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ON THE OUTCOME OF EPIDEMICS WITH DETECTIONS

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Abstract

The classical SIR epidemic model is generalized to incorporate a detection process of infectives in the course of time. Our purpose is to determine the exact distribution of the population state at the first detection instant and the following ones. An extension is also discussed that allows the parameters to change with the number of detected cases. The followed approach relies on simple martingale arguments and uses a special family of Abel–Gontcharoff polynomials.

Keywords: General epidemic; detection of infectives; change of parameters; stopping time; Abel–Gontcharoff polynomial

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1. Introduction

The detection of infected cases is an effective measure in slowing down the propagation of an epidemic. It is applied nowadays to control the spread of certain infectious diseases in animals (e.g. swine fever and foot and mouth) and in humans (e.g. influenza or AIDS).

Our paper is concerned with an SIR (susceptible→infected→removed) epidemic model in which infectives can be detected in the course of time. A closed population contains initially n susceptibles and m infectives. Each infective is infectious during a period of time that is exponentially distributed with parameter μ. At the end of the infectious period, it becomes removed (dead or immune). An infective, however, can be detected by a health control system. The time needed for its detection is exponentially distributed with parameter α (independently of the infectious period). As long as it is not detected, an infective can contact any susceptible still present according to a Poisson process with rate β (independently of the other infectives). Removed and detected individuals can no longer transmit the infection.

Let S_t, I_t, R_t, and D_t be the number of susceptibles, infectives, removed, and detected cases at time $t \geq 0$. The process $\{S_t, I_t, R_t, D_t,\ t \geq 0\}$ is Markovian with infinitesimal transition probabilities given by

$$\begin{aligned}&\mathbb{P}[S_{t+\mathrm{d}t} = s-1, I_{t+\mathrm{d}t} = i+1, R_{t+\mathrm{d}t} = r, D_{t+\mathrm{d}t} = d \mid S_t = s, I_t = i, R_t = r, D_t = d]\\ &\quad = \beta s i\,\mathrm{d}t + o(\mathrm{d}t), \qquad (1)\\ &\mathbb{P}[S_{t+\mathrm{d}t} = s, I_{t+\mathrm{d}t} = i-1, R_{t+\mathrm{d}t} = r+1, D_{t+\mathrm{d}t} = d \mid S_t = s, I_t = i, R_t = r, D_t = d]\\ &\quad = \mu i\,\mathrm{d}t + o(\mathrm{d}t), \qquad (2)\end{aligned}$$

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and

$$\mathbb{P}[S_{t+\mathrm{d}t} = s, I_{t+\mathrm{d}t} = i-1, R_{t+\mathrm{d}t} = r, D_{t+\mathrm{d}t} = d+1 \mid S_t = s, I_t = i, R_t = r, D_t = d] = \alpha i \,\mathrm{d}t + o(\mathrm{d}t), \tag{3}$$

with $S_t + I_t + R_t + D_t = n + m$ (the population being closed). When $\alpha = 0$, i.e. without a possible detection process, this model is a classical general epidemic.

The books of Daley and Gani (1999) and Andersson and Britton (2000) provide a comprehensive overview of the epidemic theory. The above model with detections was introduced by Trapman and Bootsma (2009). In that work, the epidemic was approximated by a branching process and a relationship with queueing was exploited to determine the distribution of the number of infectives at the first detection time. This approximation was used in Bootsma *et al.* (2011) and an in-depth study was then made by Lambert and Trapman (2013). Other links between epidemiology and queueing can be found in Hernández-Suárez *et al.* (2010) and the references therein.

Consider the instant T_1' of the first detection of an infective. It can be infinite (no detection is made) with positive probability. If this arises, the epidemic will terminate at the time T' (finite) when the infectives are all removed. Thus, the associated stopping time T_1 is given by $T_1 = \min(T_1', T')$. This is the situation discussed so far in the literature. More generally, however, control measures could be decided only at the instant T_j' when there are j detections ($j \geq 1$). In that case, the stopping time T_j is given by $T_j = \min(T_j', T')$. Several real-life applications are given in Trapman and Bootsma (2009). For instance, with animal diseases in farms, the whole herd will be culled to prevent contagion at the first or jth detection of an infected animal. Similarly, for the spread of nosocomial pathogens (e.g. MRSA) in a hospital, all infectious patients will be isolated as soon as the symptoms appear in one or j carriers.

It is important to know the distribution of the population state at the stopping times T_j. This is the object of the present paper. In contrast to previous works, we will consider the epidemic model itself and not an approximation by a branching model. In Section 2 we deal with the stopping time T_1 that was considered earlier. In Section 3 we extend the analysis to the stopping times T_j with $j \geq 1$. In Section 4 we discuss a generalized model in which the epidemic parameters change with the actual number of detections. For each section, the main result provides us with the joint probability generating function (PGF) of the number of susceptibles, infectives, and detected cases (see Propositions 3, 7, and 10). Our approach relies on the construction of hidden martingales and their stopping at the detection times. This leads us to a set of simple identities that can be then exploited using a special family of Abel–Gontharoff polynomials.

The present study focuses on exact results. Deriving large population approximations from the exact results does not seem to be easy. In a forthcoming work, however, we will apply a similar martingale method to a branching process with detections, which will yield the asymptotic result obtained by Trapman and Bootsma (2009). As indicated by these authors, an interesting extension would be an SIR epidemic in which the infectious periods are independent with a general distribution. The analysis of such a model remains an open problem.

2. At the first detection

Consider the epidemic model defined by (1)–(3). In this section we determine the distribution of the population state at time T_1, when the first detection arises if ever. We will work with the vector $(S_{T_1}, I_{T_1}, D_{T_1})$, as $R_{T_1} = n + m - S_{T_1} - I_{T_1} - D_{T_1}$.

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To begin with, we look for a real function $a(s, i, d)$, where $0 \le s \le n,\ 0 \le i \le n+m,\ d = 0, 1$, such that the process

$$M_t \equiv a(S_t, I_t, D_t), \qquad t \ge 0, \tag{4}$$

forms a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{(S_u, I_u, D_u),\ 0 \le u \le t\}$.

Lemma 1. *It holds that $\{M_t,\ t \ge 0\}$ is a martingale if the $a(s, i, d)$ satisfy the conditions*

$$\begin{aligned} &\beta si[a(s-1, i+1, d) - a(s, i, d)] + \mu i[a(s, i-1, d) - a(s, i, d)] \\ &\quad + \alpha i[a(s, i-1, d+1) - a(s, i, d)] \\ &\quad = 0, \qquad i \ge 0, \end{aligned} \tag{5}$$

where $a(-1, i, d) = 0$.

Proof. Define $m_t = \mathbb{E}(M_t \mid \mathcal{F}_{t_0}),\ t \ge t_0 \ge 0$. By the Markov property, we have

$$\begin{aligned} m_{t+\mathrm{d}t} = \mathbb{E}\{&\beta S_t I_t a(S_t - 1, I_t + 1, D_t)\,\mathrm{d}t + \mu I_t a(S_t, I_t - 1, D_t)\,\mathrm{d}t \\ &+ \alpha I_t a(S_t, I_t - 1, D_t + 1)\,\mathrm{d}t \\ &+ a(S_t, I_t, D_t)[1 - \beta S_t I_t\,\mathrm{d}t - \mu I_t\,\mathrm{d}t - \alpha I_t\,\mathrm{d}t] + o(\mathrm{d}t) \mid \mathcal{F}_{t_0}\}. \end{aligned}$$

Thus, the right derivative m_t' exists and is given by

$$\begin{aligned} m_t' = \mathbb{E}\{&\beta S_t I_t[(a(S_t - 1, I_t + 1, D_t) - a(S_t, I_t, D_t)] + \mu I_t[a(S_t, I_t - 1, D_t) \\ &- a(S_t, I_t, D_t)] + \alpha I_t[a(S_t, I_t - 1, D_t + 1) - a(S_t, I_t, D_t)] \mid \mathcal{F}_{t_0}\}. \end{aligned}$$

If $a(s, i, d)$ satisfies (5) then $m_t' = 0$, so that $\{M_t,\ t \ge 0\}$ is a martingale. □

We now derive $n + 1$ solutions, indexed by k, to the relations (5).

Lemma 2. *For each integer $k \in [0, n]$, $a_k(s, i, d)$ defined by*

$$a_k(s, i, d) = \binom{s}{k} \rho^{s+i-k} \sigma^d, \tag{6}$$

is a solution to (5) if the parameters ρ and σ satisfy the relation

$$-\beta k + \mu\left(\frac{1}{\rho} - 1\right) + \alpha\left(\frac{\sigma}{\rho} - 1\right) = 0. \tag{7}$$

Proof. Substituting (6) into (5) yields

$$\binom{s}{k} \rho^{s+i-k} \sigma^d i\left[-\beta k + \mu\left(\frac{1}{\rho} - 1\right) + \alpha\left(\frac{\sigma}{\rho} - 1\right)\right] = 0,$$

hence condition (7). □

For any stopping time T, the martingales (4) then yield the $n + 1$ identities below.

Lemma 3. *It holds that*

$$\mathbb{E}\left\{\binom{S_T}{k} \rho^{S_T + I_T} \sigma^{D_T}\right\} = \binom{n}{k} \rho^{n+m}, \qquad 0 \le k \le n, \tag{8}$$

where the parameters ρ and σ satisfy (7).

Proof. The state space being bounded, we may apply the stopping time theorem to each martingale $\{M_{k,t} = a_k(S_t, I_t, D_t),\ t \ge 0\}$. This gives the relations (8). □

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Let us choose for T the stopping time T_1. Our goal is to derive the final state distribution from the relations (8). A convenient mathematical tool for that is the family of Abel–Gontcharoff (AG) polynomials. A review of these polynomials can be found in Lefèvre and Picard (2015).

In short, consider any given real sequence $U = \{u_i,\ i \geq 0\}$. To U we attach a unique family of AG polynomials $\{G_k(x \mid U),\ k \geq 0\}$, of degree k in x, defined by

$$\frac{\mathrm{d}G_k(x \mid U)}{\mathrm{d}x} = G_{k-1}(x \mid \mathcal{E}U), \qquad k \geq 1, \tag{9}$$

where $\mathcal{E}U = \{u_{i+1},\ i \geq 0\}$, together with the conditions

$$G_0(x \mid U) = 1 \quad \text{and} \quad G_k(u_0 \mid U) = 0, \qquad k \geq 1.$$

A key property is that any polynomial $R(x)$ of degree n has an Abelian type expansion

$$R(x) = \sum_{k=0}^{n} R^{(k)}(u_k) G_k(x \mid U). \tag{10}$$

Choosing $R(x) = x^n,\ n \geq 1$, yields a simple recursive method to compute the polynomials G_k. We also recall that for any reals a and b with $aU + b \equiv \{au_i + b,\ i \geq 0\}$,

$$G_k(ax + b \mid aU + b) = a^n G_k(x \mid U), \qquad k \geq 0. \tag{11}$$

Coming back to the epidemic model, we first examine the case where $T_1 = T'$, i.e. when no infective is detected until the end of the contagion process. Set $n_{[k]} = n(n-1)\cdots(n-k+1)$, with $n_{[0]} = 1$.

Proposition 1. *We have*

$$\mathbb{E}\left\{ \binom{S_{T_1}}{k} \rho_k^{S_{T_1}} \mathbf{1}_{\{T_1 = T'\}} \right\} = \binom{n}{k} \rho_k^{n+m}, \qquad 0 \leq k \leq n, \tag{12}$$

where ρ_k is given by

$$\rho_k = \frac{\mu}{\mu + \alpha + \beta k}. \tag{13}$$

Thus, for any real x,

$$\mathbb{E}(x^{S_{T_1}} \mathbf{1}_{\{T_1 = T'\}}) = \sum_{k=0}^{n} n_{[k]} \rho_k^{n+m-k} G_k(x \mid \{\rho_i,\ i \geq 0\}) \equiv H_1(x). \tag{14}$$

Proof. In (7), let us choose $\sigma = 0$, so that $\rho \equiv \rho_k$ is given by (13). Then (8) becomes

$$\mathbb{E}\left\{ \binom{S_{T_1}}{k} \rho_k^{S_{T_1}+I_{T_1}} 0^{D_{T_1}} \right\} = \binom{n}{k} \rho_k^{n+m}. \tag{15}$$

By definition, either $T_1 = T_1'$ and then $D_{T_1} = 1$ (so that $0^{D_{T_1}} = 0$), or $T_1 = T'$ and then $D_{T_1} = 0$ (so that $0^{D_{T_1}} = 1$) and $I_{T_1} = 0$. Thus, (15) reduces to (12).

Now, let us consider the AG polynomials defined from the family $U = \{u_i = \rho_i,\ i \geq 0\}$. By expanding the polynomial $x^{S_{T_1}}$ by means of (10), we obtain

$$x^{S_{T_1}} = \sum_{k=0}^{n} S_{T_1,[k]} \rho_k^{S_{T_1}-k} G_k(x \mid \{\rho_i,\ i \geq 0\}).$$

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We can then write

$$\mathbb{E}(x^{S_{T_1}}\,\mathbf{1}_{\{T_1=T'\}}) = \sum_{k=0}^{n} \mathbb{E}(S_{T_1,[k]}\rho_k^{S_{T_1}-k}\,\mathbf{1}_{\{T_1=T'\}})G_k(x \mid \{\rho_i,\ i \geq 0\}).$$

Inserting the relations (12) in the right-hand side, we obtain the announced formula (14). □

Remark 1. (i) Without detection, i.e. if $\alpha = 0$ then $T_1 = T'$ and (14) reduces to

$$\mathbb{E}(x^{S_{T'}}) = \sum_{k=0}^{n} n_{[k]} q_k^{n+m-k} G_k(x \mid \{q_i,\ i \geq 0\}), \tag{16}$$

in which $q_i = \mu/(\mu + \beta i k)$ from (13). This is the PGF of the final number of susceptibles in the standard general epidemic (e.g. Picard and Lefèvre (1990) and Ball and O'Neill (1999)).

(ii) When $\alpha > 0$, consider the aggregated process $\{S_t,\ I_t,\ R_t + D_t,\ t \geq 0\}$. Clearly, this is also a general epidemic model but with now the parameters $\mu + \alpha$ substituted for μ. Thus, the PGF of $S_{T'}$ is still given by (16) in which, this time,

$$q_i = \frac{\mu + \alpha}{\mu + \alpha + \beta i}, \qquad i \geq 0. \tag{17}$$

This can be checked from (7) and (8) by taking $\sigma = 1$.

We now examine the case where $T_1 = T_1'$, i.e. when the epidemic is stopped by the first detection of an infective.

Proposition 2. *For any real ρ,*

$$\mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}+I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}}\right\} = \frac{1}{\sigma_k(\rho)}\left[\binom{n}{k}\rho^{n+m} - \mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}}\,\mathbf{1}_{\{T_1=T'\}}\right\}\right], \qquad 0 \leq k \leq n, \tag{18}$$

where $\sigma_k(\rho)$ is given by

$$\sigma_k(\rho) = \frac{\rho(\mu + \alpha + \beta k) - \mu}{\alpha}. \tag{19}$$

Thus, for any reals x and ρ,

$$\mathbb{E}(x^{S_{T_1}}\rho^{I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}}) = \sum_{k=0}^{n} \frac{1}{\sigma_k(\rho)}\left[\binom{n}{k}\rho^{n+m-k} - \frac{1}{k!}H_1^{(k)}(\rho)\right](x-\rho)^k \equiv K_1(x,\rho). \tag{20}$$

Proof. In (7), the parameter ρ is left arbitrary, so that $\sigma \equiv \sigma_k(\rho)$ is given by (19). By the definition of T_1, (8) then becomes

$$\mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}}\,\mathbf{1}_{\{T_1=T'\}}\right\} + \mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}+I_{T_1}}\sigma_k(\rho)\,\mathbf{1}_{\{T_1=T_1'\}}\right\} = \binom{n}{k}\rho^{n+m},$$

which yields (18).

Let us now go back to (14). Differentiating k times at $x = \rho$ yields

$$\mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}}\,\mathbf{1}_{\{T_1=T'\}}\right\} = H_1^{(k)}(\rho)\frac{\rho^k}{k!}.$$

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From (18), we then have

$$\mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}+I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}}\right\}=\frac{1}{\sigma_k(\rho)}\left[\binom{n}{k}\rho^{n+m}-H_1^{(k)}(\rho)\frac{\rho^k}{k!}\right]. \tag{21}$$

Moreover, by Taylor's expansion of $x^{S_{T_1}}$ around $x=1$,

$$\mathbb{E}(x^{S_{T_1}}\rho^{S_{T_1}+I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}})=\sum_{k=0}^{n}\mathbb{E}\left\{\binom{S_{T_1}}{k}\rho^{S_{T_1}+I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}}\right\}(x-1)^k.$$

Inserting the relations (21) in the right-hand side, we then obtain

$$\mathbb{E}(x^{S_{T_1}}\rho^{S_{T_1}+I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}})=\sum_{k=0}^{n}\frac{1}{\sigma_k(\rho)}\left[\binom{n}{k}\rho^{n+m}-H_1^{(k)}(\rho)\frac{\rho^k}{k!}\right](x-1)^k. \tag{22}$$

Finally, we substitute x/ρ for x in (22) to obtain

$$\mathbb{E}(x^{S_{T_1}}\rho^{I_{T_1}}\,\mathbf{1}_{\{T_1=T_1'\}})=\sum_{k=0}^{n}\frac{1}{\sigma_k(\rho)}\left[\binom{n}{k}\rho^{n+m}-H_1^{(k)}(\rho)\frac{\rho^k}{k!}\right]\left(\frac{x}{\rho}-1\right)^k,$$

hence the announced (20). □

Remark 2. (i) To use (20), we have to compute at first the polynomials $H_1^{(k)}(\rho)$. For that, we differentiate (12) giving $H_1(\rho)$ and apply (9) defining the AG polynomials. This provides us with the expansion

$$H_1^{(k)}(\rho)=\sum_{j=k}^{n}n_{[j]}\rho_j^{n+m-j}G_{j-k}(\rho\mid\{\rho_{k+i},\,i\geq 0\}),\qquad 0\leq k\leq n. \tag{23}$$

So, we see that the $H_1^{(k)}(\rho)$ can be easily evaluated from the AG polynomials G_{j-k}.

(ii) Equation (20) is valid for arbitrary ρ except for some values ρ_j such that $\sigma_j(\rho_j)=0$, i.e. for $\rho_j=\mu/(\mu+\alpha+\beta j)$. When $\rho=\rho_j$, however, it suffices to consider (20) for $\rho\neq\rho_j$ and then to take the limit $\rho\to\rho_j$ using L'Hospital's rule. In (20), the function $K_1(x,\rho)$ at $\rho=\rho_j$ then becomes

$$\begin{aligned}K_1(x,\rho_j)=&\sum_{k=0,k\neq j}^{n}\frac{1}{\sigma_k(\rho_j)}\left[\binom{n}{k}\rho_j^{n+m-k}-\frac{1}{k!}H_1^{(k)}(\rho_j)\right](x-\rho_j)^k\\&+\frac{\alpha}{\alpha+\mu+\beta j}\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\left[\binom{n}{j}\rho^{n+m-j}-\frac{1}{j!}H_1^{(j)}(\rho)\right](x-\rho)^j\right\}_{\rho=\rho_j}.\end{aligned}$$

By combining the results of Propositions 1 and 2, we obtain the joint PGF of the vector $(S_{T_1},I_{T_1},D_{T_1})$.

Proposition 3. *For any reals x, ρ, ζ,*

$$\mathbb{E}(x^{S_{T_1}}\rho^{I_{T_1}}\zeta^{D_{T_1}})=H_1(x)+\zeta K_1(x,\rho). \tag{24}$$

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This yields, in particular,

$$\mathbb{P}(\textit{no detection until } T_1) = H_1(1), \tag{25}$$

$$\mathbb{E}(S_{T_1,[k]}) = \frac{\alpha}{\alpha+\beta k} n_{[k]} + \frac{\beta k}{\alpha+\beta k} H_1^{(k)}(1), \qquad 1 \le k \le n, \tag{26}$$

$$\mathbb{E}(S_{T_1} + I_{T_1}) = n + m - \frac{\alpha+\mu}{\alpha}[1 - H_1(1)]. \tag{27}$$

Proof. Equation (24) is immediate from (14) and (20). For $x = \rho = 1$ and $\zeta = 0$, (24) reduces to $\mathbb{P}(D_{T_1} = 0) = H_1(1)$, which corresponds to (25).

Setting $\rho = \zeta = 1$ yields $\mathbb{E}(x^{S_{T_1}}) = H_1(x) + K_1(x, 1)$. Differentiating k times at $x = 1$ and using (20), we then obtain

$$\mathbb{E}(S_{T_1,[k]}) = H_1^{(k)}(1) + [K_1^{(k)}(x, 1)]_{x=1} = H_1^{(k)}(1) + \frac{1}{\sigma_k(1)}[n_{[k]} - H_1^{(k)}(1)],$$

where $\sigma_k(1) = (\alpha + \beta k)/\alpha$ from (19). This yields (26).

For $\zeta = 1$ and $\rho = x$, (20) and (24) yield

$$\mathbb{E}(x^{S_{T_1}+I_{T_1}}) = H_1(x) + K_1(x, x) = H_1(x) + \frac{1}{\sigma_0(x)}[x^{n+m} - H_1(x)],$$

where $\sigma_0(x) = [x(\mu + \alpha) - \mu]/\alpha$ from (19). Differentiating again at $x = 1$, we obtain (27), completing the proof. □

For illustration, we computed numerically $\mathbb{P}$(no detection until T_1), $\mathbb{E}(S_{T_1})$, and $\mathbb{E}(I_{T_1})$ using (25), (26) with $k = 1$, and (27). The population is of size 50 with initially $n = 47$ susceptibles and $m = 3$ infectives. The infectious period is exponential of mean $\mu = 1$. The contact rate is of the normalized form $\beta = \beta_0/(n + m)$, where β_0 is independent of n, m. In Figure 1, $\beta_0 = 0.5$ or 2, and the detection rate α varies in [0, 1]. In Figure 2, $\alpha = 0.5$ or 1, and β_0 varies in [0, 4].

As expected, $\mathbb{E}(S_{T_1})$ increases with the detection rate α (left panel of Figure 1) and decreases with the contact rate β_0 (left panel of Figure 2). Also, the probability of no detection is decreasing with α and β_0 (right panel of Figure 2). For the graph of $\mathbb{E}(I_{T_1})$, we note that when $\alpha = 0$,

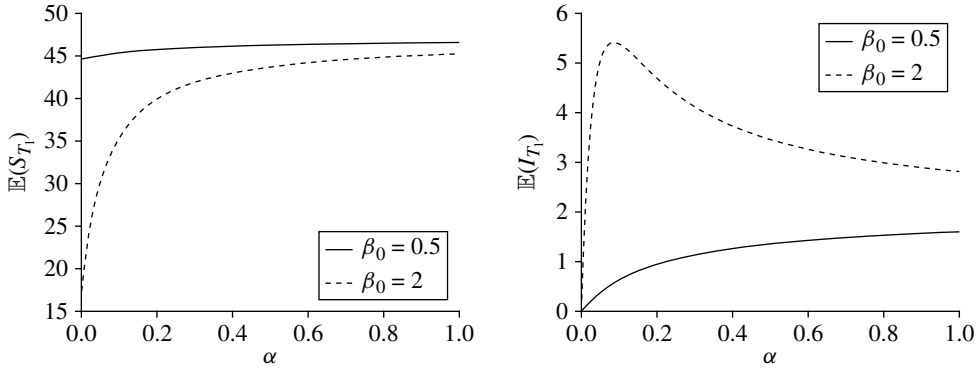


FIGURE 1: Graphs of $\mathbb{E}(S_{T_1})$ (*left*) and $\mathbb{E}(I_{T_1})$ (*right*) as functions of $\alpha \in [0, 1]$ when $n = 47$, $m = 3$, $\mu = 1$, and $\beta_0 = 0.5$ or 2.

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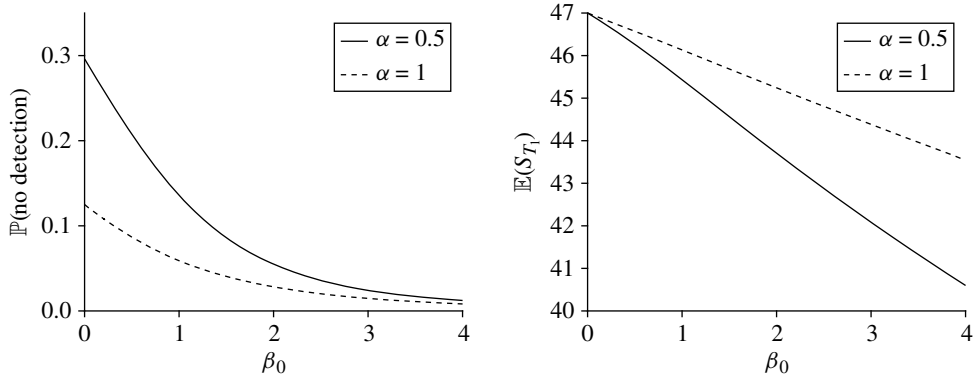


FIGURE 2: Graphs of $\mathbb{P}$(no detection until T_1) and $\mathbb{E}(S_{T_1})$ as functions of $\beta_0 \in [0, 4]$ when $n = 47, m = 3$, $\mu = 1$, and $\alpha = 0.5$ or 1.

the epidemic terminates with the removal of all infectives, i.e. $I_{T_1} = 0$ almost surely, while when $\alpha \to \infty$, a first detection of an infective is almost immediate, hence, $I_{T_1} \to m - 1$ almost surely. So, $\mathbb{E}(I_{T_1})$ starts by increasing with α and then converges to $m - 1$ after having reached possibly a maximum (left panel of Figure 1). The condition for a maximum seems to be $\beta_0 > \mu$, but this is not proved.

3. At the jth detection

In this section we extend the previous study to T_j, the time of the jth detection if ever. Let us start with the case where $T_j = T'$.

Proposition 4. *We have*

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\mathbf{1}_{\{T_j=T'\}}\right\} = \sum_{d=0}^{j-1}\left(\frac{\alpha}{\mu+\alpha}\right)^d A_{k,d}\left(\frac{\mu}{\mu+\alpha}\right), \qquad 0 \le k \le n, \tag{28}$$

where $A_{k,d}(x)$ is a polynomial in x defined by

$$A_{k,d}(x) = \mathbb{E}\left[\binom{S_{T'}}{k}\binom{n+m-S_{T'}}{d}x^{n+m-S_{T'}-d}\right]. \tag{29}$$

Thus, for any real x,

$$\mathbb{E}(x^{S_{T_j}}\mathbf{1}_{\{T_j=T'\}}) = \sum_{d=0}^{j-1}\left(\frac{\alpha}{\mu+\alpha}\right)^d\left[\sum_{k=0}^{n}(x-1)^k A_{k,d}\left(\frac{\mu}{\mu+\alpha}\right)\right] \equiv H_j(x). \tag{30}$$

Proof. Clearly, $T_j = T'$ means $D_{T'} \in \{0, 1, \ldots, j-1\}$. Thus, we can write

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\mathbf{1}_{\{T_j=T'\}}\right\} = \sum_{d=0}^{j-1}\mathbb{E}\left\{\binom{S_{T'}}{k}\mathbf{1}_{\{D_{T'}=d\}}\right\}. \tag{31}$$

By construction, for any time $t \ge 0$, conditioned upon (S_t, I_t), the number D_t has a binomial distribution bin$[n + m - S_t - I_t, \alpha/(\mu + \alpha)]$. Thus, at time T', conditioned upon $S_{T'}$,

$$D_{T'} \overset{\mathrm{D}}{=} \mathrm{bin}\left[n + m - S_{T'}, \frac{\alpha}{\mu+\alpha}\right], \tag{32}$$

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where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. So, let us condition on $S_{T'}$ in the right-hand side of (31). From (32), we then have

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\mathbf{1}_{\{T_j=T'\}}\right\} = \sum_{d=0}^{j-1}\mathbb{E}\left\{\binom{S_{T'}}{k}\mathbb{P}(D_{T'}=d \mid S_{T'})\right\}$$
$$= \sum_{d=0}^{j-1}\left(\frac{\alpha}{\mu+\alpha}\right)^d \mathbb{E}\left[\binom{S_{T'}}{k}\binom{n+m-S_{T'}}{d}\left(\frac{\mu}{\mu+\alpha}\right)^{n+m-S_{T'}-d}\right],$$

which is (28) using the notation of (29). The PGF (30) is then straightforward. □

It remains to calculate $A_{k,d}(\cdot)$ in (28). As explained in Remark 1 following Proposition 1, we know how to determine the distribution of $S_{T'}$. Thus, we are able to compute $A_{k,d}(x)$ from its definition (29). A more explicit expression for $A_{k,d}(x)$ is provided below.

Proposition 5. *We have*

$$A_{k,d}(x) = \frac{1}{k!}\sum_{s=k}^{n} n_{[s]} q_s^{n+m-s} a_{k,d,s}(x), \tag{33}$$

where $a_{k,d,s}(x)$ is a polynomial in x given by

$$a_{k,d,s}(x) = \sum_{j=0}^{s-k} x^{n+m-s-d+j}\left[\binom{n+m-s+j}{d}\frac{1}{(s-k-j)!}G_j(0 \mid \{q_{s-j+i},\ i\ge 0\})\right], \tag{34}$$

the parameters q_i being defined by (17).

Proof. From the PGF of $S_{T'}$ given by (16) with (17), we see that, for any reals x and ρ,

$$\mathbb{E}(\rho^{S_{T'}}x^{n+m-S_{T'}}) = x^{n+m}\sum_{s=0}^{n} n_{[s]} q_s^{n+m-s} G_s\left(\frac{\rho}{x} \;\middle|\; \{q_i,\ i\ge 0\}\right)$$
$$= \sum_{s=0}^{n} n_{[s]}(xq_s)^{n+m-s} G_s(\rho \mid \{xq_i,\ i\ge 0\}),$$

where the latter equality follows from (11). By differentiating k times with respect to ρ with $\rho = 1$, and using (9), we then obtain

$$\mathbb{E}\left[\binom{S_{T'}}{k}x^{n+m-S_{T'}}\right] = \frac{1}{k!}\sum_{s=k}^{n} n_{[s]}(xq_s)^{n+m-s} G_{s-k}(1 \mid \{xq_{k+i},\ i\ge 0\}). \tag{35}$$

Let us differentiate (35) d times with respect to x. The left-hand side becomes $d!\,A_{k,d}(x)$, so that

$$A_{k,d}(x) = \frac{1}{k!\,d!}\sum_{s=k}^{n} n_{[s]}\frac{\partial^d}{\partial x^d}[(xq_s)^{n+m-s}G_{s-k}(1 \mid \{xq_{k+i},\ i\ge 0\})]. \tag{36}$$

Moreover, one can show that, for any nonnegative integers a and b,

$$\frac{\partial^d}{\partial x^d}[x^a G_b(1 \mid \{xu_i,\ i\ge 0\})] = \sum_{j=0}^{b} x^{a+j-d}\frac{(a+j)_{[d]}}{(b-j)!}G_j(0 \mid \{u_{b-j+i},\ i\ge 0\}), \tag{37}$$

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by applying Taylor's expansion to $G_b(1 \mid \cdots)$ in (37). Thus, applying (37) to (36), we obtain the desired results, (33) and (34). □

Let us now examine the case where $T_j = T'_j$.

Proposition 6. *For any real ρ,*

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}+I_{T_j}}\,\mathbf{1}_{\{T_j=T'_j\}}\right\} = \frac{\rho^{n+m}}{[\sigma_k(\rho)]^j}\left[\binom{n}{k} - \sum_{d=0}^{j-1}\left(\frac{\alpha\sigma_k(\rho)}{\rho(\mu+\alpha)}\right)^d A_{k,d}\left(\frac{\mu}{\rho(\mu+\alpha)}\right)\right], \qquad 0 \le k \le n. \tag{38}$$

Thus, for any reals x and ρ,

$$\mathbb{E}(x^{S_{T_j}}\rho^{I_{T_j}}\,\mathbf{1}_{\{T_j=T'_j\}}) = \sum_{k=0}^{n}\frac{\rho^{n+m-k}}{[\sigma_k(\rho)]^j}\left[\binom{n}{k} - \sum_{d=0}^{j-1}\left(\frac{\alpha\sigma_k(\rho)}{\rho(\mu+\alpha)}\right)^d A_{k,d}\left(\frac{\mu}{\rho(\mu+\alpha)}\right)\right](x-\rho)^k \equiv K_j(x,\rho). \tag{39}$$

Proof. By the definition of T_j, (8) yields the relations

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}+I_{T_j}}[\sigma_k(\rho)]^{D_{T_j}}\,\mathbf{1}_{\{T_j=T'\}}\right\} + \mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}+I_{T_j}}[\sigma_k(\rho)]^{D_{T_j}}\,\mathbf{1}_{\{T_j=T'_j\}}\right\} = \binom{n}{k}\rho^{n+m},$$

where $\sigma_k(\rho)$ is defined by (19). As with (31), this can be expressed as

$$\sum_{d=0}^{j-1}[\sigma_k(\rho)]^d\mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}}\,\mathbf{1}_{\{T_j=T',\,D_{T'}=d\}}\right\} + [\sigma_k(\rho)]^j\mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}+I_{T_j}}\,\mathbf{1}_{\{T_j=T'_j\}}\right\} = \binom{n}{k}\rho^{n+m}.$$

Thus, we have

$$\mathbb{E}\left\{\binom{S_{T_j}}{k}\rho^{S_{T_j}+I_{T_j}}\,\mathbf{1}_{\{T_j=T'_j\}}\right\} = \frac{1}{[\sigma_k(\rho)]^j}\left[\binom{n}{k}\rho^{n+m} - \sum_{d=0}^{j-1}[\sigma_k(\rho)]^d\mathbb{E}\left\{\binom{S_{T'}}{k}\rho^{S_{T'}}\,\mathbf{1}_{\{D_{T'}=d\}}\right\}\right],$$

and from (32), the expectation on the right-hand side is given by

$$\mathbb{E}\left\{\binom{S_{T'}}{k}\rho^{S_{T'}}\binom{n+m-S_{T'}}{d}\left(\frac{\alpha}{\mu+\alpha}\right)^d\left(\frac{\mu}{\mu+\alpha}\right)^{n+m-S_{T'}-d}\right\}.$$

Equations (38) and (39) then follow easily. □

Finally, Propositions 4 and 6 provide us with the PGF of $(S_{T_j}, I_{T_j}, D_{T_j})$.

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Proposition 7. *For any reals x, ρ, and ζ,*

$$\mathbb{E}(x^{S_{T_j}}\rho^{I_{T_j}}\zeta^{D_{T_j}}) = \zeta^j K_j(x,\rho) + \sum_{d=0}^{j-1}\left(\frac{\alpha\zeta}{\mu+\alpha}\right)^d x^{n+m-d} A_{0,d}\left(\frac{\mu}{x(\mu+\alpha)}\right). \tag{40}$$

This yields, for instance,

$$\mathbb{P}(\textit{less than } j \textit{ detections until } T_j) = H_j(1). \tag{41}$$

Proof. From (39), we have

$$\begin{aligned}\mathbb{E}(x^{S_{T_j}}\rho^{I_{T_j}}\zeta^{D_{T_j}}) &= \mathbb{E}(x^{S_{T_j}}\rho^{I_{T_j}}\zeta^j\,\mathbf{1}_{\{T_j=T'_j\}}) + \mathbb{E}(x^{S_{T_j}}\rho^0\zeta^{D_{T_j}}\,\mathbf{1}_{\{T_j=T'\}})\\ &= \zeta^j K_j(x,\rho) + \sum_{d=0}^{j-1}\zeta^d\mathbb{E}(x^{S_{T'}}\,\mathbf{1}_{\{D_{T_j}=d\}}).\end{aligned}$$

Using (32), the sum on the right-hand side can be written as

$$\begin{aligned}&\sum_{d=0}^{j-1}\zeta^d\mathbb{E}(x^{S_{T'}}\,\mathbf{1}_{\{D_{T_j}=d\}})\\ &\quad= \sum_{d=0}^{j-1}\zeta^d\mathbb{E}\left[x^{S_{T'}}\binom{n+m-S_{T'}}{d}\left(\frac{\alpha}{\mu+\alpha}\right)^d\left(\frac{\mu}{\mu+\alpha}\right)^{n+m-S_{T'}-d}\right]\\ &\quad= \sum_{d=0}^{j-1}\left(\frac{\alpha\zeta}{\mu+\alpha}\right)^d x^{n+m-d}\mathbb{E}\left[\binom{n+m-S_{T'}}{d}\left(\frac{\mu}{x(\mu+\alpha)}\right)^{n+m-S_{T'}-d}\right],\end{aligned}$$

the latter expectation being $A_{0,d}(\mu/x(\mu+\alpha))$ by virtue of (29). This yields (40).

Let us take, for instance, $x=1$, $\rho=0$, and $\zeta=1$ in (40). The left-hand side of (40) then reduces to $\mathbb{E}(0^{I_{T_j}}) = \mathbb{P}(I_{T_j}=0)$. For the right-hand side of (40), we note that $K_j(x,0)=0$ from the left-hand side of (39) with $\rho=0$. Thus, we obtain

$$\mathbb{P}(I_{T_j}=0) = \sum_{d=0}^{j-1}\left(\frac{\alpha}{\mu+\alpha}\right)^d A_{0,d}\left(\frac{\mu}{\mu+\alpha}\right),$$

which is (41) using (30). □

4. Detection-dependent rates

It is reasonable to assume that as the number of detections increases, prophylactic measures will be reinforced (β decreases), possibly medical care too (μ increases), and the detection process will be more efficient (α increases). In other words, the rates depend on the actual number of detected cases, d say, and they take the form β_d, μ_d, and α_d.

The approach of Sections 2 and 3 can be extended to this generalized epidemic model. So, it is possible to determine the population state distribution at the detection times $(T_1,\dots,T_j)$. For the sake of brevity, we will only discuss the case of (T_1,T_2). We start by deriving $n+1$ identities which are valid for any stopping time T.

Lemma 4. *We have*

$$\mathbb{E}\left\{\binom{S_T}{k}\rho_{k,0}^{S_T+I_T}\,\mathbf{1}_{\{D_T=0\}}\right\} = \binom{n}{k}\rho_{k,0}^{n+m}, \qquad 0\le k\le n, \tag{42}$$

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where

$$\rho_{k,0} = \frac{\mu_0}{\mu_0 + \alpha_0 + \beta_0 k}. \tag{43}$$

For any real ρ,

$$\mathbb{E}\left\{\binom{S_T}{k}\rho^{S_T+I_T}\sigma_0\sigma_{k,1}(\rho)\cdots\sigma_{k,D_T}(\rho)\right\} = \binom{n}{k}\rho^{n+m}, \qquad 0 \le k \le n, \tag{44}$$

where $\sigma_0 = 1$ *and*

$$\sigma_{k,1+d}(\rho) = \frac{\rho(\mu_d + \alpha_d + \beta_d k) - \mu_d}{\alpha_d}. \tag{45}$$

Proof. We look for a martingale $\{M_{k,t} = a_k(S_t, I_t, D_t),\ t \ge 0\}$, where $a_k(s, i, d)$ is of the form

$$a_k(s, i, d) = \binom{s}{k}\rho^{s+i-k}\sigma_0\sigma_1\cdots\sigma_d, \tag{46}$$

with $\sigma_0 = 1$. Condition (5) requires that ρ and $\sigma_1, \ldots, \sigma_d$ satisfy

$$\sigma_0\sigma_1\cdots\sigma_d\left(-\beta_d k + \mu_d\left(\frac{1}{\rho} - 1\right) + \alpha_d\left(\frac{\sigma_{1+d}}{\rho} - 1\right)\right) = 0, \qquad d \ge 0. \tag{47}$$

Two different situations are possible. First, we choose $\sigma_u = 0$ for all $u \ge 1$. Then $\sigma_0\sigma_1\cdots\sigma_d = 0^d$ for all $d \ge 1$, so that (47) is satisfied for all $d \ge 1$. Thus, it is enough to satisfy (47) when $d = 0$, which yields $\rho = \rho_{k,0}$ defined in (43). From (46), the corresponding martingale is

$$M_{k,t} = \binom{S_t}{k}\rho_{k,0}^{S_t+I_t}0^{D_t}, \qquad t \ge 0,$$

and (42) follows.

We now consider $\sigma_u \ne 0$ for some $u \ge 1$. Then (47) will be satisfied if $\sigma_{1+d} = \sigma_{k,1+d}(\rho)$ defined in (45). From (46), the martingale becomes

$$M_{k,t} = \binom{S_t}{k}\rho^{S_t+I_t}\sigma_0\sigma_{k,1}(\rho)\cdots\sigma_{k,D_t}(\rho), \qquad t \ge 0,$$

hence, (44). □

Let us come back to the study of (T_1, T_2). Since the parameters change only after a detection, the case of T_1 is the same as in Section 2, based this time on the parameters $\mu_0, \beta_0, \alpha_0, \rho_{i,0}$. So, instead of $H_1(x)$ in (14), we write here

$$H_1(x_1) \equiv H_1(x_1;\ n, m, 0) = \sum_{k=0}^{n} n_{[k]}\rho_{k,0}^{n+m-k}G_k(x_1 \mid \{\rho_{i,0},\ i \ge 0\}), \tag{48}$$

where the three arguments $n, m, 0$ in $H_1(x_1; \ldots)$ serve to indicate that initially there are n susceptibles and m infectives and the parameters in (48) are the $\rho_{i,0}$ given by (43). Similarly, $K_1(x, \rho)$ in (20) is replaced by

$$K_1(x_1, \rho_1;\ n, m, 0) = \sum_{k=0}^{n}\frac{1}{\sigma_{k,1}(\rho_1)}\left[\binom{n}{k}\rho_1^{n+m-k} - \frac{1}{k!}H_1^{(k)}(\rho_1;\ n, m, 0)\right](x_1 - \rho_1)^k, \tag{49}$$

where the argument 0 in $K_1(\cdots)$ refers to the parameters μ_0, β_0, α_0 in $\sigma_{k,1}(\rho_1)$ and the $\rho_{i,0}$ in $H_1^{(k)}(\rho_1;\ n, m, 0)$.

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The epidemic is stopped at time T_1 if all the infectives were removed at time T_1 (i.e. if $D_{T_1} = 0$). Thus, considering T_2 has a sense only if $D_{T_1} = 1$. Specifically, we will determine the following expectations:

$$B_{1,1}(x_1, \rho_1, x_2) = \mathbb{E}(x_1^{S_{T_1}} \rho_1^{I_{T_1}} x_2^{S_{T_2}} \mathbf{1}_{\{D_{T_1}=1,\, D_{T_2}=1\}}),$$
$$B_{1,2}(x_1, \rho_1, x_2, \rho_2) = \mathbb{E}(x_1^{S_{T_1}} \rho_1^{I_{T_1}} x_2^{S_{T_2}} \rho_2^{I_{T_2}} \mathbf{1}_{\{D_{T_1}=1,\, D_{T_2}=2\}}). \tag{50}$$

Proposition 8. *We have*

$$B_{1,1}(x_1, \rho_1, x_2) = \sum_{k=0}^{n} K_1^{(k,0)}(\rho_{k,1}x_1, \rho_{k,1}\rho_1;\, n, m, 0) x_1^k G_k(x_2 \mid \{\rho_{i,1},\, i \geq 0\}), \tag{51}$$

where $\rho_{i,1} = \mu_1/(\mu_1 + \alpha_1 + \beta_1 i)$, *and* $K_1^{(k,0)}(\cdots)$ *denotes the kth derivative in* x_1 *of* $K_1(x_1, \rho_1;\, n, m, 0)$ *given by (49).*

Proof. By conditioning, $B_{1,1}(x_1, \rho_1, x_2)$ in (50) can be expressed as

$$\begin{aligned} B_{1,1}(x_1, \rho_1, x_2) &= \mathbb{E}[x_1^{S_{T_1}} \rho_1^{I_{T_1}} \mathbf{1}_{\{D_{T_1}=1\}}\, \mathbb{E}(x_2^{S_{T_2}} \mathbf{1}_{\{D_{T_2}=1\}} \mid S_{T_1}, I_{T_1}, D_{T_1} = 1)] \\ &= \mathbb{E}[x_1^{S_{T_1}} \rho_1^{I_{T_1}} \mathbf{1}_{\{D_{T_1}=1\}}\, H_1(x_2;\, S_{T_1}, I_{T_1}, 1)], \end{aligned}$$

where $H_1(x_2;\, S_{T_1}, I_{T_1}, 1)$ is defined as in (48) but this time with the parameters $\rho_{i,1} = \mu_1/(\mu_1 + \alpha_1 + \beta_1 i)$ (compare with (43)), i.e.

$$H_1(x_2;\, S_{T_1}, I_{T_1}, 1) = \sum_{k=0}^{S_{T_1}} S_{T_1,[k]} \rho_{k,1}^{S_{T_1}+I_{T_1}-k} G_k(x_2 \mid \{\rho_{i,1},\, i \geq 0\}).$$

Note that the sum above can be taken up to n since $S_{T_1,[k]} = 0$ when $k > S_{T_1}$. Therefore,

$$B_{1,1}(x_1, \rho_1, x_2) = \sum_{k=0}^{n} \mathbb{E}[S_{T_1,[k]} (\rho_{k,1}x_1)^{S_{T_1}-k} (\rho_{k,1}\rho_1)^{I_{T_1}} \mathbf{1}_{\{D_{T_1}=1\}}] x_1^k G_k(x_2 \mid \{\rho_{i,1},\, i \geq 0\}). \tag{52}$$

From (20), we see that

$$\mathbb{E}[S_{T_1,[k]} (\rho_{k,1}x_1)^{S_{T_1}-k} (\rho_{k,1}\rho_1)^{I_{T_1}} \mathbf{1}_{\{D_{T_1}=1\}}] = K_1^{(k,0)}(\rho_{k,1}x_1, \rho_{k,1}\rho_1;\, n, m, 0), \tag{53}$$

which can be calculated from (49). Inserting (53) into (52) then yields (51). □

Proposition 9. *We have*

$$\begin{aligned} &B_{1,2}(x_1, \rho_1, x_2, \rho_2) \\ &\quad = \sum_{k=0}^{n} \frac{1}{\sigma_{k,2}(\rho_2)} \frac{(x_2 - \rho_2)^k}{k!} \\ &\qquad \times \Bigg[K_1^{(k,0)}(\rho_2 x_1, \rho_2\rho_1;\, n, m, 0) x_1^k \\ &\qquad\quad - \sum_{j=k}^{n} K_1^{(j,0)}(\rho_{j,1}x_1, \rho_{j,1}\rho_1;\, n, m, 0) x_1^j G_{j-k}(\rho_2 \mid \{\rho_{k+i,1},\, i \geq 0\}) \Bigg]. \end{aligned} \tag{54}$$

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Proof. Arguing as for $B_{1,1}$, we can write $B_{1,2}(x_1, \rho_1, x_2, \rho_2)$ defined in (50) as

$$B_{1,2}(x_1, \rho_1, x_2, \rho_2) = \mathbb{E}[x_1^{S_{T_1}} \rho_1^{I_{T_1}} \mathbf{1}_{\{D_{T_1}=1\}} K_1(x_2, \rho_2;\ S_{T_1}, I_{T_1}, 1)], \tag{55}$$

where $K_1(x_2, \rho_2;\ S_{T_1}, I_{T_1}, 1)$ is defined as in (49) but with the parameters $\mu_1, \beta_1, \alpha_1, \rho_{i,1}$, i.e.

$$K_1(x_2, \rho_2;\ S_{T_1}, I_{T_1}, 1) = \sum_{k=0}^{n} \frac{1}{\sigma_{k,2}(\rho_2)} \left[\binom{S_{T_1}}{k} \rho_2^{S_{T_1}+I_{T_1}-k} - \frac{1}{k!} H_1^{(k)}(\rho_2;\ S_{T_1}, I_{T_1}, 1) \right] (x_2 - \rho_2)^k. \tag{56}$$

From (23), $H_1^{(k)}$ above can be expanded as

$$H_1^{(k)}(\rho_2;\ S_{T_1}, I_{T_1}, 1) = \sum_{j=k}^{n} S_{T_1,[j]} \rho_{j,1}^{S_{T_1}+I_{T_1}-j} G_{j-k}(\rho_2 \mid \{\rho_{k+i,1},\ i \geq 0\}). \tag{57}$$

Inserting (56) and (57) into (55), and using (20), we easily deduce (54). □

Proposition 10. *For any reals x_1, ρ_1, x_2, ρ_2, and ζ,*

$$\mathbb{E}(x_1^{S_{T_1}} \rho_1^{I_{T_1}} x_2^{S_{T_2}} \rho_2^{I_{T_2}} \zeta^{D_{T_2}}) = H_1(x_1 x_2;\ n, m, 0) + \zeta B_{1,1}(x_1, \rho_1, x_2) + \zeta^2 B_{1,2}(x_1, \rho_1, x_2, \rho_2). \tag{58}$$

Proof. From (50), we have

$$\mathbb{E}(x_1^{S_{T_1}} \rho_1^{I_{T_1}} x_2^{S_{T_2}} \rho_2^{I_{T_2}} \zeta^{D_{T_2}} \mathbf{1}_{\{D_{T_1}=1\}}) = \zeta B_{1,1}(x_1, \rho_1, x_2) + \zeta^2 B_{1,2}(x_1, \rho_1, x_2, \rho_2). \tag{59}$$

When $D_{T_1} = 0$ then $T_1 = T_2 = T'$, which implies that $I_{T_1} = I_{T_2} = I_{T'} = 0$, $S_{T_1} = S_{T_2} = S_{T'}$, and $D_{T_2} = 0$. Thus,

$$\mathbb{E}(x_1^{S_{T_1}} \rho_1^{I_{T_1}} x_2^{S_{T_2}} \rho_2^{I_{T_2}} \zeta^{D_{T_2}} \mathbf{1}_{\{D_{T_1}=0\}}) = \mathbb{E}[(x_1 x_2)^{S_{T_1}} \mathbf{1}_{\{D_{T_1}=0\}}] = H_1(x_1 x_2;\ n, m, 0), \tag{60}$$

by virtue of (14). Combining (59) and (60), we then obtain (58). □

The above analysis works well too for $(T_1, \ldots, T_j)$ with $j > 2$. The results have a similar structure and use recurrence relations that continue to be rather simple.

For illustration, we evaluated $\mathbb{P}(D_{T_1} = D_{T_2} = 1)$, the probability that a first infective is detected and the epidemic terminates before the detection of a second infective. From (50), this corresponds to $B_{1,1}(1, 1, 1)$ and is provided by (51). Note that $\mathbb{P}(D_{T_1} = 1)$, the probability that a first infective is detected before the end of the epidemic, is given by $1 - H_1(1)$ (see (25)). The population is of size 30 with initially $n = 27$ susceptibles and $m = 3$ infectives. When there are 0 or 1 detected cases, the infectious periods are exponential of mean $\mu_0 = \mu_1 = 2$, the contact rates are $\beta_0/(n+m)$ and $\beta_1/(n+m)$, and the detection rates are α_0 and α_1, respectively. In the right panel of Figure 3 we consider $\beta_0 = 10$ and a smaller value for $\beta_1 \in [1, \beta_0]$, when $\alpha_0 = \alpha_1 = 0.5$ or 2. In the left panel of Figure 3 we consider $\alpha_0 = 1$ and a larger value for $\alpha_1 \in [\alpha_0, 10]$, when $\beta_0 = \beta_1 = 1$ or 5.

If control measures allow a decrease in the infection rate β_1, then $\mathbb{P}(D_{T_1} = D_{T_2} = 1)$ will increase, that is, a second detection becomes less likely (right panel of Figure 3). On the other hand, if such measures increase the detection rate α_1, then the effect is reversed, that is, the probability of a second detection increases (left panel of Figure 3). Applying both types of measures can thus have different consequences (shown in the right panel).

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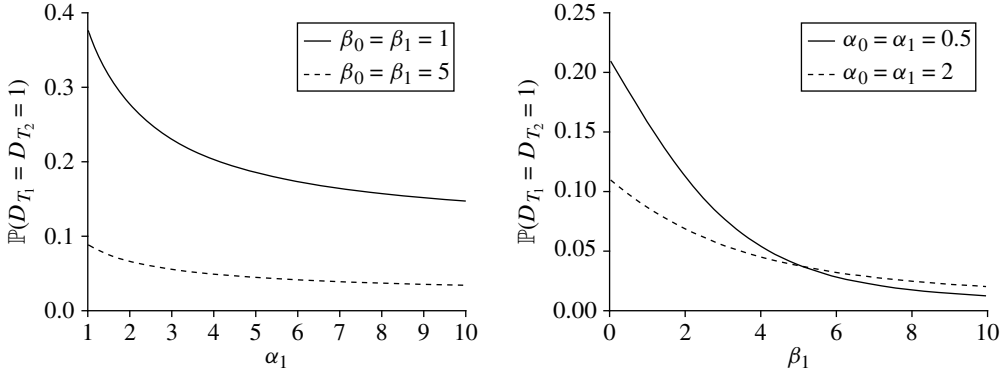


FIGURE 3: Graphs of $\mathbb{P}(D_{T_1} = D_{T_2} = 1)$ as a function of α_1 (*left*) or β_1 (*right*). For the former, $n = 27$, $m = 3$, $\mu_0 = \mu_1 = 2$, $\beta_0 = \beta_1 = 1$ or 5, and $\alpha_0 = 1$, while $\alpha_1 \in [\alpha_0, 10]$. For the latter, $n = 27$, $m = 3$, $\mu_0 = \mu_1 = 2$, $\alpha_0 = \alpha_1 = 0.5$ or 2, and $\beta_0 = 10$, while $\beta_1 \in [0, \beta_0]$.

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