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A congruence between link polynomials

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0. *Introduction*

Throughout, 'link' means 'oriented link in S^3', and 'polynomial' means 'Laurent polynomial'. Let L be a link; then $P_L(v,z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ denotes the oriented polynomial of L (see [**2**, **8**]), written in Morton's variables (see [**5**]), and $F_L(a,x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ denotes the semi-oriented polynomial of L (see [**3**]). Let L have components $K_1, \ldots, K_n$. The total linking of L, denoted $t(L)$, is the sum of the linking numbers $\mathrm{lk}(K_i, K_j)$ with $1 \leqslant i < j \leqslant n$. Let f be a framing of L (that is, f assigns an integer to each K_i); then $f(L)$ denotes the total framing of L (that is, the sum of these integers), and $A(L,f)$ denotes the naturally associated union of annuli embedded in S^3. Define the framed polynomial $\{L,f\}(v,z)$ to be $(-1)^n$ plus $(v^{-1}-v)z^{-1}$ times the sum, over all non-empty sublinks K of L, of $(-1)^{n-k} P_{\mathrm{Bd}\, A(K)}$ (where K has k components with $1 \leqslant k \leqslant n$).

CONGRUENCE THEOREM. $(1+(v^{-2}+v^2)z^{-2})F_L(v^{-2},z^2) \equiv v^{4t(L)}\{L,0\}(v,z)$ modulo 2.

Sections 1–2 are preliminary; the Congruence Theorem is proved in Section 3; and Section 4 gives two applications of the framed polynomial (to the theories of quasipositivity and the enhanced Milnor number).

1. *The oriented and semi-oriented polynomials*

A crossing in a link diagram is homogeneous if it involves a single component, and heterogeneous if it involves two components. Given three links L_+, L_0, and L_- with diagrams identical except as indicated in Figure 1, let us say that we are in Case 1 if the visible crossing is homogeneous, and in Case 2 if it is heterogeneous.

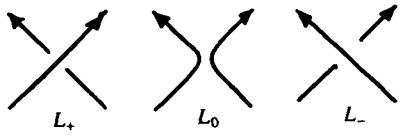


Fig. 1

We define some other links and knots as follows. In Case 1, L_∞ is the link indicated in Figure 2(i); K_+ (resp., K_-; K_∞) is the visible component of L_+ (resp. L_-; L_∞), $K_{0<}$ (resp., $K_{0>}$) is the component of L_0 which contains the right-hand (resp., left-hand) segment of L_0 in Figure 1, and $K_0 = K_{0<} \cup K_{0>}$. In Case 2, L_∞ is the link indicated in Figure 2(ii); K_0 (resp., K_∞) is the visible component of L_0 (resp. L_∞); $K_{\pm\backslash}$ (resp., $K_{\pm/}$) is the component of $L_\pm$ containing the bottom-right to top-left (resp., bottom-left to top-right) segment of $L_\pm$ in Figure 1, and $K_\pm = K_{\pm\backslash} \cup K_{\pm/}$. Let p (resp., q) be the linking number $\mathrm{lk}(K_<, L_0 - K_<)$ (resp., $\mathrm{lk}(K_{+\backslash}, L_+ - K_{+\backslash})$).

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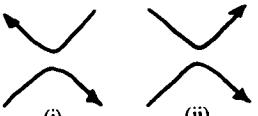


Fig. 2

According to [2] and [8], there is one and only one way to assign to each link L an element P_L of $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ (the choice of variables v and z follows [5]) so as to satisfy:

(P1) $P_O = 1$ (where O is an unknot);

(P2) $P_{L_+} = vzP_{L_0} + v^2 P_{L_-}$ for all instances of Figure 1.

There is, however, no uniqueness in the name of this invariant (one may assume that 'P' abbreviates polyonomous, if not polygenetic); I will follow [4] and call P_L the oriented polynomial of L.

Similarly, according to [3] (as expounded in [4]) there is one and only one way to assign to each link L an element F_L of $\mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ so as to satisfy

(F1) $F_O = 1$;

(F2) $aF_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4p}F_{L_\infty})$ for all instances of Case 1 (resp., $aF_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4q+2}F_{L_\infty})$ for all instances of Case 2) of Figures 1 and 2.

Following [4], we shall call F_L the semi-oriented polynomial of L.

2. *The framed polynomial*

Let L be a link in S^3 with components $K_1, \ldots, K_n$. A *framing* of L is a function $f: \{K_1, \ldots, K_n\} \to \mathbb{Z}$. A framing of L induces a framing of any sublink K of L, which will be denoted by the same symbol. Any orientable surface M embedded in S^3, on which L lies, induces a framing on L (its Seifert self-linking) independent of the orientation of L; every framing is induced by some M. The regular neighbourhood of L on M is a union of disjoint annuli; its ambient isotopy class in S^3 depends only on the framing. If (L,f) is a framed link, then $A(L,f)$ is such a union of annuli. The link type of its oriented boundary $\mathrm{Bd}\, A(L,f)$ is independent of the orientation of $A(L,f)$.

Notation. $[L,f]$ denotes $P_{\mathrm{Bd}\, A(L,f)}$.

The unlink of two components is $\mathrm{Bd}\, A(O,0)$. Write u for its oriented polynomial $[O,0] = (v^{-1} - v)z^{-1}$. Of course u is not invertible in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$; nonetheless, it can be handy to invert it formally, and (with discretion) to interpret u^{-1} as $P_\varnothing$, where $\varnothing$ is 'the empty link' (which is 'the unlink of 0 components'). The following proposition collects some consequences of (P1) and (P2) (for (5), cf. Figure 3, in which $c = 1$).

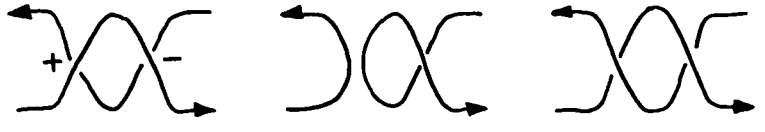
PROPOSITION 1. (1) *Let* $-L$ *denote* L *with its orientation reversed; then* $P_{-L} = P_L$.

(2) *Let* $\mathrm{Mir}\, L$ *denote the mirror image of* L; *then* $P_{\mathrm{Mir}\, L}(v,z) = P_L(-v^{-1}, z)$.

(3) *For* $n \geqslant 0$, *let* O_n *denote an unlink of* n *components; then* $P_{O_n} = u^{n-1}$.

(4) *Let* $L_1 \sqcup L_2$ *denote the split sum of* L_1 *and* L_2; *then* $P_{L_1 \sqcup L_2} = uP_{L_1}P_{L_2}$.

(5) *Let* f *and* f' *be framings of* L *which differ only on the component* K, *with* $f'(K) = f(K) + c$; *then* $[L,f'] = (1 - v^{-2c})[L-K, f] + v^{-2c}[L,f]$. ∎



$\mathrm{Bd}\, A(K,f) \subset \mathrm{Bd}\, A(L,f)$ $\quad$ $\mathrm{Bd}\, A(L-K,f) \# 0$ $\quad$ $\mathrm{Bd}\, A(K,f') \subset \mathrm{Bd}\, A(L,f')$

Fig. 3

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Definition. If (L,f) is a framed link with n components, then the framed polynomial $\{L,f\} \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is u times the sum, over sublinks K of L (including $\varnothing$), of $(-1)^{n-k}[K,f]$ (where K has k components and $0 \leqslant k \leqslant n$); $\operatorname{Bd} A(\varnothing,f)$ is of course $\varnothing$.

Remarks. Up to the normalizing factor u, $\{L,f\}$ is a 'Möbius transform' of $[L,f]$. Yamada [**18**] has made a general study of Möbius transforms of link polynomials, and (6) below is [**18**], proposition 1.

PROPOSITION 2. (1) *$\{L,f\}$ is independent of the orientation of L.*

(2) $\{\operatorname{Mir} L,f\}(v,z) = \{L,f\}(-v^{-1},z)$.

(3) $\{O,0\} = u^2 - 1$.

(4) $\{L_1 \sqcup L_2,f\} = \{L_1,f\}\{L_2,f\}$.

(5) *$v^{2f(L)}\{L,f\}$ is independent of f.*

(6) *$[L,f]$ is u^{-1} times the sum of $\{K,f\}$ over all sublinks K of L (in particular, this sum is divisible by u in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$).*

Proof. (1) follows from the first paragraph of this Section; (2)–(4) follow immediately, and (5) after an easy induction, from the definition of $\{L,f\}$ and the corresponding statements in Proposition 1; and (6) is a standard 'Möbius inversion formula'. ∎

Notation. Let 0 be the framing which assigns 0 to each component of L; then $\{L\}$ will denote $\{L,0\}$.

In this notation, Proposition 2(5) becomes the attractively simple formula $\{L,f\} = v^{-2f(L)}\{L\}$.

3. *The Congruence Theorem*

Consider a fixed diagram for a link L. The algebraic number of heterogeneous crossings is of course twice the total linking $t(L)$, independent of the diagram. The algebraic number of homogeneous crossings, however, depends on the diagram, and is, in fact, the total framing $w_L(L)$, where w_L is the diagrammatic framing which assigns to each component K of L its writhe $w(K)$ relative to the diagram, that is, the algebraic number of crossings in the induced diagram of K. (Note that $w_L(L) = w(L) + 2t(L)$.) A diagram for $\operatorname{Bd} A(L,w_L)$ can be produced by replacing each crossing of the diagram for L with a quadrangle of crossings, as in Figure 4, and each arc between crossings of the diagram for L with a pair of parallel arcs.

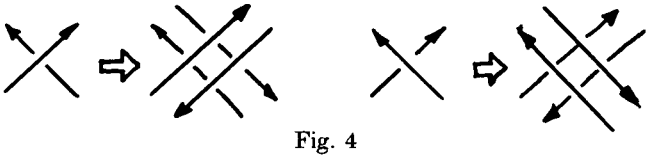


Fig. 4

Now suppose that L_+, L_0, L_-, and L_∞ are as in Figures 1 and 2(i) or 2(ii) as the case may be. Write w_+ for w_{L_+}, etc. Consider a diagram for $\operatorname{Bd} A(L_+,w_+)$ constructed as above, but modified in the vicinity of the quadrangle corresponding to the visible crossing of L_+ so as to have the appearance of the partial diagram of the link A at the root of the tree in Figure 5.

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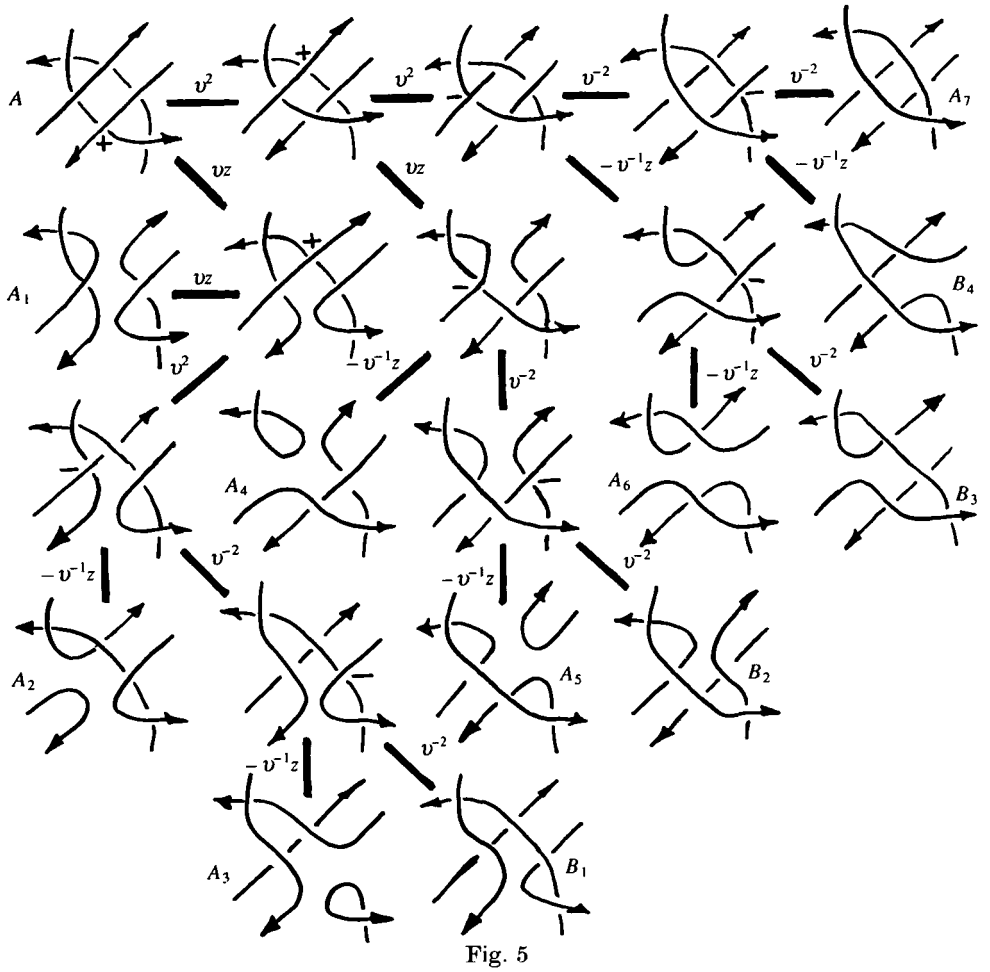


Fig. 5

This tree represents, in the usual way, a calculation of the oriented polynomial of the link A in terms of those of the links at the terminal nodes. Explicitly, each non-terminal node represents an application of $(P2)$, either directly or indirectly (in the form $P_{L_-} = -v^{-1}zP_{L_0} + v^{-2}P_{L_+}$), and the polynomial of the link at a non-terminal node is the appropriate weighted sum of the polynomials of the links at its two daughters. As an aid to the reader, at each non-terminal node the crossing to which $(P2)$ is being applied is marked, with a + or − according as the application is direct or indirect; and edges are marked with the relevant monomial weights. We have

$$P_A = v^2z^2(P_{A_1} - P_{A_2} - P_{A_4} + P_{A_6}) - z^2(P_{A_3} + P_{A_5}) + P_{A_7} + v^{-1}z(P_{B_1} + P_{B_2}) - vz(P_{B_3} + P_{B_4}) \quad (*)$$

with the terminal nodes labelled $A_1, \ldots, A_7, B_1, \ldots, B_4$ as shown.

Now, from the hypothesis $A = \mathrm{Bd}\, A(L_+, f)$ it follows that there are framed links (J_i, f_i), for $i = 1, \ldots, 7$, with $A_i = \mathrm{Bd}\, A(J_i, f_i)$. More precisely, we have the following result.

LEMMA 1. *In both cases,* $J_1 = L_0$, $J_6 = L_\infty$, *and* $J_7 = L_-$. *In Case* 1, $J_2 = J_4 = L_0 - K_{0>}$ *and* $J_3 = J_5 = L_0 - K_{0<}$. *In Case* 2, $J_2 = J_5 = L_+ - K_{+/} = L_- - K_{-/}$ *and* $J_3 = J_4 = L_+ - K_{+\backslash} = L_- - K_{-\backslash}$.

Proof. This is clear. ∎

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LEMMA 2. *For $i = 1, \ldots, 7$, for each component K of J_i, we have $f_i(K) = w(K)$, with the following exceptions: in Case 1, $f_1(K_{0>}) = w(K_{0>}) + 1$ and $f_6(K_\infty) = w(K_\infty) + 1$; in Case 2, $f_1(K_0) = w(K_0) + 1$, $f_2(K_{+\backslash}) = w(K_{+\backslash}) + 1$, $f_4(K_{+/}) = w(K_{+/}) + 1$, and $f_6(K_\infty) = w(K_\infty) + 1$.* ▮

Proof. This is again clear; the exceptional values are generated by the modifications in the upper-left and lower-right corners of the partial diagrams in Figure 5. ▮

LEMMA 3. *In both cases, $P_{B_1} = P_{B_2}$. In Case 1,*

$$P_{B_3} = P_{B_4} - vzP_{A_2} + v^{-1}zP_{A_3} - (1 - v^2)\, v^{-1}z[L_+ - K_+, w_+].$$

In Case 2, $P_{B_3} = P_{B_4}$.

Proof. In both cases, as indicated in Figure 6, B_2 is ambient isotopic to a partial reorientation of B_1; specifically, each visible component of B_2 has the opposite orientation to the corresponding visible component of B_1, while the invisible components (which we may take to be common to B_1 and B_2) are left alone. In general, of course, partial reorientation will have drastic effects on the oriented polynomial. But here, the invisible components bound annuli in pairs, so by a further isotopy (supported in tubular neighbourhoods of these annuli), we see that B_2 has the link type of $-B_1$, so that $P_{B_1} = P_{B_2}$ by Proposition 1(1).

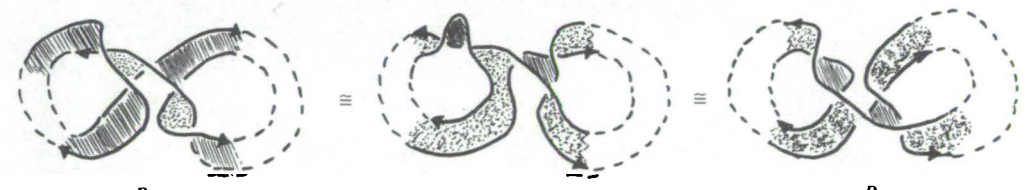


Fig. 6

In Case 2, an exactly similar argument shows that $P_{B_3} = P_{B_4}$. In Case 1, however, this argument shows instead that $P_{B_3} = P_C$, where C has the partial diagram at the root of the tree in Figure 7; the rest of the tree completes the proof. ▮

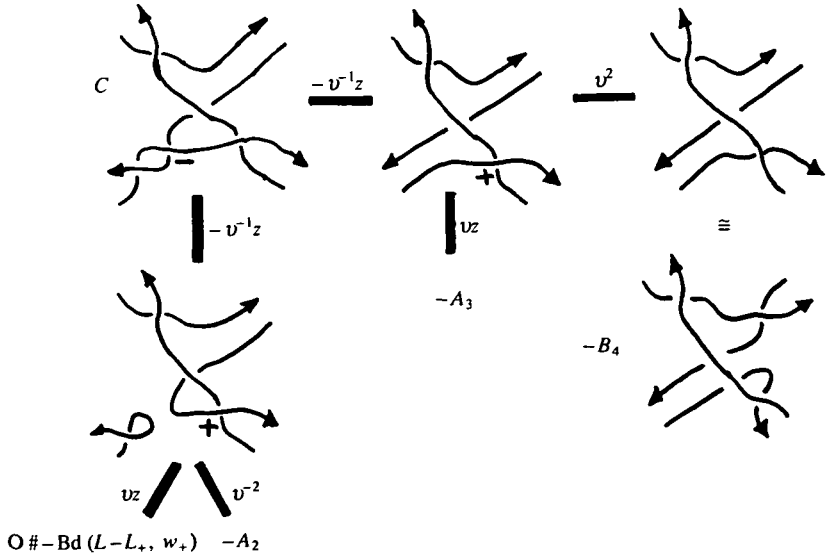


Fig. 7

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Let $\equiv$ denote congruence modulo 2.

LEMMA 4. *In Case* 1,

$$[L_+, w_+] \equiv z^2([L_0, w_0] + [L_0 - K_{0>}, w_0] + [L_0 - K_{0<}, w_0] + [L_\infty, w_\infty]) + [L_-, w_-].$$

In Case 2,

$$[L_+, w_+] \equiv z^2([L_0, w_0] + [L_\infty, w_\infty]) + [L_-, w_-].$$

Proof. Apply Proposition 1(5) and Lemmas 1, 2, 3 to (*), noting that

$$(L_\infty - K_\infty, w_\infty) = (L_0 - K_0, w_0)$$

in Case 1, and $(L_+ - K_+, w_+) = (L_0 - K_0, w_0) = (L_\infty - K_\infty, w_\infty)$ in Case 2. ∎

THEOREM. $\{L_+, w_+\} \equiv z^2(\{L_0, w_0\} + \{L_\infty, w_\infty\}) + \{L_-, w_-\}$.

Proof. Let I be the (invisible) residual link

$$L_+ - K_+ = L_0 - K_0 = L_- - K_- = L_\infty - K_\infty.$$

We apply the inversion formula Proposition 2(6) to the congruences in Lemma 4. If I is empty, then the inversion formula yields

$$\begin{aligned}\{L_+, w_+\} + \{\varnothing, w_+\} &\equiv z^2(\{L_0, w_0\} + \{K_{0>}, w_0\} + \{K_{0<}, w_0\} + \{\varnothing, w_0\} \\ &\quad + \{K_{0>}, w_0\} + \{\varnothing, w_0\} + \{K_{0<}, w_0\} + \{\varnothing, w_0\} \\ &\quad + \{L_\infty, w_\infty\} + \{\varnothing, w_\infty\}) + \{L_-, w_-\} + \{\varnothing, w_-\} \\ &\equiv z^2(\{L_0, w_0\} + \{L_\infty, w_\infty\}) + \{L_-, w_-\} + \{\varnothing, w_-\}\end{aligned}$$

in Case 1 (so that the desired congruence is immediate), and

$$\begin{aligned}&\{L_+, w_+\} + \{K_{+\backslash}, w_+\} + \{K_{+/}, w_+\} + \{\varnothing, w_+\} \\ &\quad \equiv z^2(\{L_0, w_0\} + \{\varnothing, w_0\} + \{L_\infty, w_\infty\} + \{\varnothing, w_\infty\}) \\ &\qquad + \{L_-, w_-\} + \{K_{-\backslash}, w_-\} + \{K_{-/}, w_-\} + \{\varnothing, w_-\}\end{aligned}$$

in Case 2 (so that the desired congruence follows since $(K_{+\backslash}, w_+) = (K_{-\backslash}, w_-)$ and $(K_{+/}, w_+) = (K_{-/}, w_-)$). If I is not empty, then there are extra terms on both sides of the corresponding congruences, but the ones which do not cancel immediately (modulo 2) can be grouped so as to cancel on applying the theorem to links $L_+ - J$ with J a non-empty sublink of I. By induction this completes the proof. ∎

COROLLARY. *In Case* 1, *if* $r = \mathrm{lk}\,(K_{0<}, K_{0>})$, *then*

$$v^{-2}\{L_+\} + v^2\{L_-\} \equiv z^2(v^{4r}\{L_0\} + v^{8r}\{L_\infty\}).$$

In Case 2, *if* $s = \mathrm{lk}\,(K_{+/}, K_{+\backslash})$, *then*

$$\{L_+\} + \{L_-\} \equiv z^2(v^{2-4s}\{L_0\} + v^{4s-2}\{L_\infty\}).$$

Proof. Use Proposition 2(5) to adjust the total framings to 0, taking due account of changes in crossings. ∎

I thank the referee for observing that the Theorem can be rephrased as follows:

CONGRUENCE THEOREM. $(1 + (v^{-2} + v^2)\,z^{-2})\,F_L(v^{-2}, z^2) \equiv v^{4t(L)}\{L\}(v, z)$.

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Proof. From the Corollary (and the definitions of the integers p, q, r, s, and $t(L)$), if we set $a = v^{-2}$, $x = z^2$, then $v^{4t(L)}\{\dot{L}\}(v,z)$ satisfies the characteristic identity $(F2)$ of the semi-oriented polynomial modulo 2, and so, by [**3**], $v^{4t(L)}\{L\}(v,z)$ must be congruent modulo 2 to $v^{4t(O)}\{O\}(v,z)\,F_L(v^{-2},z^2)$. But by Proposition 2(3), $v^{4t(O)}\{O\}(v,z) = u^2 - 1 \equiv 1 + (v^{-2}+v^2)\,z^{-2}$. ∎

Remark. The referee also notes that the Congruence Theorem is 'a generalization, *to all values of v and z*, but only modulo 2,' of a formula of Yamada [**18**], proposition 10, which relates, by equality, not congruence, a (one-variable) specialization of the semi-oriented polynomial and a Möbius transform of the original Jones polynomial.

4. *Two applications of the framed polynomial*

Review of quasipositivity

A quasipositive braid is any product of conjugates of the standard generators of the braid group, and a quasipositive link is the closure of a quasipositive braid. (These first arose in a study of closed braids associated to complex algebraic functions (see [**9**]), and are related to interesting open questions in the knot theory of complex plane curves, such as the 'Thom Conjecture' on the genus of smooth surfaces in the complex projective plane and 'Milnor's Question' on the unknotting number of links of singularities; for more details, the reader is referred to the surveys [**12, 15**].) Although the original definition was quite different (involving 'braided surfaces'), according to [**16**] one may define a quasipositive Seifert surface to be any Seifert surface which is a full subsurface of a fibre surface of some torus link $O\{m,n\}$ with $mn > 0$. (We say S is a full subsurface of T if no component of $T \backslash S$ is contractible.) A strongly quasipositive link is the boundary of a quasipositive Seifert surface; strongly quasipositive implies quasipositive.

The oriented polynomial has already been applied to the theory of quasipositivity. Until the advent of P it was unknown whether any links were not quasipositive; indeed, it was conceivable that every link was *strongly* quasipositive, and it was known that every Seifert matrix did occur for some quasipositive Seifert surface (so, e.g. the Alexander polynomial even of a strongly quasipositive link was undistinguished among all Alexander polynomials; see [**11**]). However, it is an immediate corollary [**15**] of a theorem of Morton [**5**] and Franks and Williams [**1**] that many links are not quasipositive. Here is their result, in the present language.

COROLLARY TO THE MORTON–FRANKS–WILLIAMS THEOREM. *If S is a 'quasipositive Seifert ribbon' (defined in* [**10**]*) (in particular, if S is a quasipositive Seifert surface), and if exactly d components of S are disks, then* $\operatorname{ord}_v P_{\mathrm{Bd}\,S} \geqslant 1-d$.

(For any coefficient ring R and indeterminates x, y, if

$$Q(x,y) = \sum_{i=m}^{M} Q_i(y)\,x^i \in R[x^{\pm 1}, y^{\pm 1}] = R[y^{\pm 1}][x^{\pm 1}],$$

where $Q_m, Q_M \in R[y^{\pm 1}]$ are non-zero, then $\operatorname{ord}_x Q := m$, $\deg_x Q := M$. Trivially, if R/I is a quotient ring of R, and $Q^*(x,y) \in (R/I)[x^{\pm 1}, y^{\pm 1}]$ is the reduction of Q modulo I, then $\operatorname{ord}_x Q^* \geqslant \operatorname{ord}_x Q$.)

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First application

In [**13**], we proved that every knot K has a modulus of quasipositivity $q(K) > -\infty$ such that $A(K,f)$ fails to be quasipositive if and only if $f > q(K)$. The possibility that $q(K)$ might equal $+\infty$ for some K was left open.

PROPOSITION 3. *If $A(L,f)$ is quasipositive, then* $\mathrm{ord}_v\{L,f\} \geqslant 0$.

Proof. If $A(L,f)$ is quasipositive and K is a non-empty sublink of L, then $A(K,f)$ is quasipositive. (This is immediate from the original definition in terms of braided surfaces, but also follows easily from the characterization of quasipositive Seifert surfaces as full subsurfaces of fibre surfaces of positive torus links.) By the Morton–Franks–Williams Theorem, $\mathrm{ord}_v[K,f] \geqslant 1$. Also $\mathrm{ord}_v u = -1$. The proposition then follows from elementary properties of ord_v. ∎

COROLLARY. *For every knot K, $q(K) \leqslant \frac{1}{2}\mathrm{ord}_v\{K\} \leqslant -1 - \deg_a F_K^*(a,x) < +\infty$, where F_K^* is the reduction of F_K modulo 2.*

Proof. Combine Proposition 1(5), Proposition 3, and the Congruence Theorem. ∎

Example. The negative trefoil knot is $O\{2,-3\}$. From the table of semi-oriented polynomials in [**3**], and the Corollary, we find that $q(O\{2,-3\}) \leqslant -6$. A direct construction [**13**] shows that $A(O\{2,-3\},-6)$ is quasipositive, so in fact $q(O\{2,-3\}) = -6$. (The corresponding calculation using $[O\{2,-3\},f]$, and not reducing the coefficients, can be done by hand but is lengthy; the forbidding prospect of similar calculations, for knots with more crossings than the trefoil's mere three, was the original motivation of the present investigation.)

Review of the enhancement

The enhancement $\lambda(L)$ of (the Milnor number of) a fibred link L was introduced in [**14**] (see also [**6**, **7**] and the survey [**15**]). As L varies, $\lambda(L)$ takes on all values in $\mathbb{Z}$. The enhancement of L is not determined by the Seifert form (or algebraic monodromy) of L. In the special case that L is Hopf-plumbed, that is, its fibre surface is an iterated Murasugi sum of positive and negative Hopf annuli $A(O, \mp 1)$ (e.g. if L is a fibred arborescent link or a closed generalized strict homogeneous braid), then $\lambda(L)$ equals the number of negative Hopf annuli in any such iterated Murasugi sum. According to [**17**], a Hopf-plumbed fibred link L is strongly quasipositive if and only if $\lambda(L) = 0$.

Second application

Let $K = O\{m_1, n_1; \ldots; m_k, n_k\}$ be an iterated torus knot. Then K is fibred, but need not be Hopf-plumbed, e.g. $O\{2,3;2,1\}$ (see [**6**]). Nonetheless, we have the following (see [**15**]).

PROPOSITION 4. *K is strongly quasipositive if and only if $\lambda(K) = 0$.*

Sketch proof. According to [**7**], $\lambda(K) \geqslant 0$, and $\lambda(K) > 0$ if and only if $m_j > 1$ and $n_j < 0$ for some j. If $\lambda(K) = 0$, then K is strongly quasipositive for algebro-geometric reasons (it is a 'good link-at-infinity'; see [**6**]). Otherwise, let S be the fibre surface of K. Given the pair (m_j, n_j) with $m_j > 1$ and $n_j < 0$, one can draw a picture of S which

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shows that S contains $A(O\{2, -3\}, f)$ as a full subsurface for some $f \geqslant -5$. By the Example and the characterization of quasipositive Seifert surfaces, S is not quasipositive. Let S' be any non-fibre Seifert surface for K; then S' is compressible, and so S' contains $A(O, 0)$ as a full subsurface. But the Morton–Franks–Williams Theorem shows that $A(O, 0)$ is not quasipositive, and so S' is not quasipositive. Thus K is not strongly quasipositive. ∎

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