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Correlated Matroids

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Semple and Welsh [5] introduced the concept of correlated matroids, which relate to conjectures by Grimmett and Winkler [2], and Pemantle [4], respectively, that the uniformly random forest and the uniformly random connected subgraph of a finite graph have the edge-negative-association property. In this paper, we extend results of Semple and Welsh, and show that the Grimmett and Winkler, and Pemantle conjectures are equivalent to statements about correlated graphic matroids. We also answer some open questions raised in [5] regarding correlated matroids, and in particular show that the 2-sum of correlated matroids is correlated.

1. Introduction

Motivated by a property of electrical networks, in a recent paper Choe and Wagner [1] introduced the class of Rayleigh matroids. These are defined as follows.

Let M be a matroid with ground set E. Let $\mathbf{y} : E \to \mathbb{R}^+$ be a positive real-valued weighting of E. For a subset A of E, set $\mathbf{y}(A) = 1$ if A is the empty set and set $\mathbf{y}(A) = \prod_{a \in A} \mathbf{y}(a)$ if A is not the empty set. For disjoint subsets U and V of E, set

$$B_U^V(M; \mathbf{y}) = \sum \mathbf{y}(A),$$

where the summation is over all bases of M that contain U and avoid V, that is, contain all elements in U but no elements in V.

A matroid M with ground set E is *Rayleigh* if, for all distinct $e, f \in E$ and positive real-valued weightings $\mathbf{y}$ of E,

$$\Delta_B(M; \mathbf{y}) = B_e(M; \mathbf{y})B_f(M; \mathbf{y}) - B_{ef}(M; \mathbf{y})B(M; \mathbf{y}) \geqslant 0.$$

Analogously, Semple and Welsh [5] define the classes of independence- and spanning-correlated matroids. For disjoint subsets U and V of E, set

$$I_U^V(M; \mathbf{y}) = \sum \mathbf{y}(A)$$

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and

$$S_U^V(M;\mathbf{y}) = \sum \mathbf{y}(A),$$

where the summations are over all independent sets and spanning sets of M, respectively, that contain U and avoid V. A matroid M is *independence-correlated* if, for all distinct $e, f \in E$ and positive real-valued weightings $\mathbf{y}$ of E,

$$\Delta_I^{ef}(M;\mathbf{y}) = I_e(M;\mathbf{y})I_f(M;\mathbf{y}) - I_{ef}(M;\mathbf{y})I(M;\mathbf{y}) \geqslant 0.$$

A matroid is *spanning-correlated* if, for all distinct $e, f \in E$ and positive real-valued weightings $\mathbf{y}$ of E,

$$\Delta_S^{ef}(M;\mathbf{y}) = S_e(M;\mathbf{y})S_f(M;\mathbf{y}) - S_{ef}(M;\mathbf{y})S(M;\mathbf{y}) \geqslant 0.$$

Semple and Welsh call a matroid *correlated* if it is both independence-correlated and spanning-correlated.

These concepts are related to the following counting conjectures about graphs, due to Grimmett and Winkler [2], and Pemantle [4], respectively. For a finite graph G and subset A of edges of G, let $F_A(G)$ denote the number of forests of G containing A and let $C_A(G)$ denote the number of connected subgraphs of G containing A.

Conjecture 1.1. *For any finite graph G and distinct edges e and f of G,*

$$F_{\{e\}}(G)F_{\{f\}}(G) - F_{\{e,f\}}(G)F(G) \geqslant 0.$$

Conjecture 1.2. *For any finite graph G and distinct edges e and f of G,*

$$C_{\{e\}}(G)C_{\{f\}}(G) - C_{\{e,f\}}(G)C(G) \geqslant 0.$$

A main result of this paper is a proof of the following equivalences.

Proposition 1.3.

(i) *The class of graphic matroids is independence-correlated if and only if Conjecture 1.1 holds for all graphs.*
(ii) *The class of graphic matroids is spanning-correlated if and only if Conjecture 1.2 holds for all graphs.*

In this paper we also answer some questions left open in [5]. In particular, we show the following.

Proposition 1.4.

(i) *The 2-sum of two independence-correlated matroids is also independence-correlated.*
(ii) *The 2-sum of two spanning-correlated matroids is also spanning-correlated.*

We also partly answer the question of whether every rank-3 matroid is correlated, and show that every rank-3 matroid is independence-correlated.

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2. The 2-sum of correlated matroids

In this section we show that the 2-sum of correlated matroids is correlated. The 2-sum of matroids K and L at g where $g \in K$ and $g \in L$ but K and L are otherwise disjoint is written $K\Delta_g L$. The elements of $K\Delta_g L$ are the elements of K together with the elements of L, but excluding g. The bases are the union of bases in K that contain g with bases in L that do not contain g together with the union of bases in L that contain g and bases in K that do not; in both cases the element g is omitted. For graphic matroids the 2-sum corresponds to joining two graphs at a common edge g, and then deleting that edge.

We begin the proof with the following lemma.

Lemma 2.1. *Suppose g is the element on which K and L are to be joined. Suppose the sum of products of element weights over all independent sets $I(K;\mathbf{y})$ can be written as $U + Vg$, where U,V are sums of products of element weights not involving g, and we identify the element g with its weight. Similarly, suppose that $I(L;\mathbf{y})$ can be written in the form $R + Sg$.*

Then $I(K\Delta_g L) = UR - (U - V)(R - S)$, where the 2-sum is taken to inherit the natural weights from its two components.

Proof. The independent sets in $K\Delta_g L$ can be formed as any combination of independent sets in K not involving g with any independent set in L not involving g but excluding any pair where g is dependent upon those sets in both K and L.

The first of these two terms is given by UR.

The independent sets in K which cannot be extended to an independent set by appending g have sum of products of element weights equal to $U - V$. Similarly, the independent sets in L which cannot be extended to an independent set by appending g have sum of products of element weights equal to $R - S$. Thus the pairs of sets where g is dependent upon those sets in both K and L have sum of products of element weights equal to $(U - V)(R - S)$. □

We can now prove the following result.

Proposition 2.2. *If K, L are independence-correlated matroids then so is their 2-sum $K\Delta_g L$.*

Proof. We need to show that if $M = K\Delta_g L$, then, for any edges e,f and for all positive real-valued weightings $\mathbf{y}$,

$$\Delta_I^{ef}(M;\mathbf{y}) = I_e(M;\mathbf{y})I_f(M;\mathbf{y}) - I_{ef}(M;\mathbf{y})I(M;\mathbf{y}) \geqslant 0.$$

The proof is in two parts, depending upon whether the elements e and f are in the same component of K, L or not. In what follows we drop explicit reference to the weighting $\mathbf{y}$ where there is no ambiguity.

Case 1: elements e, f are in K, L respectively.

We have established above that

$$I(M) = UR - (U - V)(R - S),$$

where U, V, R, S are as defined in Lemma 2.1 above.

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Similarly,

$$I_e(M) = U_eR - (U_e - V_e)(R - S),$$
$$I_f(M) = UR_f - (U - V)(R_f - S_f),$$
$$I_{ef}(M) = U_eR_f - (U_e - V_e)(R_f - S_f),$$

where U_e are the terms in U that contain e, *etc.*

Hence,

$$\Delta_I^{ef}(M) = (R_fS - RS_f)(U_eV - UV_e).$$

But $\Delta_I^{fg}(L) = g(R_fS - RS_f)$ and so is non-negative because L is independence-correlated by assumption. Also, $\Delta_I^{eg}(K) = g(U_eV - UV_e)$ and so is also non-negative.

Hence $\Delta_I^{ef}(M) \geqslant 0$.

Case 2: elements e, f are (without loss of generality) both in K.

By a similar argument we can derive the formulae for $I(M), I_e(M), I_f(M), I_{ef}(M)$, from which it follows that

$$\Delta_I^{ef}(M) = (U_eS + V_e(R - S))(U_fS + V_f(R - S)) - (U_{ef}S + V_{ef}(R - S))(US + V(R - S)).$$

Thus,

$$\Delta_I^{ef}(M; \mathbf{y}) = S^2\Delta_I^{ef}(K; \mathbf{y}^*),$$

where $\mathbf{y}^* = \mathbf{y}$ except that $\mathbf{y}^*(g) = (R - S)/S$.

Every monomial in S is also in R, so $(R - S)/S$ is non-negative. Thus, since K is independence correlated it follows that $\Delta_I^{ef}(K)$, and hence $\Delta_I^{ef}(M)$ are non-negative. □

Corollary 2.3. *If K, L are spanning-correlated matroids then so is their 2-sum $K\Delta_g L$.*

Proof. This follows by duality, since

$$K\Delta_g L = (K^*\Delta_g L^*)^*,$$

where K^* is the dual of K. But K^* and L^* are independence-correlated since the dual of a spanning-correlated matroid is independence-correlated [5].

So $K^*\Delta_g L^*$ is independence-correlated and hence $K\Delta_g L$ is spanning-correlated. □

Also, note that by following the approach in the proof of Proposition 2.2, we get the result that, in the case that e, f both lie in K,

$$\Delta_S^{ef}(M; \mathbf{y}) = R^2\Delta_S^{ef}(K; \mathbf{y}^*),$$

where $\mathbf{y}^* = \mathbf{y}$, except that $\mathbf{y}^*(g) = (S - R)/R$, where R and S are defined in an analogous way to the above by writing $S(L; \mathbf{y})$ in the form $R + Sg$.

Proposition 2.4 follows immediately from the definition of correlated.

Proposition 2.4. *If K, L are independence-correlated matroids, then so is their 2-sum $K\Delta_g L$.*

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3. Conjectures of Grimmett, Winkler and Pemantle

The following result is given in [5].

Proposition 3.1.

(i) *If the class of graphic matroids is independence-correlated, then Conjecture 1.1 holds for all graphs.*
(ii) *If the class of graphic matroids is spanning-correlated, then Conjecture 1.2 holds for all graphs.*

In fact, the following stronger result holds.

Proposition 3.2.

(i) *The class of graphic matroids is independence-correlated if and only if Conjecture 1.1 holds for all graphs.*
(ii) *The class of graphic matroids is spanning-correlated if and only if Conjecture 1.2 holds for all graphs.*

Proof. To see this, suppose first that there exists some graph G such that $M(G)$ is not independence-correlated. Then, there will be a pair of elements e, f and a positive weighting $\mathbf{y}$ such that

$$\Delta_I^{ef}(M;\mathbf{y}) = I_e(M;\mathbf{y})I_f(M;\mathbf{y}) - I_{ef}(M;\mathbf{y})I(M;\mathbf{y}) < 0.$$

Because the set of rational numbers of the form $m/(2^n - 1)$, with m, n integer, is dense in the reals, there will be a weighting $\mathbf{y}$ with values of this form such that $\Delta_I(M;\mathbf{y}) < 0$.

We create a new graph H by replacing each edge g (apart from edges e and f) with m_g parallel copies of n_g edges in series, where $m_g/(2^{n_g} - 1)$ is the edge weighting $\mathbf{y}(g)$ for edge g that makes $\Delta_I(M;\mathbf{y}) < 0$.

This can be constructed through a series of 2-sum operations on the corresponding matroid $M(G)$, joining G at edge g with the graph consisting of edge g in parallel with m_g copies of n_g edges in series. It is straightforward that if the independent sets of the latter graph are written as the formal sum $R + Sg$, then S has $(2^{n_g} - 1)^{m_g}$ terms and $R - S$ has $m_g(2^{n_g} - 1)^{(m_g - 1)}$ terms.

Thus, following the proof of Proposition 2.2 above, it follows that $\Delta_I(M(H))$ with edge weighting 1 equals, up to a square factor, $\Delta_I(M(G);\mathbf{y})$, and so is negative. Thus the derived graph H violates Conjecture 1.1.

The proof when $M(G)$ is not spanning-correlated is similar. We need the observation that if L is a graph consisting of an edge g in parallel with m copies of n edges in series, then if the spanning sets are written in the form $R + Sg$, S has $(m+1)^n$ terms and $S - R$ has m^n terms. Thus $(S - R)/R = ((1 + 1/m)^n - 1)^{-1}$, a set of numbers that as m, n varies is dense in the non-negative reals. So, if $M(G)$ is not spanning-correlated there will be a pair of elements e, f and an edge weighting $\mathbf{y}$ taking values of the form $((1 + 1/m)^n - 1)^{-1}$ such that $\Delta_S^{ef}(M;\mathbf{y}) < 0$.

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Then we can create a derived graph H by replacing each edge g (apart from edges e and f) with m_g parallel copies of n_g edges in series, where $((1+1/m_g)^{n_g}-1)^{-1}$ is the edge weighting $\mathbf{y}(g)$ for edge g.

Thus $\Delta_S(M(H)$ with edge weighting 1 equals, up to a square factor, $\Delta_S(M(G);\mathbf{y})$, and so is negative, and thus we have created a graph H that violates Conjecture 1.2. □

We can combine Conjectures 1.1 and 1.2 into the following conjecture.

Conjecture 3.3. *The class of graphic matroids is correlated.*

4. $M(K_5)$ is correlated

We would therefore like to extend the set of graphic matroids that is known to be correlated. In [5] it is shown that $M(K_4)$ is correlated.

With extensive use of the computer algebra package Maple it is possible to show that $M(K_5)$ is correlated. This paper merely contains a sketch of the method.

In order to show that $\Delta_I^{ef}(M;\mathbf{y})$ and $\Delta_S^{ef}(M;\mathbf{y})$ are non-negative for positive weightings y, there are only two cases to consider. These are: e,f adjacent and e,f non-adjacent.

If e, f are adjacent, then all of the monomials in these two expressions have positive coefficients, and the non-negativity of $\Delta_I^{ef}(M;\mathbf{y})$ and $\Delta_S^{ef}(M;\mathbf{y})$ is immediate.

In the case that e and f are not adjacent there are both positive and negative terms. However, it is sufficient to show that if we collect together all the terms of each degree, then the individual homogeneous expressions are each separately non-negative.

Non-negativity can be demonstrated as follows. Each of the homogeneous expressions can be written as a sum of terms, each of which is either a monomial in the original expression $\Delta_I^{ef}(M;\mathbf{y})$ or $\Delta_S^{ef}(M;\mathbf{y})$ with positive coefficient, or is the square of sums of monomials. Although we considered using linear programming to assign terms into each square to achieve this property, for $M(K_5)$ it was possible to do this by hand. However, a better method will be needed to extend this approach to $M(K_6)$ or beyond.

5. Rank-3 matroids

Wagner [6] showed that every rank-3 matroid is Rayleigh. Semple and Welsh [5] ask whether every rank-3 matroid is correlated. We give a partial answer to their question.

Proposition 5.1. *Every rank-3 matroid is independence-correlated.*

Proof. Suppose that the elements of a rank-3 matroid M are $\{e_i\}$. Write the formal algebraic sum of bases as $Ae_1+Be_2+Ce_1e_2+D$, where A,B,C,D are formal sums of products of elements, with each product corresponding to a base. Thus D has degree 3, A and B have degree 2, and C is a sum of monomials.

A weighting $\mathbf{y}$ on the elements will map each of these sums into the real numbers, but for ease of notation in the proof we will identify each element e_i with the (positive) weight

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function **y** at e_i. Then

$$\Delta_B^{e_1e_2}(M) = e_1e_2(AB - CD),$$

and for positive element weights $AB - CD \geqslant 0$ as M is Rayleigh by [6].

Now consider the independent sets. We can assume that all pairs $\{e_i, e_j\}$ are independent, otherwise we can collapse any such pair and create a smaller matroid which is independence-correlated if and only if M is, and can therefore establish the result by induction.

So the formal algebraic sum of independent sets can be written as

$$(A + S + 1)e_1 + (B + S + 1)e_2 + (C + 1)e_1e_2 + D + T + S + 1,$$

where $S = \sum_{j>2} e_j$ and $T = \sum_{i>j>2} e_ie_j$.

It follows that $\Delta_I^{e_1e_2}(M) = ((A + S + 1)(B + S + 1) - (C + 1)(D + T + S + 1))e_1e_2$.

Hence $\Delta_I^{e_1e_2}(M) = (e_1e_2)(L4 + L3 + L2 + L1)$, where

$$L4 = AB - CD, \quad \text{the degree 4 terms,}$$
$$L3 = BS + AS - CT - D, \quad \text{the degree 3 terms,}$$
$$L2 = A + B + S^2 - CS - T, \quad \text{the degree 2 terms,}$$
$$L1 = S - C, \quad \text{the degree 1 terms.}$$

We show that each of $L1, L2, L3, L4$ are non-negative, given non-negative edge weights. We know already that $L4$ is non-negative because M is Rayleigh.

Degree 1: C is a subset of S, so $S - C$ is non-negative.

Degree 2: $L2 = A + B + (S - C)S - T$.

Consider each term of the form e_ie_j. If it is in either A or$B$ then it occurs with non-negative multiplicity in $L2$. But, if the term e_ie_j is in neither A nor B then neither $\{e_1, e_i, e_j\}$ nor $\{e_2, e_i, e_j\}$ are bases, so both sets must have rank 2. Hence, by the standard properties of matroids neither $\{e_1, e_2, e_i\}$ nor $\{e_1, e_2, e_j\}$ are bases so e_i and e_j are not in C. Thus e_i and e_j are in $S - C$ and so e_ie_j occurs with coefficient 2 in $(S - C)S$.

Degree 3: $L3 = BS + AS - CT - D$.

Consider first each term of the form $e_i^2e_j$ in $L3$. The maximum number of occurrences in CT is clearly 1, and in D is 0. So the coefficient of the term will be non-negative unless there are no such terms in $(A + B)S$.

But in that case e_ie_j is not a term in either A or B, and thus, as above, neither e_i nor e_j are in C. In that case there are no terms of the form $e_i^2e_j$ in $L3$.

Finally, we consider each term of the form $e_ie_je_k$ with i, j, k distinct. It is easily seen that the most number of occurrences of this term are: $\{3, 3, 3, 1\}$ in $\{AS, BS, CT, D\}$, respectively. So the coefficient of $e_ie_je_k$ will be positive unless at least three of the possible terms in AS and BS are not present, and thus without loss of generality at least two of these missing terms are in AS, and so $\{e_1, e_i, e_j\}$ and $\{e_1, e_i, e_k\}$ are not bases. Therefore $\{e_i, e_j, e_k\}$ cannot be a basis and thus D does not have a term of this form.

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So to have a negative coefficient in $L3$ there can be at most two terms $e_ie_je_k$ in $AS + BS$. Thus there will be a pair, say e_i, e_j (without loss of generality) such that neither $\{e_1, e_i, e_j\}$ nor $\{e_2, e_i, e_j\}$ are bases of the matroid. Thus, as above, neither $\{e_1, e_2, e_i\}$ nor $\{e_1, e_2, e_j\}$ are bases so e_i and e_j will not be in C, and hence the coefficient of $e_ie_je_k$ in $CT + D$ can be at most 1. Thus a negative coefficient in $L3$ would require no terms of the form $e_ie_je_k$ in $(A + B)S$, but we can extend the argument above to show that would imply none of e_i, e_j, e_k would be in C and so there could be no term of the form $e_ie_je_k$ in $CT + D$.

Thus the coefficients of all the terms in $L3$ are non-negative. □

6. Conclusion

The paper has extended the set of matroids known to be correlated. Many open questions remain, including the following.

(1) Is $M(K_n)$ correlated for all n? This is equivalent to Conjecture 3.3.
(2) Are rank-3 matroids necessarily spanning-correlated?
(3) We note that the question raised in [5] as to whether all Rayleigh matroids are correlated is still open. A positive result to this would also settle the first two questions.

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References

[1] Choe, Y.-B. and Wagner, D. G. (2006) Rayleigh matroids. *Combin. Probab. Comput.* **15** 765–781.
[2] Grimmett, G. R. and Winkler, S. N. (2004) Negative association in uniform forests and connected graphs. *Random Struct. Alg.* **24** 444–460.
[3] Oxley, J. G. (1992) *Matroid Theory*, Oxford University Press.
[4] Pemantle, R. (2000) Towards a theory of negative independence. *J. Math. Phys.* **41** 1371–1390.
[5] Semple, C. and Welsh, D. (2008) Negative correlation in graphs and matroids. *Combin. Probab. Comput.* **17** 423–435.
[6] Wagner, D. G. (2005) Rank-three matroids are Rayleigh. *Electron. J. Combin.* **12** #N8 1–11.

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