Ergod. Th. & Dynam. Sys. (2003), **23**, 1551–1574
DOI: 10.1017/S014338570200175X
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Printed in the United Kingdom

On strong shift equivalence of symbolic matrix systems

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(*Received* 7 *July* 2002 *and accepted in revised form* 17 *January* 2003)

Abstract. A symbolic matrix system and a λ-graph system for a subshift are generalizations of a symbolic matrix and a λ-graph (= finite symbolic matrix) for a sofic shift respectively. In [K. Matsumoto. (Presentations of subshifts and their topological conjugacy invariants. *Doc. Math*. **4** (1999), 285–340)], two equivalence relations between symbolic matrix systems were formulated, properly strong shift equivalence and strong shift equivalence, which are related to topological conjugacy of subshifts. In this paper, we first prove that these two equivalence relations are equivalent to each other for a large class of symbolic matrix systems. State splitting of a λ-graph system is introduced as a generalization of state splitting of a finite directed graph. State splitting gives rise to a strong shift equivalence for the corresponding symbolic matrix systems. We prove that two subshifts are topologically conjugate if and only if the canonical λ-graph system for one subshift is obtained by a finite sequence of merged state splittings and merged state amalgamations from the canonical λ-graph system for the other subshift.

1. *Introduction*

Williams [**Wi**] proved that two shifts of finite type are topologically conjugate if and only if their defining nonnegative matrices are strong shift equivalent. The 'if' part is immediate. In his proof of the only if part, state splittings and state amalgamations play a key role. They are procedures to construct new finite directed graphs from the finite directed graphs defining the shifts of finite type. Let A, B be finite square matrices with entries in the nonnegative integers, simply called nonnegative matrices. Let G_A, G_B be their finite directed graphs respectively. Let Λ_A, Λ_B be the shifts of finite type defined by the sets of all bi-infinite sequences of allowed edges in the graphs G_A, G_B, respectively. He in fact proved that if Λ_A, Λ_B are topologically conjugate, then the graph G_A is obtained by a finite sequence of state splittings and state amalgamations from the other graph G_B. State splittings are methods to split vertices by partitions of the outgoing edges and

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the in coming edges at the vertices. State splitting by outgoing (respectively incoming) edges is called an out-splitting (respectively in-splitting). The nonnegative matrices for the finite directed graphs obtained by state splittings are strong shift equivalent to the original matrices.

Nasu [**N**] generalized the Williams result above to sofic shifts. He introduced the notions of bipartite λ-graphs for sofic shifts and bipartite codes for topological conjugacy between general subshifts. He proved that a topological conjugacy between subshifts is decomposed into compositions of a finite sequence of bipartite codes. He also introduced the notions of strong shift equivalence of symbolic matrices and showed that two sofic shifts are topologically conjugate if and only if the symbolic matrices for their Krieger covers are strong shift equivalent ([**N, N2**], cf. [**BK**]).

In [**Ma2**], the present author introduced the notion of a symbolic matrix system and a λ-graph system from an idea in the theory of C^*-algebras [**Ma, Ma3**]. The symbolic matrix system is a generalization of a symbolic matrix, and a λ-graph system is a generalization of a λ-graph. We henceforth denote by $\mathbb{Z}_+$ the set of all nonnegative integers. Let Σ be a finite set, called an alphabet. A symbolic matrix system over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have entries which are formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. They satisfy the following commutation relations:

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+. \tag{1.1}$$

The matrices $I_{l,l+1}, l \in \mathbb{Z}_+$, have one 1 in each column and at least one 1 in each row. We denote the symbolic matrix system by $(\mathcal{M}, I)$. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling $\lambda : E \to \Sigma$ and a surjective map $\iota\ (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. It naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The edges from a vertex $v_i^l \in V_l$ to a vertex v_j^{l+1} are given by the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of the matrix $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$. Then the commutation relations (1.1) prescribe a certain property on $\mathfrak{L}$, which we call the local property of $\mathfrak{L}$. The symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts by gathering all the label sequences appearing in the labeled Bratteli diagram. A canonical method to construct a symbolic matrix system and a λ-graph system from an arbitrary subshift has been introduced in [**Ma2**]. The symbolic matrix system and the λ-graph system obtained are said to be canonical for the subshift. The notions of strong shift equivalence for nonnegative matrices and symbolic matrices have been generalized to symbolic matrix systems in two equivalences: properly strong shift equivalence and strong shift equivalence. The properly strong shift equivalence exactly corresponds to bipartite λ-graph systems. It implies strong shift equivalence. For the canonical symbolic matrix systems, these two equivalences coincide. As a result, the above-mentioned Williams and Nasu results have been generalized to topological conjugacy between subshifts. That is, two subshifts are topologically conjugate if and only if their canonical symbolic matrix systems are strong shift equivalent [**Ma2**].

In this paper, we will first show that these two equivalences, properly strong shift equivalence and strong shift equivalence, coincide for a large class of symbolic matrix

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systems that contain the class of the canonical symbolic matrix systems. If the vertex set V_0 at level 0 for a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of only one vertex, it is said to be a single zero-vertex set. If, for each $l \in \mathbb{Z}_+$, the predecessor sets of label sequences at vertices of V_l are different, $\mathfrak{L}$ is said to be predecessor-separated. If different edges with the same label have different terminals, $\mathfrak{L}$ is said to be left-resolving. We will prove the following.

THEOREM A. (Theorem 4.3) *Strong shift equivalence implies properly strong shift equivalence for symbolic matrix systems whose λ-graph systems are left-resolving, predecessor-separated with a single zero-vertex.*

We will introduce state splitting of λ-graph systems as a generalization of state splitting of finite directed graphs. The converse procedure to state splitting is called state amalgamation. We will also introduce a procedure to merge vertices with the same predecessors of a λ-graph system. The resulting λ-graph system is called the merged λ-graph system. We will prove the following.

THEOREM B. (Theorem 7.1) *Let $\mathfrak{L}^{\Lambda}$ and $\mathfrak{L}^{\Lambda'}$ be the canonical λ-graph systems for subshifts Λ and Λ' respectively. Then Λ is topologically conjugate to Λ' if and only if $\mathfrak{L}^{\Lambda}$ is obtained by a finite procedure of merged out-splittings, merged in-splittings, merged out-amalgamations and merged in-amalgamations from $\mathfrak{L}^{\Lambda'}$.*

The merged out-splittings and merged in-splittings of λ-graph systems yield strong shift equivalences between the corresponding symbolic matrix systems. This theorem clarifies a procedure to obtain one canonical λ-graph system from the other canonical λ-graph system under the condition that the presenting subshifts are topologically conjugate.

We constructed C^*-algebras from λ-graph systems as a generalization of the C^*-algebras associated with subshifts and the Cuntz–Krieger algebras for shifts of finite type ([**Ma3**], cf. [**CK, C, Ma, CM**]). A relationship between the C^*-algebras constructed from strong shift equivalent λ-graph systems has been clarified in [**Ma4**].

2. *The λ-graph systems as presentations of subshifts*

Let Σ be a finite set with its discrete topology. We call it an alphabet. Each element of Σ is called a symbol or a label. Let $\Sigma^{\mathbb{Z}}$ be the infinite product spaces $\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $(\sigma(x_i))_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$, and is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$, i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ and write the subshift as Λ for brevity. We denote by $X_\Lambda\big(\subset \prod_{i\in\mathbb{N}}\Sigma_i\big)$ the set of all right-infinite sequences that appear in Λ. A finite sequence $\mu = (\mu_1, \ldots, \mu_k)$ of symbols $\mu_j \in \Sigma$ is called a block or a word of length k. We write the empty symbol $\emptyset$ in Σ as 0. Let $\mathfrak{S}_\Sigma$ be the set of all finite formal sums of symbols of Σ. By a symbolic matrix $\mathcal{A}$ over Σ we mean a finite matrix with entries in $\mathfrak{S}_\Sigma$. It naturally gives rise to a labeled directed graph, called a λ-graph, which we denote by $G_\mathcal{A}$. The labeled directed graph defines a subshift over Σ which consists of all infinite labeled sequences following the labeled edges in $G_\mathcal{A}$. Such a subshift is called a sofic shift presented by $G_\mathcal{A}$

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[**Fi, Kr, Kr2, We, Ki, LM**]. If, in particular, different edges have different labels, the sofic shift is called a shift of finite type.

Let $\mathcal{A}$, $\mathcal{A}'$ be symbolic matrices over Σ, Σ' respectively, and ϕ a bijection from a subset of Σ onto a subset of Σ'. Following Nasu in [**N, N2**], we say that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent under the specification ϕ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by replacing every symbol a appearing in the components of $\mathcal{A}$ by $\phi(a)$. We write it as $\mathcal{A} \overset{\phi}{\simeq} \mathcal{A}'$.

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$ permutation matrix P_l for each $l \in \mathbb{Z}_+$ such that

$$P_l \mathcal{M}_{l,l+1} \overset{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \quad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \quad \text{for } l \in \mathbb{Z}_+.$$

By a λ-graph over Σ we mean a finite-directed graph with labeled edges in Σ [**N2**]. Recall that a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is a directed Bratteli diagram [**Bra**] with vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$ and edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}$ that is labeled by a map $\lambda(= \lambda_{l,l+1}) : E_{l,l+1} \to \Sigma$ with symbols in Σ for $l \in \mathbb{Z}_+$, and that is supplied with a sequence of surjective maps $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here both $V_l, l \in \mathbb{Z}_+$ and $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1}. Every vertex in V has a successor and every vertex in V, except the vertices in V_0 at level 0, has a predecessor. It is then required that there exists an edge in $E_{l,l+1}$ with label α whose terminal is $v_l \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α whose terminal is $\iota(v_l) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E^{\iota}(u, v) = \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},$$
$$E_{\iota}(u, v) = \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective correspondence between $E^{\iota}(u, v)$ and $E_{\iota}(u, v)$ that preserves labels for all pairs of vertices $u \in V_{l-1}, v \in V_{l+1}$. This property is called the local property of the λ-graph system. We call an edge $e \in E_{l,l+1}$ a λ-edge and a connecting finite sequence of λ-edges a λ-path. For $u \in V_l$ and $v \in V_{l+1}$, if $\iota(v) = u$, we say that there exists an ι-edge from u to v. For a vertex $v \in V_l$, put

$$\Gamma^-_{\mathfrak{L},l}(v) = \{(\lambda(e_1), \ldots, \lambda(e_l)) \in \Sigma^l \mid \exists e_n \in E_{n-1,n},\ n = 1, 2, \ldots, l;$$
$$t(e_n) = s(e_{n+1}) \text{ for } n = 1, \ldots, l-1, t(e_l) = v\}.$$

We say that $\mathfrak{L}$ is predecessor-separated if for $u, v \in V_l$ the condition $\Gamma^-_{\mathfrak{L},l}(u) = \Gamma^-_{\mathfrak{L},l}(v)$ implies $u = v$. If the vertex set V_0 at level 0 is a singleton, we say that $\mathfrak{L}$ has single zero-vertex. Two λ-graph systems (V, E, λ, ι) over Σ and $(V', E', \lambda', \iota')$ over Σ' are said to be isomorphic if there exist bijections $\Phi_V : V_l \to V'_l$, $\Phi_E : E_{l,l+1} \to E'_{l,l+1}$ for $l \in \mathbb{Z}_+$, and a specification $\phi : \Sigma \to \Sigma'$ such that $\Phi_V(s(e)) = s(\Phi_E(e))$, $\Phi_V(t(e)) = t(\Phi_E(e))$, $\lambda'(\Phi_E(e)) = \phi(\lambda(e))$ for $e \in E$, and $\iota'(\Phi_V(v)) = \Phi_V(\iota(v))$ for $v \in V$. There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ-graph systems.

We know that a λ-graph defines a λ-graph system in the following way, where a λ-graph and also a λ-graph system are said to be left-resolving if different edges with the same label must have different terminals.

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LEMMA 2.1. *A λ-graph defines a λ-graph system such that the presenting subshifts coincide, and if the λ-graph is left-resolving (respectively predecessor-separated), the λ-graph system is left-resolving (respectively predecessor-separated).*

Proof. Let $\mathcal{A}$ be a symbolic matrix over alphabet Σ and (G, λ) its λ-graph with finite-directed graph G and labeling λ. Let $\{v_1, \dots, v_n\}$ be the vertex set of G. Put $V_l = \{v_1, \dots, v_n\}$ for all $l \in \mathbb{Z}_+$. We define the ι-map from V_{l+1} to V_l by $\iota(v_i) = v_i$ for $i = 1, \dots, n$. Write labeled edges from V_l to V_{l+1} for $l \in \mathbb{Z}_+$ following the edges of the directed graph G with labeling λ. It is obvious that the resulting labeled Bratteli diagram with the ι-map becomes a λ-graph system. □

We henceforth assume for λ-graph systems that there are no two edges e and f such that $\lambda(e) = \lambda(f), s(e) = s(f), t(e) = t(f)$.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. Let L_k for $k \in \mathbb{N}$ be the set $\bigcup_{v \in V_k} \Gamma^-_{\mathfrak{L},k}(v)$ of all label sequences of paths from V_0 to V_k. We set

$$X_{\mathfrak{L}} = \{(\lambda(e_1), \lambda(e_2), \dots) \in \Sigma^{\mathbb{N}} \mid e_i \in E_{i-1,i}, t(e_i) = s(e_{i+1}) \text{ for } i \in \mathbb{N}\},$$

the set of all right infinite labeled sequences of infinite λ-paths in $\mathfrak{L}$. The topology on $X_{\mathfrak{L}}$ is defined from open sets of the form

$$U_{(\mu_1,\dots,\mu_k)} = \{(\alpha_1, \alpha_2, \dots) \in X_{\mathfrak{L}} \mid \alpha_i = \mu_i \text{ for } i = 1, \dots, k\}$$

for $(\mu_1, \dots, \mu_k) \in L_k$ so that $X_{\mathfrak{L}}$ is a compact Hausdorff space. If $(\alpha_1, \alpha_2, \dots) \in X_{\mathfrak{L}}$, we have $(\alpha_2, \alpha_3, \dots) \in X_{\mathfrak{L}}$. For $(\alpha_1, \alpha_2, \dots) \in X_{\mathfrak{L}}$, we may find a symbol $\alpha_0 \in \Sigma$ such that $(\alpha_0, \alpha_1, \alpha_2, \dots) \in X_{\mathfrak{L}}$. Hence the map $S : (\alpha_1, \alpha_2, \alpha_3, \dots) \in X_{\mathfrak{L}} \to (\alpha_2, \alpha_3, \dots) \in X_{\mathfrak{L}}$ is well-defined, continuous and surjective. We set

$$\Lambda_{\mathfrak{L}} = \varprojlim\{S : X_{\mathfrak{L}} \to X_{\mathfrak{L}}\},$$

the projective limit of the compact Hausdorff spaces $X_{\mathfrak{L}}$ along S. Thus $\Lambda_{\mathfrak{L}}$ is identified with the set of all bi-infinite sequences arising from the sequences in $X_{\mathfrak{L}}$. That is

$$\Lambda_{\mathfrak{L}} = \{(\dots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots) \mid (\alpha_n, \alpha_{n+1}, \dots) \in X_{\mathfrak{L}} \text{ for all } n \in \mathbb{Z}\}.$$

The map S induces a homeomorphism σ on $\Lambda_{\mathfrak{L}}$ that satisfies $\sigma((\alpha_i)_{i\in\mathbb{Z}}) = (\alpha_{i+1})_{i\in\mathbb{Z}}$. Therefore we obtain a subshift $(\Lambda_{\mathfrak{L}}, \sigma)$.

Conversely, for a subshift (Λ, σ) over Σ and a number $k \in \mathbb{Z}_+$, let Λ^k be the set of all words of length k in Σ^k occurring in some $x \in \Lambda$, where Λ^0 denotes the empty word $\{\emptyset\}$. For $l \in \mathbb{Z}_+$, two points $x, y \in X_\Lambda$ are said to be l-past equivalent if $\{\mu \in \Lambda^l \mid \mu x \in X_\Lambda\} = \{\nu \in \Lambda^l \mid \nu y \in X_\Lambda\}$. Let $F^l_i, i = 1, 2, \dots, m(l)$ be the set of all l-past equivalence classes of X_Λ. We define two rectangular $m(l) \times m(l+1)$ matrices $I^\Lambda_{l,l+1}, \mathcal{M}^\Lambda_{l,l+1}$ with entries in $\{0, 1\}$ and entries in $\mathfrak{S}_\Sigma$, respectively, in the following way. For $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, the (i, j)-component $I^\Lambda_{l,l+1}(i, j)$ of $I^\Lambda_{l,l+1}$ is 1 if F^l_i includes F^{l+1}_j, otherwise it is 0. Let $a_1, \dots, a_n$ be the set of all symbols in Σ for which $a_k x \in F^l_i$ for some $x \in F^{l+1}_j$. We then define the (i, j)-component of the matrix $\mathcal{M}^\Lambda_{l,l+1}(i, j)$ as $\mathcal{M}^\Lambda_{l,l+1}(i, j) = a_1 + \cdots + a_n \in \mathfrak{S}_\Sigma$. We can show that the pair $(\mathcal{M}^\Lambda, I^\Lambda)$ becomes a symbolic matrix system. We call it the *canonical* symbolic

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matrix system for Λ. We denote its λ-graph system by $\mathfrak{L}^{\Lambda} = (V^{\Lambda}, E^{\Lambda}, \lambda^{\Lambda}, \iota^{\Lambda})$ and call it the *canonical* λ-graph system for Λ. We note that the canonical λ-graph systems are left-resolving and predecessor-separated with a single zero-vertex. The subshift $\Lambda_{\mathfrak{L}^{\Lambda}}$ presented by $\mathfrak{L}^{\Lambda}$ coincides with the original subshift Λ.

Therefore, subshifts yield a class of λ-graph systems, the class of canonical λ-graph systems, whereas all λ-graph systems give rise to subshifts. An abstract characterization of the canonical λ-graph systems is given in [**KM**].

Definition 1. [**Ma2**] A symbolic matrix system $(\mathcal{M}, I)$ over Σ is said to be *bipartite* if there exist disjoint subsets $C, D \subset \Sigma$ and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{Z}_+$ with $m(l) = n(l) + n'(l)$ for $l \in \mathbb{N}$ such that $\Sigma = C \cup D$ and for each $l \in \mathbb{Z}_+$, there exist an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_{l,l+1}$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_{l,l+1}$ over D, an $n(l) \times n(l+1)$ matrix $X_{l,l+1}$ over $\{0,1\}$ and an $n'(l) \times n'(l+1)$ matrix $X'_{l,l+1}$ over $\{0,1\}$ satisfying the following matrix equations for $l \in \mathbb{N}$:

$$\mathcal{M}_{l,l+1} = \begin{bmatrix} 0 & \mathcal{P}_{l,l+1} \\ \mathcal{Q}_{l,l+1} & 0 \end{bmatrix}, \quad I_{l,l+1} = \begin{bmatrix} X_{l,l+1} & 0 \\ 0 & X'_{l,l+1} \end{bmatrix}.$$

Definition 2. [**Ma2**] A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is said to be *bipartite* if there exist disjoint subsets $C, D \subset \Sigma$ such that $\Sigma = C \cup D$ and disjoint subsets $V^C_{D,l}, V^D_{C,l} \subset V_l$ for each $l \in \mathbb{N}$ such that $V^C_{D,l} \cup V^D_{C,l} = V_l$ and for $l \in \mathbb{N}$:
(1) for each $e \in E_{l,l+1}$

$$s(e) \in V^D_{C,l}, \quad r(e) \in V^C_{D,l+1} \text{ if and only if } \lambda(e) \in C,$$
$$s(e) \in V^C_{D,l}, \quad r(e) \in V^D_{C,l+1} \text{ if and only if } \lambda(e) \in D;$$

(2) $\iota(V^D_{C,l+1}) = V^D_{C,l}, \iota(V^C_{D,l+1}) = V^C_{D,l}$.
It is obvious that a symbolic matrix system is bipartite if and only if its λ-graph system is bipartite. For alphabets C, D, put $CD = \{cd \mid c \in C, d \in D\}$. For $x = \sum_j c_j \in \mathfrak{S}_C$ and $y = \sum_k d_k \in \mathfrak{S}_D$, define xy by $\sum_{j,k} c_j d_k \in \mathfrak{S}_{CD}$.

We then see the following.

LEMMA 2.2. [**Ma2**] *For a bipartite symbolic matrix system $(\mathcal{M}, I)$ as above, set*

$$\mathcal{P}_l = \mathcal{P}_{l,l+1}, \quad \mathcal{Q}_l = \mathcal{Q}_{l,l+1}, \quad X_l = X_{l,l+1}, \quad X'_l = X'_{l,l+1}$$

and

$$\mathcal{M}^{CD}_{l,l+1} = \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \quad \mathcal{M}^{DC}_{l,l+1} = \mathcal{Q}_{2l}\mathcal{P}_{2l+1},$$
$$I^{CD}_{l,l+1} = X_{2l}X_{2l+1}, \quad I^{DC}_{l,l+1} = X'_{2l}X'_{2l+1}.$$

Then both pairs $(\mathcal{M}^{CD}, I^{CD})$ and $(\mathcal{M}^{DC}, I^{DC})$ are symbolic matrix systems over alphabets CD and DC, respectively.

In this case, we say that a pair $(\mathcal{M}^{CD}, I^{CD})$ and $(\mathcal{M}^{DC}, I^{DC})$ is *bipartite* or *bipartitely related* via a bipartite symbolic matrix system $(\mathcal{M}, I)$. We denote by $\mathfrak{L}^{CD}$ and $\mathfrak{L}^{DC}$ the corresponding λ-graph systems to the symbolic matrix systems $(\mathcal{M}^{CD}, I^{CD})$ and

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$(\mathcal{M}^{DC}, I^{DC})$, respectively. We similarly say that a pair $\mathfrak{L}^{CD}$ and $\mathfrak{L}^{DC}$ is bipartite or bipartitely related.

Let M, N be two square nonnegative matrices. They naturally give rise to finite directed graphs for each. Regard them as λ-graphs by labeling different symbols in all edges. It is easy to see that M and N are strong shift equivalent in one step as nonnegative matrices if and only if their associated λ-graph systems $\mathfrak{L}(M)$ and $\mathfrak{L}(N)$ constructed in Lemma 2.1 are bipartitely related.

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over Σ and Σ', respectively.

Definition 3. [**Ma2**] $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent in one step* if there exist alphabets C, D and specifications $\varphi : \Sigma \to CD$, $\phi : \Sigma' \to DC$ and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{Z}_+$ such that for each $l \in \mathbb{Z}_+$, there exists an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_l$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_l$ over D, an $n(l) \times n(l+1)$ matrix X_l over $\{0, 1\}$ and an $n'(l) \times n'(l+1)$ matrix X'_l over $\{0, 1\}$ satisfying the following equations:

$$\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \quad \mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+1},$$
$$I_{l,l+1} = X_{2l}X_{2l+1}, \quad I'_{l,l+1} = X'_{2l}X'_{2l+1}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+1}, \quad X'_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+1}.$$

We write this situation as $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}', I')$.

$(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent* if there exists a finite sequence $(\mathcal{M}^{(1)}, I^{(1)}), \ldots, (\mathcal{M}^{(N-1)}, I^{(N-1)})$ of symbolic matrix systems such that

$$(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{\text{1-pr}}{\approx} \cdots \underset{\text{1-pr}}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{\text{1-pr}}{\approx} (\mathcal{M}', I').$$

Hence one sees the following.

LEMMA 2.3. *$(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are properly strong shift equivalent in one step if and only if they are isomorphic to symbolic matrix systems that are bipartitely related.*

In [**Ma2**], the following theorem was proved.

THEOREM 2.4. *Two subshifts Λ and Λ' are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are properly strong shift equivalent.*

In the proof of the only if part given in [**Ma2**], Nasu's factorization theorem for topological conjugacy between subshifts into bipartite codes has been used ([**N**, Theorem 2.4]). Let G, H be finite directed graphs and M_G, M_H their adjacency matrices. Williams [**Wi**] proved that the edge shifts Λ_G, Λ_H defined by the sets of all bi-infinite sequences of the connected edges in G, H, respectively, are topologically conjugate if and only if the nonnegative matrices M_G, M_H are strong shift equivalent. In the proof of the only if part, the directed graphs G and H are related to a finite sequence of procedures of state splittings and state amalgamations.

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

In this paper, we will introduce the notions of state splitting and state amalgamation of λ-graph systems and introduce a procedure to merge the λ-graph system. We will then prove that topological conjugacy between subshifts comes from finite sequences of merged state splittings and merged state amalgamations of their canonical λ-graph systems.

3. *The merged λ-graph systems*

We note that the canonical λ-graph systems are left-resolving and predecessor-separated. This section is devoted to constructing a predecessor-separated λ-graph system from a λ-graph system. This method is a generalization of the method to construct the merged λ-graph from a λ-graph [**LM**, p. 78]. We will henceforth treat λ-graph systems that are not necessarily canonical. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. For a vertex $v \in V_l$ with $l \in \mathbb{N}$, let $\mathrm{Pred}(v)$ be the set of all pairs $(s(e), \lambda(e)) \in V_{l-1} \times \Sigma$ of vertices and symbols such that $t(e) = v$. Two vertices $u_1, u_2 \in V_l$ are said to be equivalent if $\iota(u_1) = \iota(u_2)$ and $\mathrm{Pred}(u_1) = \mathrm{Pred}(u_2)$. For $l = 0$, all vertices of V_0 are defined to be equivalent to each other. For a fixed $l \in \mathbb{Z}_+$, this equivalence relation partitions V_l into disjoint equivalence classes that we denote by $\{\mathfrak{v}_1, \ldots, \mathfrak{v}_r\}$. Write an edge from $u \in V_{l-1}$ to $\mathfrak{v}_i$ with label α if there exist a vertex $v_j \in \mathfrak{v}_i$ and a λ-edge from u to v_j with label α. Write an edge from $\mathfrak{v}_i$ to $w \in V_{l+1}$ with label β if there exist a vertex $v_k \in \mathfrak{v}_i$ and a λ-edge from v_k to w with label β. Write ι-edges from $u \in V_{l-1}$ to $\mathfrak{v}_i$, and from $\mathfrak{v}_i$ to $w \in V_{l+1}$ if there exist a vertex $v_j \in \mathfrak{v}_i$ and an ι-edge from u to v_j, and there exist vertex $v_k \in \mathfrak{v}_i$ and an ι-edge from v_k to w respectively. The resulting graph becomes a λ-graph system. We call it the merged λ-graph system of $\mathfrak{L}$ at level l. Let $\mathfrak{L}_{(0)}$ be the merged λ-graph system of $\mathfrak{L}$ at level 0, and $\mathfrak{L}_{(1)}$ be the merged λ-graph system of $\mathfrak{L}_{(0)}$ at level 1. For $k > 1$, we denote by $\mathfrak{L}_{(k)}$ the merged λ-graph system of $\mathfrak{L}_{(k-1)}$ at level k. Then we define for each $l \in \mathbb{Z}_+$:

$$
\begin{aligned}
\overline{V}_l &= \text{the vertex set at level } l \text{ of } \mathfrak{L}_{(n)} \text{ for } n \geq l;\\
\overline{E}_{l,l+1} &= \text{the edge set from level } l \text{ to level } l+1 \text{ of } \mathfrak{L}_{(n)} \text{ for } n \geq l+1;\\
\overline{\lambda}_{l,l+1} &= \text{ the labeling of the edges of } \overline{E}_{l,l+1} \text{ induced from } \mathfrak{L}_{(n)} \text{ for } n \geq l+1;\\
\overline{\iota}_{l,l+1} &= \text{the map defined by } \iota\text{-edges from the level } l \text{ vertices}\\
&\qquad \text{to the level } l+1 \text{ vertices of } \mathfrak{L}_{(n)} \text{ for } n \geq l+1.
\end{aligned}
$$

LEMMA 3.1. *If $\mathfrak{L}$ is left-resolving, the resulting graph $\overline{\mathfrak{L}} = (\overline{V}_l, \overline{E}_{l,l+1}, \overline{\lambda}_l, \overline{\iota}_{l,l+1})_{l\in\mathbb{Z}_+}$ is a left-resolving and predecessor-separated λ-graph system over Σ with single zero-vertex.*

Proof. As $\mathfrak{L}_{(n)}$ is a left-resolving λ-graph system for all $n \in \mathbb{Z}_+$, it is clear that $\overline{\mathfrak{L}}$ is left-resolving. It remains to show that $\overline{\mathfrak{L}}$ is predecessor-separated. Recall that, for $v_i^l \in V_l$, $\Gamma^-_{\mathfrak{L},l}(v_i^l)$ denotes the set of all l-length label sequences of λ-paths in $\mathfrak{L}$ terminating at v_i^l. For $v_i^l \in V_l$, we denote by $\bar{v}_i{}^l$ its equivalence class in $\overline{V}_l$. We will prove the following assertion $(*)$:

$$
\Gamma^-_{\overline{\mathfrak{L}},l}(\bar{v}_i{}^l) = \Gamma^-_{\overline{\mathfrak{L}},l}(\bar{v}_j{}^l) \quad \text{implies} \quad \bar{v}_i{}^l = \bar{v}_j{}^l. \tag{$*$}
$$

For $l = 1$, since the vertex set at the level zero of $\mathfrak{L}_{(0)}$ is one point, the condition $\Gamma^-_{\overline{\mathfrak{L}},1}(\bar{v}_i{}^1) = \Gamma^-_{\overline{\mathfrak{L}},1}(\bar{v}_j{}^1)$ implies $\mathrm{Pred}(v_i^1) = \mathrm{Pred}(v_j^1)$ in $\mathfrak{L}_{(0)}$. Hence $\bar{v}_i{}^1 = \bar{v}_j{}^1$ in $\overline{V}_1$. Assume that the assertion $(*)$ holds for all $l \leq k$. Now we suppose that

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$\Gamma^-_{\overline{\mathfrak{L}},k+1}(\bar{v_i}^{k+1}) = \Gamma^-_{\overline{\mathfrak{L}},k+1}(\bar{v_j}^{k+1})$. By the local property of the λ-graph system, we have $\Gamma^-_{\overline{\mathfrak{L}},k}(\bar{\iota}(\bar{v_i}^{k+1})) = \Gamma^-_{\overline{\mathfrak{L}},k}(\bar{\iota}(\bar{v_j}^{k+1}))$. By the hypothesis $(*)$ for all $l \leq k$, we have $\bar{\iota}(\bar{v_i}^{k+1}) = \bar{\iota}(\bar{v_j}^{k+1})$ in $\overline{V}_k$. Hence $\bar{\iota}(\bar{v_i}^{k+1}) = \bar{\iota}(\bar{v_j}^{k+1})$ in $\mathfrak{L}_{(k)}$. For $\alpha \in \Sigma$ and an edge e in $\overline{E}$ with $\overline{\lambda}(e) = \alpha, t(e) = \bar{v_i}^{k+1}$, as $\Gamma^-_{\overline{\mathfrak{L}},k+1}(\bar{v_i}^{k+1}) = \Gamma^-_{\overline{\mathfrak{L}},k+1}(\bar{v_j}^{k+1})$, we may find an edge e' in $\overline{E}$ with $\overline{\lambda}(e') = \alpha, t(e') = \bar{v_j}^{k+1}$. Since $\overline{\mathfrak{L}}$ is left-resolving, such edges e and e' are unique and we have $\Gamma^-_{\overline{\mathfrak{L}},k}(s(\bar{e})) = \Gamma^-_{\overline{\mathfrak{L}},k}(s(\bar{e'}))$. By the hypothesis for $l = k$, we have $s(e) = s(e')$ in $\overline{V}_k$. This implies $\mathrm{Pred}(v_i^{k+1}) = \mathrm{Pred}(v_j^{k+1})$ in $\mathfrak{L}_{(k)}$. Hence v_i^{k+1} is equivalent to v_j^{k+1} in $\mathfrak{L}_{(k)}$ so that $\bar{v_i}^{k+1} = \bar{v_j}^{k+1}$ in $\mathfrak{L}_{(k+1)}$. Therefore we obtain $\bar{v_i}^{k+1} = \bar{v_j}^{k+1}$ in $\overline{\mathfrak{L}}$. Thus $\overline{\mathfrak{L}}$ is predecessor-separated. □

The λ-graph system $\overline{\mathfrak{L}}$ is called the merged λ-graph system of $\mathfrak{L}$. It is evident to show that the subshift presented by $\overline{\mathfrak{L}}$ is the same as the subshift presented by $\mathfrak{L}$.

For a vertex $v_i^l \in V_l$, we denote by $\bar{v_i}^l \in \overline{V}_l$ its equivalence class in $\overline{\mathfrak{L}}$. As in the proof of Lemma 3.1, we know that $\bar{v_i}^l = \bar{v_j}^l$ if and only if $\Gamma_l^-(v_i^l) = \Gamma_l^-(v_j^l)$ by induction on l. Hence one can interpret the merged vertex set $\overline{V}_l$ as the equivalence class of V_l of the equivalence relation between v_i^l and v_j^l defined by $\Gamma_l^-(v_i^l) = \Gamma_l^-(v_j^l)$.

LEMMA 3.2. *For the merged λ-graph system $\overline{\mathfrak{L}} = (\overline{V}_l, \overline{E}_{l,l+1}, \overline{\lambda}_l, \overline{\iota}_{l,l+1})_{l\in\mathbb{Z}_+}$ of a left-resolving, predecessor-separated λ-graph system $\mathfrak{L} = (V_l, E_{l,l+1}, \lambda_l, \iota_{l,l+1})_{l\in\mathbb{Z}_+}$, we have $\overline{V}_l = V_l$ for $l \in \mathbb{N}$. If, in particular, $\mathfrak{L}$ has single zero-vertex, we have $\overline{\mathfrak{L}} = \mathfrak{L}$.*

Proof. For $l = 1$, the condition $\Gamma^-_{\mathfrak{L},1}(v_1) \neq \Gamma^-_{\mathfrak{L},1}(v_2)$ implies that $\mathrm{Pred}(v_1) \neq \mathrm{Pred}(v_2)$ so that $\overline{V}_1 = V_1$. Assume that the condition $\overline{V}_l = V_l$ holds for all $l \leq k$. Suppose that for $v_1, v_2 \in V_{k+1}$ with $v_1 \neq v_2$, $\iota(v_1) = \iota(v_2)$ in V_k. If $\mathrm{Pred}(v_1) = \mathrm{Pred}(v_2)$, we see $\Gamma^-_{\mathfrak{L},k+1}(v_1) = \Gamma^-_{\mathfrak{L},k+1}(v_2)$ a contradiction for the hypothesis that $\mathfrak{L}$ is a predecessor-separated. This concludes $\overline{V}_l = V_l$ for $l = k + 1$. □

We will next show that the procedure to merge a λ-graph system preserves properly strong shift equivalence. Let $(\mathcal{M}, I), (\mathcal{M}', I')$ be symbolic matrix systems and $\mathfrak{L}, \mathfrak{L}'$ their λ-graph systems, respectively. We denote by $(\overline{\mathcal{M}}, \overline{I}), (\overline{\mathcal{M}'}, \overline{I'})$ the symbolic matrix systems for their merged λ-graph systems $\overline{\mathfrak{L}}, \overline{\mathfrak{L}'}$ of $\mathfrak{L}, \mathfrak{L}'$, respectively.

PROPOSITION 3.3. *If $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are properly strong shift equivalent in one step, then $(\overline{\mathcal{M}}, \overline{I})$ and $(\overline{\mathcal{M}'}, \overline{I'})$ are properly strong shift equivalent in one step.*

Proof. Let $\widehat{\mathfrak{L}} = (\widehat{V}, \widehat{E}, \hat{\lambda}, \hat{\iota})$ be the bipartite λ-graph system over $C \cup D$ for which $\mathfrak{L}, \mathfrak{L}'$ are bipartitely related. Hence there exist disjoint subsets $\widehat{V}^C_{D,l}, \widehat{V}^D_{C,l}$ of $\widehat{V}_l$ for each $l \in \mathbb{N}$ such that $\widehat{V}^C_{D,l} \cup \widehat{V}^D_{C,l} = \widehat{V}_l$ and

(1) for each $e \in \widehat{E}_{l,l+1}$

$$s(e) \in \widehat{V}^D_{C,l}, \quad r(e) \in \widehat{V}^C_{D,l+1} \quad \text{if and only if } \hat{\lambda}(e) \in C,$$
$$s(e) \in \widehat{V}^C_{D,l}, \quad r(e) \in \widehat{V}^D_{C,l+1} \quad \text{if and only if } \hat{\lambda}(e) \in D;$$

(2) $\hat{\iota}_{l,l+1}(\widehat{V}^D_{C,l+1}) = \widehat{V}^D_{C,l}, \hat{\iota}_{l,l+1}(\widehat{V}^C_{D,l+1}) = \widehat{V}^C_{D,l}$.

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

We put $V_l^{CD} = \widehat{V}_{C,2l}^D$, $V_l^{DC} = \widehat{V}_{D,2l}^C$ and

$$E_{l,l+1}^{CD} = \{(e, f) \in \widehat{E}_{2l,2l+1}^C \times \widehat{E}_{2l+1,2l+2}^D \mid t(e) = s(f)\},$$
$$E_{l,l+1}^{DC} = \{(e, f) \in \widehat{E}_{2l,2l+1}^D \times \widehat{E}_{2l+1,2l+2}^C \mid t(e) = s(f)\},$$

where $\widehat{E}_{l,l+1}^C = \{e \in E_{l,l+1} \mid \hat{\lambda}(e) \in C\}$, $\widehat{E}_{l,l+1}^D = \{e \in E_{l,l+1} \mid \hat{\lambda}(e) \in D\}$. We set $V^{CD} = \bigcup_{l=0}^{\infty} V_l^{CD}$, $V^{DC} = \bigcup_{l=0}^{\infty} V_l^{DC}$ and $E^{CD} = \bigcup_{l=0}^{\infty} E_{l,l+1}^{CD}$, $E^{DC} = \bigcup_{l=0}^{\infty} E_{l,l+1}^{DC}$. The labelings of E^{CD}, E^{DC} are defined by

$$\lambda^{CD}(e, f) = (\hat{\lambda}(e), \hat{\lambda}(f)) \in CD, \quad \lambda^{DC}(e', f') = (\hat{\lambda}(e'), \hat{\lambda}(f')) \in DC,$$

for $(e, f) \in E^{CD}, (e', f') \in E^{DC}$. The inclusion maps $\iota_{l,l+1}^{CD} : V_{l+1}^{CD} \to V_l^{CD}$ and $\iota_{l,l+1}^{DC} : V_{l+1}^{DC} \to V_l^{DC}$ are defined by the restrictions of $\hat{\iota}_{2l,2l+1} \circ \hat{\iota}_{2l+1,2l+2}$ to $\widehat{V}_{C,2l+2}^D$ and $\widehat{V}_{D,2l+2}^C$, respectively. Then we have λ-graph systems

$$\mathfrak{L}^{CD} = (V^{CD}, E^{CD}, \lambda^{CD}, \iota^{CD}), \quad \mathfrak{L}^{DC} = (V^{DC}, E^{DC}, \lambda^{DC}, \iota^{DC}),$$

over CD, DC, respectively. It is easy to see that $\mathfrak{L}^{CD}$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{L}^{DC}$ is isomorphic to $\mathfrak{L}'$. As in the construction of the merged λ-graph system, let $(\widehat{\mathfrak{L}})_{(0)}$ be the merged λ-graph system of $\widehat{\mathfrak{L}}$ at level 0 and inductively $(\widehat{\mathfrak{L}})_{(l+1)}$ the merged λ-graph system of $(\widehat{\mathfrak{L}})_{(l)}$ at level l for $l \in \mathbb{Z}_+$. It is straightforward to see that $(\mathfrak{L}^{CD})_{(l)}$ is isomorphic to $[(\widehat{\mathfrak{L}})_{(2l)}]^{CD}$ and $(\mathfrak{L}^{DC})_{(l)}$ is isomorphic to $[(\widehat{\mathfrak{L}})_{(2l)}]^{DC}$. This implies that the merged λ-graph system $\overline{(\widehat{\mathfrak{L}})}$ of $\widehat{\mathfrak{L}}$ gives rise to a bipartite pair between the merged λ-graph systems $\overline{(\mathfrak{L}^{CD})}$ and $\overline{(\mathfrak{L}^{DC})}$ of $\mathfrak{L}^{CD}$ and $\mathfrak{L}^{DC}$, respectively. □

4. *Strong shift equivalence and properly strong shift equivalence*

Properly strong shift equivalence exactly corresponds to a finite sequence of bipartite decompositions of symbolic matrix systems and of λ-graph systems. The definition of properly strong shift equivalence for symbolic matrix systems, however, needs rather more complicated formulations than that of strong shift equivalence for nonnegative matrices. In [**Ma2**], the notion of strong shift equivalence between two symbolic matrix systems has also been introduced. Its formulation is simpler than the formulation of properly strong shift equivalence. It is also a generalization of the notion of strong shift equivalence between nonnegative matrices defined by Williams [**Wi**] and between symbolic matrices defined by Nasu [**N**]. We take crucial algebraic relations in the formulation of properly strong shift equivalence to formulate the strong shift equivalence in the following way. Let $(\mathcal{M}, I)$, $(\mathcal{M}', I')$ be two symbolic matrix systems over Σ, Σ', respectively. Let $m(l)$, $m'(l)$ be the sequences for which $\mathcal{M}_{l,l+1}$, $I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}$, $I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices respectively.

Definition 4. [**Ma2**] $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *strong shift equivalent in one step* if there exist alphabets C, D and specifications $\varphi : \Sigma \to CD$, $\phi : \Sigma' \to DC$ such that for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix $\mathcal{H}_l$ over C and an $m'(l-1) \times m(l)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$I_{l-1,l}\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{H}_l\mathcal{K}_{l+1}, \quad I'_{l-1,l}\mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{K}_l\mathcal{H}_{l+1},$$

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

and

$$\mathcal{H}_l I'_{l,l+1} = I_{l-1,l}\mathcal{H}_{l+1}, \quad \mathcal{K}_l I_{l,l+1} = I'_{l-1,l}\mathcal{K}_{l+1}.$$

We write this situation as $(\mathcal{M}, I) \underset{\text{1-st}}{\approx} (\mathcal{M}', I')$.

$(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *strong shift equivalent* if there exists a finite sequence $(\mathcal{M}^{(1)}, I^{(1)}), \ldots, (\mathcal{M}^{(N-1)}, I^{(N-1)})$ of symbolic matrix systems such that

$$(\mathcal{M}, I) \underset{\text{1-st}}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{\text{1-st}}{\approx} \ldots \underset{\text{1-st}}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{\text{1-st}}{\approx} (\mathcal{M}', I').$$

Let $\mathcal{P}_l, \mathcal{Q}_l, X_l$ and X'_l be the matrices as in the definition of properly strong shift equivalence in one step between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$. By setting

$$\mathcal{H}_l = X_{2(l-1)}\mathcal{P}_{2l-1}, \quad \mathcal{K}_l = X'_{2(l-1)}\mathcal{Q}_{2l-1},$$

we know that they give rise to a strong shift equivalence in one step between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$. Hence properly strong shift equivalence in one step implies strong shift equivalence in one step [**Ma2**, Proposition 4.12].

In this section we prove the converse implication for the symbolic matrix systems whose λ-graph systems are left-resolving and predecessor-separated with single zero-vertices. That is, it will be proved that, for symbolic matrix systems, strong shift equivalence in one step implies properly strong shift equivalence in one step. Hence we will know that strong shift equivalence is equivalent to properly strong shift equivalence in the symbolic matrix systems whose λ-graph systems are left-resolving and predecessor-separated with single zero-vertices.

For a symbolic matrix system $(\mathcal{M}, I) = (\mathcal{M}_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ over Σ, we set

$$\begin{aligned} {\mathcal{M}^{\mathrm{ev}}}_{l,l+1} &= I_{2l,2l+1}\mathcal{M}_{2l+1,2l+2}\ (= \mathcal{M}_{2l,2l+1}I_{2l+1,2l+2}), \\ {I^{\mathrm{ev}}}_{l,l+1} &= I_{2l,2l+1}I_{2l+1,2l+2}, \ \text{for } l \in \mathbb{Z}_+. \end{aligned}$$

It is clear to see that $({\mathcal{M}^{\mathrm{ev}}}_{l,l+1}, {I^{\mathrm{ev}}}_{l,l+1})_{l\in\mathbb{Z}_+}$ is a symbolic matrix system that presents the subshift presented by $(\mathcal{M}, I)$. Put $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}}) = ({\mathcal{M}^{\mathrm{ev}}}_{l,l+1}, {I^{\mathrm{ev}}}_{l,l+1})_{l\in\mathbb{Z}_+}$.

LEMMA 4.1. $(\mathcal{M}, I) \underset{\text{1-st}}{\approx} (\mathcal{M}', I')$ *implies* $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}}) \underset{\text{1-pr}}{\approx} (\mathcal{M}'^{\mathrm{ev}}, I'^{\mathrm{ev}})$.

Proof. Let $\mathcal{H}_l, \mathcal{K}_l, l \in \mathbb{Z}_+$ be the symbolic matrices as in the definition of strong shift equivalence in one step between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$. We put for $l \in \mathbb{Z}_+$

$$\mathcal{P}_l = \mathcal{H}_{l+1}, \quad \mathcal{Q}_l = \mathcal{K}_{l+1}, \quad X_l = I_{l,l+1}, \quad X'_l = I'_{l,l+1}.$$

They satisfy the following equations:

$$\begin{aligned} {\mathcal{M}^{\mathrm{ev}}}_{l,l+1} &\overset{\varphi}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \quad {\mathcal{M}'^{\mathrm{ev}}}_{l,l+1} \overset{\phi}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+1}, \\ {I^{\mathrm{ev}}}_{l,l+1} &= X_{2l}X_{2l+1}, \quad {I'^{\mathrm{ev}}}_{l,l+1} = X'_{2l}X'_{2l+1} \end{aligned}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+1}, \quad X'_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+1}.$$

Hence we have $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}}) \underset{\text{1-pr}}{\approx} (\mathcal{M}'^{\mathrm{ev}}, I'^{\mathrm{ev}})$. □

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Let $\mathfrak{L}^{\mathrm{ev}}$ be the λ-graph system for $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}})$.

LEMMA 4.2. *Let $(\mathcal{M}, I)$ be a symbolic matrix system whose λ-graph system $\mathfrak{L}$ is left-resolving and predecessor-separated with a single zero-vertex. Then the symbolic matrix system $(\overline{\mathcal{M}^{\mathrm{ev}}}, \overline{I^{\mathrm{ev}}})$ of the merged λ-graph system of $\mathfrak{L}^{\mathrm{ev}}$ is isomorphic to $(\mathcal{M}, I)$.*

Proof. For $k \in \mathbb{Z}_+$, let $(\mathfrak{L}^{\mathrm{ev}})_{(k)}$ be the merged λ-graph system of $(\mathfrak{L}^{\mathrm{ev}})_{(k-1)}$ at level k as in the previous section. Denoted by $({\mathcal{M}^{\mathrm{ev}}}_{(k)}, {I^{\mathrm{ev}}}_{(k)})$ its symbolic matrix system. It is immediate to see that

$$
{\mathcal{M}^{\mathrm{ev}}}_{(1),l,l+1} = \begin{cases} \mathcal{M}_{0,1} & (l=0), \\ \mathcal{M}_{1,2}I_{2,3}I_{3,4} & (l=1), \\ {\mathcal{M}^{\mathrm{ev}}}_{l,l+1} & (l \geq 2), \end{cases}
$$

$$
{I^{\mathrm{ev}}}_{(1),l,l+1} = \begin{cases} I_{0,1} & (l=0), \\ I_{1,2}I_{2,3}I_{3,4} & (l=1), \\ {I^{\mathrm{ev}}}_{l,l+1} & (l \geq 2). \end{cases}
$$

Generally it follows that

$$
{\mathcal{M}^{\mathrm{ev}}}_{(k),l,l+1} = \begin{cases} \mathcal{M}_{l,l+1} & (l=0,1,\ldots,k-1), \\ \mathcal{M}_{k,k+1}I_{k+1,k+2}I_{k+2,k+3}\cdots I_{2k+1,2k+2} & (l=k), \\ {\mathcal{M}^{\mathrm{ev}}}_{l,l+1} & (l \geq k+1), \end{cases}
$$

$$
{I^{\mathrm{ev}}}_{(k),l,l+1} = \begin{cases} I_{l,l+1} & (l=0,1,\ldots,k-1), \\ I_{k,k+1}I_{k+1,k+2}I_{k+2,k+3}\cdots I_{2k+1,2k+2} & (l=k), \\ {I^{\mathrm{ev}}}_{l,l+1} & (l \geq k+1). \end{cases}
$$

Since we have, from Lemma 3.2, $\overline{\mathcal{M}^{\mathrm{ev}}}_{l,l+1} = {\mathcal{M}^{\mathrm{ev}}}_{(k),l,l+1}$, $\overline{I^{\mathrm{ev}}}_{l,l+1} = {I^{\mathrm{ev}}}_{(k),l,l+1}$ for $k > l$, we obtain $\overline{\mathcal{M}^{\mathrm{ev}}}_{l,l+1} = \mathcal{M}_{l,l+1}$, $\overline{I^{\mathrm{ev}}}_{l,l+1} = I_{l,l+1}$ for $l \in \mathbb{Z}_+$. □

We thus have the following.

THEOREM 4.3. *In the symbolic matrix systems whose λ-graph systems are left-resolving and predecessor-separated with single zero-vertices, strong shift equivalence in one step implies properly strong shift equivalence in one step. Hence strong shift equivalence is equivalent to properly strong shift equivalence in the symbolic matrix systems whose λ-graph systems are left-resolving and predecessor-separated with single zero-vertices.*

Proof. Let $(\mathcal{M}, I), (\mathcal{M}', I')$ be symbolic matrix systems whose λ-graph systems are predecessor-separated with single zero-vertices. Suppose that they are strong shift equivalent in one step. By Lemma 4.1 and Proposition 3.3, we have $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}}) \underset{\text{1-pr}}{\approx} (\mathcal{M}'^{\mathrm{ev}}, I'^{\mathrm{ev}})$ and hence $(\overline{\mathcal{M}^{\mathrm{ev}}}, \overline{I^{\mathrm{ev}}}) \underset{\text{1-pr}}{\approx} (\overline{\mathcal{M}'^{\mathrm{ev}}}, \overline{I'^{\mathrm{ev}}})$. From Lemma 4.2, we conclude that $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}', I')$. □

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Remark. For $(\mathcal{M}, I) = (\mathcal{M}_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ over Σ, we set

$$\mathcal{M}^{\mathrm{od}}{}_{l,l+1} = I_{2l+1,2l+2}\mathcal{M}_{2l+2,2l+3}\ (= \mathcal{M}_{2l+1,2l+2}I_{2l+2,2l+3}),$$
$$I^{\mathrm{od}}{}_{l,l+1} = I_{2l+1,2l+2}I_{2l+2,2l+3} \quad \text{for } l \in \mathbb{Z}_+.$$

Similarly to $(\mathcal{M}^{\mathrm{ev}}{}_{l,l+1}, I^{\mathrm{ev}}{}_{l,l+1})_{l\in\mathbb{Z}_+}$, we see that $(\mathcal{M}^{\mathrm{od}}{}_{l,l+1}, I^{\mathrm{od}}{}_{l,l+1})_{l\in\mathbb{Z}_+}$ is a symbolic matrix system. We denote it by $(\mathcal{M}^{\mathrm{od}}, I^{\mathrm{od}})$. We then know that $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}})$ and $(\mathcal{M}^{\mathrm{od}}, I^{\mathrm{od}})$ are properly strong shift equivalent in one step, as follows. Let E_n be the $n \times n$ identity matrix. For $l \in \mathbb{Z}_+$ put

$$\mathcal{P}_{2l} = I_{2l,2l+1}, \quad \mathcal{P}_{2l+1} = I_{2l+2,2l+3}, \quad \mathcal{Q}_{2l} = \mathcal{Q}_{2l+1} = \mathcal{M}_{2l+1,2l+2},$$
$$X_{2l} = I_{2l,2l+1}I_{2l+1,2l+2}, \quad X_{2l+1} = E_{m(2l+2)},$$
$$X'_{2l} = E_{m(2l+1)}, \quad X'_{2l+1} = I_{2l+1,2l+2}I_{2l+2,2l+3}.$$

They satisfy the following equations:

$$\mathcal{M}^{\mathrm{ev}}{}_{l,l+1} = \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \quad \mathcal{M}^{\mathrm{od}}{}_{l,l+1} = \mathcal{Q}_{2l}\mathcal{P}_{2l+1},$$
$$I^{\mathrm{ev}}{}_{l,l+1} = X_{2l}X_{2l+1}, \quad I^{\mathrm{od}}{}_{l,l+1} = X'_{2l}X'_{2l+1},$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+1}, \quad X'_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+1}.$$

This shows that $(\mathcal{M}^{\mathrm{ev}}, I^{\mathrm{ev}}) \underset{\text{1-pr}}{\approx} (\mathcal{M}^{\mathrm{od}}, I^{\mathrm{od}})$. Also, if $(\mathcal{M}, I) \underset{\text{1-st}}{\approx} (\mathcal{M}', I')$, then we have $(\mathcal{M}^{\mathrm{od}}, I^{\mathrm{od}}) \underset{\text{1-pr}}{\approx} (\mathcal{M}'^{\mathrm{od}}, I'^{\mathrm{od}})$ by putting

$$\mathcal{P}_l = \mathcal{H}_{l+2}, \quad \mathcal{Q}_l = \mathcal{K}_{l+2}, \quad X_l = I_{l+1,l+2}, \quad X'_l = I'_{l+1,l+2},$$

where $\mathcal{H}_{l+2}, \mathcal{K}_{l+2}$ are symbolic matrices as in the definition of strong shift equivalence $(\mathcal{M}, I) \underset{\text{1-st}}{\approx} (\mathcal{M}', I')$ in one step.

One may have a similar relationship between properly shift equivalence and shift equivalence of symbolic matrix systems to the relationship between properly strong shift equivalence and strong shift equivalence in a parallel way. Shift equivalence of a symbolic matrix system has been defined in [**Ma2**, p. 309]. We will give here a definition of properly shift equivalence between symbolic matrix systems in the following way. It is slightly different from the definition given in [**Ma2**, p. 312]. The following formulation is more reasonable in our context than the definition given in [**Ma2**, p. 312].

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ', respectively. For $N \in \mathbb{N}$, we denote by $(\Sigma)^N = \Sigma\cdots\Sigma$, $(\Sigma')^N = \Sigma'\cdots\Sigma'$: N-times products.

Definition 5. If there exist alphabets C_N, D_N, specifications $\psi_1 : \Sigma C_N \to C_N\Sigma'$, $\psi_2 : \Sigma' D_N \to D_N\Sigma$ and $\varphi_1 : (\Sigma)^N \to C_N D_N$, $\varphi_2 : (\Sigma')^N \to D_N C_N$, and, for $l \in \mathbb{Z}_+$, an $n(l) \times n'(l+N)$ symbolic matrix $\mathcal{P}_l$ over C_N, an $n'(l) \times n(l+N)$ symbolic matrix $\mathcal{Q}_l$ over D_N, an $n(l) \times n(l+1)$ matrix X_l over $\{0, 1\}$ and an $n'(l) \times n'(l+1)$ matrix

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

X'_l over $\{0, 1\}$ satisfying the following equations,

$$\mathcal{M}_{l,l+N} \overset{\varphi_1}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+N}, \quad \mathcal{M}'_{l,l+N} \overset{\varphi_2}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+N},$$

$$\mathcal{M}_{l,l+1}\mathcal{P}_{2l+2}X'_{2l+N+2,2l+2N+2} \overset{\psi_1}{\simeq} \mathcal{P}_{2l}X'_{2l+N,2l+2N}\mathcal{M}'_{l+N,l+N+1},$$

$$\mathcal{M}'_{l,l+1}\mathcal{Q}_{2l+2}X_{2l+N+2,2l+2N+2} \overset{\psi_2}{\simeq} \mathcal{Q}_{2l}X_{2l+N,2l+2N}\mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+1} = X_{2l}X_{2l+1}, \quad I'_{l,l+1} = X'_{2l}X'_{2l+1}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+N}, \quad X'_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+N},$$

where $X_{l,l+k} = X_l \cdots X_{l+k-1}$, $X'_{l,l+k} = X'_l \cdots X'_{l+k-1}$ and $n(2l) = m(l)$, $n'(2l) = m'(l)$, then $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly shift equivalent of lag N*.

Shift equivalence for symbolic matrices was formulated by Boyle and Krieger [**BK**]. The above formulation is a generalization of the Boyle–Krieger formulation for symbolic matrices. Properly strong shift equivalence implies properly shift equivalence. Properly shift equivalence of lag 1 coincides with properly strong shift equivalence in one step. It is easy to see that properly shift equivalence of lag N implies shift equivalence of lag N in the sense of [**Ma2**, p. 309]. The converse implication holds for symbolic matrix systems whose λ-graph systems are left-resolving and predecessor-separated with single zero-vertices in a similar way to the preceding discussions for strong shift equivalences.

5. *State splitting*

Two subshifts are topologically conjugate if and only if their canonical symbolic matrix systems are (properly) strong shift equivalent. Hence their canonical λ-graph systems are connected by a finite sequence of bipartitely related λ-graph systems. If the two subshifts are topological Markov shifts that are defined by finite directed graphs, one directed graph is obtained by a finite procedure of state splittings and state amalgamations from the other directed graph. State splitting of a finite directed graph is a procedure for constructing new graphs from a given graph. For a given partition of edges, whose sources (or terminals) are the same vertex, each vertex is split into a number of derived vertices. The converse procedure is called state amalgamation. We will generalize these constructions for finite-directed graphs to λ-graph systems.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. Let $\mathcal{P}$ be a partition of Σ. We denote by $\{v_1^l, v_2^l, \ldots, v_{m(l)}^l\}$ the vertex set V_l at level $l \in \mathbb{Z}_+$ of $\mathfrak{L}$. For a vertex $v_i^l \in V_l$, let

$$\mathcal{E}_{l,i} = \{e \in E_{l,l+1} \mid s(e) = v_i^l\}$$

be the set of all edges whose source is v_i^l. We identify the partition $\mathcal{P}$ of Σ with the equivalence classes of Σ induced by $\mathcal{P}$. For a symbol $\alpha \in \Sigma$, we denote by $[\alpha] \in \mathcal{P}$ its equivalence class. Let

$$\mathcal{P}_{l,i} = \{[\alpha] \in \mathcal{P} \mid \alpha \in \lambda(\mathcal{E}_{l,i})\}$$

be the set of all equivalence classes of symbols starting at the vertex v_i^l, where $\lambda(\mathcal{E}_{l,i}) = \{\lambda(e) \in \Sigma \mid e \in \mathcal{E}_{l,i}\}$. For $p \in \mathcal{P}$, let

$$\mathcal{E}_{l,i}^p = \{e \in \mathcal{E}_{l,i} \mid \lambda(e) \in p\}$$

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

be the set of all edges whose sources are v_i^l and labels belong to p. Hence we have a disjoint union $\mathcal{E}_{l,i} = \bigcup_{p\in\mathcal{P}} \mathcal{E}_{l,i}^p$ We will construct a new λ-graph system $\mathfrak{L}^{[\mathcal{P}]} = (V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ over an alphabet $\Sigma^{[\mathcal{P}]}$.

(i) Alphabet $\Sigma^{[\mathcal{P}]}$. Set $\Sigma^{[\mathcal{P}]} = \Sigma \times \mathcal{P}$.

(ii) Vertices $V^{[\mathcal{P}]}$. Put $v_{i,p}^l = (v_i^l, p)$ for $p \in \mathcal{P}_{l,i}$, and set for $l \in \mathbb{Z}_+$

$$V_l^{[\mathcal{P}]} = \{v_{i,p}^l \mid p \in \mathcal{P}_{l,i}, i = 1, 2, \ldots, m(l)\}.$$

That is, the new vertex set at level l consists of all equivalence classes $p \in \mathcal{P}_{l,i}$ of symbols starting at the original vertices v_i^l, $i = 1, 2, \ldots, m(l)$.

(iii) Edges $E^{[\mathcal{P}]}$. For an edge $e \in \mathcal{E}_{l,i}^p$, take $j(e) = 1, 2, \ldots, m(l+1)$ such that $t(e) = v_{j(e)}^{l+1}$. For any $q \in \mathcal{P}_{l+1,j(e)}$, define a new edge e^q by

$$s(e^q) = v_{i,p}^l \in V_l^{[\mathcal{P}]}, \quad t(e^q) = v_{j(e),q}^{l+1} \in V_{l+1}^{[\mathcal{P}]}.$$

Let the new edge set be

$$E_{l,l+1}^{[\mathcal{P}]} = \{e^q \mid e \in E_{l,l+1}, q \in \mathcal{P}_{l+1,j(e)}\}.$$

(iv) Labeling $\lambda^{[\mathcal{P}]} : E^{[\mathcal{P}]} \to \Sigma^{[\mathcal{P}]}$. Define the labeling $\lambda^{[\mathcal{P}]}$ by setting

$$\lambda^{[\mathcal{P}]}(e^q) = \lambda(e) \times q \in \Sigma \times \mathcal{P} = \Sigma^{[\mathcal{P}]} \quad \text{for } e^q \in E^{[\mathcal{P}]}.$$

(v) Inclusion $\iota^{[\mathcal{P}]} : V_{l+1}^{[\mathcal{P}]} \to V_l^{[\mathcal{P}]}$, $l \in \mathbb{Z}_+$. Define $\iota^{[\mathcal{P}]}$ by setting

$$\iota^{[\mathcal{P}]}(v_{j,q}^{l+1}) = v_{i,q}^l \quad \text{for } v_{j,q}^{l+1} \in V_{l+1}^{[\mathcal{P}]},$$

where $\iota(v_j^{l+1}) = v_i^l$. By the local property of $\mathfrak{L}$, $\mathcal{P}_{l+1,j}$ is regarded as a subset of $\mathcal{P}_{l,i}$ when $\iota(v_j^{l+1}) = v_i^l$, so that $v_{i,q}^l$ belongs to $V_l^{[\mathcal{P}]}$ for $v_{j,q}^{l+1} \in V_{l+1}^{[\mathcal{P}]}$.

LEMMA 5.1. *$(V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ is a λ-graph system over $\Sigma^{[\mathcal{P}]}$.*

Proof. For $v_{i,p}^l \in V_l^{[\mathcal{P}]}$, $v_{j,p}^{l+1} \in V_{l+1}^{[\mathcal{P}]}$, $v_{k,q}^{l+2} \in V_{l+2}^{[\mathcal{P}]}$, let $E_{l,l+1}^{[\mathcal{P}]}(v_{i,p}^l, \iota^{[\mathcal{P}]}(v_{k,q}^{l+2}))$ and $E_{l+1,l+2}^{[\mathcal{P}]}(v_{j,p}^{l+1}, v_{k,q}^{l+2})$ be the set of all edges in $E_{l,l+1}^{[\mathcal{P}]}$ whose sources are the vertex $v_{i,p}^l$ and terminals are the vertex $\iota^{[\mathcal{P}]}(v_{k,q}^{l+2})$, and the set of all edges in $E_{l+1,l+2}^{[\mathcal{P}]}$ whose sources are the vertex $v_{j,p}^{l+1}$ and terminals are the vertex $v_{k,q}^{l+2}$, respectively. There is a bijective correspondence between $E_{l,l+1}^{[\mathcal{P}]}(v_{i,p}^l, \iota^{[\mathcal{P}]}(v_{k,q}^{l+2}))$ and $E_{l,l+1}(v_i^l, \iota(v_k^{l+2}))_p$, the set of all edges with label p in $E_{l,l+1}$ whose sources are v_i^l and terminals are $\iota(v_k^{l+2})$. There is also a bijective correspondence between $E_{l+1,l+2}^{[\mathcal{P}]}(v_{j,p}^{l+1}, v_{k,q}^{l+2})$ and $E_{l+1,l+2}(v_j^{l+1}, v_k^{l+2})_p$. Hence we know that the sets

$$\bigcup_{\substack{1\le j\le m(l+1)\\ v_{i,p}^l = \iota^{[\mathcal{P}]}(v_{j,p}^{l+1})}} E_{l+1,l+2}^{[\mathcal{P}]}(v_{j,p}^{l+1}, v_{k,q}^{l+2}) \quad \text{and} \quad \bigcup_{\substack{1\le j\le m(l+1)\\ v_i^l = \iota(v_j^{l+1})}} E_{l+1,l+2}(v_j^{l+1}, v_k^{l+2})_p$$

bijectively correspond. By the local property of λ-graph system, the sets

$$E_{l,l+1}(v_i^l, \iota(v_k^{l+2}))_p \quad \text{and} \quad \bigcup_{\substack{1\le j\le m(l+1)\\ v_i^l = \iota(v_j^{l+1})}} E_{l+1,l+2}(v_j^{l+1}, v_k^{l+2})_p$$

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press

bijectively correspond. Therefore, we have a bijective correspondence between

$$E^{[\mathcal{P}]}_{l,l+1}(v^l_{i,p}, \iota^{[\mathcal{P}]}(v^{l+2}_{k,q})) \quad \text{and} \quad \bigcup_{\substack{1\le j\le m(l+1)\\ v^l_{i,p}=\iota^{[\mathcal{P}]}(v^{l+1}_{j,p})}} E^{[\mathcal{P}]}_{l+1,l+2}(v^{l+1}_{j,p}, v^{l+2}_{k,q}).$$

Hence $(V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ is a λ-graph system. □

Definition 6. We call $(V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ the *out-split λ-graph system formed from* $\mathfrak{L} = (V, E, \lambda, \iota)$ *using a partition* $\mathcal{P}$ *of* Σ. We write it as $\mathfrak{L}^{[\mathcal{P}]}$.

We note that if $\mathfrak{L}$ is left-resolving, so is $\mathfrak{L}^{[\mathcal{P}]}$. Let $(\mathcal{M}, I)$ be the symbolic matrix system for $\mathfrak{L}$. We denote by $(\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$ the symbolic matrix system for $\mathfrak{L}^{[\mathcal{P}]}$.

PROPOSITION 5.2. $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$.

Proof. We will construct a bipartite λ-graph system $(\widetilde{V}, \widetilde{E}, \tilde{\lambda}, \tilde{\iota})$ whose symbolic matrix system gives rise to a properly strong shift equivalence between $(\mathcal{M}, I)$ and $(\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$.

(i) Alphabet $\widetilde{\Sigma}$. Put $C = \Sigma$ and $D = \mathcal{P}$. Set $\widetilde{\Sigma} = C \cup D$, a disjoint union.

(ii) Vertices $\widetilde{V}$. Put, for $k \in \mathbb{Z}_+$,

$$\widetilde{V}^C_l = V_{k+1}, \quad (l = 2k, 2k+1) \quad \text{and} \quad \widetilde{V}^D_l = \begin{cases} V^{[\mathcal{P}]}_k, & (l = 2k), \\ V^{[\mathcal{P}]}_{k+1}, & (l = 2k+1). \end{cases}$$

Set $\widetilde{V}_l = \widetilde{V}^C_l \cup \widetilde{V}^D_l$, a disjoint union for $l \in \mathbb{Z}_+$.

(ii) Edges $\widetilde{E}$. For $p \in \mathcal{P}_{l,i}$, define an edge whose source is v^l_i and terminal is $v^l_{i,p}$. We denote it by $e^l_{i,p}$. We write this as $\tilde{s}(e^l_{i,p}) = v^l_i, \tilde{t}(e^l_{i,p}) = v^l_{i,p}$. Put

$$\mathcal{E}^l_{\mathcal{P}} = \{e^l_{i,p} \mid p \in \mathcal{P}_{l,i}, i = 1, 2, \ldots, m(l)\}.$$

For $e \in \mathcal{E}^{[\mathcal{P}]}_{l,i}$ with $t(e) = v^{l+1}_j$, define an edge $\tilde{e}$ whose source is $v^l_{i,p}$ and whose terminal is v^{l+1}_j. We write this as $\tilde{s}(\tilde{e}) = v^l_{i,p}, \tilde{t}(\tilde{e}) = v^{l+1}_j$. Put

$$\mathcal{E}^{\mathcal{P}}_{l,l+1} = \{\tilde{e} \mid e \in E_{l,l+1}\}.$$

Set, for $l \in \mathbb{Z}_+$,

$$\widetilde{E}_{l,l+1} = \begin{cases} \mathcal{E}^{\mathcal{P}}_{k,k+1} \cup \mathcal{E}^{k+1}_{\mathcal{P}}, & (l = 2k), \\ \mathcal{E}^{k+1}_{\mathcal{P}} \cup \mathcal{E}^{\mathcal{P}}_{k+1,k+2}, & (l = 2k+1). \end{cases}$$

(iii) Labeling $\tilde{\lambda} : \widetilde{E} \to \widetilde{\Sigma}$. Put for $\tilde{e} \in \widetilde{E}$,

$$\tilde{\lambda}(\tilde{e}) = \begin{cases} p \in \mathcal{P} = D, & (e = e^l_{i,p} \in \mathcal{E}^l_{\mathcal{P}}), \\ \lambda(e) \in C, & (e \in \mathcal{E}^{\mathcal{P}}_{l,l+1}). \end{cases}$$

Hence, for $\tilde{e} \in \widetilde{E}_{l,l+1}$, we have

$$\tilde{s}(\tilde{e}) \in \widetilde{V}^D_l, \quad \tilde{t}(\tilde{e}) \in \widetilde{V}^C_{l+1} \quad \text{if and only if} \quad \tilde{\lambda}(\tilde{e}) \in C,$$
$$\tilde{s}(\tilde{e}) \in \widetilde{V}^C_l, \quad \tilde{t}(\tilde{e}) \in \widetilde{V}^D_{l+1} \quad \text{if and only if} \quad \tilde{\lambda}(\tilde{e}) \in D.$$

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(iv) Inclusion $\tilde{\iota} : \widetilde{V}_{l+1} \to \widetilde{V}_l$. For $l = 2k$, we have

$$\widetilde{V}_{l+1} = \widetilde{V}^C_{2k+1} \cup \widetilde{V}^D_{2k+1} = V_{k+1} \cup V^{[\mathcal{P}]}_{k+1}, \quad \widetilde{V}_l = \widetilde{V}^C_{2k} \cup \widetilde{V}^D_{2k} = V_{k+1} \cup V^{[\mathcal{P}]}_k.$$

Define $\tilde{\iota} : \widetilde{V}_{2k+1} \to \widetilde{V}_{2k}$ by

$$\tilde{\iota}(v) = \begin{cases} v & \text{if } v \in V_{k+1}, \\ \iota^{[\mathcal{P}]}(v) & \text{if } v \in V^{[\mathcal{P}]}_{k+1}, \end{cases}$$

and $\tilde{\iota} : \widetilde{V}_{2k} \to \widetilde{V}_{2k-1}$ by

$$\tilde{\iota}(v) = \begin{cases} \iota(v) & \text{if } v \in V_{k+1}, \\ v & \text{if } v \in V^{[\mathcal{P}]}_k. \end{cases}$$

Hence we have

$$\tilde{\iota}(\widetilde{V}^D_{l+1}) = \widetilde{V}^D_l, \quad \tilde{\iota}(\widetilde{V}^C_{l+1}) = \widetilde{V}^C_l \quad \text{for } l \in \mathbb{Z}_+.$$

LEMMA 5.3. *$(\widetilde{V}, \widetilde{E}, \tilde{\lambda}, \tilde{\iota})$ is a λ-graph system over alphabet $\widetilde{\Sigma}$.*

Proof. For $u \in \widetilde{V}_l$, $w \in \widetilde{V}_{l+2}$, We have two cases.

Case 1: $l = 2k - 1$. We have four subcases:

(i) $u \in V_k$, $w \in V^{[\mathcal{P}]}_{k+1}$;
(ii) $u \in V_k$, $w \in V_{k+1}$;
(iii) $u \in V^{[\mathcal{P}]}_k$, $w \in V^{[\mathcal{P}]}_{k+1}$;
(iv) $u \in V^{[\mathcal{P}]}_k$, $w \in V_{k+1}$.

Case 2: $l = 2k$. We have four subcases:

(v) $u \in V^{[\mathcal{P}]}_k$, $w \in V_{k+2}$;
(vi) $u \in V^{[\mathcal{P}]}_k$, $w \in V^{[\mathcal{P}]}_{k+1}$;
(vii) $u \in V_k$, $w \in V_{k+1}$;
(viii) $u \in V_k$, $w \in V^{[\mathcal{P}]}_{k+1}$.

For cases (ii), (iii), (vi) and (vii) we see that $\widetilde{E}_{\tilde{\iota}}(u, w) = \emptyset = \widetilde{E}^{\tilde{\iota}}(u, w)$.

For case (i) we put $v = v^k_i$, $w = v^{k+1}_{j,q}$ for $q \in \mathcal{P}_{k+1,j}$. It then follows that $\widetilde{E}^{\tilde{\iota}}(u, w) \neq \emptyset$ if and only if $\iota(v^{k+1}_j) = v^k_i$, and $\widetilde{E}_{\tilde{\iota}}(u, w) \neq \emptyset$ if and only if $\iota^{[\mathcal{P}]}(v^{k+1}_{j,q}) = v^k_{i,q}$. Since the condition $\iota(v^{k+1}_j) = v^k_i$ is equivalent to the condition $\iota^{[\mathcal{P}]}(v^{k+1}_{j,q}) = v^k_{i,q}$, we have that $\widetilde{E}^{\tilde{\iota}}(u, w) \neq \emptyset$ is equivalent to $\widetilde{E}_{\tilde{\iota}}(u, w) \neq \emptyset$. In this case, each of the sets $\widetilde{E}^{\tilde{\iota}}(u, w)$ and $\widetilde{E}_{\tilde{\iota}}(u, w)$ consists of one edge with the same label.

For case (iv), we put $u = v^k_{i,p}$ for $q \in \mathcal{P}_{k,i}$ and $w = v^{k+1}_j$. It is clear that

$$\widetilde{E}^{\tilde{\iota}}(u, w) = \{e \in E_{k,k+1} \mid s(e) = v^k_i, t(e) = v^{k+1}_j, \lambda(e) = p\} = \widetilde{E}_{\tilde{\iota}}(u, w).$$

For case (v), we put $u = v^k_{i,p}$ for $q \in \mathcal{P}_{k,i}$ and $w = v^{k+2}_j$. It follows that

$$\widetilde{E}^{\tilde{\iota}}(u, w) = \{\tilde{e} \in \mathcal{E}^{\mathcal{P}}_{k+1,k+2} \mid \iota(s(e)) = v^k_i, t(e) = v^{k+2}_j, \lambda(e) = p\},$$
$$\widetilde{E}_{\tilde{\iota}}(u, w) = \{\tilde{e} \in \mathcal{E}^{\mathcal{P}}_{k,k+1} \mid s(e) = v^k_i, t(e) = \iota(v^{k+2}_j), \lambda(e) = p\}.$$

By the local property of the λ-graph system $\mathfrak{L}$, $\widetilde{E}^{\tilde{\iota}}(u, w)$ bijectively corresponds to $\widetilde{E}_{\tilde{\iota}}(u, w)$.

For case (viii), the equality $\widetilde{E}^{\tilde{\iota}}(u, w) = \widetilde{E}_{\tilde{\iota}}(u, w)$ is easy.

Therefore $(\widetilde{V}, \widetilde{E}, \tilde{\lambda}, \tilde{\iota})$ satisfies the local property of λ-graph system and hence it is a λ-graph system. □

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Let $(\widetilde{\mathcal{M}}, \widetilde{I})$ be the symbolic matrix system for the λ-graph system $(\widetilde{V}, \widetilde{E}, \tilde{\lambda}, \tilde{\iota})$. It is straightforward to see that $(\widetilde{\mathcal{M}}, \widetilde{I})$ gives rise to a properly strong shift equivalence in one step between $(\mathcal{M}, I)$ and $(\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$. Hence $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$. Thus we end the proof of Proposition 5.2. □

For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, take a partition $\mathcal{P}$ as Σ, that is, $\mathcal{P}$ partitions Σ into the singleton sets of individual symbols. Then the out-split λ-graph system

$$\mathfrak{L}^{[\Sigma]} = (V^{[\Sigma]}, E^{[\Sigma]}, \lambda^{[\Sigma]}, \iota^{[\Sigma]})$$

over the alphabet $\Sigma \times \Sigma$ is called the *complete out-split* λ-graph system of $\mathfrak{L}$.

We will next construct in-split λ-graph system that is symmetrically defined to the out-split of the λ-graph system, by inverting sources and terminals of edges.

Let $\mathcal{P}$ be a partition of Σ. For $l \in \mathbb{N}$ and $v_i^l \in V_l$, let

$$\mathcal{E}^{l,i} = \{e \in E_{l-1,l} \mid t(e) = v_i^l\}$$

be the set of all edges whose terminals are v_i^l. Let

$$\mathcal{P}^{l,i} = \{[\alpha] \in \mathcal{P} \mid \alpha \in \lambda(\mathcal{E}^{l,i})\}$$

be the set of all equivalence classes of symbols terminating at the vertex v_i^l, where $\lambda(\mathcal{E}^{l,i}) = \{\lambda(e) \in \Sigma \mid e \in \mathcal{E}^{l,i}\}$. Let

$$\mathcal{E}_p^{l,i} = \{e \in \mathcal{E}^{l,i} \mid \lambda(e) \in p\}$$

be the set of all edges whose terminals are v_i^l and labels belong to p. Hence we have a disjoint union $\mathcal{E}^{l,i} = \bigcup_{p \in \mathcal{P}} \mathcal{E}_p^{l,i}$. We will construct a new λ-graph system $\mathfrak{L}_{[\mathcal{P}]} = (V_{[\mathcal{P}]}, E_{[\mathcal{P}]}, \lambda_{[\mathcal{P}]}, \iota_{[\mathcal{P}]})$ over an alphabet $\Sigma_{[\mathcal{P}]}$.

(i) Alphabet $\Sigma_{[\mathcal{P}]}$. Set $\Sigma_{[\mathcal{P}]} = \mathcal{P} \times \Sigma$.

(ii) Vertices $V_{[\mathcal{P}]}$. Put $v_i^{l,p} = (v_i^l, p)$ for $p \in \mathcal{P}^{l,i}$, and set, for $l \in \mathbb{Z}_+$,

$$V_{[\mathcal{P}],l-1} = \{v_i^{l,p} \mid p \in \mathcal{P}^{l,i}, i = 1, 2, \ldots, m(l)\}.$$

That is, the new vertex set at level l consists of all equivalence classes $p \in \mathcal{P}^{l,i}$ of symbols terminating at the original vertices v_i^l, $i = 1, 2, \ldots, m(l)$.

(iii) Edges $E_{[\mathcal{P}]}$. For an edge $e \in \mathcal{E}_p^{l+1,j}$, take $h(e) = 1, 2, \ldots, m(l)$ such that $s(e) = v_{h(e)}^l$. For any $q \in \mathcal{P}^{l,h(e)}$, define a new edge e_q by

$$s(e_q) = v_{h(e)}^{l,q} \in V_{[\mathcal{P}],l-1}, \quad t(e_q) = v_j^{l+1,p} \in V_{[\mathcal{P}],l}.$$

We let the new edge set be

$$E_{[\mathcal{P}],l-1,l} = \{e_q \mid e \in E_{l,l+1}, q \in \mathcal{P}^{l,h(e)}\}.$$

(iv) Labeling $\lambda_{[\mathcal{P}]} : E_{[\mathcal{P}]} \to \Sigma_{[\mathcal{P}]}$: Define the labeling $\lambda_{[\mathcal{P}]}$ by setting

$$\lambda_{[\mathcal{P}]}(e_q) = q \times \lambda(e) \in \mathcal{P} \times \Sigma = \Sigma_{[\mathcal{P}]} \quad \text{for } e_q \in E_{[\mathcal{P}]}.$$

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(v) Inclusion $\iota_{[\mathcal{P}]} : V_{[\mathcal{P}],l+1} \to V_{[\mathcal{P}],l}$, $l \in \mathbb{Z}_+$. Define $\iota_{[\mathcal{P}]}$ by setting

$$\iota_{[\mathcal{P}]}(v_k^{l+2,p}) = v_{i_k}^{l+1,p} \quad \text{for } v_k^{l+2,p} \in V_{[\mathcal{P}],l+1},$$

where $\iota(v_k^{l+2}) = v_{i_k}^{l+1}$. By the local property of $v_{i_k}^{l+1,p}$, $\mathcal{P}^{l+2,k}$ is regarded as a subset of $\mathcal{P}^{l+1,i_k}$ when $\iota(v_k^{l+2}) = v_{i_k}^{l+1}$ so that $v_{i_k}^{l+1,p}$ belongs to $V_{[\mathcal{P}],l}$ for $v_k^{l+2,p} \in V_{[\mathcal{P}],l+1}$.

Similarly to Lemma 5.1, we have the following.

LEMMA 5.4. *$(V_{[\mathcal{P}]}, E_{[\mathcal{P}]}, \lambda_{[\mathcal{P}]}, \iota_{[\mathcal{P}]})$ is a λ-graph system over $\Sigma_{[\mathcal{P}]}$.*

Definition 7. We call $(V_{[\mathcal{P}]}, E_{[\mathcal{P}]}, \lambda_{[\mathcal{P}]}, \iota_{[\mathcal{P}]})$ the *in-split λ-graph system formed from $\mathfrak{L} = (V, E, \lambda, \iota)$ using a partition $\mathcal{P}$ of Σ*. We write it as $\mathfrak{L}_{[\mathcal{P}]}$.

We note that if $\mathfrak{L}$ is left-resolving, so is $\mathfrak{L}_{[\mathcal{P}]}$. Let $(\mathcal{M}, I)$ be the symbolic matrix system for $\mathfrak{L}$. We denote by $(\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$ the symbolic matrix system for $\mathfrak{L}_{[\mathcal{P}]}$.

Similarly to Proposition 5.2, we have the following.

PROPOSITION 5.5. $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$.

We similarly define the *complete in-split* λ-graph system $\mathfrak{L}_{[\Sigma]}$ of $\mathfrak{L}$ by taking the partition $\mathcal{P}$ as Σ to be the in-split λ-graph system

$$\mathfrak{L}_{[\Sigma]} = (V_{[\Sigma]}, E_{[\Sigma]}, \lambda_{[\Sigma]}, \iota_{[\Sigma]})$$

over the alphabet $\Sigma \times \Sigma$.

We will briefly study the converse operation to splitting, called amalgamation. Let $\mathfrak{L}' = (V', E', \lambda', \iota')$ be a λ-graph system over Σ'. Let $\varphi : \Sigma' \to \Sigma \times \mathcal{P}$ be a specification from Σ' to the product of a finite set Σ and a partition $\mathcal{P}$ of Σ. We denote it by $\varphi(\alpha) = [\varphi_\Sigma(\alpha), \varphi_\mathcal{P}(\alpha)] \in \Sigma \times \mathcal{P}$ for $\alpha \in \Sigma'$. Suppose that the specification φ satisfies the following condition: for $e \in E'_{l-1,l}$, $f \in E'_{l,l+1}$,

$$t(e) = s(f) \quad \text{if and only if} \quad \varphi_\mathcal{P}(\lambda'(e)) = [\varphi_\Sigma(\lambda'(f))]_\mathcal{P},$$

where $[*]_\mathcal{P}$ denotes the equivalence class of Σ by the partition $\mathcal{P}$. Hence, for a vertex $v \in V'_l$, the partition element $\varphi_\mathcal{P}[\lambda'(e)] \in \mathcal{P}$ for the incoming edges e to v is unique. We denote it by $p(v) \in \mathcal{P}$. We now assume that $\mathfrak{L}'$ is left-resolving. For $v, u \in V'_{l+1}$, we define $v \sim u$ if $p(v) \neq p(u)$ and for $w \in V'_l$, $\alpha \in \Sigma$ there exists an edge $e \in E'_{l,l+1}$ such that $s(e) = w, t(e) = v, \varphi_\Sigma(\lambda'(e)) = \alpha$ if and only if there exists an edge $f \in E'_{l,l+1}$ such that $s(f) = w, t(f) = u, \varphi_\Sigma(\lambda'(f)) = \alpha$. We note that $v \sim u$ implies $\iota'(v) \sim \iota'(u)$. For $e, f \in E'_{l,l+1}$, we define $e \sim f$ if $s(e) = s(f), t(e) \sim t(f), \varphi_\Sigma(\lambda'(e)) = \varphi_\Sigma(\lambda'(f))$. We set, for $l \in \mathbb{Z}_+$, $V_l = V'_l/\sim, E_{l,l+1} = E'_{l,l+1}/\sim$, $\lambda_\Sigma([e]) = \varphi_\Sigma(\lambda'(e))$ for $[e] \in E_{l,l+1}$ and $\iota([v]) = [\iota'(v)]$ for $[v] \in V_l$. It is direct to see that $(V, E, \lambda_\Sigma, \iota)$ defines a λ-graph system $\mathfrak{L}$ over Σ. The resulting λ-graph system $\mathfrak{L}$ over Σ is called the *out-amalgamation of $\mathfrak{L}'$ using a specification $\varphi : \Sigma' \to \Sigma \times \mathcal{P}$*. Then the out-splitting $\mathfrak{L}^{[\mathcal{P}]}$ of $\mathfrak{L}$ by the partition $\mathcal{P}$ of Σ is $\mathfrak{L}'$. We may similarly define the in-amalgamation of $\mathfrak{L}'$ using a specification $\psi : \Sigma' \to \mathcal{P} \times \Sigma$.

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6. *The merged state splittings of λ-graph systems*

For a left-resolving λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ and a partition $\mathcal{P}$ of Σ, we define the merged out-split λ-graph system $\mathfrak{L}^{\overline{[\mathcal{P}]}}$ and the merged in-split λ-graph system $\mathfrak{L}_{\overline{[\mathcal{P}]}}$ as the merged λ-graph system $\overline{\mathfrak{L}^{[\mathcal{P}]}}$ of the out-split λ-graph system $\mathfrak{L}^{[\mathcal{P}]}$ and the merged λ-graph system $\overline{\mathfrak{L}_{[\mathcal{P}]}}$ of the in-split λ-graph system $\mathfrak{L}_{[\mathcal{P}]}$, respectively. Then we have the following by the previous discussions.

PROPOSITION 6.1. *Assume that $\mathfrak{L}$ is left-resolving and predecessor-separated with single zero-vertex. Both $\mathfrak{L}^{\overline{[\mathcal{P}]}}$ and $\mathfrak{L}_{\overline{[\mathcal{P}]}}$ are left-resolving and predecessor-separated λ-graph systems whose symbolic matrix systems are properly strong shift equivalent to the symbolic matrix system of $\mathfrak{L}$.*

Proof. Let $(\mathcal{M}, I)$ be the symbolic matrix system for $\mathfrak{L}$. Since $\mathfrak{L}$ is predecessor-separated with single zero-vertex, its merged λ-graph system is the same as $\mathfrak{L}$ and hence $(\overline{\mathcal{M}}, \overline{I}) = (\mathcal{M}, I)$. It follows that $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\overline{\mathcal{M}^{[\mathcal{P}]}}, \overline{I^{[\mathcal{P}]}})$ by Proposition 3.3 and Proposition 5.2, and similarly $(\mathcal{M}, I) \underset{\text{1-pr}}{\approx} (\overline{\mathcal{M}_{[\mathcal{P}]}}, \overline{I_{[\mathcal{P}]}})$ by Proposition 5.5. □

We note that both the zero vertices $V_0^{\overline{[\mathcal{P}]}}$ and $V_{\overline{[\mathcal{P}]}0}$ are one point.

Let Λ be a subshift over Σ and $\mathcal{P}$ a partition of Σ. For $a \in \Sigma$, we denote by $[a] \in \mathcal{P}$ the equivalence class of a induced by the partition $\mathcal{P}$. The out-split subshift of Λ by $\mathcal{P}$ is defined to be the subshift $\Lambda^{[\mathcal{P}]}$ over alphabet $\Sigma^{[\mathcal{P}]} = \Sigma \times \mathcal{P}$:

$$\Lambda^{[\mathcal{P}]} = \{(a_i[a_{i+1}])_{i\in\mathbb{Z}} \in (\Sigma^{[\mathcal{P}]})^{\mathbb{Z}} \mid (a_i)_{i\in\mathbb{Z}} \in \Lambda\}.$$

The in-split subshift of Λ by $\mathcal{P}$ is defined to be the subshift $\Lambda_{[\mathcal{P}]}$ over alphabet $\Sigma_{[\mathcal{P}]} = \mathcal{P} \times \Sigma$:

$$\Lambda_{[\mathcal{P}]} = \{([a_i]a_{i+1})_{i\in\mathbb{Z}} \in (\Sigma_{[\mathcal{P}]})^{\mathbb{Z}} \mid (a_i)_{i\in\mathbb{Z}} \in \Lambda\}$$

(cf. [**N, LM**, Exercise 7.1.10]). We will prove the following.

PROPOSITION 6.2. *The canonical λ-graph system of the out-split (respectively in-split) subshift $\Lambda^{[\mathcal{P}]}$ (respectively $\Lambda_{[\mathcal{P}]}$) by partition $\mathcal{P}$ is the merged out-split (respectively in-split) λ-graph system of the canonical λ-graph system of Λ. That is,*

$$\mathfrak{L}^{(\Lambda^{[\mathcal{P}]})} = (\mathfrak{L}^{\Lambda})^{\overline{[\mathcal{P}]}} \quad (\textit{respectively } \mathfrak{L}^{(\Lambda_{[\mathcal{P}]})} = (\mathfrak{L}^{\Lambda})_{\overline{[\mathcal{P}]}}).$$

Proof. We first study the out-split λ-graph system. Since the canonical λ-graph system $\mathfrak{L}^{\Lambda}$ is predecessor-separated, so is its out-split $(\mathfrak{L}^{\Lambda})^{[\mathcal{P}]}$. Hence, by Lemma 3.2, the merged out-split $(\mathfrak{L}^{\Lambda})^{\overline{[\mathcal{P}]}}$ is the same as $(\mathfrak{L}^{\Lambda})^{[\mathcal{P}]}$ except the zero-vertices. Consider the right one-sided subshift $X_{\Lambda^{[\mathcal{P}]}}$ for $\Lambda^{[\mathcal{P}]}$:

$$X_{\Lambda^{[\mathcal{P}]}} = \{(a_1[a_2], a_2[a_3], a_3[a_4], \dots) \mid (a_i)_{i\in\mathbb{Z}} \in \Lambda\}.$$

The vertex set $V_l^{(\Lambda^{[\mathcal{P}]})}$ of level l of the λ-graph system $\mathfrak{L}^{(\Lambda^{[\mathcal{P}]})}$ is the l-past equivalence classes $X_{\Lambda^{[\mathcal{P}]}}/\sim_l$ of $X_{\Lambda^{[\mathcal{P}]}}$. Suppose that $(a_i[a_{i+1}])_{i\in\mathbb{N}} \sim_l (b_i[b_{i+1}])_{i\in\mathbb{N}}$ in $X_{\Lambda^{[\mathcal{P}]}}$. This condition is equivalent to the condition for $(\mu_{-l}, \dots, \mu_{-1}, \mu_0) \in \Lambda^{l+1}$:

$$(\mu_{-l}[\mu_{-l+1}], \mu_{-l+1}[\mu_{-l+2}], \dots, \mu_{-1}[\mu_0], a_1[a_2], a_2[a_3], a_3[a_4], \dots) \in X_{\Lambda^{[\mathcal{P}]}}$$

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if and only if

$$(\mu_{-l}[\mu_{-l+1}], \mu_{-l+1}[\mu_{-l+2}], \dots, \mu_{-1}[\mu_0], b_1[b_2], b_2[b_3], b_3[b_4], \dots) \in X_{\Lambda^{[\mathcal{P}]}}.$$

This is equivalent to the condition that $(\mu_{-l}, \mu_{-l+1}, \dots, \mu_{-1}, a_1, a_2, a_3, \dots)$ belongs to X_Λ and $[\mu_0] = [a_1]$ if and only if $(\mu_{-l}, \mu_{-l+1}, \dots, \mu_{-1}, b_1, b_2, b_3, \dots)$ belongs to X_Λ and $[\mu_0] = [b_1]$. It is equivalent to the condition that $(a_i)_{i\in\mathbb{N}} \sim_l (b_i)_{i\in\mathbb{N}}$ in X_Λ and $[a_1] = [b_1]$. Hence we can identify $V_l^{(\Lambda^{[\mathcal{P}]})}$ with $(V^\Lambda)_l^{[\mathcal{P}]}$ through the correspondence between the l-past equivalence class $[(a_i[a_{i+1}])_{i\in\mathbb{N}}]_l$ of $(a_i[a_{i+1}])_{i\in\mathbb{N}}$ in $X_{\Lambda^{[\mathcal{P}]}}$ and the l-past equivalence class $([(a_i)_{i\in\mathbb{N}}]_l, [a_1])$ of $(a_i)_{i\in\mathbb{N}}$ in X_Λ with the equivalence class $[a_1] \in \mathcal{P}$. This identification naturally induces an isomorphism of the λ-graph systems between $\mathfrak{L}^{(\Lambda^{[\mathcal{P}]})}$ and $(\mathfrak{L}^\Lambda)^{\overline{[\mathcal{P}]}}$.

We next discuss on the in-split λ-graph system. Consider the right one-sided subshift $X_{\Lambda_{[\mathcal{P}]}}$ for $\Lambda_{[\mathcal{P}]}$:

$$X_{\Lambda_{[\mathcal{P}]}} = \{([a_1]a_2, [a_2]a_3, [a_3]a_4, \dots) \mid (a_i)_{i\in\mathbb{Z}} \in \Lambda\}.$$

The vertex set $V_l^{(\Lambda_{[\mathcal{P}]})}$ of level l of the λ-graph system $\mathfrak{L}^{(\Lambda_{[\mathcal{P}]})}$ is the l-past equivalence class $X_{\Lambda_{[\mathcal{P}]}}/\sim_l$ of $X_{\Lambda_{[\mathcal{P}]}}$. Suppose that $([a_i]a_{i+1})_{i\in\mathbb{N}} \sim_l ([b_i]b_{i+1})_{i\in\mathbb{N}}$ in $X_{\Lambda_{[\mathcal{P}]}}$. This condition is equivalent to the condition for $(\mu_{-l}, \dots, \mu_{-1}, \mu_0) \in \Lambda^{l+1}$:

$$([\mu_{-l}]\mu_{-l+1}, [\mu_{-l+1}]\mu_{-l+2}, \dots, [\mu_{-1}]\mu_0, [a_1]a_2, [a_2]a_3, [a_3]a_4, \dots) \in X_{\Lambda_{[\mathcal{P}]}}$$

if and only if

$$([\mu_{-l}]\mu_{-l+1}, [\mu_{-l+1}]\mu_{-l+2}, \dots, [\mu_{-1}]\mu_0, [b_1]b_2, [b_2]b_3, [b_3]b_4, \dots) \in X_{\Lambda_{[\mathcal{P}]}}.$$

To study this condition, we will construct a λ-graph system $(V'_{[\mathcal{P}]}, E'_{[\mathcal{P}]}, \lambda'_{[\mathcal{P}]}, \iota'_{[\mathcal{P}]})$ in the following way. We fix $l = 2, 3, \dots$; for $v_i^l \in V_l$ and $p \in \mathcal{P}$ with $p \in \mathcal{P}^{l,i}$, put

$$\Gamma^-_{p,l-1}(v_i^l) = \{([\mu_1], \mu_2, \mu_3, \dots, \mu_l) \in \mathcal{P} \times \Lambda^{l-1} \mid (\mu_1, \mu_2, \mu_3, \dots, \mu_l) \in \Gamma^-_{\mathfrak{L},l}(v_i^l), [\mu_l] = p\},$$

where $\Gamma^-_{\mathfrak{L},l}(v_i^l)$ is the set of all label sequences of length l terminating at v_i^l. For $v_i^{l,p}, v_j^{l,q} \in V_{[\mathcal{P}],l-1}$, we say $v_i^{l,p}$ is equivalent to $v_j^{l,q}$ if $p = q$ and $\Gamma^-_{p,l-1}(v_i^l) = \Gamma^-_{q,l-1}(v_j^l)$. We denote by $[v_i^{l,p}]$ the equivalence class of $v_i^{l,p}$. Let $V'_{[\mathcal{P}],l-1}$ be the set of all such equivalence classes of $v_i^{l,p}$, $p \in \mathcal{P}^{l,i}$, $i = 1, 2, \dots, m(l)$. For $l = 1$, we define $V'_{[\mathcal{P}],0}$ to be one point. Put $V'_{[\mathcal{P}]} = \bigcup_{l=0}^\infty V'_{[\mathcal{P}],l}$. We define a labeled edge e_q from $[v_j^{l+1,q}]$ to $[v_i^{l+2,[\lambda(e)]}]$ if there exists an edge e in $\mathfrak{L}$ such that $s(e) = v_j^{l+1}, t(e) = v_i^{l+2}$. The labeling $\lambda'_{[\mathcal{P}]}$ of e_q is defined by $(q, \lambda(e)) \in \mathcal{P} \times \Sigma$. We denote by $E'_{[\mathcal{P}],l,l+1}$ the set of all such labeled edges. Put $E'_{[\mathcal{P}]} = \bigcup_{l=0}^\infty E'_{[\mathcal{P}],l,l+1}$. The inclusion map $\iota'_{[\mathcal{P}]} : V'_{[\mathcal{P}],l+1} \to V'_{[\mathcal{P}],l}$ is defined as follows. For $[v_i^{l+2,p}] \in V'_{[\mathcal{P}],l+1}$ there exists an edge $e \in E_{l+1,l+2}$ such that $t(e) = v_i^{l+2}, [\lambda(e)] = p$. Put $v_{i'}^{l+1} = \iota(v_i^{l+2})$. By the local property of $\mathfrak{L}$, there exists an edge $e' \in E_{l,l+1}$ such that $t(e') = v_{i'}^{l+1}, \lambda(e') = \lambda(e)$. Hence $v_{i'}^{l+1,p} \in V'_{[\mathcal{P}],l}$. We then define $\iota'_{[\mathcal{P}]}([v_i^{l+2,p}]) = [v_{i'}^{l+1,p}]$. Therefore, we obtain

$$(V'_{[\mathcal{P}]}, E'_{[\mathcal{P}]}, \lambda'_{[\mathcal{P}]}, \iota'_{[\mathcal{P}]}).$$

We denote it by $\mathfrak{L}'_{[\mathcal{P}]}$. Then we have the following lemma.

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LEMMA 6.3. *$\mathfrak{L}'_{[\mathcal{P}]}$ is a λ-graph system which is naturally isomorphic to $\mathfrak{L}_{\overline{[\mathcal{P}]}}$. In particular, $(\mathfrak{L}^{\Lambda})'_{[\mathcal{P}]} = (\mathfrak{L}^{\Lambda})_{\overline{[\mathcal{P}]}}$ for the canonical λ-graph system $\mathfrak{L}^{\Lambda}$ for a subshift Λ.*

Proof. For $([\mu_1], \mu_2, \mu_3, \dots, \mu_l) \in \mathcal{P} \times \Lambda^{l-1}$ we see that $([\mu_1], \mu_2, \mu_3, \dots, \mu_l)$ belongs to $\Gamma^-_{p,l-1}(v_i^l)$ if and only if $([\mu_1]\mu_2, [\mu_2]\mu_3, \dots, [\mu_{l-1}]\mu_l)$ is a predecessor of $v_i^{l,p}$ in the graph $\mathfrak{L}_{[\mathcal{P}]}$. That is,

$$([\mu_1], \mu_2, \mu_3, \dots, \mu_l) \in \Gamma^-_{p,l-1}(v_i^l)$$

if and only if

$$([\mu_1]\mu_2, [\mu_2]\mu_3, \dots, [\mu_{l-1}]\mu_l) \in \Gamma^-_{\mathfrak{L}_{[\mathcal{P}]},l-1}(v_i^{l,p}).$$

Hence the vertex set $V'_{[\mathcal{P}],l-1}$ is the merged vertex set $\overline{V}_{[\mathcal{P}],l-1}$ of $V_{[\mathcal{P}],l-1}$ so that we have $V'_{[\mathcal{P}]} = \overline{V}_{[\mathcal{P}]}$. It is straightforward to see that the edge sets $E'_{[\mathcal{P}],l,l+1}$ and $\overline{E}_{[\mathcal{P}],l,l+1}$ naturally correspond with each other such that the correspondence is compatible to the identification between $V'_{[\mathcal{P}]}$ and $\overline{V}_{[\mathcal{P}]}$. Hence we see that $\mathfrak{L}'_{[\mathcal{P}]} = \mathfrak{L}_{\overline{[\mathcal{P}]}}$. □

LEMMA 6.4. $(\mathfrak{L}^{\Lambda})'_{[\mathcal{P}]} = \mathfrak{L}^{(\Lambda_{[\mathcal{P}]})}$.

Proof. For $(a_n)_{n\in\mathbb{N}} \in X_\Lambda$, put $v_i^l = [(a_{n+1})_{n\in\mathbb{N}}]_l \in X_\Lambda/\sim_l = V_l^\Lambda$ and $p = [a_1] \in \mathcal{P}^{l,i}$. Then

$$([\mu_1], \mu_2, \mu_3, \dots, \mu_l) \in \Gamma^-_{p,l-1}(v_i^l)$$

if and only if

$$([\mu_1]\mu_2, [\mu_2]\mu_3, \dots, [\mu_{l-1}]\mu_l) \in \Gamma^-_{\mathfrak{L}_{[\mathcal{P}]},l-1}(v_i^{l,p}).$$

Hence we can regard $(V^\Lambda)'_{[\mathcal{P}],l-1}$ as the vertex set $V^{(\Lambda_{[\mathcal{P}]})}{}_{l-1}$ so that we have $(V^\Lambda)'_{[\mathcal{P}]} = V^{(\Lambda_{[\mathcal{P}]})}$. This induces a natural identification between $(\mathfrak{L}^\Lambda)'_{[\mathcal{P}]}$ and $\mathfrak{L}^{(\Lambda_{[\mathcal{P}]})}$. □

By Lemmas 6.3 and 6.4 with the previous discussion on the out-splitting of $\mathfrak{L}^\Lambda$, we complete the proof of Proposition 6.2. □

7. *Decomposition of topological conjugacy between subshifts and state splittings of λ-graph systems*

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be λ-graph systems over Σ and Σ', respectively. If $\mathfrak{L}'$ is the merged out-split (respectively in-split) λ-graph system of $\mathfrak{L}$ for a partition $\mathcal{P}$ of Σ, we say that $\mathfrak{L}$ is the merged out-amalgamation (respectively in-amalgamation) of $\mathfrak{L}'$.

We will show the following theorem.

THEOREM 7.1. *Let $\mathfrak{L}^\Lambda$ and $\mathfrak{L}^{\Lambda'}$ be the canonical λ-graph systems for subshifts Λ and Λ' respectively. Then Λ is topologically conjugate to Λ' if and only if $\mathfrak{L}^\Lambda$ is obtained by a finite procedure of merged out-splittings, merged in-splittings, merged out-amalgamations and merged in-amalgamations from $\mathfrak{L}^{(\Lambda')}$.*

Since merged splittings and merged amalgamations yield properly strong shift equivalences between the corresponding symbolic matrix systems, the above theorem is a detailed version of Theorem 2.4. We will show it by using a Nasu's idea in the proof of the decomposition theorem of topological conjugacy into bipartite codes [**N**, Theorem 2.4] with Proposition 6.2.

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The merged λ-graph system $\overline{\mathfrak{L}^{[\Sigma]}}$ of the complete out-splitting of $\mathfrak{L}$ is said to be the complete merged out-splitting of $\mathfrak{L}$ and is written as $\mathfrak{L}^{\overline{[\Sigma]}}$. For $N \in \mathbb{N}$, we define the merged N-higher λ-graph system $\mathfrak{L}^{\overline{[N]}}$ to be the complete merged out-splitting of $\mathfrak{L}^{\overline{[N-1]}}$, where $\mathfrak{L}^{\overline{[2]}}$ is the complete merged out-splitting $\mathfrak{L}^{\overline{[\Sigma]}}$ of $\mathfrak{L}$, and $\mathfrak{L}^{\overline{[1]}} = \mathfrak{L}$. Let $\Lambda^{[N]}$ be the N-higher block presentation of Λ [**LM**]. By Proposition 6.2, we have the following.

LEMMA 7.2. *The canonical λ-graph system of $\Lambda^{[N]}$ is isomorphic to the merged N-higher λ-graph system of the canonical λ-graph system of Λ. That is, $\mathfrak{L}^{(\Lambda^{[N]})} = (\mathfrak{L}^{\Lambda})^{\overline{[N]}}$.*

The complete merged in-splitting $\mathfrak{L}_{\overline{[\Sigma]}}$ and the merged N-higher λ-graph system $\mathfrak{L}_{\overline{[N]}}$ of $\mathfrak{L}$ are similarly defined. By Proposition 6.2, one sees that $\mathfrak{L}^{(\Lambda^{[N]})} = (\mathfrak{L}^{\Lambda})_{\overline{[N]}}$.

For a partition $\mathcal{P}$ of Σ, consider the out-split $\Lambda^{[\mathcal{P}]}$ of Λ. Then the map $\psi^{[\mathcal{P}]} : \Lambda \to \Lambda^{[\mathcal{P}]}$ defined by $\psi^{[\mathcal{P}]}[(a_i)_{i\in\mathbb{Z}}] = (a_i[a_{i+1}])_{i\in\mathbb{Z}}$ has memory 0 and anticipation 1 [**LM**]. It is called an out-splitting code. Its inverse $\alpha^{[\mathcal{P}]} : \Lambda^{[\mathcal{P}]} \to \Lambda$ is a 1-block code and is called an out-amalgamation code. Similarly we define an in-splitting code having memory 1 and anticipation 0, and an in-amalgamation code.

Let $\psi : \Lambda \to \Lambda'$ be a topological conjugacy. By [**LM**, Proposition 1.5.12], there exists a number $N \in \mathbb{N}$ and a 1-block conjugacy $\tilde{\psi} : \Lambda^{[N]} \to \Lambda'$. The following lemma is essentially the same as [**N**, Lemma 2.5] (see also [**LM**, Lemma 7.1.3]). We omit its proof.

LEMMA 7.3. *Let $\phi : \Lambda \to \Lambda'$ be a 1-block conjugacy whose inverse ϕ^{-1} has memory $m \geq 0$ and anticipation $n \geq 1$. Then there exists a partition $\mathcal{P}$ of Σ and a 1-block conjugacy $\tilde{\phi} : \Lambda^{[\mathcal{P}]} \to \Lambda'^{[\Sigma']}$ whose inverse $\tilde{\phi}^{-1}$ has memory m and anticipation $n-1$ such that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\psi^{[\mathcal{P}]}} & \Lambda^{[\mathcal{P}]} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\tilde{\phi}} \\ \Lambda' & \xleftarrow{\alpha^{[\Sigma']}} & \Lambda'^{[\Sigma']} \end{array}$$

commutes.

Proof of Theorem 7.1 Since the merged out-splittings and merged in-splittings of the canonical λ-graph systems yield out-splittings and in-splittings of the subshifts presented by the λ-graph systems, they give rise to out-split codes and in-split codes between the subshifts. Hence the if part of the theorem is immediate. For the proof of the only if part, by the same argument as the proof of Nasu [**N**, Theorem 2.4] (see also [**LM**, Theorem 7.1.2]), we can establish the assertion by Proposition 6.2, Lemma 7.3 and the in-splitting version of Lemma 7.3. □

By Proposition 3.3, 5.2, 5.5 and 6.2, the symbolic matrix systems of the λ-graph systems for merged out-splittings, merged in-splittings, merged out-amalgamations and merged in-amalgamations of the canonical λ-graph system for a subshift Λ are (properly) strong shift equivalent to the canonical symbolic matrix system for the subshift. Hence we have the following.

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COROLLARY 7.4. [**Ma2**, Theorem A] *Let* Λ *and* Λ' *be two subshifts and* $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ *and* $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ *their canonical symbolic matrix systems, respectively. Then* Λ *and* Λ' *are topologically conjugate if and only if* $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ *and* $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ *are strong shift equivalent.*

Acknowledgement. The author would like to thank the referee for valuable comments and suggestions on the first version of this paper.

REFERENCES

[BK] M. Boyle and W. Krieger. Almost Markov and shift equivalent sofic systems. *Proc.of Maryland Special Year in Dynamics 1986–87 (Springer-Verlag Lecture Notes in Mathematics, 1342)*. Springer, 1988, pp. 33–93.

[Bra] O. Bratteli. Inductive limits of finite-dimensional C^*-algebras. *Trans. Amer. Math. Soc.* **171** (1972), 195–234.

[CM] T. M. Carlsen and K. Matsumoto. Some remarks on the C^*-algebras associated with subshifts. *Preprint*, 2001.

[C] J. Cuntz. A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext- functor for C^*-algebras. *Invent. Math.* **63** (1980), 25–40.

[CK] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), 251–268.

[Fi] R. Fischer. Sofic shifts and graphs. *Monats. Math.* **80** (1975), 179–186.

[Ki] B. P. Kitchens. *Symbolic Dynamics*. Springer, Berlin 1998.

[Kr] W. Krieger. On sofic systems I. *Israel J. Math.* **48** (1984), 305–330.

[Kr2] W. Krieger. On sofic systems II. *Israel J. Math.* **60** (1987), 167–176.

[KM] W. Krieger and K. Matsumoto. Shannon graphs, subshifts and lambda-graph systems. *J. Math. Soc. Japan* **54** (2002), 877–899.

[LM] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[Ma] K. Matsumoto. On C^*-algebras associated with subshifts. *Int. J. Math.* **8** (1997), 357–374.

[Ma2] K. Matsumoto. Presentations of subshifts and their topological conjugacy invariants. *Doc. Math.* **4** (1999), 285–340.

[Ma3] K. Matsumoto. C^*-algebras associated with presentations of subshifts. *Doc. Math.* **7** (2002), 1–30.

[Ma4] K. Matsumoto. Strong shift equivalence of symbolic dynamical systems and Morita equivalence of C^*-algebras. *Ergod. Th. & Dynam. Sys.* to appear.

[N] M. Nasu. Topological conjugacy for sofic shifts. *Ergod. Th. & Dynam. Sys.* **6** (1986), 265–280.

[N2] M. Nasu. Textile systems for endomorphisms and automorphisms of the shift. *Mem. Amer. Math. Soc.* **546** (1995).

[We] B. Weiss. Subshifts of finite type and sofic systems. *Monats. Math.* **77** (1973), 462–474.

[Wi] R. F. Williams. Classification of subshifts of finite type. *Ann. Math.* **98** (1973), 120–153; *Ann. Math.* **99** (1974), 380–381 (Erratum).

https://doi.org/10.1017/S014338570200175X Published online by Cambridge University Press