102.46 Showing the divergence of the harmonic series using integration

The harmonic series is a divergent series defined by

$$\sum \frac{1}{n}.$$

The first proof of the divergence of the harmonic series was given by Oresme (c.1360), and can be found with other nice proofs in the famous book of John Derbyshire [1]. In this Note we give a proof using simple properties of infinite sums and integrals.

Proof: It is easy to see that for $s > 0$

$$\int_0^\infty e^{-st}dt = \frac{1}{s}. \tag{1}$$

Summing equation (1) over values of s and interchanging the summation and integration gives

$$\sum_{s=1}^{N} \frac{1}{s} = \sum_{s=1}^{N} \left(\int_0^\infty e^{-st}dt \right) = \int_0^\infty \left(\sum_{s=1}^{N} e^{-st} \right) dt. \tag{2}$$

Applying the formula for the sum of a geometric progression shows that the sum $\sum_{s=1}^{N} e^{-st}$ in the bracket above converges monotonically to $\frac{1}{e^t - 1}$ as $N \to \infty$. By using the Monotone Convergence Theorem (see [2, pp. 95-96]) we deduce that, as $N \to \infty$, the last expression in (2) tends to

$$\int_0^\infty \frac{dt}{e^t - 1}. \tag{3}$$

Substituting $e^t = u$, so that $dt = e^{-t}\,du$, and also $v = 1 - \frac{1}{u}$, so that $du = u^2\,dv$, in the integral in (3), gives

$$\int_0^\infty \frac{dt}{e^t - 1} = \int_1^\infty \frac{du}{(u-1)u} = \int_0^1 \frac{dv}{v} = \left[\ln v\right]_0^1$$

which is clearly divergent. Since formally

$$\int_0^\infty \frac{dt}{e^t - 1} = \sum_{s=1}^{\infty} \frac{1}{s}$$

we conclude that the harmonic series diverges.

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102.47 An equilateral triangle associated with a hexagon

In [1], Dao Thanh Oai discusses two equilateral triangles associated with an arbitrary hexagon. Our aim in this paper is to display two more equilateral triangles based on the same configuration.

Theorem

Let $A_1A_2A_3A_4A_5A_6$ be a hexagon, with equilateral triangles constructed externally on the sides with apexes B_i (for $1 \leqslant i \leqslant 6$) as shown in Figure 1. Let C_i be the midpoint of B_iB_{i+1} (working throughout with indices modulo 6) and let P_1, P_2 and P_3 be the midpoints of C_1C_4, C_2C_5 and C_3C_6 respectively. Then

(a) the segments C_1C_4, C_2C_5 and C_3C_6 are equal in length;

(b) the triangle $P_1P_2P_3$ is equilateral.

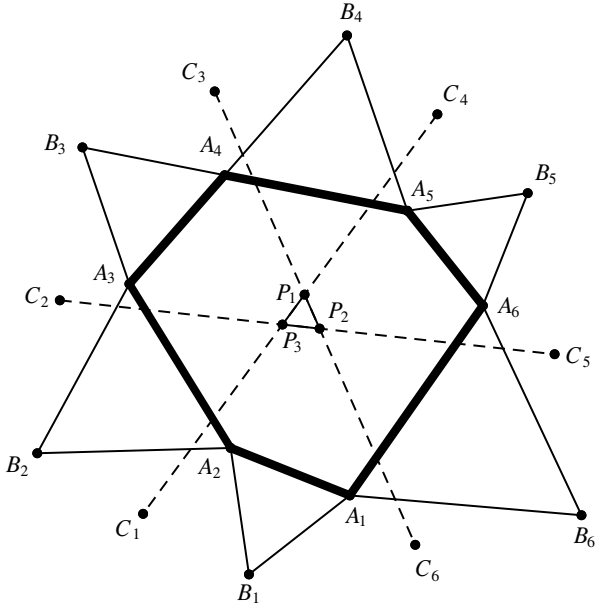


FIGURE 1

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