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The Kahn–Priddy theorem

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1. *Introduction.* Let $\pi_r^S(X)$ be the stable homotopy group

$$\operatorname*{Lim}_{n\to\infty} \pi_{n+r}(S^n X),$$

where $S^n X$ means the n-fold suspension of X. For example, the groups $\pi_r^S(S^0)$ are the stable homotopy groups of spheres. Let

$$O = \operatorname*{Lim}_{n\to\infty} O(n)$$

be the 'infinite-dimensional' orthogonal group. Then topologists are familiar with the 'stable J-homomorphism'

$$J : \pi_r(O) \to \pi_r^S(S^0).$$

G. W. Whitehead observed that J factors through an 'even more stable' J-homomorphism

$$J' : \pi_r^S(O) \to \pi_r^S(S^0);$$

he conjectured that J' is epi (for $r > 0$).

It is well known that there is a canonical embedding of real projective space RP^∞ in the orthogonal group O. On the basis of calculations, M. E. Mahowald conjectured that $\pi_r^S(RP^\infty)$ maps onto the 2-primary component of $\pi_r^S(S^0)$ (for $r > 0$). Of course this implies the 2-primary part of G. W. Whitehead's conjecture. Mahowald's conjecture has been proved by Kahn and Priddy (4), and their work has several features of interest which I will not mention here. I am grateful to D. S. Kahn for a number of letters on this subject.

Two further points should be mentioned. The statement above refers to the prime 2, but Kahn and Priddy realized that there is an analogous statement for any odd prime p. The statement also refers to a particular map $RP^\infty \to O$, and this map induces a 'stable map' from RP^∞ to S^0. (The notion of a 'stable map' will be made precise below.) However, it is rather hard to construct a proof which uses the fact that the stable map from RP^∞ to S^0 is this particular one and no other. Kahn and Priddy quite rightly constructed a general proof, which applies to all stable maps from RP^∞ to S^0 with a given property. More precisely, they consider stable maps ϕ from RP^∞ to S^0 which induce an isomorphism of the first stable homotopy group π_1^S; this group is of course Z_2 both for RP^∞ and for S^0.

I have recently given a treatment of the Kahn–Priddy theorem in lectures. One of the key features of this treatment was as follows. By working in a suitable category of spectra, one can make the Kahn–Priddy theorem assert not only the existence

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and properties of a stable map ϕ from RP^∞ to S^0, but also the essential uniqueness of ϕ. Because of the uniqueness clause, it then becomes sufficient to do all the hard work not for a general map ϕ, but for one particular map of one's own choice. Now it was already known to Kahn and Priddy that there is available a 'convenient' choice for ϕ which simplifies the proof and reduces the hard work; one therefore makes this choice and uses the simpler proof.

As a digression, it is not obvious that the 'convenient' choice for ϕ is the same as the one obtained via O. Indeed, when one replaces the prime 2 by an odd prime, one still has a 'convenient' choice of ϕ; but no choice of ϕ will factor through O, even stably. It may be that the role of O in all this is that of a red herring. Perhaps one should replace it by the space 'Im J'.

To continue, my treatment included four points.

(i) The formulation of a 'strong' form of the Kahn–Priddy theorem, including a uniqueness clause. This formulation will be given as (2·3) below.

(ii) The formulation of a 'weak' form of the Kahn–Priddy theorem, in which it is sufficient to prove the result for one map of one's own choice. Equivalent formulations will be given as (2·4) and (2·5) below.

(iii) The proof of the 'weak' form of the Kahn–Priddy theorem. This I did only for $p = 2$, and therefore $p = 2$ is the only case for which I personally can swear that the result is true. There is no doubt, however, that the result is true for all primes. My proof followed Kahn and Priddy with simplifications, and so I will not give it here. (After taking the favourable choice of ϕ, I used different subgroups in the transfer argument. Instead of the Sylow subgroup of the symmetric group Σ_{2^n} on 2^n letters, it is sufficient to consider a 'wreath product' subgroup of Σ_{2m}, namely an extension with kernel $(Z_2)^m$ and quotient Σ_m.)

(iv) The proof of the following result.

THEOREM 1·1. *The 'weak' form of the Kahn–Priddy theorem,* (2·4) *or* (2·5), *is equivalent to the 'strong' form* (2·3).

This can be proved by standard techniques, and it is the object of this paper to record the proof.

The remainder of this paper is arranged as follows. Section 2 explains the various forms of the Kahn–Priddy theorem. Section 3 contains a few easy proofs postponed from Section 2. Section 4 contains the proof of Theorem 1·1.

2. *Formulation of the statements.* The object of this section is to explain the 'weak' and 'strong' forms of the Kahn–Priddy theorem.

When we see a result about homotopy groups, it is natural to ask what geometrical result about spaces underlies the result about homotopy groups. For example, a well-known theorem of James(3) asserts that if n is odd, we have an exact sequence

$$\ldots \to \pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \xrightarrow{H} \pi_{r+1}(S^{2n+1}) \xrightarrow{\Delta} \pi_{r-1}(S^n) \to \ldots.$$

The underlying geometrical fact is that there is a fibering $F \to E \to B$ in which the spaces F, E and B are (up to weak equivalence) S^n, ΩS^{n+1} and ΩS^{2n+1} respectively. The exact sequence due to James is the exact homotopy sequence of this fibering. It

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is arguable that in such cases the geometrical fact should be treated as the main theorem; the consequent results about homotopy groups (or other invariants) should be treated as corollaries. We will apply this philosophy to the theorem of Kahn and Priddy.

As a matter of algebra, the cheapest way to prove that a homomorphism α is epi is to construct a one-sided inverse for it, say β, so that $\alpha\beta = 1$ and α is a split epimorphism. The proof of Kahn and Priddy may be put in this form. In this case, the groups which occur are stable homotopy groups; they may be considered either as homotopy groups of spectra, or as homotopy groups of spaces. The homomorphism α may be induced by a map of spectra or by a map of spaces. The homomorphism β can be induced by a map of spaces, but not by a map of spectra. The underlying geometrical fact is that one space is a retract in another. Thus, although the problem solved by Kahn and Priddy appears to lie in stable homotopy-theory, their solution uses methods which in a sense are unstable; this is one reason for interest in it.

To make all this precise, we need some details on spaces and spectra. Let $\mathscr{C}$ be the category in which the objects are CW-complexes with base-point, and the morphisms are homotopy classes of maps (where maps and homotopies preserve the base-point). Let $\mathscr{S}$ be Boardman's category of spectra (1, 6), so that the objects are spectra $\mathbf{E}$ and the morphisms are homotopy classes (in a suitable sense.) (The reader is assured that we need only the simplest properties of this category.) $[X, Y]$ will mean the set of morphisms from X to Y in the appropriate category. The functor $S^\infty: \mathscr{C} \to \mathscr{S}$ assigns to each complex the corresponding suspension spectrum; for example, $S^\infty S^0$ is the sphere-spectrum. If X and Y are complexes and X is finite-dimensional, then

$$[S^\infty X, S^\infty Y]_{\mathscr{S}} = \operatorname*{Lim}_{n\to\infty} [S^n X, S^n Y]_{\mathscr{C}}.$$

(Here the limit is attained for n sufficiently large.) The words 'stable map from X to Y' in section 1 may be interpreted as meaning a morphism $S^\infty X \to S^\infty Y$ in $\mathscr{S}$.

The functor S^∞ has an adjoint $\Omega^\infty: \mathscr{S} \to \mathscr{C}$, so that

$$[S^\infty X, \mathbf{E}]_{\mathscr{S}} = [X, \Omega^\infty \mathbf{E}]_{\mathscr{C}},$$

where $X \in \mathscr{C}$, $\mathbf{E} \in \mathscr{S}$. Crudely speaking, a spectrum $\mathbf{E}$ is a sequence of spaces E_n, and the complex $\Omega^\infty \mathbf{E}$ is $\operatorname*{Lim}_{n\to\infty} \Omega^n E_n$. Alternatively, it is easy to show (using Brown's Representability Theorem (2)) that S^∞ has an adjoint; one can then take this adjoint as the definition of Ω^∞, and show that under suitable hypotheses, $\Omega^n E_n$ approximates to $\Omega^\infty \mathbf{E}$ in a suitable sense.

If $r \geqslant 0$, the rth homotopy group of a spectrum is given by

$$\begin{aligned} \pi_r(\mathbf{E}) &= [S^\infty S^r, \mathbf{E}]_{\mathscr{S}} \\ &= [S^r, \Omega^\infty \mathbf{E}]_{\mathscr{C}}. \end{aligned}$$

The stable homotopy groups of a complex X are the homotopy groups of the corresponding suspension spectrum. That is, if $r \geqslant 0$ we have

$$\begin{aligned} \pi_r^S(X) &= \operatorname*{Lim}_{n\to\infty} [S^n S^r, S^n X] \\ &= [S^\infty S^r, S^\infty X]_{\mathscr{S}} \\ &= [S^r, \Omega^\infty S^\infty X]_{\mathscr{C}}. \end{aligned}$$

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In particular, the stable homotopy groups of spheres are the homotopy groups of the space $\Omega^\infty S^\infty S^0$. This space has pathwise-components in (1–1) correspondence with the integers; we write $(\Omega^\infty S^\infty S^0)_0$ for the component of the base-point. If W is a connected CW-complex, we have

$$[W, \Omega^\infty S^\infty S^0] = [W, (\Omega^\infty S^\infty S^0)_0].$$

Suppose given a prime p. We say that a complex W is p-primary if the homology groups $H_r(W)$ are p-primary for $r > 0$. Up to equivalence, we can decompose $(\Omega^\infty S^\infty S^0)_0$ as a product $X \times Y$, where the homotopy groups of X are p-primary and those of Y are of order prime to p. We see by obstruction-theory that if W is p-primary, then

$$[W, Y] = 0.$$

(Y is not simply connected if $p \neq 2$, but its fundamental group acts trivially on its higher homotopy groups.) So (if W is p-primary) we have

$$[W, X \times Y] = [W, X].$$

Let us write ${}_p(\Omega^\infty S^\infty S^0)_0$ for the p-primary factor X; then (if W is p-primary) we have

$$[W, (\Omega^\infty S^\infty S^0)_0] = [W, {}_p(\Omega^\infty S^\infty S^0)_0].$$

For the prime 2, then, the basic geometrical fact is that the complex ${}_2(\Omega^\infty S^\infty S^0)_0$ is a retract of $\Omega^\infty S^\infty RP^\infty$. We must now explain what space replaces RP^∞ when we replace the prime 2 by an odd prime p.

Let Z_p be a cyclic group of order p, embedded as a Sylow subgroup in the symmetric group Σ_p on p letters; and let BG be the classifying space of G. Then the inclusion $i: Z_p \to \Sigma_p$ induces $Bi: BZ_p \to B\Sigma_p$. When we specialize to the case $p = 2$, both BZ_p and $B\Sigma_p$ specialize to RP^∞. We might consider generalizing RP^∞ either to BZ_p or to $B\Sigma_p$. It turns out that although Z_p is a smaller group than Σ_p, $B\Sigma_p$ is better than BZ_p because $B\Sigma_p$ is 'smaller' than BZ_p in a homological sense. More precisely, let x be the generator in $H^1(BZ_p; Z_p)$, and let β be the Bockstein boundary; then we know that $y = \beta x$ is the generator in $H^2(BZ_p; Z_p)$, and that $H^*(BZ_p; Z_p)$ has a base consisting of the monomials $x^\epsilon y^n$, where $\epsilon = 0, 1$ and $n = 0, 1, 2, 3, \ldots$.

LEMMA 2·1. *The induced homomorphism*

$$(Bi)^*: H^*(B\Sigma_p; Z_p) \to H^*(BZ_p; Z_p)$$

is mono, and a base for its image is given by the monomials $x^\epsilon y^{m-\epsilon}$ *with* $\epsilon = 0, 1$ *and* $m \equiv 0 \bmod (p-1)$.

In order not to interrupt the exposition in this section, we defer the proofs of even the most trivial results, such as this one.

If $p > 2$, the space $B\Sigma_p$ is better than BZ_p, but it is not p-primary; for example, $H_1(B\Sigma_p) = Z_2$. We must repair this defect; first we need a definition. Let Q_p be the ring of fractions a/b with b prime to p; a map $f: X \to Y$ is said to be a p-equivalence if the map

$$f_*: H_*(X; Q_p) \to H_*(Y; Q_p)$$

is iso. If X and Y are connected and the homology groups $H_r(X)$, $H_r(Y)$ are finite for $r > 0$, it is equivalent to ask that the p-components of $H_r(X)$ and $H_r(Y)$ be isomorphic under f_* (for each $r > 0$).

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LEMMA 2·2. (i) *If $p > 2$ there is a CW-complex $L = L(p)$ which is 1-connected, p-primary and admits a p-equivalence $f: B\Sigma_p \to L$.*

(ii) *Such a complex L is unique up to canonical equivalence.*

(iii) *The first non-zero homotopy group of $S^\infty L$ is Z_p in dimension $2p-3$.*

It follows immediately that we can realize L as a CW-complex, having one cell in each dimension n such that $n \geqslant 0$ and $n \equiv 0$ or $-1 \bmod 2(p-1)$, and having no cells in any other dimension. (Since L is simply-connected, we can use a 'homology decomposition' of L.) (The attaching map for the cell of dimension $2(p-1)n$ has degree p if $n > 0$.) We then write L^n for the skeleton of L consisting of the cells of dimension $\leqslant 2(p-1)n$. We interpret L^∞ as L.

If $p = 2$, we interpret L as RP^∞ and L^n as RP^{2n}.

Let ${}_pG$ be the p-primary component of the finite Abelian group G. When we go from the case $p = 2$ to the case $p > 2$, we have to replace the group $\pi_1^S(S^0) = Z_2$ by the p-component ${}_p\pi^S_{2p-3}(S^0) = Z_p$.

We can now formulate a strong form of the Kahn–Priddy theorem.

Formulation 2·3. (i) *For $n = 1, 2, 3, \ldots$ and for $n = \infty$ there is a morphism*

$$\phi_n: S^\infty L^n \to S^\infty S^0$$

(in $\mathscr{S}$) which induces an isomorphism of ${}_p\pi_{2p-3}$.

(ii) *Such a morphism ϕ_n is unique up to composition with an equivalence*

$$\epsilon: S^\infty L^n \to S^\infty L^n$$

in $\mathscr{S}$.

(iii) *For any such morphism $\phi = \phi_\infty$, the induced map*

$$\Omega^\infty\phi: \Omega^\infty S^\infty L \to {}_p(\Omega^\infty S^\infty S^0)_0$$

is, up to equivalence of spaces, the projection $X \times Y \to X$ of a product on one factor.

Various comments are in order. In part (ii), it is clear that we cannot hope to do better; if ϕ_n induces an isomorphism of ${}_p\pi_{2p-3}$ and if ϵ is an equivalence, then $\phi_n\epsilon$ induces an isomorphism of ${}_p\pi_{2p-3}$. In part (iii), the map $\Omega^\infty\phi$ runs in the first instance from $\Omega^\infty S^\infty L$ to $\Omega^\infty S^\infty S^0$; but since $\Omega^\infty S^\infty L$ is connected and p-primary, we may regard $\Omega^\infty\phi$ as mapping into ${}_p(\Omega^\infty S^\infty S^0)_0$.

In the original work of Kahn and Priddy, the functors Ω^∞ and S^∞ do not occur separately, but only as the composite $Q = \Omega^\infty S^\infty$. One justification for introducing the category $\mathscr{S}$ and the separate functors Ω^∞, S^∞ is that this allows one to formulate the uniqueness statement (2·3ii) in a simple and conceptual form. I concede that many of the benefits which one obtains from the pair of adjoint functors Ω^∞, S^∞ can also be obtained by exploiting the natural transformations $X \to QX$ and $QQX \to QX$; but by the time one gets to $QQQX$ such manipulations tend to obscure the real issues.

We can also formulate a weak form of the Kahn–Priddy theorem.

Formulation 2·4. *For $n = 1, 2, 3, \ldots$ there is a morphism $\phi_n: S^\infty L^n \to S^\infty S^0$ in $\mathscr{S}$ such that the induced homomorphism*

$$\phi_{n*}: [S^\infty W, S^\infty L^n]_{\mathscr{S}} \to [S^\infty W, S^\infty S^0]_{\mathscr{S}}$$

is epi whenever W is a connected p-primary CW-complex of dimension $< 2(p-1)n$.

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Formulation 2·5. *For* $n = 1, 2, 3, \ldots$ *and for* $m > 2(p-1)n$ *there is a map*

$$\phi_n : S^m L^n \to S^m$$

in $\mathscr{C}$ *such that the induced homomorphism*

$$\phi_{n*} : [S^m W, S^m L^n]_{\mathscr{C}} \to [S^m W, S^m]_{\mathscr{C}}$$

is epi whenever W *is a connected* p*-primary* CW*-complex of dimension* $< 2(p-1)n$.

Various comments are in order. The two formulations are strictly equivalent; if we take (2·4) and translate the statements about spectra into elementary terms, we obtain (2·5). The track groups $[X, Y]$ which occur in (2·5) are stable. As remarked above, the main advantage of (2·4) or (2·5) is that in order to prove it, it is sufficient to consider one map ϕ_n, which can be chosen to simplify the proof. It is not even necessary to have any relation between the choices of ϕ_n for different values of n.

Some readers may be willing to believe that

$$\phi_{n*} : [S^m S^r, S^m L^n] \to {}_p[S^m S^r, S^m]$$

is epi for $r < 2(p-1)n < m$, without being willing to replace the sphere S^r by a complex W. The answer is to look at the final steps of the Kahn–Priddy proof. Here we find the following diagram.

$$\begin{array}{ccccc} S^m X & \xrightarrow{f} & S^m \Omega^m S^m L^n & \xrightarrow{S^m \Omega^m \phi_n} & S^m (\Omega^m S^m)_0 \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ & & S^m L^n & \xrightarrow{\phi_n} & S^m \end{array}$$

The natural transformation ϵ is defined by 'evaluation'. It is not asserted that the map f is an m-fold suspension; but the complex X and the map f are so arranged that the composite

$$S^m X \xrightarrow{(S^m \Omega^m \phi_n) f} S^m (\Omega^m S^m)_0$$

is a p-equivalence up to dimension $m + 2(p-1)n$ approximately. Suppose given a connected p-primary complex W and a map

$$g : S^m W \to S^m.$$

By considering the adjoint

$$g' : W \to (\Omega^m S^m)_0$$

we can factor g in the form

$$S^m W \xrightarrow{S^m g'} S^m (\Omega^m S^m)_0 \xrightarrow{\epsilon} S^m.$$

By the choice of X we can lift $S^m g'$ to $S^m X$, and so factor g through ϕ_n. This proves that ϕ_{n*} is epi; and in this proof, a complex W gives us no more trouble than a sphere.

3. *Some easy proofs.* This section contains a few proofs postponed from section 2.

Proof of Lemma 2·1. This induced homomorphism

$$(Bi)^* : H^*(B\Sigma_p; Z_p) \to H^*(BZ_p; Z_p)$$

is mono because the transfer provides a one-sided inverse for it. It can easily be shown that the image is contained in the subspace spanned by the monomials $x^\epsilon y^{m-\epsilon}$ with $\epsilon = 0, 1$ and $m \equiv 0 \bmod (p-1)$; see (5).

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Let ρ be the permutation representation of Σ_p; then its Chern class $c_{p-1}(\rho)$ is an element of $H^{2(p-1)}(B\Sigma_p)$ which restricts to a non-zero multiple of y^{p-1} in

$$H^{2(p-1)}(BZ_p; Z_p).$$

So all the powers y^m with $m \equiv 0 \bmod (p-1)$ lie in the image. Finally, since $H_r(B\Sigma_p)$ has no elements of order p^2 and is zero for $r = 2s > 0$, the Bockstein boundary

$$\beta: H^{2s-1}(B\Sigma_p; Z_p) \to H^{2s}(B\Sigma_p; Z_p)$$

is iso for $s > 0$. This proves (2·1).

Proof of Lemma 2·2. (i) Assume $p > 2$. Let Z_2 be the subgroup of Σ_p generated by a single transposition, say (12). The inclusion $i: Z_2 \to \Sigma_p$ induces $Bi: BZ_2 \to B\Sigma_p$; form the mapping-cone

$$X = B\Sigma_p \cup_{Bi} CBZ_2.$$

Then X is simply-connected and its homology groups are finite. It is a standard deduction that X is equivalent to a product $L \times M$, where the homotopy groups of X are p-primary and those of M are of order prime to p. The composite map

$$B\Sigma_p \to X \to L$$

is a p-equivalence.

(ii) Suppose L, L' are 1-connected and p-primary, while $B\Sigma_p \to L$, $B\Sigma_p \to L'$ are p-equivalences. Without loss of generality we may assume $B\Sigma_p \subset L$. Then obstruction-theory shows that we can extend the map $B\Sigma_p \to L'$ over L.

(iii) Finally, the first non-zero homology group of L is Z_p in dimension $2p-3$. This proves (2·2).

Next we wish to prove that the 'strong' form of the Kahn–Priddy theorem does indeed imply the 'weak' one. For later use we record the first step as a lemma.

LEMMA 3·1. *Assume the statement* (2·3iii). *Then for any such* ϕ *the induced homomorphism*

$$\phi_*: [S^\infty W, S^\infty L] \to [S^\infty W, S^\infty S^0]$$

is epi whenever W *is a connected* p*-primary* CW*-complex.*

Proof. Assuming (2·3iii), the function

$$(\Omega^\infty \phi)_*: [W, \Omega^\infty S^\infty L] \to [W, \Omega^\infty S^\infty S^0]$$

is a surjection of sets. By adjointness,

$$\phi_*: [S^\infty W, S^\infty L] \to [S^\infty W, S^\infty S^0]$$

is epi. This proves (3·1).

If we assume further that W is of dimension $< 2(p-1)n$, then $[S^\infty W, S^\infty L]$ can be replaced by $[S^\infty W, S^\infty L^n]$, and ϕ can be replaced by its restriction to $S^\infty L^n$. This yields (2·4).

4. *Proof of the main theorem.* In this section we will see that the 'weak' form of the Kahn–Priddy theorem implies the 'strong' form. We build up the proof in steps.

LEMMA 4·1. *There is a morphism* $\phi_1: S^\infty L^1 \to S^\infty S^0$ *(in* $\mathscr{S}$*) which induces an isomorphism from* $\pi_{2p-3}(S^\infty L^1) = Z_p$ *to* ${}_p\pi_{2p-3}(S^\infty S^0) = Z_p$.

This is immediate; a map $f: S^{2p} \to S^3$ representing a generator of ${}_p\pi_{2p}(S^3) = Z_p$ can be extended to $S^{2p} \cup_p e^{2p+1}$.

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From this point on, we assume (2·4).

LEMMA 4·2. *The existence statement* (2·3i) *is true for* $n < \infty$.

Proof. L^1 is a connected p-primary complex. So assuming 2·4, the morphism

$$\phi_1 : S^\infty L^1 \to S^\infty S^0$$

of (4·1) can be factored through the map

$$\phi_n : S^\infty L^n \to S^\infty S^0$$

of (2·4). So $$\phi_{n*} : \pi_{2p-3}(S^\infty L^n) \to {}_p\pi_{2p-3}(S^\infty S^0)$$

is epi. Since both groups are Z_p, this homomorphism ϕ_{n*} is iso.

LEMMA 4·3. *Let*

$$\dots \leftarrow E_n \leftarrow E_{n+1} \leftarrow \dots$$

be an inverse system of non-empty finite sets. Then $\varprojlim_{\leftarrow} E_n$ *is non-empty.*

This is a standard triviality.

LEMMA 4·4. *The existence statement* (2·3(i)) *is true for* $n = \infty$.

Proof. Let E_n be the subset of elements in $[S^\infty L^n, S^\infty S^0]$ which induce an isomorphism of ${}_p\pi_{2p-3}$. The sets E_n form an inverse system; they are non-empty by (4·2), and finite because $[S^\infty L^n, S^\infty S^0]$ is finite. So $\varprojlim_{\leftarrow} E_n$ is non-empty by (4·3). But

$$[S^\infty L^\infty, S^\infty S^0] \to \varprojlim_{\leftarrow n} [S^\infty L^n, S^\infty S^0]$$

is epi, so we can lift any element of $\varprojlim_{\leftarrow} E_n$ to a morphism $\phi^\infty : S^\infty L^\infty \to S^\infty S^0$. This morphism induces an isomorphism of ${}_p\pi_{2p-3}$.

LEMMA 4·5. *Let* $n = 1, 2, 3, \dots$ *or* $n = \infty$. *Then a morphism* $f : S^\infty L^n \to S^\infty L^n$ *which induces an isomorphism of* π_{2p-3} *is an equivalence in* $\mathscr{S}$.

Proof. Let us use the expression $x^e y^{m-e}$ from (2·1) to mean also the corresponding element in $H^*(S^\infty L^n; Z_p)$. If $f : S^\infty L^n \to S^\infty L^n$ induces an isomorphism of π_{2p-3}, then $f^* x = cx$ for some $c \neq 0$ in Z_p. Now f^* commutes with the Steenrod operations $Q_0 = \beta$ and $Q_1 = P_1\beta - \beta P_1$ (where P_1 is interpreted as Sq^2 if $p = 2$). We have the formulae

$$Q_0 x y^{m-1} = y^m,$$
$$Q_1 x y^{m-1} = y^{m+(p-1)}.$$

We see by induction that

$$f^* : H^*(S^\infty L^n; Z_p) \to H^*(S^\infty L^n; Z_p)$$

is multiplication by c (at each step we use Q_1 to go up $2p-1$ dimensions and Q_0 to go down 1). So f is an equivalence, by the theorem of J. H. C. Whitehead in $\mathscr{S}$ (1, 6).

LEMMA 4·6. *The uniqueness statement* (2·3(ii)) *is true for* $n < \infty$.

Proof. Let $\psi : S^\infty L^n \to S^\infty S^0$ be any map inducing an isomorphism of ${}_p\pi_{2p-3}$. Let $\phi_{n+1} : S^\infty L^{n+1} \to S^\infty S^0$ be as in (2·4). Then by (2·4), we can factor ψ in the form

$$S^\infty L^n \xrightarrow{f} S^\infty L^{n+1} \xrightarrow{Q_{n+1}} S^\infty S^0.$$

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Here f induces a monomorphism of ${}_p\pi_{2p-3}$, and therefore an isomorphism since both $\pi_{2p-3}(S^\infty L^n)$ and $\pi_{2p-3}(S^\infty L^{n+1})$ are Z_p. We can also assume, of course, that f maps $S^\infty L^n$ into $S^\infty L^n$. Then f is an equivalence from $S^\infty L^n$ to $S^\infty L^n$, by (4·5).

LEMMA 4·7. *Let $\phi: S^\infty L^\infty \to S^\infty S^0$ be any map as in* (2·3i). *Then the induced map*

$$\phi_*: [S^\infty W, S^\infty L^\infty] \to [S^\infty W, S^\infty S^0]$$

is epi where W is connected, p-primary and finite-dimensional.

Proof. Choose n finite but large compared with the dimension of W. Then by (2·4), there is a map

$$\phi_n: S^\infty L^n \to S^\infty S^0$$

such that

$$\phi_{n*}: [S^\infty W, S^\infty L^n] \to [S^\infty W, S^\infty S^0]$$

is epi. Let $\psi = \phi | S^\infty L^n$. Then by (4·6), $\psi = \phi_n f$, where f is an equivalence. Therefore

$$\psi_*: [S^\infty W, S^\infty L^n] \to [S^\infty W, S^\infty S^0]$$

is epi. This proves (4·7).

LEMMA 4·8. *We may assume without loss of generality that the complex $X = {}_p(\Omega^\infty S^\infty S^0)_0$ is the union of an increasing sequence of subcomplexes X_n which are finite, connected and p-primary.*

Proof. If $p > 2$ then ${}_p(\Omega^\infty S^\infty S^0)$ is 1-connected, and we can simply use a homology decomposition. It remains to consider the case $p = 2$. In this case we have a map

$$RP^\infty \to \Omega^\infty S^\infty S^0$$

(adjoint to ϕ) inducing an isomorphism of π_1; we may suppose it maps into

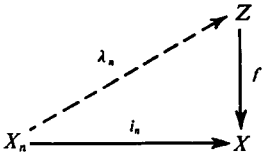
$${}_2(\Omega^\infty S^\infty S^0)_0 = \Gamma,$$

say. Let $\tilde{\Gamma}$ be the universal cover of Γ. Since Γ is an H-space, we can construct the map

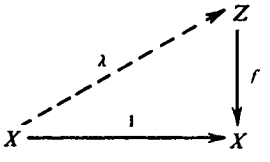
$$RP^\infty \times \tilde{\Gamma} \to \Gamma \times \Gamma \to \Gamma.$$

It is an equivalence, by J. H. C. Whitehead's Theorem. Now filter RP^∞ by using the subcomplexes RP^{2n} and use a homology decomposition of the 1-connected space $\tilde{\Gamma}$.

LEMMA 4·9. *Let Z be a connected CW-complex with finite homotopy groups, X a CW-complex which is the union of an increasing sequence of finite subcomplexes X_n, and $f: Z \to X$ a map such that each diagram*



can be completed up to homotopy. Then the diagram



can be completed up to homotopy.

Proof. Let E_n be the subset of $[X_n, Z]$ consisting of maps λ_n such that $f\lambda_n \sim i_n$.

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By hypothesis, it is non-empty, and it is finite since $[X_n, Z]$ is finite. By (4·3), $\varprojlim E_n$ is non-empty. But

$$[X, Z] \to \varprojlim_n [X_n, Z]$$

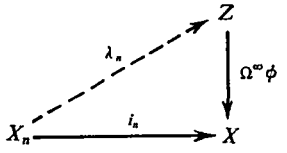
is onto; so we can choose a map $\gamma: X \to Z$ which projects to an element of $\varprojlim E_n$. Then the composite $f\gamma$ induces the identity map of homotopy groups, because each map $S^r \to X$ factors through some X_n. By the theorem of J. H. C. Whitehead, $f\gamma$ is an equivalence. Then $\lambda = \gamma(f\gamma)^{-1}$ has $f\lambda \sim 1$. This proves (4·9).

LEMMA 4·10. (2·3(iii)) *is true.*

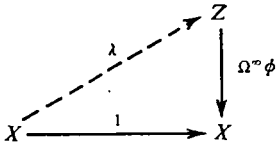
Proof. In (4·9) we take Z to be $\Omega^\infty S^\infty L$, X to be as in (4·8), and f to be $\Omega^\infty\phi$. By (4·7), the homomorphism

$$(\Omega^\infty\phi)_*: [X_n, \Omega^\infty S^\infty L] \to [X_n, \Omega^\infty S^\infty S^0]$$

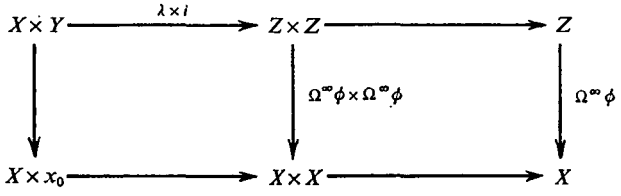
is epi, so we can complete the following diagram.



By (4·9), we can complete the following diagram.



Let Y be a CW-complex weakly equivalent to the fibre of $\Omega^\infty\phi$. Since Z and X are H-spaces and $\Omega^\infty\phi$ is an H-map, we can construct the following diagram, in which x_0 is the base-point.



The top line is an equivalence by the theorem of J. H. C. Whitehead, and the bottom line is homotopic to the identity map. This proves (4·10).

At this stage we have the data for Lemma 3·1, and therefore we have its conclusion.

LEMMA 4·11. *The uniqueness statement* (2·3(ii)) *is true for* $n = \infty$.

Proof. Let $\phi, \psi: S^\infty L^\infty \to S^\infty S^0$ be two maps both inducing an isomorphism of ${}_p\pi_{2p-3}$. By (3·1) applied (say) to ϕ, we can express ψ as a composite

$$S^\infty L \xrightarrow{\epsilon} S^\infty L \xrightarrow{\phi} S^\infty S^0.$$

Here ϵ must induce an isomorphism of π_{2p-3}, and so must be an equivalence by (4·5).

This completes the deduction of all the parts of (2·3), and so proves Theorem 1·1.

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REFERENCES

(1) Adams, J. F. *Stable Homotopy*, lecture notes, University of Chicago 1972.
(2) Brown, E. H. Cohomology theories. *Ann. of Math.* (2) **75** (1962), 467–484.
(3) James, I. M. On the suspension sequence. *Ann. of Math.* (2) **65** (1957), 74–107.
(4) Kahn, D. S. and Priddy, S. B. Applications of the transfer to stable homotopy theory (preprint).
(5) Steenrod, N. E. Cyclic reduced powers of cohomology classes. *Proc. Nat. Acad. Sci. USA* **39** (1953), pp. 217–223, especially Theorem 4.8, pp. 218–219.
(6) Vogt, R. *Boardman's Stable Homotopy Category*, Aarhus University (Lecture Notes series No. 21), 1970.

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