that C_1 and C_2 are $t^{1+\alpha}$-close, where $\alpha > 0$ depends only on $\log(|\lambda_{r+1}(0)|)/\log(\lambda_r(0))$ and $\log(|\beta_{r+1}(0)|)/\log(\beta_r(0))$, see the calculations in [**D1**, §3.2].

This proximity between C_1 and C_2 allows us to carry our constructions into the general case as follows. We first replace $F_t^{uu}(f_t^n(x))$ by $C(f_t^n(x), f_t^n(F_t^{uu}(x)) \cap \mathcal{U}_Q)$. In particular, one needs to use a new projection, say $\tilde{\pi}_t^3$, instead of $\pi_t^3 = \pi_t^1 \circ \pi_t^2$, defined as follows:

$$\tilde{\pi}_t^3 = C(f_t^{N_t}(\pi_t^2(x)), f_t^{N_t}(H(\pi_t^2(x))) \cap \mathcal{B}) \cap S(Q_t),$$
$$H(\pi_t^2(x)) = C(\pi_t^2(x), f_t^{N_t+k_n}(F_t^{uu}(x)) \cap \mathcal{U}_Q).$$

The comments before guarantee that $\tilde{\pi}_t^3$ is $t^{1+\alpha}$-close to π_t^3.

We also need to introduce the same kind of modifications in the definitions of the projections ϱ^J and ϕ^i in §5.2.3 in the construction of the blender. To do this, we define new projections $\tilde{\varrho}^J$ and $\tilde{\phi}^i$ which are $t^{1+\alpha}$-close to ϱ^J and ϕ^i. More precisely and inductively, consider a projection ϱ_0 associated to a family of disks $\mathcal{G}_0 = (G_0(x))$ (in our case $\mathcal{G}_0 = \mathcal{F}_t^{uu}$) and to a return n_0, that is

$$\varrho_0(x) = C(f^{n_0}(x), f^{n_0}(G_0(x)) \cap \mathcal{B}) \cap S(Q_t).$$

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In the next step we consider the family $\mathcal{G}_1$ given by $G_1(x) = C(\varrho_0(x), f_t^{n_0}(G_0(x)) \cap \mathcal{U}_Q)$ instead of $\mathcal{G}_0$, as well as the projection ϱ_1 associated to $\mathcal{G}_1$ and the corresponding return n_1 of the construction. Inductively we have projections ϱ_i associated to returns n_i and disks $\mathcal{G}_i$. As before, we have that $C(\varrho_i(x), f_t^{n_i}(G_i(x)) \cap \mathcal{U}_Q)$ and $F_t^{uu}(\varrho_i(x))$ are $t^{1+\alpha}$-close.

We claim that we can ignore the deviations introduced by the earlier replacements. To see why this is so fix t_n corresponding to a secondary cycle and the s_n close to t_n. Note that in the construction of the cube C_{ν_n} we work with a margin of error of the order of t_n, i.e. of the size of a fixed negative iterate of the scaled fundamental domain $D_{t_n}^+$, see (3.13). Thus our construction also works considering perturbations of the projections of order much smaller than t_n. Replacing the projections along the leaves of $\mathcal{F}_t^{uu}$ by the new projections, we introduce an error (deviation from our affine model) of the order of $t_n^{1+\alpha} \ll t_n$, $\alpha > 0$. Thus we can carry our construction to the general setting without extra difficulties. For details see [**D1**, §3.2].

6.1.2. *Regularity of the projections along the leaves of $\mathcal{F}_t^{ss}$ and $\mathcal{F}_t^{uu}$.* On the one hand, the projection from $S(P_t)$ to $S(Q_t)$ along the leaves of the $\mathcal{F}_t^{uu}$ in general fails to be C^1. To guarantee the regularity of this projection one needs to assume some (open but not dense) conditions on the eigenvalues of $D_{P_t} f_t$. In fact, the unique point where such a projection may fail to be differentiable is P_t: the derivative of the projection at P_t in the direction $F_t^{ss}(P_t)$ may not exist. On the other hand, our construction does not involve points in $F_t^{ss}(P_t)$. In fact, in our proof we only consider points in a sector $\mathcal{S}$ of $S(P_t)$ of the form

$$\mathcal{S} = \{(x_s, x_c, 0_u) : \|x_s\| < \|x_c\|\},$$

and in $\mathcal{S}$ the projection along the leaves of $\mathcal{F}_t^{uu}$ is differentiable. So we lose no generality assuming such a differentiability.

6.2. *Projections (AP).* In general, the transverse homoclinic point x_t of P_t in the definition of unfolding of the cycle (see (GU1) and (GU2)) is of the form

$$x(t) = (a(t)_s, tC(t), 0_u), \quad \text{where } C(0) \neq 0$$

and the projections

$$\pi_t^u : \hat{\gamma}_t \cap \mathcal{U}_P = \{(0_s, x_c, 0_u)\} \to S(Q(t)), \quad \pi_t^s : S(Q_t) \to \{(0_s, x_c, 0_u)\} \subset \mathcal{U}_Q,$$

are not isometries. Let now us explain how one argues in the general case.

Let $W_t^u = W^u(P_t) \cup W^u(Q_t)$ and $\Gamma_t = C(x_t, W_t^u \cap S(Q_t))$. Define K^u and K^s as the derivatives at P_0 and x_0 of the projections

$$\pi_0^u : \hat{\gamma}_0 \to S(Q_0) \quad \text{and} \quad \pi_0^s : \Gamma_0 \to \{(0_s, x_c, 0_u)\},$$

respectively. For $t > 0$ there is a transverse homoclinic point y_t of Q_t which is the continuation of x_0, that is the analogous of x_t for Q_t. Write

$$w_t = \pi_t^s(x_t) \in \gamma_t, \quad z_t = (\pi_t^u)^{-1}(y_t) \in \gamma_t,$$

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and consider the restrictions $\tilde{\pi}_t^{s,u}$ of $\pi_t^{s,u}$ to the segments $J(t) = [z_t, P_t] \subset \gamma_t$ and $L(t) = [y_t, x_t] \subset \Gamma_t$,

$$\tilde{\pi}_t^u : J(t) = [z_t, P_t] \to [y_t, x_t] \subset \Gamma_t, \quad \tilde{\pi}_t^s : L(t) = [y_t, x_t] \to [Q_t, w_t] \subset \gamma_t.$$

Denote by $K_t^{u,s}$ and $k_t^{u,s}$ the maximum and the minimum of the derivatives of $\tilde{\pi}_t^{u,s}$. Then these constants satisfy (see the arguments in [**D1**, p. 298])

$$K_t^{u,s}, k_t^{u,s} \to K^{u,s}, \quad \text{in particular, } \frac{K_t^{u,s}}{k_t^{u,s}} \to 1. \tag{6.1}$$

Moreover, one immediately gets

$$(z_t)_c \in [(K_t^u)^{-1} tC(t), (k_t^u)^{-1} tC(t)], \quad (w_t)_c \in [k_t^s tC(t), K_t^s tC(t)].$$

Recall that in our affine model we had $(z_t)^c = -t$ and $(w_t)^c = t$.

In the general case the scaled fundamental domains $D_t^{\pm}$ are defined by

$$\begin{aligned} D_t^+ &= f_t^{-N_t^+}(\tilde{D}^+), \quad N_t^+ \text{ minimum } i \geq 0 \text{ such that } f_t^i(w_t) \in \tilde{D}^+, \\ D_t^- &= f_t^{N_t^-}(\tilde{D}^-), \quad N_t^- \text{ minimum } i \geq 0 \text{ such that } f_t^{-i}(z_t) \in \tilde{D}^-, \end{aligned} \tag{6.2}$$

where the fundamental domains $\tilde{D}^{\pm}$ are exactly as in §3.2. The estimates before give

$$\frac{tC(t)\lambda_t}{K_t^u} \leq \lambda_t^{N_t^-} \leq \frac{tC(t)}{k_t^u}, \quad \frac{1}{tC(t)\beta_t K_t^s} \leq \beta_t^{N_t^+} \leq \frac{1}{tC(t)k_t^s}.$$

As in §3.4, see (3.14), let

$$\tilde{\sigma}_t = \lambda_t^{N_t^-} \beta_t^{N_t^+} \quad \text{where } \frac{\beta_0^{-2}\lambda_0^2}{K_t^u K_t^s} \leq \tilde{\sigma}_t \leq \frac{\beta_0^2 \lambda_0^{-2}}{k_t^u k_t^s}.$$

Then

$$\tilde{\sigma}_t = \frac{1}{\tilde{k}_t^u \tilde{k}_t^s}\sigma_t, \quad \text{for some } \sigma_t \in (\lambda_0^2\beta_0^{-2}, \lambda_0^{-2}\beta_0^2) \text{ and } \tilde{k}_t^u \tilde{k}_t^s \in [k_t^u k_t^s, K_t^u K_t^s]. \tag{6.3}$$

Using that, by definition, $|D_x \tilde{\pi}_t^{s,u}| \in [k_t^{s,u}, K_t^{s,u}]$, we can obtain the analog of the estimates in (5.7) for the distortion of the derivatives (in the central direction) of the new projections. For instance, consider π_t^1 and $x \in f_t^{-N_t^+}(U_i)$, where U_i is an interval of the covering $\mathcal{U}$ of $\tilde{D}_t^+$. As in §5.1.2

$$m_i \tilde{\sigma}_t k_t^s k_t^u \leq (\pi_t^s \pi_t^1)'(x) \leq M_i \tilde{\sigma}_t K_t^s K_t^u.$$

From (6.1) and (6.3) we have

$$\begin{gathered} m_i k_t^- \sigma_t \leq (\pi_t^s \pi_t^1)'(x) \leq M_i K_t^+ \sigma_t, \\ k_t^- = \frac{k_t^s k_t^u}{K_t^s K_y^u} \quad \text{and} \quad k_t^+ = \frac{K_t^s K_t^u}{k_t^s k_t^+}, \quad k_t^{\pm} \to 1 \text{ as } t \to 0. \end{gathered}$$

That is, if t is small, we have essentially the estimates we found in the affine case assuming (AP). It is now clear that the general construction follows exactly as in the simplified case (assuming (IF) and (AP)).

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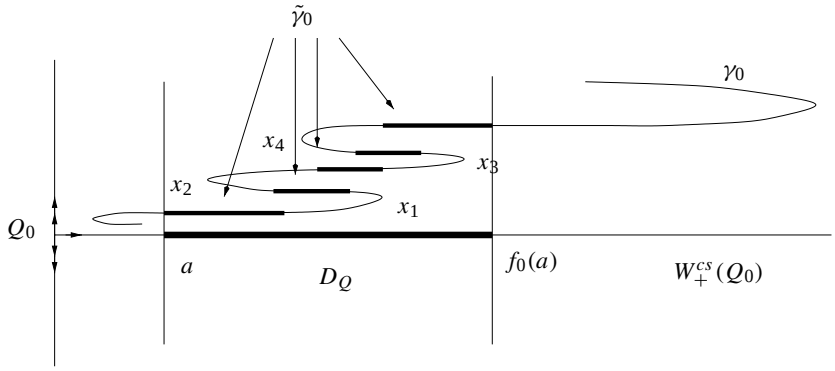


FIGURE 7. Non-critical levels.

6.3. *The critical case: assumption (NC).* Let us now explain how we eliminate the non-criticalies assumption (NC) in the general case. Suppose that $\Gamma(f_0, P_0, Q_0)$ is a connected cycle and let γ_0 be a periodic connection (in our case we are assuming that $f_0(\gamma_0) = \gamma_0$).

Consider the foliations $\mathcal{F}^{ss}_0$ of $W^s(P_0)$ and $\mathcal{F}^{uu}_0$ of $W^u(Q_0)$ induced by the linearizing coordinates. Denote by $\mathcal{C}$ the set of points $x \in \gamma_0$ such that either $\mathcal{F}^{ss}_0(x)$ or $\mathcal{F}^{uu}_0$ is tangent to γ_0 at x. Take fundamental domains $D_P \subset \mathcal{U}_P$ and $D_Q \subset \mathcal{U}_Q$ of f_0 in γ_0. Generically, there are $x_1, \ldots, x_r$ such that

$$\mathcal{C} \cap (D_P \cup D_Q) = \{x_1, \ldots, x_r\},$$

and the contact at each x_i is quadratic. Using the coordinates in $\mathcal{U}_{P,Q}$, consider the separatrizes of the centre stable manifold of P_0 and of the centre unstable manifold of Q_0 given by

$$W^{cs}_-(P_0) = \{(0_s, x_c, 0_u), x_c < 0\} \quad \text{and} \quad W^{cs}_+(Q_0) = \{(0_s, x_c, 0_u), x_c > 0\}.$$

Take a neighborhood $U_{\mathcal{C}}$ of $\{x_1, \ldots, x_m\}$ in γ_0 and define

$$\tilde{\gamma}_0 = \gamma_0 \setminus \left(\bigcup_{i \in \mathbb{Z}} f_0^i(U_{\mathcal{C}}) \right).$$

If $U_{\mathcal{C}}$ is small enough then

$$\pi_0^{ss}(\tilde{\gamma}_0 \cap \mathcal{U}_P) = W^{cs}_-(P_0) \quad \text{and} \quad \pi_0^{uu}(\tilde{\gamma}_0 \cap \mathcal{U}_Q) = W^{cu}_+(Q_0),$$

where π_0^{ss} and π_0^{uu} are the projections along the leaves of $\mathcal{F}^{ss}_0$ and $\mathcal{F}^{uu}_0$, respectively. Moreover, for small t we can define a non-critical continuation $\tilde{\gamma}_t$ of $\tilde{\gamma}_0$ satisfying this covering property (in the language in §2.3, $\tilde{\gamma}_t$ contains fundamental domains).

Using the arguments in [**DR1**] we can repeat the constructions in the non-critical case replacing γ_t by $\tilde{\gamma}_t$ and considering non-critical subsets $\tilde{W}^s_t$ of $W^s(P_t)$ as follows:

$$\tilde{W}^s_t = \bigcup_{x \in \tilde{\gamma}_t} C(x, F^{ss}_t(x) \cap V).$$

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Now we can define projections (which are multivalued at some points) from $\tilde{W}^s_t \cap \mathcal{U}_P$ to $\tilde{W}^s_t \cap \mathcal{U}_Q$ along the leaves of $\mathcal{F}^{uu}_t$. More precisely, at this stage we need to take the ξ_n small enough and ρ_2 close to 1^+ in such that

$$\frac{\beta_0^{-12}\lambda_0^{11}m^2}{(\xi_n/\nu_n)+\lambda_0^{-2}\beta_0^2 M(1-\rho_2^{-1})} \geq 2(r+1),$$

where r is the number of critical points. Observe that (in each fundamental domain of γ_t) $\tilde{W}^s_t$ has r non-critical levels, say $\tilde{W}^s_t(i)$. Consider now the projections ϕ^i_t as in Lemma 5.14. Then for every curve α in a rectangle R_i, $i \geq 1$, there is some j such that $((\phi^i_t)(\alpha) \cap W^s_t(j))_c > 2(\alpha)_c$. See [**DR1**] for details. The result now follows as in the non-critical case.

Remark 6.1. The arguments before also work if the cycle contains fundamental domains: consider $\tilde{\gamma}_0 \subset W^s(P_0) \pitchfork W^u(Q_0)$ consisting of countable many components without criticalities and such that $\pi_0^{ss}(\tilde{\gamma}_0)$ and $\pi_0^{uu}(\tilde{\gamma}_0)$ cover the separatrizes $W^{sc}_-(P_0)$ and $W^{cu}_+(Q_0)$, respectively.

7. *Proofs of the auxiliary results: Theorem 2.2 and Corollary 2.3*

In this section we end the proofs of the auxiliary results in §2.3. Observe that Corollary 2.3 follows immediately from Theorem 2.2.

In view of the proof of Theorem 2.1 and Remark 6.1, to state the first part of Theorem 2.2 we are left to consider the case in which the projection $\pi^s_0 \circ \pi^u_0$ does not preserve the ordering $<$. In this case one considers parameters t_n corresponding to secondary cycles and cycles $\Gamma(f_{t_n}, P_{t_n}, Q_{t_n})$. It can be immediately checked that these new cycles preserve the ordering $<$ in the continuation γ_{t_n} of γ_0. Applying the arguments of the proof of Theorem 2.1, for each n we get an interval B'_n arbitrarily close to t_n of parameters corresponding to non-hyperbolic diffeomorphisms.

Observe that we do not know if these intervals B'_n have positive density at 0: in the proof of Theorem 2.1 we fix compact parts of the leaves of the strong stable and unstable foliations (the parts corresponding to the continuations of $W^s_{\mathrm{loc}}(Q_0)$ and $f_0^{h_0}(W^u_{\mathrm{loc}}(Q_0))$). This allows us to control the distortion of the holonomy and thus the size of the bifurcation intervals. In this general case the size of the compact part of $W^u(P_{t_n})$ we need to consider goes to infinity as n; just observe that the secondary cycle is associated to the intersection $W^s_{\mathrm{loc}}(Q_{t_n}) \cap f_{t_n}^{N_{t_n}}(W^u_{\mathrm{loc}}(P_{t_n}))$, $N_{t_n} \to \infty$, as in the definition of the secondary cycle, see (3.15).

Finally, to get the neighborhood of the cycle we argue as in the proof of the critical case, see §6.3. It suffices to replace V in the definition of the neighborhood of the cycle by some suitable W obtained as follows: define the critical set $\mathcal{C}$ and its neighborhood $U_{\mathcal{C}}$ exactly as in §6.3. Now it is enough to take $W = (V \setminus U_{\mathcal{C}})$.

8. *Proofs of the theorems*

In this section we prove the theorems in the introduction. Observe that Theorem E is an immediate consequence of (iii) in Theorem 2.1; just define $\mathcal{A}_3$ as the union of the neighborhoods $\mathcal{W}$ in Theorem 2.1, where the $(f_t)_{t\in I}$ run over $\mathcal{JE}(M) \cap \mathcal{NIE}(M)$.

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8.1. *Proof of Theorem D.* Consider an arc $(f_t)_{t\in I}$ with a cycle $\Gamma(f_0, P_0, Q_0)$. Denote by λ_c and β_c, $|\lambda_c| < 1 < |\beta_c|$, the central eigenvalues of the cycle.

8.1.1. *Proof of (i).* To prove the first part of the theorem we classify the cycles according to the character of the central eigenvalues (real and complex) and the connections (the existence of critical components of the intersection $W^s(P_0) \cap W^u(Q_0)$). We consider generic arcs and observe that there are the following possibilities:

(1) either λ_c or β_c is in $(\mathbb{C} \setminus \mathbb{R})$;
(2) the cycle has real central eigenvalues:
  (2.1) the cycle is connected,
  (2.2) the cycle is non-connected (every component of $W^s(P_0) \cap_t W^u(Q_0)$ is non-periodic),
    (2.2.1) the cycle is critical: there is a connected component of $W^s(P_0) \pitchfork W^u(Q_0)$ having quadratic criticalities (related to either the strong stable foliation $\mathcal{F}_0^{ss}$ of $W^s(P_0)$ or to the strong unstable foliation $\mathcal{F}_0^{uu}$ of $W^u(Q_0)$),
    (2.2.2) the cycle is non-critical: every component of $W^s(P_0) \pitchfork W^u(Q_0)$ is non-critical (transverse to $\mathcal{F}_0^{ss}$ and $\mathcal{F}_0^{uu}$).

Cases (1) and (2.2.1). In these cases there is a sequence (t_n) of parameters, $t_n \to 0$, such that f_{t_n} has a homoclinic tangency associated to P_{t_n} or Q_{t_n} for every t_n. More precisely, in Case (1) and if β_c (respectively λ_c) is complex then the tangencies are associated to P_{t_n} (respectively Q_{t_n}). On the other hand, in Case (2.2.1), if the critical component of $W^s(P_0) \pitchfork W^u(Q_0)$ is tangent to $\mathcal{F}_0^{ss}$ (respectively to $\mathcal{F}^{uu}$) then the tangencies are associated to Q_{t_n} (respectively to P_{t_n}).

From [**R**, Theorem C] and [**MV**, Theorem A], for each large n there is a set of positive measure J_n, J_n close to t_n, such that f_t has a strange saddle (roughly speaking, the product of a Hénon-like attractor/repeller by a hyperbolic dynamics) for all $t \in J_n$. In particular, f_t is non-hyperbolic for every $t \in J_n$. This completes the proof in Cases (1) and (2.2.1).

Case (2.1). This case follows immediately from Theorem 2.2 using the following remark.

Remark 8.1. Let $\Gamma(f, P, Q)$ a connected cycle with real central eigenvalues. Then it contains infinitely many fundamental domains.

Case (2.2.2). We claim that one can perturb the arc to get either a critical cycle or a cycle containing infinitely many fundamental domains. In the first case the conclusion follows from the arguments in Cases (1) and (2.2.1); in the second it follows from Theorem 2.2.

LEMMA 8.2. *Let f be a $\mathcal{C}^k$-diffeomorphism, $k \geq 1$, with a cycle $\Gamma(f, P, Q)$ such that its central eigenvalues are both real and $W^s(P) \pitchfork W^u(Q)$ has a connected component γ which is non-periodic and non-critical. Then there is g $\mathcal{C}^k$-close to f with a cycle $\Gamma(g, P_g, Q_g)$ such that either:*

- *$\Gamma(g, P_g, Q_g)$ is critical; or*
- *$\Gamma(g, P_g, Q_g)$ contains infinitely many fundamental domains.*

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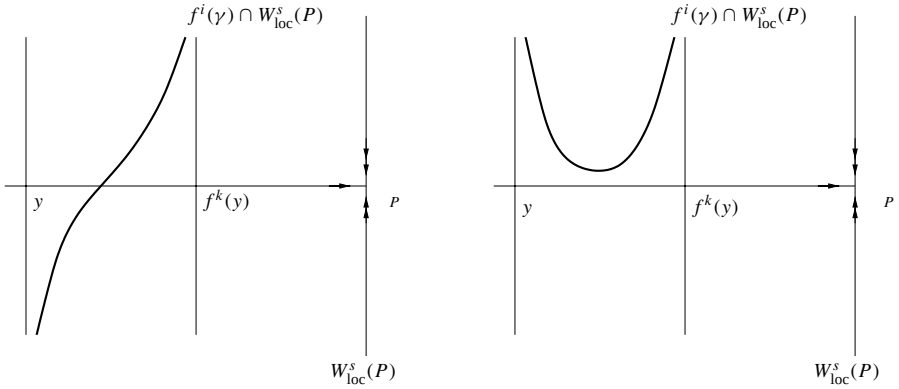


FIGURE 8. The intersection $f^i(\gamma) \cap W^s_{\mathrm{loc}}(P)$.

Proof. After perturbation we can assume that f is $\mathcal{C}^1$-linearizable at P and Q. We argue by contradiction. Suppose that for every connected component γ of $W^s(P) \pitchfork W^u(Q)$ one has that $\pi^{ss}(\gamma)$ and $\pi^{uu}(\gamma)$ contain only finitely many fundamental domains of $W^{cs}(P)$ and $W^{cu}(Q)$, respectively. If we can perturb f to get a critical cycle we are done. Thus let us suppose that it is not the case and that for every g close to f the cycle $\Gamma(g, P_g, Q_g)$ is non-critical. Fixing a component γ of $W^s(P) \pitchfork W^u(Q)$ we have the following.

CLAIM 8.3. *Either $f^i(\gamma)$ is not contained in $W^s_{\mathrm{loc}}(P)$ for all $i \geq 0$ or $f^{-i}(\gamma)$ is not contained in $W^s_{\mathrm{loc}}(Q)$ for all $i \geq 0$.*

Proof. Suppose that the claim is false, then there is $k \geq 0$ such that $\gamma^s = f^k(\gamma) \subset W^s_{\mathrm{loc}}(P)$ and $\gamma^u = f^{-k}(\gamma) \subset W^u_{\mathrm{loc}}(Q)$. The curve γ^s has extremes a^s and b^s, $a^s, b^s \notin \gamma^s$. We can assume that $a^s, b^s \notin W^u(Q)$, otherwise there is a tangency between $W^s(P)$ and $W^u(Q)$ at either a^s or b^s, and after perturbation one gets a critical cycle. Let $\gamma' = \gamma^s \cup \{a^s, b^s\}$. Then $f^{-i}(\gamma')$ is not contained in $W^u_{\mathrm{loc}}(Q)$ for every $i \geq 0$, otherwise a^s and b^s belong to $W^u(Q)$. This implies that $f^{-k}(\gamma)$ is not contained in $W^u_{\mathrm{loc}}(Q)$, contradicting our hypotheses. □

Now suppose, for instance, that $f^i(\gamma)$ is not contained in $W^s_{\mathrm{loc}}(P)$ for every $i \geq 0$. Since $\pi^{ss}(f^i(\gamma))$ contains only finitely many fundamental domains of $W^{cs}(P)$, we have that γ does not intersect $W^{ss}(P)$. There are essentially the two possibilities depicted in Figure 8 for the intersection $f^i(\gamma) \cap W^s_{\mathrm{loc}}(P)$.

Consider any $x \in \gamma$ and the vector v_x tangent to γ at x. Since γ is non-critical the component of v_x in the central direction (the direction spanned by the eigenvector v^c of λ_c) is different from zero. Thus using the linearizing coordinates we get that the vectors $D_x f^i(v_x) = v_{f^i(x)}$ converge to a vector parallel to v^c. Now, since $f^i(x)$ converges to P, $\pi^{ss}(f^i(\gamma))$ contains only finitely many fundamental domains, and $f^i(\gamma)$ is not contained in $W^s_{\mathrm{loc}}(P)$, one immediately gets sequences of points (z_i), $z_i \in (f^i(\gamma) \cap W^s_{\mathrm{loc}}(P))$, and

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of unitary vectors w_i, $w_i = ((w_i)_s, (w_i)_c, 0_u) \in T_{z_i} f^i(\gamma)$, such that

$$\frac{\|(w_i)_s\|}{\|(w_i)_c\|} \to \infty.$$

That is, if i is large enough w_i is almost parallel to $W^{ss}(P)$. It is now clear that we can perturb f to obtain g with a critical cycle $\Gamma(g, P_g, Q_g)$. This concludes the proof of the lemma. □

8.1.2. *Proof of (ii).* There are two possibilities. First, if the cycle $\Gamma(f_0, P_0, Q_0)$ has central real eigenvalues the result follows from (i) in Theorem 2.2.

Suppose that the cycle has a complex (non-real) central eigenvalue. In this case the result follows from [**DU**]: for an open and dense subset of arcs $(f_t)_{t\in I}$ exhibiting a connected heterodimensional cycle $\Gamma(f_0, P_0, Q_0)$ and having only one central real eigenvalue there is an interval $[-\varepsilon, \varepsilon]$ in the parameter space such that:

- the set of parameters t corresponding to diffeomorphisms f_t with homoclinic tangencies associated either to P_t or Q_t is dense in $[-\varepsilon, \varepsilon]$; and
- for every $t \in [-\varepsilon, \varepsilon]$ the connection γ_t of the cycle is contained in $L(f_t)$.

Note that in this case one has $b_n^+ = \varepsilon$ and $b_n^- = -\varepsilon$ for all n. Let us also observe that in the proofs in [**DU**] the authors assume some conditions on the central eigenvalues, but these assumptions can be removed exactly as we did in §6.1.

8.1.3. *Proof of (iii).* Consider an arc $(f_t)_{t\in I}$ with a cycle $\Gamma(f_0, P_0, Q_0)$ corresponding to a first bifurcation, i.e. $(f_t)_{t\in I} \in \mathcal{F}b\mathcal{C}(M)$. Then (for an open and dense subset of such arcs) one has that:

- the central eigenvalues of the cycle are both real and positive;
- the projection $\pi_0^s \circ \pi_0^u$ along compact parts of the strong unstable and stable foliations in §3.1.3 is order preserving;

see the arguments in [**DR2**, §3]. The result now follows arguing exactly as in the proof of Theorem 2.1. □

8.2. *Proof of Theorem A.* Consider a heterodimensional cycle $\Gamma(f, P, Q)$. After perturbation of f we can assume that:

- f is $\mathcal{C}^1$-linearizable at P and Q;
- the central eigenvalues are both real (otherwise we can approximate f by a diffeomorphism with homoclinic tangencies associated either to P or Q);
- the foliations $\mathcal{F}^{ss}$ of $W^s(P)$ and $\mathcal{F}^{uu}$ of $W^u(Q)$ are defined and every connected component of $W^s(P) \pitchfork W^u(Q)$ is transverse to these foliations (otherwise we obtain homoclinic tangencies).

We now have two possibilities.

- The cycle is connected. The result follows from Corollary 2.3 recalling that the cycle contains infinitely many fundamental domains, see Remark 8.1.
- The cycle is non-connected. Observe that we can not perturb $\Gamma(f, P, Q)$ to get a critical cycle $\Gamma(g, P_g, Q_g)$ (recall that the cycle is far from tangencies). Now, from Lemma 8.2, after a new perturbation (if necessary) we get a cycle

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containing infinitely many fundamental domains and the result follows as before from Corollary 2.3. □

8.3. *Proof of Theorem B.* Since the cycle $\Gamma(f, P, Q)$ is connected it contains infinitely many fundamental domains (Remark 8.1). From Corollary 2.3, fixing any neighborhood $\mathcal{U}_f$ of f there is an open subset $\mathcal{V}$ of $\mathcal{U}_f$ such that for every $g \in \mathcal{V}$ there is a transitive set Λ_g containing P_g and Q_g. In particular, P_g and Q_g are persistently connected in $\mathcal{V}$.

By hypothesis, P and A (respectively Q and B) are homoclinically related; thus P_g and A_g (respectively Q_g and B_g) are persistently connected in $\mathcal{V}$ (small $\mathcal{V}$). Since to be persistently connected defines a relation of equivalence, see [**BD2**, Theorem 3.1], A_g and B_g are persistently connected in $\mathcal{V}$. Finally, [**BD2**, Theorem C] implies that there is a residual subset $\mathcal{R}$ in $\mathcal{V}$ of diffeomorphisms with infinitely many sinks or sources. This completes the proof of the theorem. □

8.4. *Proof of Corollary C.* Suppose that f has a bicritical cycle $\Gamma(f, P, Q)$. Denote by λ_s, λ_c, and λ_u, $|\lambda_s| < |\lambda_c| < |\lambda_u|$, (respectively β_s, β_c and β_u, $|\beta_s| < |\beta_c| < |\beta_u|$) the eigenvalues of $D_P f^k$, k the period of P (respectively $D_Q f^m$, with m the period of Q). Observe that $\Gamma(f, P, Q)$ contains infinitely many fundamental domains. Given any $\mathcal{C}^1$-neighborhood $\mathcal{U}_f$ of f there is an open subset $\mathcal{V}$ consisting of g such that:

- there is a transitive set Λ_g containing P_g and Q_g, recall Corollary 2.3;
- $W^s(P_g) \pitchfork W^u(Q_g)$ has a bicritical periodic connected component.

After perturbation we can assume that every g in $\mathcal{V}$ satisfies either (1) or (2) below:

(1) either P_g is sectionally dissipative (i.e. $|\lambda_c(g)\lambda_u(g)| < 1$) or Q_g is sectionally expansive (i.e. $|\beta_c(g)\beta_s(g)| > 1$),

(2) $|\lambda_c(g)\lambda_u(g)| > 1 > |\beta_c(g)\beta_s(g)|$.

Let us first prove the corollary in the second case, which is the hardest one. For that we see that there is g $\mathcal{C}^1$-close to f with a cycle $\Gamma(g, P_g, Q_g)$ satisfying the hypotheses in Theorem B. That is, we get periodic points A_g and B_g homoclinically related to P_g and Q_g, respectively, such that their derivatives have complex (non-real) eigenvalues.

Since P_g and Q_g are in the same transitive set, using the connecting lemma in [**H**] we get $g_1 \in \mathcal{V}$ such that $W^u(P_{g_1})$ and $W^s(Q_{g_1})$ have a quasi-transverse intersection. In particular, $\Gamma(g_1, P_{g_1}, Q_{g_1})$ is a bicritical connected cycle. Unfolding this cycle we get $g_2 \in \mathcal{V}$ with a homoclinic tangency associated to Q_{g_2}. Since Q_{g_2} is not sectionally expansive, unfolding this tangency we get g_3 with a saddle P'_{g_3} such that its derivative has a complex contracting eigenvalue (see [**R**] and [**YA**]). We claim that P'_{g_3} can be chosen satisfying:

(i) $W^s(P'_{g_3}) \cap W^u(Q_{g_3}) \neq \emptyset$;

(ii) $W^u(P'_{g_3})$ intersects transversely $W^s(P_{g_3})$.

The second assertion follows from the standard cone field construction. In this way we get that $W^u(P'_{g_3})$ is close to $W^{uu}_{\mathrm{loc}}(Q_{g_3})$ and $W^s(P_{g_3})$ accumulates on $W^s(Q_{g_3})$. From these observations it is now not difficult to get (ii). The first assertion follows from [**R**], we also refer the reader to [**BD2**, Lemmas 2.4–2.6]. Let us sketch the proof of this fact. From [**R**] we can think of this tangency as a two-dimensional one associated to arcs σ^s and σ^u, $\sigma^{s,u} \subset W^{s,u}(Q_{g_2})$, in a normally hyperbolic surface Σ which is locally g_2-invariant

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and such that g_2 is dissipative in Σ (this follows from $|\beta_c(g_2)\beta_s(g_2)| < 1$, see [**R**]). Unfolding this tangency (in Σ) we get g_3 such that $\sigma^s(g_3)$ and $\sigma^u(g_3)$ bound a disk $D \subset \Sigma$ containing a point P'_{g_3} which is an attractor of the restriction of g_3 to Σ, see [**YA**]. Since the restriction of g_3 to Σ is area contracting, $W^s(P'_{g_3})$ is not contained in D: just consider the backward iterations of $W^s_{\text{loc}}(P'_{g_3})$ and observe that the area of these iterations grows exponentially. Thus $W^s(P'_{g_3})$ intersects the boundary of D. Since the stable manifolds of different periodic points are disjoint, $W^s(P'_{g_3})$ necessarily intersects $\sigma^u(g_3) \subset W^u(Q_{g_3})$. This ends the proof of the claim.

We are now ready to end the proof of the corollary in case (2). We claim that (after a new perturbation if necessary) we can assume that $W^u(P_{g_3})$ intersects transversely $W^s(P'_{g_3})$. For that observe that since P_{g_3} and Q_{g_3} are in the same transitive set after perturbation we can get a cycle $\Gamma(g_4, P_{g_4}, Q_{g_4})$. By (i), $W^s(P'_{g_4})$ accumulates on $W^s(Q_{g_4})$. Now, unfolding $\Gamma(g_4, P_{g_4}, Q_{g_4})$ we get g_5 such that $W^u(P_{g_5})$ intersects $W^s(P'_{g_5})$ transversely.

The claim and (ii) now imply that P_{g_5} and P'_{g_5} are in the same homoclinic class. We now have an open and dense subset $\mathcal{W}$ of $\mathcal{V}$ such that for every $g \in \mathcal{W}$:

- there is a saddle P'_g homoclinically related to P_g such that its derivative has a contracting complex eigenvalue.

Applying the previous arguments to Q_g we get an open and dense subset $\mathcal{Y}$ of $\mathcal{W}$ of diffeomorphisms h with a saddle Q_h homoclinically related Q'_h such that the derivative of Q_h has a non-real expanding eigenvalue. By construction, every h in $\mathcal{Y}$ satisfies the hypotheses of Theorem B. This concludes the proof of the corollary in case (2).

In the first case, suppose, for instance, that P is sectionally dissipative and consider the subsets $\mathcal{V}_k$ of $\mathcal{V}$ defined by

$$\mathcal{V}_k = \{g \in \mathcal{V} \text{ such that } g \text{ has at least } k \text{ hyperbolic sinks}\}.$$

By definition, the $\mathcal{V}_k$ are open. We claim that every $\mathcal{V}_k$ is dense in $\mathcal{V}$. In fact, it is enough to see that $\mathcal{V}_k$ is dense in $\mathcal{V}_{k-1}$. To get the density of $\mathcal{V}_k$ in $\mathcal{V}_{k-1}$, fix g in $\mathcal{V}_{k-1}$. As before, we first perturb g to get a bicritical cycle. After a new perturbation, we get a homoclinic tangency associated to P_g. Since P_g is sectionally dissipative, this leads to the creation of a new sink, see [**MV**].

Finally, we let $\mathcal{R} = \cap \mathcal{V}_k$. By construction, $\mathcal{R}$ is a residual subset of $\mathcal{V}$ consisting of diffeomorphisms with infinitely many sinks. This completes the proof of the corollary. □

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