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A SIMPLICIAL POLYTOPE THAT MAXIMIZES THE ISOTROPIC CONSTANT MUST BE A SIMPLEX

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Abstract. The isotropic constant L_K is an affine-invariant measure of the spread of a convex body K. For a d-dimensional convex body K, L_K can be defined by $L_K^{2d} = \det(A(K))/(\mathrm{vol}(K))^2$, where $A(K)$ is the covariance matrix of the uniform distribution on K. It is an open problem to find a tight asymptotic upper bound of the isotropic constant as a function of the dimension. It has been conjectured that there is a universal constant upper bound. The conjecture is known to be true for several families of bodies, in particular, highly symmetric bodies such as bodies having an unconditional basis. It is also known that maximizers cannot be smooth. In this work we study bodies that are neither smooth nor highly symmetric by showing progress towards reducing to a highly symmetric case among non-smooth bodies. More precisely, we study the set of maximizers among simplicial polytopes and we show that if a simplicial polytope K is a maximizer of the isotropic constant among d-dimensional convex bodies, then when K is put in isotropic position it is symmetric around any hyperplane spanned by a $(d-2)$-dimensional face and the origin. By a result of Campi, Colesanti and Gronchi, this implies that a simplicial polytope that maximizes the isotropic constant must be a simplex.

§1. *Introduction.* Let a d-dimensional *convex body* be a compact convex set in $\mathbb{R}^d$ with non-empty interior. For a d-dimensional convex body K, let $A(K)$ be the covariance matrix of the uniform distribution on K, that is, for X random in K, let $\mu_K = \mathbb{E}(X)$ and let $A(K) = \mathbb{E}_K((X-\mu_K)(X-\mu_K)^T)$. The *isotropic constant* L_K of K can be defined by $L_K^{2d} = \det(A(K))/(\mathrm{vol}(K))^2$. An open problem in asymptotic geometric analysis, the *slicing problem*, is to find a tight asymptotic upper bound of the isotropic constant as a function of the dimension only. The *slicing conjecture* (also known as the *hyperplane conjecture*) states that there is a universal constant upper bound of the isotropic constant. The slicing problem seems to be mentioned for the first time by Bourgain [3] and some equivalent formulations are discussed by Ball in [1]. Milman and Pajor [12] studied the problem systematically. It is one of the outstanding open problems in convex geometry; among the reasons are its connections with classical problems in convexity, like the Busemann–Petty problem and Sylvester's problem [12, 7, 4].

The isotropic constant is invariant under affine transformations. About the asymptotic behavior, it is known that L_K is bounded below by a universal constant and this is tight [3]. Bourgain also showed that $L_K \leqslant O(\sqrt[4]{d}\log d)$.

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The best known upper bound at this time is by Klartag, who showed that $L_K \leqslant O(\sqrt[4]{d})$ [9].

A natural question is to understand bodies that maximize L_K in every dimension. We review now some results that give a partial understanding of the nature of maximizers. There are subfamilies of all convex bodies where universal constant upper bounds to L_K are known:

- zonoids, that is, limits of zonotopes (which are Minkowski sums of segments) (proved by Bourgain and in [12]; see [12] for a discussion),
- bodies having an unconditional basis, that is, convex bodies K such that $(x_1, \ldots, x_d) \in K \iff (|x_1|, \ldots, |x_d|) \in K$ [3, 12].

Also [5] showed that if K has a non-empty subset of its boundary of class C^2 with positive principal curvatures, then it cannot be a maximizer of L_K. It also showed that if K is of the form $K = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in \pi_{\mathbb{R}^{d-1}}(K), -f(x) \leqslant y \leqslant f(x)\}$ where $\pi_{\mathbb{R}^{d-1}}$ is the projection onto the first $d-1$ coordinates (i.e. K has a hyperplane of symmetry) and f is not affine, then K is not a maximizer of the isotropic constant (see Theorem 3 in this paper for a formal statement).

The results above tell us that for bodies having certain symmetries or having a smooth boundary we have some understanding of the behavior of the isotropic constant. This leaves unexplored bodies that are not smooth and have no symmetries. Can we argue that maximizing bodies must have certain symmetries?

A polytope is *simplicial* if every facet is a simplex. The first result of this paper is the following necessary condition on simplicial polytopes that are local extrema of the isotropic constant, Theorem 1. We first recall a few definitions and basic facts. In the context of this paper, a d-dimensional convex body is a *local extremum* of a functional if and only if the body is a local maximum or a local minimum of the functional in the space of all convex bodies with respect to the Hausdorff topology. A d-dimensional polytope is called *isohedral* or a *isohedron* if and only if it is facet-transitive, that is, for any pair of facets F_1, F_2 there is an element of the (full) symmetry group of the polytope that maps F_1 to F_2. (The *full symmetry group* of a polytope is the group under composition of all isometries of $\mathbb{R}^d$ that map the polytope to itself.) A d-dimensional convex body K is *isotropic* if and only if $\mu_K = 0$ and $A(K) = I$. Given any d-dimensional convex body, there is an affine transformation that maps it to an isotropic convex body. As the isotropic constant is invariant under affine transformations, it is enough to study it over isotropic convex bodies.

THEOREM 1. *Let P be a d-dimensional isotropic simplicial polytope that is a local extremum of $P \mapsto L_P$. Let H be any hyperplane spanned by a $(d-2)$-dimensional face of P and the origin. Then P is symmetric around H. In particular, P is isohedral and all facets are congruent.*

The second and main result of this paper is Theorem 2, a corollary of Theorem 1 and of the result from [5] mentioned before (which is stated in this paper as Theorem 3).

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THEOREM 2. *Let P be a d-dimensional simplicial polytope that is a maximizer of $P \mapsto L_P$. Then P is a simplex.*

It is worth mentioning a parallel situation for Mahler's problem: a similar gap between smooth bodies and symmetric bodies. For simplicity, we only discuss the *symmetric* Mahler problem: given a centrally symmetric convex body K, define the *volume product* as $f(K) = \mathrm{vol}(K)\,\mathrm{vol}(K^\circ)$, where K° is the polar body of K, given by

$$K^\circ := \{x \in \mathbb{R}^d : x \cdot y \leqslant 1 \ \forall y \in K\}.$$

The problem is to determine the minimizers of the volume product. The cube is conjectured to be a minimizer. This conjecture has been verified among unconditional bodies [16, 11] (which correspond to symmetries around coordinate hyperplanes) and this result has been extended to more general symmetries [2]. On the other hand, [14] (improving earlier work, [20]), showed that if a convex body K has a boundary point with positive generalized Gauss curvature, then K cannot minimize the volume product.

The rest of the paper is devoted to the proofs of Theorems 1 and 2.

§2. *Preliminaries.* For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, \ldots, n\}$. For a probability distribution $\mathcal{D}$ and a random variable X, let $X \sim \mathcal{D}$ denote that X is distributed according to $\mathcal{D}$.

If K is a d-dimensional isotropic convex body and X is uniformly random in K, then $\mathbb{E}(X_i^2) = 1$ and therefore $\mathbb{E}(\|X\|^2) = d$.

A polytope in $\mathbb{R}^d$ is the convex hull of a finite number of points. For a polytope P with non-empty interior, the unique irredundant H-representation of P is the unique set of closed halfspaces $\{H_i : i = 1, \ldots, n\}$ such that $P = \bigcap_{i=1}^n H_i$. See [21, Chs 0, 1, 2] for background.

We use a result from [5] that rules out many bodies having a hyperplane of symmetry as potential maximizers of the isotropic constant. We state the result, Theorem 3, after a few definitions. Let K be a d-dimensional convex body. Let $H \subseteq \mathbb{R}^d$ be a hyperplane through the origin. Let $\pi_H : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection onto H. Let $f_H, g_H : \pi_H(K) \to \mathbb{R}$ be convex functions such that (identifying H with $\mathbb{R}^{d-1}$)

$$K = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in \pi_H(K),\ f_H(x) \leqslant y \leqslant -g_H(x)\}. \tag{1}$$

THEOREM 3 [5]. *If a d-dimensional convex body K is symmetric with respect to a hyperplane through the origin H and the corresponding functions f_H, g_H are not affine, then K is not a maximizer of the isotropic constant.*

(Note that in this case we have $f_H = g_H$.)

Proof. The isotropic constant is related to random simplices by the following formula (see [12] and [5] for a discussion of this connection):

$$L_K^{2d} = \frac{\det A(K)}{\mathrm{vol}(K)^2} = \frac{d!}{d+1} M_2(K; d+1) \tag{2}$$

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where $M_2(K; d+1)$ is the second moment of the volume of a random simplex in K, normalized so that it is affinely invariant. That is,

$$M_2(K; d+1) = \frac{1}{\mathrm{vol}(K)^{d+3}} \int_{x_1 \in K} \cdots \int_{x_{d+1} \in K} (\mathrm{vol}\,\mathrm{conv}(x_1, \ldots, x_{d+1}))^2 \, dx_{d+1} \cdots dx_1.$$

The statement to be proved then follows immediately from (2) and results on random simplices [**5**, Theorems 3.2 and 3.6]. □

§3. *Derivative with respect to hinging.* In this section we compute the derivative of the isotropic constant of a polytope with respect to hinging of one of its facets. The first step (the next lemma) is to show conditions under which integrals of functions over polytopes are differentiable as a facet hinges. This is slightly non-trivial; similar differentiability issues have been discussed before [**10**, Lemma 2.4].

LEMMA 4. *Let $P \subseteq \mathbb{R}^d$ be a closed polyhedron with non-empty interior such that the origin is in the relative interior of a facet of P. Let Z be a $(d-2)$-dimensional linear subspace that does not intersect the interior of P. Let $\theta : \mathbb{R}^d \to [-\pi, \pi)$ be an angle in a system of polar coordinates (see Figure 1) on $Z^\perp$ so that:*

(1) *there exist $a, b \in (-\pi, \pi)$ satisfying $\theta(P) = [a, b]$ and $a < 0 < b$;*
(2) *$P \cap \{x : a \leqslant \theta(x) \leqslant 0\}$ is bounded.*

Let $f : P \to \mathbb{R}$ be a continuous function. Let $K_t = \{x \in P : \theta(x) \in [-\pi, t]\}$. Then $g(t) = \int_{K_t} f(x)\, dx$ is differentiable at 0.

Proof. Let $r : S^{d-1} \to [0, \infty]$ be the radial function of P. That is, $r(u) = \max\{a \in [0, \infty] : au \in P\}$. Write $g(t)$ in hyperspherical coordinates:

$$g(t) = \int_{\theta=-\pi}^{t} \int_{\phi_{d-2}=0}^{\pi} \cdots \int_{\phi_1=0}^{\pi} \int_{\rho=0}^{r(v(\phi,\theta))} f(\rho v(\phi, \theta)) \times J(\rho, \phi, \theta)\, d\rho\, d\phi_1\, d\phi_2 \cdots d\phi_{d-2}\, d\theta, \tag{3}$$

where $\phi = (\phi_1, \ldots, \phi_{d-1})$ and $J(\rho, \phi, \theta)$ is the volume element, given by

$$J(\rho, \phi, \theta) = \rho^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \cdots \sin(\phi_{d-2}),$$

and $v(\phi, \theta)$ is the change of variable that gives a point on S^{d-1} given the angles, that is, $v(\phi, \theta)_1 = \cos(\phi_i)$, $v(\phi, \theta)_2 = \sin(\phi_1)\cos(\phi_2), \ldots, v(\phi, \theta)_{d-1} = \sin(\phi_1) \cdots \sin(\phi_{d-2})\cos(\theta)$, $v(\phi, \theta)_d = \sin(\phi_1) \cdots \sin(\phi_{d-2})\sin(\theta)$.

Let $h(\theta)$ be the integrand of the outermost integral in (3). To establish that g is differentiable at 0, it is enough to show that $h(\theta)$ is continuous in a neighborhood of 0.

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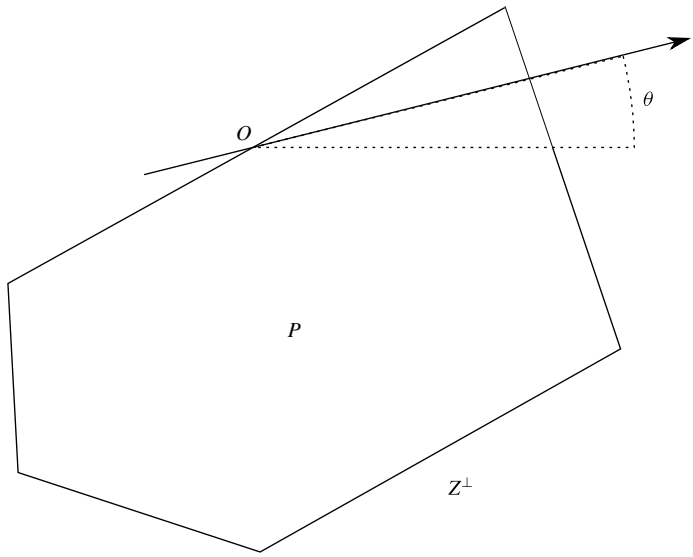


Figure 1: Coordinates for the proof of Lemma 4.

To simplify the argument we can go to a fixed domain of integration for h by the change of variable $\lambda = \rho/r(v(\phi,\theta))$ so that

$$h(\theta) = \int_{\phi_{d-2}=0}^{\pi} \cdots \int_{\phi_1=0}^{\pi} \int_{\lambda=0}^{1} f(\lambda r(v(\phi,\theta))v(\phi,\theta)) \times J(\lambda r(v(\phi,\theta)),\phi,\theta)\, r(v(\phi,\theta))\, d\lambda\, d\phi_1\, d\phi_2 \cdots d\phi_{d-2}.$$

We will now deduce the continuity of h via Lebesgue's dominated convergence theorem. To be able to use it, we will now show continuity and uniform boundedness of the integrand.

The region of S^{d-1} where r is finite is as follows: $r(v) = \infty$ if and only if v is in the recession cone of P (denoted $\operatorname{rec}(P)$ and given by $\operatorname{rec}(P) = \{x \in \mathbb{R}^d : x + P \subseteq P\}$; see [**19**, §1.4] or [**15**, §8]).

We can understand where r is continuous more precisely in the following way: Let T be the unique closed halfspace that contains P and contains the origin on its boundary. Note that $r(v) = 1/p(v)$, where $p(v) := \inf\{\lambda > 0 : v \in \lambda P\}$ is the (slightly generalized) Minkowski functional of P. We have $p : \mathbb{R}^d \to [0,\infty]$ and p is finite, convex and piecewise affine in T. Therefore, p is continuous in T. Moreover, $p(v) > 0$ for v in the complement of $\operatorname{rec} P$. This establishes that r is continuous and finite in $S^{d-1} \cap (T \setminus \operatorname{rec} P)$.

Finally, we establish that r is continuous in a compact neighborhood of $\{\theta = 0\}$ on the sphere and use this to complete the argument. Assumption (2) implies that $r(v)$ is finite for $\theta = 0$ and any ϕ. Since P is a closed convex body, its recession cone is closed (see [**19**, §1.4] or [**15**, Theorem 8.2]). The following simple topological argument is written relative to $S^{d-1} \cap T$, which is compact.

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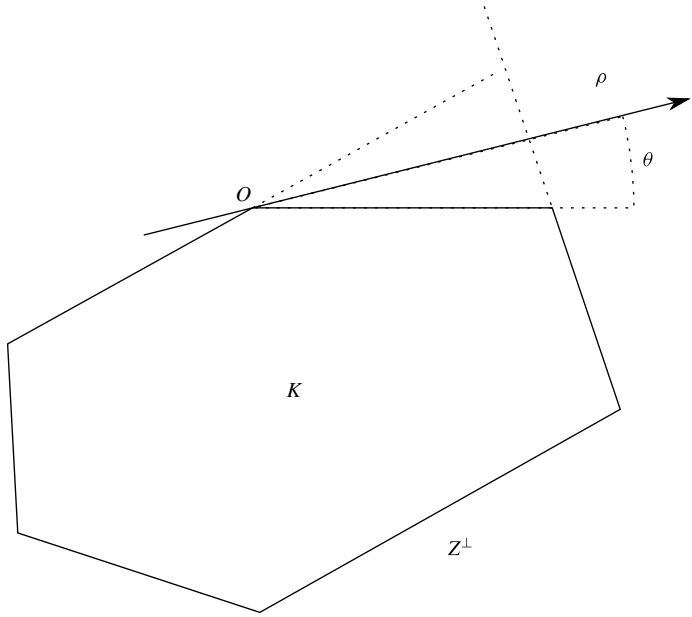


Figure 2: Cylindrical coordinates for the proof of Lemma 5.

In $S^{d-1} \cap T$, we have that $\{\theta = 0\} \cap S^{d-1}$ is compact and $\mathrm{rec}(P) \cap S^{d-1}$ is also compact, and they are disjoint. Therefore, there is an open neighborhood $A \subseteq S^{d-1} \cap T$ containing $\{\theta = 0\} \cap S^{d-1}$ and such that $A \cap \mathrm{rec}(P) = \emptyset$. There is also a smaller compact neighborhood $C \subseteq S^{d-1} \cap T$ satisfying $\{\theta = 0\} \cap S^{d-1} \subseteq C \subseteq A$. As C and rec P are disjoint, the argument in the previous paragraph implies that r is finite and continuous in C. Thus, the integrand of $h(\theta)$ is continuous and bounded over a compact domain of integration and the boundedness is uniform in a neighborhood of 0 over θ. It follows from Lebesgue's dominated convergence theorem that $h(\theta)$ is continuous in the same neighborhood of 0. □

LEMMA 5 (Derivative with respect to hinging). *Let $K \subseteq \mathbb{R}^d$ be an isotropic simplicial polytope with unique irredundant H-representation $\{H_i : i = 1, \ldots, m\}$ so that $K = \bigcap_{i=1}^m H_i$. Fix a facet i' and let P be the polyhedron obtained by relaxing $H_{i'}$, that is,*

$$P = \bigcap_{i \neq i'} H_i.$$

Let $v_1, \ldots, v_d$ be the vertices of facet i'. Let $v_{j'}$ be one of those vertices. Consider a system of cylindrical coordinates as follows (see Figure 2): a point $x \in \mathbb{R}^d$ is parameterized by (ρ, θ, z). Let $Z = \mathrm{aff}\{v_j : j \neq j'\}$. The origin of the new system of coordinates is an arbitrary point in $Z \cap K$. Parameters $\rho \in [0, \infty)$, $\theta \in [-\pi, \pi)$ are polar coordinates in the two-dimensional plane orthogonal to Z. Parameter z is a vector in Z. Polar coordinates ρ, θ are oriented so that

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points in facet i' have $\theta = 0$ and all of K has $\theta \in (-\pi, 0]$. Let

$$K_t = \{x \in P : \theta(x) \in [-\pi, t]\},$$
$$S_t = P \cap \{x : \theta(x) = t\}.$$

Let $\mathcal{D}$ be the distribution of a random vector X supported on S_0 with density proportional to $\rho(X)$. Then

$$\frac{d}{dt} L_{K_t}^{2d}\Big|_{t=0} = (\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2) - d - 2)\frac{(d/dt)\operatorname{vol} K_t|_{t=0}}{(\operatorname{vol} K)^3}.$$

Proof. We have $L_{K_t}^{2d} = \det A(K_t)/\operatorname{vol}(K_t)^2$. Given this formula, the differentiability of $t \mapsto L_{K_t}^{2d}$ and of the varied integrals below at $t = 0$ follows from Lemma 4. Without changing anything else, we can move the origin within $Z \cap K$ so that it lies in the relative interior of $Z \cap K$ to satisfy the assumptions of Lemma 4.

We first compute $(d/dt)\det A(K_t)|_{t=0}$. Part of the argument is the same as the proof of [**13**, Proposition 15]: an identical argument shows that $(d/dt)\det A(K_t)|_{t=0} = (d/dt)\,\mathbb{E}_{X\in K_t}(\|X\|^2)|_{t=0}$, then we compute this last expression for the current lemma. To make the argument more readable and self-contained, we repeat the relevant parts of the argument from [**13**] here.

We have

$$\begin{aligned} A(K_t) &= \mathbb{E}_{X\in K_t}((X - \mu_{K_t})(X - \mu_{K_t})^T) \\ &= \mathbb{E}_{X\in K_t}(XX^T) - \mu_{K_t}\mu_{K_t}^T. \end{aligned}$$

By isotropy, $\mu_K = 0$ and this implies

$$\frac{d}{dt} A(K_t)\Big|_{t=0} = \frac{d}{dt}\,\mathbb{E}_{X\in K_t}(XX^T)\Big|_{t=0}. \tag{4}$$

Use the identity $(d/dM)\det M = (M^{-1})^T \det M$ to conclude

$$\begin{aligned} \frac{d}{dt}\det A(K_t)\Big|_{t=0} &= \frac{d}{dM}\det M\Big|_{M=A(K_t)} \cdot \frac{d}{dt} A(K_t)\Big|_{t=0} \\ &= \det(A(K_t))(A(K_t)^{-1})^T \cdot \frac{d}{dt} A(K_t)\Big|_{t=0} \end{aligned}$$

where the dot "$\cdot$" represents the Frobenius inner product of matrices, $M \cdot N = \sum_{ij} M_{ij}N_{ij}$. This, isotropy and (4) give

$$\begin{aligned} \frac{d}{dt}\det A(K_t)\Big|_{t=0} &= I \cdot \frac{d}{dt}\,\mathbb{E}_{X\in K_t}(XX^T)\Big|_{t=0} \\ &= \frac{d}{dt}\,\mathbb{E}_{X\in K_t}(\|X\|^2)\Big|_{t=0}. \end{aligned}$$

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Let $S_{r,t} = P \cap \{x : \rho(x) = r, \theta(x) = t\}$. We have

$$\mathbb{E}_{X\in K_t}(\|X\|^2) = \frac{1}{\operatorname{vol} K_t}\int_{-\pi}^{t}\int_0^\infty \mathbb{E}_{X\in S_{\rho,\theta}}(\|X\|^2)\operatorname{vol}_{d-2}(S_{\rho,\theta})\rho\, d\rho\, d\theta.$$

Differentiating at $t = 0$,

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}_{X\in K_t}(\|X\|^2)\bigg|_{t=0} &= -\frac{(d/dt)\operatorname{vol} K_t}{\operatorname{vol} K_t}\,\mathbb{E}_{X\in K_t}(\|X\|^2)\\ &\quad + \frac{1}{\operatorname{vol} K_t}\int_0^\infty \mathbb{E}_{X\in S_{\rho,t}}(\|X\|^2)\operatorname{vol}_{d-2}(S_{\rho,t})\rho\, d\rho\bigg|_{t=0}.\end{aligned} \tag{5}$$

We have

$$\begin{aligned}\frac{d}{dt}\operatorname{vol}(K_t)\bigg|_{t=0} &= \frac{d}{dt}\int_{-\pi}^{t}\int_0^\infty \operatorname{vol}_{d-2}(S_{\rho,\theta})\rho\, d\rho\, d\theta\bigg|_{t=0}\\ &= \int_0^\infty \operatorname{vol}_{d-2}(S_{\rho,0})\rho\, d\rho,\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2) &= \frac{\int_{S_0}\|x\|^2\rho(x)\,dx}{\int_{S_0}\rho(x)\,dx}\\ &= \frac{\int_0^\infty\int_{S_{\rho,0}}\|x\|^2\rho\,dx\,d\rho}{\int_0^\infty \operatorname{vol}_{d-2}(S_{\rho,0})\rho\,d\rho}\end{aligned} \tag{7}$$

and

$$\mathbb{E}_{X\in S_{\rho,t}}(\|X\|^2) = \frac{\int_{S_{\rho,t}}\|x\|^2\,dx}{\operatorname{vol}_{d-2}(S_{\rho,t})}. \tag{8}$$

That is, using (8), (7) and (6), we get

$$\begin{aligned}\int_0^\infty \mathbb{E}_{X\in S_{\rho,0}}(\|X\|^2)\operatorname{vol}_{d-2}(S_{\rho,0})\rho\,d\rho &= \int_0^\infty\int_{S_{\rho,0}}\|x\|^2\,dx\rho\,d\rho\\ &= \mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)\int_0^\infty \operatorname{vol}_{d-2}(S_{\rho,0})\rho\,d\rho\\ &= \mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)\frac{d}{dt}\operatorname{vol}(K_t)\bigg|_{t=0}.\end{aligned}$$

This in (5) and the fact that $\mathbb{E}_{X\in K}(\|X\|^2) = d$ (which follows from the isotropy of K; see §2) give

$$\frac{d}{dt}\,\mathbb{E}_{X\in K_t}(\|X\|^2)\bigg|_{t=0} = \frac{(d/dt)\operatorname{vol} K_t|_{t=0}}{\operatorname{vol} K}[-d + \mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)].$$

This implies

$$\frac{d}{dt}\det A(K_t)\bigg|_{t=0} = (\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2) - d)\frac{(d/dt)\operatorname{vol} K_t|_{t=0}}{\operatorname{vol} K}.$$

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Thus,

$$\begin{aligned}\frac{d}{dt}L_{K_t}^{2d}\bigg|_{t=0} &= \frac{d}{dt}\frac{\det A(K_t)}{(\operatorname{vol} K_t)^2}\bigg|_{t=0}\\ &= \frac{-2\det A(K_t)}{(\operatorname{vol} K_t)^3}\frac{d}{dt}\operatorname{vol} K_t + \frac{1}{(\operatorname{vol} K_t)^2}\frac{d}{dt}\det A(K_t)\bigg|_{t=0}\\ &= \frac{-2}{(\operatorname{vol} K)^3}\frac{d}{dt}\operatorname{vol} K_t\bigg|_{t=0} + \frac{1}{(\operatorname{vol} K)^2}(\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)-d)\\ &\quad\times\frac{(d/dt)\operatorname{vol} K_t|_{t=0}}{\operatorname{vol} K}\\ &= (\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)-d-2)\frac{(d/dt)\operatorname{vol} K_t|_{t=0}}{(\operatorname{vol} K)^3}.\end{aligned}$$

□

§4. *Proof of Theorem 1.* We start with a sketch of the proof. The actual proof is at the end of this section.

Sketch of the proof of Theorem 1. For a given facet F of P, we compute the derivative of L_P^{2d} as F "hinges" around one of its $(d-2)$-dimensional facets. To preserve convexity, this is done by hinging the halfspace inducing that facet. Setting those derivatives to zero gives d nonlinear equations on the d vertices of F. We will show that, if we fix $d-1$ vertices $v_1,\dots,v_{d-1}$, the system of equations determines the last vertex uniquely up to reflection around the hyperplane spanned by $v_1,\dots,v_{d-1}$. This implies that the two facets of P sharing $v_1,\dots,v_{d-1}$ are reflections of each other around that span. Applying the same argument to all pairs of adjacent facets, we conclude that all of P is invariant under reflection around that span.

LEMMA 6 (First-order necessary condition). *Let P be a d-dimensional simplicial polytope that is a local extremum of $P\mapsto L_P$. Let F be a facet of P with vertices $v_1,\dots,v_d$. Then, for $k=1,\dots,d$, we have*

$$\sum_{1\leqslant i\leqslant j\leqslant d}(1+\delta_{ik}+\delta_{jk})v_i\cdot v_j = \frac{(d+1)(d+2)^2}{2}.$$

Proof idea. To first-order terms and after taking care of preserving convexity, the infinitesimal hinging of a facet $F=\operatorname{conv}(v_1,\dots,v_d)$ around one of its facets $F''=\operatorname{conv}(v_2,\dots,v_d)$ is the same as adding or removing an infinitesimal layer of mass on the facet with weight proportional to barycentric coordinate x_1 (that is, 0 at F'', 1 at v_1 and interpolate affinely in F). On the other hand, the derivative of L_P^{2d}, when adding infinitesimal mass to an isotropic convex body P, is proportional to

$$\mathbb{E}(\|X\|^2)-d-2$$

where X is random according to the added mass. This is shown in Lemma 5 for our particular case. An explicit computation of $\mathbb{E}(\|X\|^2)$ completes the argument.

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Proof. We apply Lemma 5 to facet F and vertex v_k. This implies the necessary condition

$$\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2) = d+2$$

where $\mathcal{D}$ is the following distribution (from Lemma 5) of a random vector X supported on F: Let $b_1,\dots,b_d \in \mathbb{R}$ be the barycentric coordinates of a point x in F given by $x = \sum_i b_i v_i$ and $\sum_i b_i = 1$. Then $\mathcal{D}$ has density proportional to b_k.

We now compute $\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2)$ using the following known representation of the uniform distribution over a simplex using exponential random variables (see, for example, [18, Lemma 1] or [6, III.3]): let Y be a random vector, distributed uniformly in the standard simplex $\Delta^{d-1} = \operatorname{conv}\{e_i\}_{i=1}^d$. Let T be an independent random scalar distributed as the sum of d mutually independent exponential random variables with rate 1. (An exponential random variable with rate 1 has density $t \mapsto e^{-t}$ supported on $[0,\infty)$.) Then $Z := TY$ is a random vector with independent coordinates, each distributed as exponential with rate 1.

Let V be the matrix having columns $v_1,\dots,v_d$. Then VY is a uniformly random vector on $F = \operatorname{conv}\{v_i\}_{i=1}^d$. We complete now the connection with distribution $\mathcal{D}$ from the beginning of the proof. Note that $Y_1,\dots,Y_d$ are precisely barycentric coordinates of $VY \in F$. In particular, if Y has density $f(y) = 1/\mathrm{vol}_{d-1}(\Delta^{d-1})$ and Y' has density proportional to $f(y')y'_k$ supported on Δ^{d-1}, then the distribution of X, namely $\mathcal{D}$, is the same as the distribution of VY'. Thus

$$\begin{aligned}\mathbb{E}_{X\sim\mathcal{D}}(\|X\|^2) &= \frac{\int_{\Delta^{d-1}} \|Vy'\|^2 y'_k f(y')\,dy'}{\int_{\Delta^{d-1}} y'_k f(y')\,dy'}\\ &= \frac{\mathbb{E}(\|VY\|^2 Y_k)}{\mathbb{E}(Y_k)},\end{aligned}$$

where $\mathbb{E}(Y_k) = 1/d$. Also, using $Z = TY$ and the fact that Y and T are independent,

$$\mathbb{E}(\|VY\|^2 Y_k) = \frac{\mathbb{E}(\|VZ\|^2 Z_k)}{\mathbb{E}(T^3)},$$

where $\mathbb{E}(T^3) = d(d+1)(d+2)$. Finally,

$$\begin{aligned}\mathbb{E}(\|VZ\|^2 Z_1) &= \mathbb{E}\left(\left\|\sum_{i=1}^d Z_i v_i\right\|^2 Z_1\right)\\ &= \mathbb{E}\left(\sum_{i=1}^d Z_i^2 v_i\cdot v_i Z_1 + 2\sum_{i<j} Z_i Z_j v_i\cdot v_j Z_1\right)\\ &= \mathbb{E}(Z_1^3)v_1\cdot v_1 + \sum_{i=2}^d \mathbb{E}(Z_1 Z_i^2) v_i\cdot v_i\\ &\quad + 2\sum_{j=2}^d \mathbb{E}(Z_1^2 Z_j) v_1\cdot v_j + 2\sum_{1<i<j\leqslant d} \mathbb{E}(Z_1 Z_i Z_j) v_i\cdot v_j\end{aligned}$$

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$$= 6v_1 \cdot v_1 + 2\sum_{i=2}^{d} v_i \cdot v_i + 4\sum_{j=2}^{d} v_1 \cdot v_j + 2\sum_{1<i<j\leqslant d} v_i \cdot v_j$$

$$= 2\left(3v_1 \cdot v_1 + 2\sum_{j=2}^{d} v_1 \cdot v_j + \sum_{1<i\leqslant j\leqslant d} v_i \cdot v_j\right)$$

$$= 2\sum_{1\leqslant i\leqslant j\leqslant d} (1+\delta_{1i}+\delta_{1j}) v_i \cdot v_j.$$

We get

$$\mathbb{E}(\|VY\|^2 Y_k) = \frac{2}{d(d+1)(d+2)} \sum_{1\leqslant i\leqslant j\leqslant d} (1+\delta_{ki}+\delta_{kj}) v_i \cdot v_j.$$

The lemma follows. □

Proof of Theorem 1. Let P be a polytope as in the theorem. Let F and F' be two adjacent facets of P with common vertices $\{v_2, \ldots, v_d\}$ and additional vertices v_1 and v_1', respectively. The first-order necessary condition from Lemma 6 implies conditions on v_1 and v_1' given $v_2, \ldots, v_d$. Let v_1 be denoted x in those equations, with $v_2, \ldots, v_d$ as parameters. We get two types of equations: For $k = 1$, we get

$$\frac{(d+1)(d+2)^2}{2} = 3\|x\|^2 + 2\sum_{i=2}^{d} x \cdot v_j + \sum_{2\leqslant i\leqslant j\leqslant d} v_i \cdot v_j.$$

For $k = 2, \ldots, d$, we get

$$\frac{(d+1)(d+2)^2}{2} = \|x\|^2 + \sum_{i=2}^{d} x \cdot v_j + \sum_{2\leqslant i\leqslant j\leqslant d} v_i \cdot v_j + x \cdot v_k + \sum_{k\leqslant j\leqslant d} v_k \cdot v_j + \|v_k\|^2.$$

Decompose x into its components in the hyperplane spanned by $\{v_2, \ldots, v_d\}$ and orthogonal to that span. That is, let $x = x_{\mathrm{IN}} + x_{\mathrm{OUT}}$, with $x_{\mathrm{IN}} \in \mathrm{span}\{v_2, \ldots, v_d\}$ and $x_{\mathrm{OUT}} \in \mathrm{span}\{v_2, \ldots, v_d\}^{\perp}$. To get that F and F' are reflections of each other around $\mathrm{span}\{v_2, \ldots, v_d\}$, it is enough to show that the equations on x determine x_{IN} uniquely and $\|x_{\mathrm{OUT}}\|$ uniquely. In the new variables, the equations on x become

$$\frac{(d+1)(d+2)^2}{2} = 3\|x_{\mathrm{IN}}\|^2 + 3\|x_{\mathrm{OUT}}\|^2 + 2\sum_{i=2}^{d} x_{\mathrm{IN}} \cdot v_j + \sum_{2\leqslant i\leqslant j\leqslant d} v_i \cdot v_j \quad (9)$$

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and for $k = 2, \dots, d$, we get

$$\frac{(d+1)(d+2)^2}{2} = \|x_{\mathrm{IN}}\|^2 + \|x_{\mathrm{OUT}}\|^2 + \sum_{i=2}^{d} x_{\mathrm{IN}} \cdot v_j + \sum_{2 \leqslant i \leqslant j \leqslant d} v_i \cdot v_j + x_{\mathrm{IN}} \cdot v_k + \sum_{k \leqslant j \leqslant d} v_k \cdot v_j + \|v_k\|^2. \tag{10}$$

Note that the only nonlinear part is $\|x_{\mathrm{IN}}\|^2 + \|x_{\mathrm{OUT}}\|^2 = \|x\|^2$. Use (9) in (10) to eliminate $\|x\|^2$. We get

$$\|x\|^2 = \frac{1}{3}\frac{(d+1)(d+2)^2}{2} - \frac{2}{3}\sum_{i=2}^{d} x_{\mathrm{IN}} \cdot v_j - \frac{1}{3}\sum_{2 \leqslant i \leqslant j \leqslant d} v_i \cdot v_j$$

and, for $k = 2, \dots, d$,

$$\frac{1}{3} x_{\mathrm{IN}} \cdot \sum_{j=2}^{d} v_j + x_{\mathrm{IN}} \cdot v_k + \frac{2}{3}\sum_{2 \leqslant i \leqslant j \leqslant d} v_i \cdot v_j + \sum_{k \leqslant j \leqslant d} v_k \cdot v_j + \|v_k\|^2 = \frac{2}{3}\frac{(d+1)(d+2)^2}{2}. \tag{11}$$

As x_{IN} lies in $\mathrm{span}\{v_2, \dots, v_d\}$ and $v_2, \dots, v_d$ are linearly independent, we can interpret linear system of equations (11) as a system expressing x_{IN} in the unknown coordinates $y_j := x_{\mathrm{IN}} \cdot v_j$ ($j = 2, \dots, d$), and x_{IN} will be unique as long as the matrix of the linear system is invertible. In unknowns y_j, equations $k = 2, \dots, d$ in system (11) become

$$\frac{1}{3}\sum_{j=2}^{d} y_j + y_k + \frac{2}{3}\sum_{2 \leqslant i \leqslant j \leqslant d} v_i \cdot v_j + \sum_{k \leqslant j \leqslant d} v_k \cdot v_j + \|v_k\|^2 = \frac{2}{3}\frac{(d+1)(d+2)^2}{2}.$$

The matrix of the system in unknowns y_j is $\mathbb{1}\mathbb{1}^T/3 + I$, which is invertible. Therefore, there is at most one x_{IN}. Given x_{IN}, (9) determines $\|x_{\mathrm{OUT}}\|$ uniquely, as claimed. □

§5. *Proof of Theorem 2.*

Proof idea. An extremal isotropic simplicial polytope has a hyperplane of symmetry by Theorem 1. Given that the polytope is simplicial but not a simplex, one can show (Theorem 3, a restatement of results in [**5**]) that in this case Blaschke's Schüttelung process strictly increases the isotropic constant.

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Proof. Assume, without loss of generality, that P is in isotropic position. By Theorem 1, P has a hyperplane of symmetry H containing the origin. Let f_H, g_H be a representation of P as in (1). The fact that H is a hyperplane of symmetry implies $g_H = f_H$. By Theorem 3, f_H is linear in $\pi_H(P)$. This implies that $\{(x, f_H(x)) : x \in \pi_H(P)\}$ is a facet of P, and therefore a $(d-1)$-dimensional simplex. Thus, $\pi_H(P)$ is a $(d-1)$-dimensional simplex. If $d \leqslant 2$ the argument below does not work (as being simplicial is no restriction for $d = 2$), but the claim of the theorem is obvious for $d = 1$ and known to be true for $d = 2$, as triangles are the only maximizers of the isotropic constant in that case (see, for example, [**5**] for a proof that triangles are maximizers, and [**17**] for a proof that they are the only maximizers, which generalizes a result from [**8**]). If $d > 2$, the result follows from Lemma 7, with $S = \pi_H(P)$ and $f = f_H$. □

LEMMA 7. *Let $d \geqslant 3$. Let P be a d-dimensional simplicial polytope of the form $\{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in S, -f(x) \leqslant y \leqslant f(x)\}$ where $S \subseteq \mathbb{R}^{d-1}$ is a $(d-1)$-dimensional simplex with vertices $v_1, \ldots, v_d \in \mathbb{R}^{d-1}$ and $f : S \to \mathbb{R}$ is an affine function. Then P is a simplex.*

Proof. It is enough to show that $f(v_i)$ is non-zero for exactly one value of i. We will show this now. Let $F_1, \ldots, F_d$ be the facets of S, where $F_i = \mathrm{conv}(\{v_1, \ldots, v_d\}\backslash\{v_i\})$. Let $H = \{(x, 0) \in \mathbb{R}^{d-1} \times \mathbb{R}\}$. For $i \in [d]$, let H_i be the affine hull of $\pi_H^{-1}(F_i)$. In other words, H_i is the affine hyperplane orthogonal to H and containing F_i. Now for every i, $P \cap H_i$ is a face of P. If $\dim P \cap H_i < d-1$, then $f(v_j) = 0$ for all j different from i. If $\dim P \cap H_i = d-1$, then $P \cap H_i$ is a facet and therefore a simplex, and this implies that $f(v_j)$ is non-zero for at most one value of j in $[d]\backslash\{i\}$. In any case, for every set of $d-1$ values in $[d]$, there can be at most one j in it such that $f(v_j) \neq 0$. Assuming $d \geqslant 3$, this necessarily implies that there is at most one $i \in [d]$ such that $f(v_i) \neq 0$. □

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