A simple approach to solving cubic equations

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1. *Introduction*

Finding the roots of cubic equations has been the focus of research by many mathematicians. Omar Khayyam, the 11th century Iranian mathematician, astronomer, philosopher and poet, discovered a geometrical method for solving cubic equations by intersecting conic sections [1]. In more recent times, various methods have been presented to find the roots of cubic equations. Some methods require complex number calculations, a number of techniques use graphical methods to find the roots [e.g. 2, 3] and some other techniques use trigonometric functions [e.g. 4]. The method presented in this paper does not use graphical techniques as in [2] and [3], does not involve complex number calculations, and does not require using trigonometric functions. By using this fairly simple method, the roots of cubic equations can be found in a short time without using complicated formulas.

2. *Method*

A cubic equation can be written as:

$$f(x) = x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 = 0. \tag{1}$$

The equation must have at least one real root. There are three possible situations:

(a) There is one real root occurring three times:

$$f(x) = (x - r_1)^3 = x^3 - 3r_1 x^2 + 3r_1^2 x - r_1^3,$$

where r_1 is the repeated real root.

In this case, $\alpha_2 = 3r_1^2$, therefore if $\alpha_2 > 0$, then $\gamma = \pm\sqrt{\frac{1}{3}\alpha_2}$. Then if $\alpha_1 = -3\gamma$ and $\alpha_3 = -\gamma^3$, $\gamma = r_1$.

(b) There are two real roots, one of them repeated.

$$f(x) = (x - r_1)^2(x - r_2) = x^3 + x^2(-2r_1 - r_2) + x(r_1^2 + 2r_1 r_2) - r_1^2 r_2.$$

Therefore

$$\alpha_1 = -2r_1 - r_2,$$

$$\alpha_2 = r_1^2 + 2r_1 r_2,$$

$$\alpha_3 = -r_1^2 r_2.$$

From these equations $r_2 = -2r_1 - \alpha_1$, which results in the following equation with r_2 substituted in terms of r_1 in the equation for α_2:

$$3r_1^2 + 2\alpha_1 r_1 + \alpha_2 = 0$$

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which yields

$$\gamma_1, \gamma_2 = -\frac{\alpha_1}{3} \pm \frac{\sqrt{\alpha_1^2 - 3\alpha_2}}{3}. \tag{2}$$

For the answer to be real,

$$\alpha_2 \leqslant \tfrac{1}{3}\alpha_1^2.$$

In this case, let $\lambda_1 = -2\gamma_1 - \alpha_1$ and $\lambda_2 = -2\gamma_2 - \alpha_1$. Now if $\alpha_3 = -\gamma_1^2\lambda_1$, then $r_1 = \gamma_1$ and $r_2 = \lambda_1$. Alternatively, if $\alpha_3 = -\gamma_2^2\lambda_2$, then $r_1 = \gamma_2$ and $r_2 = \lambda_2$.

(c) The third and more general situation is that $f(x)$ has three distinct roots, with at least one of them a real number and the other two either real or complex conjugate pairs. In that case

$$f(x) = (x - r_1)(x^2 + \beta_1 x + \beta_2) \tag{3}$$

where r_1 is a real root of the cubic equation and

$$\alpha_3 = -r_1\beta, \tag{4}$$

$$\alpha_2 = \beta_2 - r_1\beta_1, \tag{5}$$

$$\alpha_1 = \beta_1 - r_1. \tag{6}$$

Rearranging (4) results in $\beta_2 = -\dfrac{\alpha_3}{r_1}$ and, by substituting β_2 in (5) in terms of α_3,

$$\alpha_2 = -r_1\beta_1 - \frac{\alpha_3}{r_1}, \tag{7}$$

$$\alpha_1 = \beta_1 - r_1. \tag{8}$$

These equations can be rewritten as

$$\beta_1 = -\frac{\alpha_3}{r_1^2} - \frac{\alpha_2}{r_1}, \tag{9}$$

$$r_1 = \beta_1 - \alpha_1. \tag{10}$$

So (9) can be rearranged as

$$\beta_1 r_1^2 + \alpha_2 r_1 + \alpha_3 = 0. \tag{11}$$

Therefore

$$r_1 = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_3\beta_1}}{2\beta_1}. \tag{12}$$

In order for r_1 to be a real number, it is necessary that

$$\beta_1\alpha_3 \leqslant \frac{\alpha_2^2}{4}.$$

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Since the right side of this inequality is positive, it will hold if α_3 and β_1 have opposite signs. If α_3 and β_1 are both negative, and both sides of the inequality are divided by α_3

$$\beta_1 \geqslant \frac{\alpha_2^2}{4\alpha_3} \text{ (both sides of this inequality will be negative).}$$

If α_3 and β_1 are both positive, then

$$\beta_1 \leqslant \frac{\alpha_2^2}{4\alpha_3} \text{ (both sides of this inequality will be positive).}$$

The last two conditions can be combined into:

$$|\beta_1| \leqslant \frac{\alpha_2^2}{4|\alpha_3|}.$$

This inequality can be rewritten as follows to get an acceptable range for r_1:

Since $\beta_1 = r_1 + \alpha_1$ from (10):

$$|r_1 + \alpha_1| \leqslant \frac{\alpha_2^2}{4|\alpha_3|}$$

or

$$\frac{-\alpha_2^2}{4|\alpha_3|} - \alpha_1 \leqslant r_1 \leqslant \frac{\alpha_2^2}{4|\alpha_3|} - \alpha_1. \tag{13}$$

Therefore

$$x_1 \leqslant r_1 \leqslant x_2 \tag{14}$$

where $x_1 = \frac{-\alpha_2^2}{4|\alpha_3|} - \alpha_1$ and $x_2 = \frac{\alpha_2^2}{4|\alpha_3|} - \alpha_1$. Then

$$y_1 = f(x_1) \qquad \text{and} \qquad y_2 = f(x_2)$$

(y_1 and y_2 should have opposite signs)*.

A line can be drawn between x_1 and x_2 that will cross the real axis near r_1. The equation of that line is

$$y = y_1 + k(x - x_1),$$

where $k = \frac{y_2 - y_1}{x_2 - x_1}$.

* If y_1 and y_2 have the same sign, α_3 and β_1 have opposite signs. If $\alpha_3 < 0$, then $\beta_1 > 0$ and $r_1 > -\alpha_1$. In that case, we can choose a new x_1 to be equal to x_2 and choose a new x_2 greater than the new x_1 by a margin of an integer multiple of $|\alpha_1|$ to get $f(x_2)$ of opposite sign to $f(x_1)$. If $\alpha_3 > 0$, then $\beta_1 < 0$ and $r_1 < -\alpha_1$. In that case, we can choose a new x_2 to be equal to x_1 and choose a new x_1 less than the new x_2 by an integer multiple of $|\alpha_1|$ to get $f(x_1)$ of opposite sign of $f(x_2)$ and continue the procedure.

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This line crosses the x-axis at

$$x = x_1 - \frac{y_1}{k}. \tag{15}$$

The first choice for r_1 can be equal to x from (15). Then the actual r_1 can be found by using (9) and (10) iteratively.

3. *Procedure*

The procedure of finding r_1 can now be summarised as follows:

- If $\alpha_2 > 0$ then calculate $\gamma = \pm\sqrt{\frac{\alpha_2}{3}}$, if $\alpha_1 = -3\gamma$ and $\alpha_3 = -\gamma^3$, then $r_1 = \gamma$ and occurs three times.
- If $\alpha_2 \leqslant \frac{\alpha_1^2}{3}$ then calculate $\gamma_1, \gamma_2 = -\frac{\alpha_1}{3} \pm \frac{\sqrt{\alpha_1^2 - 3\alpha_2}}{3}$ and $\lambda_1 = -2\gamma_1 - \alpha_1$ and $\lambda_2 = -2\gamma_2 - \alpha_1$.
  If $\alpha_3 = -\gamma_1^2\lambda_1$ then $r_1 = \gamma_1$ and $r_2 = \lambda_1$. If $\alpha_3 = -\gamma_2^2\lambda_2$, then $r_1 = \gamma_2$ and $r_2 = \lambda_2$.
- The above two situations are considered as 'special conditions'. If they are not met, continue with the general procedure below.
- Calculate $x_1 = -\frac{\alpha_2^2}{4|\alpha_3|} - \alpha_1$.
- Calculate $x_2 = \frac{\alpha_2^2}{4|\alpha_3|} - \alpha_1$ $(x_1 \leqslant r_1 \leqslant x_2)$.
- Calculate $y_1 = f(x_1)$ and $y_2 = f(x_2)$ (y_1 and y_2 should have opposite signs*).
- Calculate $k = \frac{y_2 - y_1}{x_2 - x_1}$.
- Choose an initial $r_1 = x_1 - \frac{y_1}{k}$.
- Use the following equations ((9) and (10)) iteratively to find r_1:

$$\beta_1 = -\frac{\alpha_3}{r_1^2} - \frac{\alpha_2}{r_1}, \qquad r_1 = \beta_1 - \alpha_1.$$

- If the iterations do not tend to converge or oscillate, the average of r_1 in two successive iterations and an upper or lower limit of r_1 that produces $f(r_1)$ of opposite sign should be used iteratively to find the final value of r_1.
- Once r_1 is determined, the other roots of the cubic can be found as follows:

$$r_2, r_3 = -\frac{k_1}{2} \pm \frac{\sqrt{k_1^2 + \frac{4\alpha_3}{r_1}}}{2} \tag{16}$$

where $k_1 = r_1 + \alpha_1$.

* See the previous footnote.

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4. *Examples*

Example 1:

$$f(x) = x^3 - 12x^2 + 48x - 64 = 0.$$

Therefore $\alpha_1 = -12$, $\alpha_2 = 48$ and $\alpha_3 = -64$.

By following the Procedure, it is seen that $\alpha_2 = 48 > 0$, γ_1, $\gamma_2 = \pm 4$, $\alpha_1 = -3\gamma_1 = -12$ and $\alpha_3 = -\gamma_1^3 = -64$. Therefore, the conditions for the first special case are satisfied and $r_1 = \gamma_1 = 4$ which is a root of the cubic equation repeated three times.

Example 2:

$$f(x) = x^3 - 8x^2 + 21x - 18 = 0, \qquad (\alpha_1 = -8, \alpha_2 = 21, \alpha_3 = -18).$$

By going through the Procedure, it is seen that the conditions for the first special case are not satisfied. Therefore, the equation does not have one real root occurring three times. It is then seen that $\alpha_2 \leqslant \frac{1}{3}\alpha_1^2$, γ_1 is calculated to be 3 and $\gamma_2 \approx 2.33$. Then $\lambda_1 = -2\gamma_1 - \alpha_1 = 2$ and $\lambda_2 = -2\gamma_2 - \alpha_1 \approx 3.33$. Since $\alpha_3 = -\gamma_1^2\lambda_1 = -18$, the conditions of the second special case are satisfied. Therefore, $r_1 = \gamma_1 = 3$ which is repeated twice and $r_2 = \lambda_1 = 2$.

Example 3:

$$f(x) = x^3 + 5x^2 + 6x + 3 = 0, \qquad (\alpha_1 = 5, \alpha_2 = 6, \alpha_3 = 3).$$

Special conditions do not apply. So following the above procedure:

- $x_1 = \dfrac{-\alpha_2^2}{4|\alpha_3|} - \alpha_1 = -8.$
- $x_2 = \dfrac{\alpha_2^2}{4|\alpha_3|} - \alpha_1 = -2$, (therefore $-8 \leqslant r_1 \leqslant -2$).
- $y_1 = f(x_1) = -237.$
- $y_2 = f(x_2) = 19.$
- $k = \dfrac{y_2 - y_1}{x_2 - x_1} \approx 42.66.$
- first choice for $r_1 = x_1 - \dfrac{y_1}{k} \approx -2.445.$
- Then start iterations with:

$$\beta_1 = -\frac{\alpha_3}{r_1^2} - \frac{\alpha_2}{r_1}, \qquad r_1 = \beta_1 - \alpha_1.$$

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Iteration	β_1	r_1
1	1.952	−3.0477
2	1.6457	−3.354
3	1.522	−3.477
4	1.4776	−3.52
5	1.46	−3.54

$r_1 \approx -3.54$ is one of the roots of the cubic equation. The other roots can be found by using (16) as:

$$r_2, r_3 \approx -0.73 \pm 0.56i.$$

Figure 1 shows a graphical representation of this cubic function.

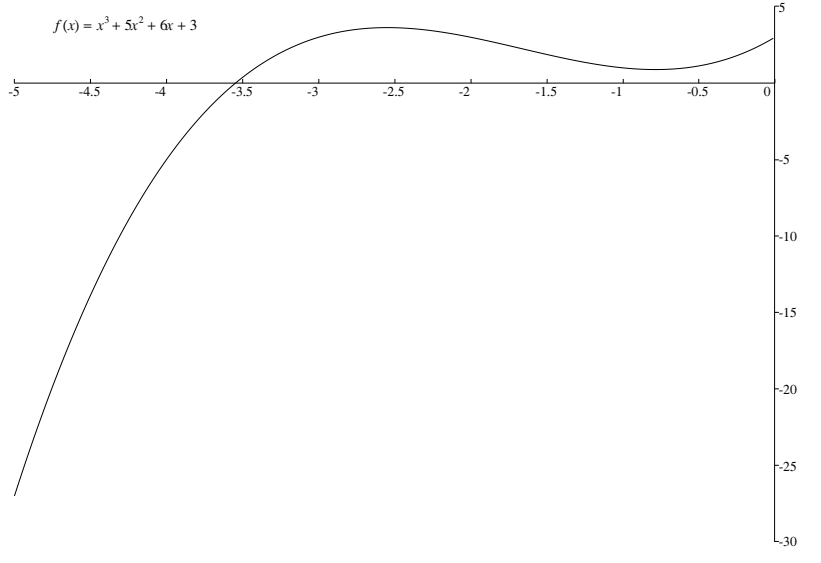


FIGURE 1

Example 4:

$$f(x) = x^3 - 4x^2 + 2x + 1 = 0.$$

Special conditions do not apply. So following the above procedure:

- $x_1 = 3, x_2 = 5$ therefore $3 \leqslant r_1 \leqslant 5$.
- $y_1 = f(x_1) = -2, y_2 = f(x_2) = 36$ and $k = 19$.

Therefore we choose $r_1 = 3 + \dfrac{2}{19} \approx 3.1$ and start the iterations using (9) and (10):

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Iteration	β_1	r_1
1	−0.749	3.25
2	−0.71	3.29
3	−0.7	3.299
4	−0.6979	3.30

$r_1 \approx 3.30$ is a root of the cubic equation. The other roots are found by using (16) as:

$$r_2 = 1, r_3 \approx -0.30.$$

Figure 2 is a graphical representation of this cubic function.

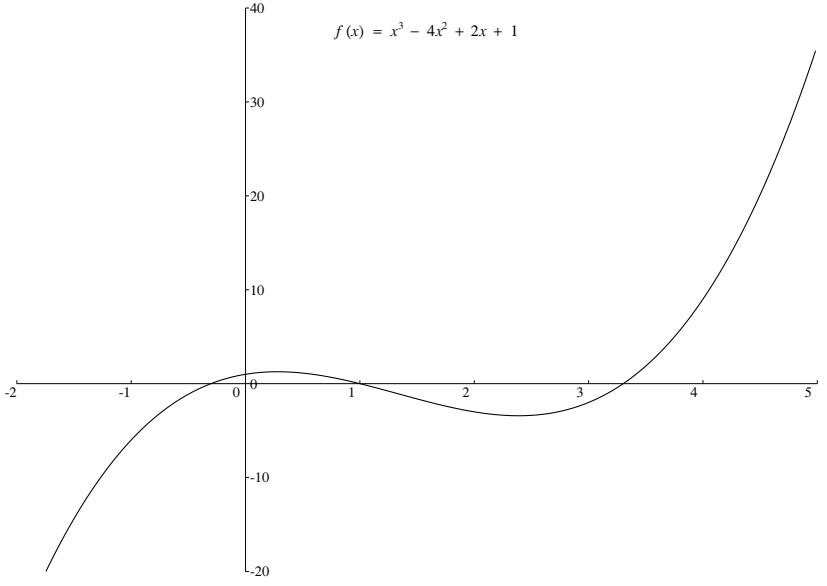


FIGURE 2

5. *Conclusions*

Various methods have been proposed in the literature to find the roots of cubic equations. Some of those techniques use graphs while some others involve the use of complex number calculations or trigonometric formulas. The procedure described in this paper is simple and yet effective in providing the roots of a cubic equation without the need for graphs or complicated formulas. This technique that can be regarded as a numerical procedure can be programmed on a computer. However, the range for the first root used in the procedure tends to be narrow enough that the procedure can normally produce that root by using a small number of iterations by hand in a relatively short period of time without using a computer.

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Request for assistance

As was stated on page 92 of the March 2016 *Gazette*, Richard Kirby has given up his role of checking the annually published *Index*. As was stated, this creates a vacancy which needs to be filled. The *Index* is constructed by Bill Richardson soon after the November issue is completed and then sent for checking. Errors at this stage might include incorrect alphabetical order, omitting a book author and such like. There is no financial reward, just our thanks and a feeling of satisfaction. If you are interested please contact me.

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