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Galerkin approximations in several parameter bifurcation problems

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1. The purpose of this paper is to prove a theorem giving conditions yielding global bifurcation of the solutions of a family of parameterized nonlinear equations, the domain and the range of which lie in Banach spaces, where the parameter is allowed to be a vector in $\mathbb{R}^k$, k a positive integer. The basic contribution is that the parameter is vector valued and that the nonlinearities allowed are very general; however, even for scalar parameters, our results extend those of previous authors.

Let X and Y be real Banach spaces, let k be a positive integer, and let $\mathcal{O}$ be an open subset of $\mathbb{R}^k \times X$ which contains $(0, 0)$. Suppose $f: \overline{\mathcal{O}} \to Y$, and there is an open subset V of $0 \in \mathbb{R}^k$ such that $f(\lambda, 0) = 0$ when $\lambda \in \mathcal{O}$. We regard $V \times \{0\}$ as constituting the trivial solutions of the equation

$$f(\lambda, x) = 0 \quad (\lambda, x) \in \overline{\mathcal{O}}, \tag{1·1}$$

and we are interested in the nontrivial solutions of equation (1·1) which bifurcate from $(0, 0)$. To make things precise it is convenient to formulate the following definition.

Definition 1·1. Let X, $\mathcal{O}$, V and f be as above, and let $\Omega = \{(\lambda, x) \in \mathcal{O} | f(\lambda, x) = 0, (\lambda, x) \notin V \times \{0\}\}$. We say that *global bifurcation from* $(0, 0)$ *of the solutions of equation* (1·1) *from the trivial solutions* occurs if there exists a connected subset Ω_0 of Ω, with $(0, 0) \in \overline{\Omega}_0$, and at least one of the following three properties holds:

(*a*) Ω_0 is unbounded in $\mathbb{R}^k \times X$,

(*b*) $\overline{\Omega}_0 \cap \{(\mathbb{R}^k \times X) \backslash \mathcal{O}\} \neq \varnothing$,

(*c*) $\overline{\Omega}_0 \cap \{\partial V \times \{0\}\} \neq \varnothing$, where ∂V denotes the boundary of V in $\mathbb{R}^k$.

Remark 1·1. The above definition of global bifurcation is stronger than the corresponding notion considered elsewhere (**5, 1, 3, 20**) inasmuch as here we require Ω_0 to be a connected subset of Ω with $(0, 0) \in \overline{\Omega}_0$, in contrast to requiring Ω_0 to be a connected subset of $\overline{\Omega}$ with $(0, 0) \in \Omega_0$.

Remark 1·2. Frequently, the term bifurcation is used in connection with the study of the *local* description of solutions of equation (1·1) near $(0, 0)$, and in this connection $(0, 0)$ is called a local bifurcation point of equation (1·1) provided that there is a sequence $\{(\lambda_n, x_n)\}$ in $\mathcal{O}$, with $x_n \neq 0$, for each n, and each (λ_n, x_n) is a solution of equation (1·1), such that $\{(\lambda_n, x_n)\}$ converges to $(0, 0)$. For the study of local bifurcation the Implicit Function Theorem has played a crucial role. The study of global bifurcation requires more topological ideas. In particular, when λ is a scalar, Rabinowitz,

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in (20), used topological degree to obtain the first general global results when $X = Y$ and

$$f(\lambda, x) = x - \lambda C(x) + R(\lambda, x),$$

where $\lambda C + R(\lambda, x)$ is compact, for each λ, C is bounded and linear and

$$\|R(\lambda, x)\| = 0(\|x\|), \quad \text{as} \quad \|x\| \to 0.$$

(Recall that a compact map is one which maps bounded sets into precompact sets and is continuous.) At the end of this section we shall describe other global bifurcation results for various classes of maps, both with scalar and vector parameters.

One of the fundamental methods for solving an operator equation in a setting where the domain and range of the operator lie in infinite dimensional linear spaces is to approximate the equation in an appropriate sense by an equation in a finite dimensional space, solve this new equation, and then try to verify that as the approximations become better the approximating solutions will converge to a solution of the original equation. Of course, in general the approximating solutions will not form a compact set, so it is necessary to impose further conditions on the class of operator equations to which one applies this procedure. To be precise, we need some definitions.

Let $\{X_n\}$ and $\{Y_n\}$ be sequences of finite dimensional sub-spaces of X and Y, respectively. Moreover, for each $n \in N$ let P_n and Q_n be projections of X onto X_n and of Y onto Y_n, respectively, and suppose $\dim X_n = \dim Y_n$. Then the quadruple

$$\Gamma = \langle \{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\} \rangle$$

is called a *complete projection scheme* for mappings from X to Y provided that

$$\{P_n(x)\} \to x,$$

for each $x \in X$ and $\{Q_n(y)\} \to y$ for each $y \in Y$. From now on Γ will denote such a scheme. When X and Y have Schauder bases such a projection scheme exists, so that for most of the important Banach spaces occurring in applications such a Γ exists.

Let Λ be a topological space and let X, Y and Γ be as above: Suppose $K \subseteq \Lambda \times X$ and $h: K \to Y$. Then h is called *A-proper* (approximation-proper) *with respect to* Γ at 0 provided that whenever $\langle n_k \rangle$ is an increasing sequence of positive integers with $\{(\lambda_{n_k}, x_{n_k})\} \subseteq K$ bounded,

$$(\lambda_{n_k}, x_{n_k}) \in \Lambda \times X_{n_k}$$

for each k, and $Q_{n_k} h(\lambda_{n_k}, x_{n_k}) \to 0$, then (λ_{n_k}, x_{n_k}) has a subsequence which converges to $(\bar{\lambda}, \bar{x}) \in K$ and $h(\bar{\lambda}, \bar{x}) = 0$.

The class of A-proper mappings, whose study was initiated by Petryshyn (see (18)), has proven to be very useful class of mappings within which one could extend, sometimes in a constructive manner, a number of finite dimensional results to an infinite dimensional setting. A large variety of abstract nonlinear operators and differential operators are A-proper. For example, if $Y = X^*$, the dual space of X, various types of monotone operators are A-proper (18), when $X = Y$, ball-condensing perturbations of the identity are A-proper ((17), (22)), so that compact perturbations of the identity are A-proper. Moreover, quite general classes of elliptic boundary value problems, both

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on bounded (18) and on unbounded (21) domains, as well as certain functional differential equations (15) and integral equations (11) have been formulated as zeros of A-proper mappings. A detailed description of A-proper mappings may be found in (18).

The various concepts can be generalized, e.g. by working with spaces more general than Banach spaces, or by using projection schemes indexed by more general ordered sets than the positive integers. Our results generalize to those situations, but we have formulated our results as we have since they cover almost all applications.

To prove our global bifurcation result we will impose the following hypotheses:

(A 1) *There is a complete projection scheme* Γ *for mappings from* X *to* Y, *with respect to which*

$$f: \overline{\mathcal{O}} \subseteq \mathbb{R}^k \times X \to Y \text{ is } A\text{-proper at } 0.$$

Moreover f *is continuous, uniformity with respect to* λ *on bounded subsets of* $\mathcal{O}$.

(A 2) *For each compact subset* C *of* $V\backslash\{0\}$ *there is a* $\delta = \delta(C) > 0$ *such that*

$$f(\lambda, x) \neq 0 \text{ when } 0 < \|x\| \leqslant \delta \text{ and } \lambda \in C.$$

Now choose $\lambda_0 > 0$ such that $\{\lambda \in \mathbb{R}^k | \, |\lambda| \leqslant \lambda_0\} \subseteq V$, let $S^{k-1} = \{\lambda \in \mathbb{R}^k | \, |\lambda| = \lambda_0\}$ and let $\delta_0 = \delta(S^{k-1})$. Then the A-properness of f implies that we can choose $n_1 \in N$ such that $Q_n f(\lambda, x) \neq 0$, when $x \in X_n$ with $n \geqslant n_1$, $|\lambda| = \lambda_0$, $\|x\| = \delta_0$.

Hence, for $n \geqslant n_1$, if we let $d_n = \dim(X_n) = \dim(Y_n)$, the transformation

$$(\lambda, x) \to Q_n f(\lambda, x)/\|Q_n f(\lambda, x)\| \quad \text{for} \quad |\lambda| = \lambda_0, \quad \|x\| = \delta_0,$$

yields a mapping h_n of $S^{k-1} \times S^{d_n-1} \to S^{d_n-1}$.

The adjoint of h_n is a map $\overline{h}_n: S^{k-1} \to \{S^{d_n-1}, S^{d_n-1}\}$, where $\{S^{d_n-1}, S^{d_n-1}\}$ denotes the space of maps from S^{d_n-1} to S^{d_n-1}. For $d_n \geqslant 2k$, the $(k-1)$st homotopy group of the space $\{S^{d_n-1}, S^{d_n-1}\}$ is isomorphic to the $(k-1)$st stable homotopy group of the sphere π^s_{k-1}, and the class γ_n, of h_n in π^s_{k-1} is well-defined.

THEOREM 1. *Under the above assumptions* (A 1) *and* (A 2), *global bifurcation of the solutions of equation* (1·1) *from the trivial solutions occurs at* $(0, 0)$ *provided that* γ_n *is nontrivial for infinitely many* n.

When $k = 1$, the class $\gamma_n \in \pi^s_0 \approx \mathbb{Z}$ and γ_n is the difference of the Brouwer degrees of $Q_n f(\epsilon\lambda_0, \cdot): B_n \to Y_n$, where $B_n = \{x \in X_n | \, \|x\| \leqslant \delta_0\}$ and $\epsilon = \pm 1$. Hence, in terms of the topological degree defined for A-proper mappings by Browder and Petryshyn (8), we have the following result.

COROLLARY 1. *Under the assumptions* (A 1) *and* (A 2) *and the further assumption that* $k = 1$, *global bifurcation of the solutions of equation* (1·1) *from the trivial solutions at* $(0, 0)$ *occurs provided that the Browder–Petryshyn degree of*

$$f(\epsilon\lambda_0, \cdot): \{x \in X | \, \|x\| \leqslant \delta_0\} \to Y$$

changes as ϵ *changes from* $+1$ *to* -1.

One way to check that the topological degree changes sign is to linearize the problem. Namely, suppose that the Fréchet derivative of $f(\lambda, x)$ with respect to x exists at $x = 0$, denote $(\partial f/\partial x)_{(\lambda, 0)}$ by $A(\lambda)$, and suppose the map $\lambda \to A(\lambda)$ of V into $\mathcal{L}(X, Y)$ is continuous and each $A(\lambda)$ is A-proper with respect to Γ. The assumption that $A(\lambda)$ is

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bijective for $\lambda \in V\backslash\{0\}$ will guarantee assumption (A 2) and, by the Inverse Function Theorem, in order to have any bifurcation one needs that $A(0)$ be singular. Moreover the topological degree of $f(\pm\epsilon, \cdot)$ agrees with the topological degree of $A(\pm\epsilon)$. By specifying the nature of $A(0)$ one can guarantee that the topological degree of $A(+\epsilon)$ and $A(-\epsilon)$ are different. In order to do so recall the following definitions. If $S \in \mathscr{L}(X, X)$ and $\lambda \in \mathbb{R}$ is an eigenvalue of S, then $\dim\left(\bigcup_{k=1}^{\infty} N((\lambda I - S)^k)\right)$ is called the algebraic multiplicity of λ. Also, if $C \subseteq X$ is bounded one defines the ball measure of noncompactness of C to be $\inf\{r > 0 | C$ can be covered by a finite numbers of balls, each of which has radius less than $r\} \equiv \chi(C)$. Finally, a continuous mapping $h: D \subseteq X \to Y$ is called ball-condensing provided that $\gamma(h(C)) < \gamma(C)$ whenever $C \subseteq D$ is bounded with $\bar{C}$ not compact.

Now if $T \in \mathscr{L}(X, X)$ is ball-condensing, $I - T$ is A-proper with respect to

$$\Gamma = \langle\{X_n\}, \{P_n\}, \{X_n\}, \{P_n\}\rangle,$$

provided that $\|P_n\| \leqslant 1$ for each n, and moreover the A-proper degree of $I - T$ at 0 is a single integer, $(-1)^\nu$, where ν is the sum of the algebraic multiplicities of the eigenvalues of T which are greater than one ((17), (22)). Hence, for the situation that

$$A(\lambda) = I - (\lambda + \lambda_0) T,$$

where $X = Y$, Γ is as above, and $(\lambda + \lambda_0) T$ is ball condensing when $\lambda \in V$, one gets global bifurcation provided that the algebraic multiplicity of $1/\lambda_0$ as an eigenvalue of T is odd. It is worth noting that the requirement that $A(\lambda) = I - (\lambda + \lambda_0) T$ with $(\lambda + \lambda_0) T$ ball condensing is a purely local statement about f at $x = 0$ and does not require $f(\lambda, \cdot)$ to be a condensing perturbation of the identity on all of X. In (21) global bifurcation results for certain A-proper mappings depending on a scalar parameter are obtained which, by the above discussion, are contained in our present results. We note that (21) contains an interesting example of a class of A-proper mappings which occur naturally when considering certain elliptic boundary value problems on *unbounded* domains.

Another method for calculating the topological degree is as follows. Suppose $A(\lambda) = A_0 + (\lambda + \lambda_0) A_1$, where A_0 is a bijection of X onto Y and A_1 is compact, A_0 and A_1 both being in $\mathscr{L}(X, Y)$. Choose an orientation for the projection scheme Γ so that $Q_n A_0: X_n \to Y_n$ is orientation preserving, for n sufficiently large. Then it follows from (8) that the A-proper degree of $A(\lambda)$ with respect to 0 is the Leray–Schauder degree of $I - (\lambda + \lambda_0) A_1 \circ A_0^{-1}$, so that the degree of $A(\lambda)$ changes as λ passes through 0 provided that the algebraic multiplicity of $-1/\lambda_0$ as an eigenvalue of $A_1 \circ A_0^{-1}$ is odd. Hence we get global bifurcation in this situation. In (18) it was shown that this algebraic condition implies local bifurcation.

In our discussion of the linearization of the vector parameter problem we will prove a result which, as a special case, gives other conditions under which the topological degree changes as λ passes through 0.

(A 3) The mapping $f: \bar{\mathcal{O}} \to Y$ has a Fréchet derivative with respect to x, at $x = 0$, for each $\lambda \in V$, which we denote by $A(\lambda)$. We suppose $A(\lambda)$ is continuous in λ, A-proper

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with respect to Γ and $A(\lambda)$ has the following form: there exists $A_i \in \mathscr{L}(X, Y), i = 0, \ldots, k$ such that

$$A(\lambda) = A_0 + \sum_{i=1}^{k} \lambda_i A_i + R(\lambda),$$

where $\|R(\lambda)\| = 0(|\lambda|)$ as $|\lambda| \to 0$.

(A 4) Whenever $x_0 \in X$ with $A_0(x_0) = 0$ and $x_0 \neq 0$, then

$$\sum_{i=1}^{k} \lambda_i A_i(x_0) \notin \text{Range}\,(A_0),$$

when $\lambda = (\lambda_1, \ldots, \lambda_k) \neq 0$.

(A 5) The mapping A_0 is Fredholm of index 0, so that

$$X = N(A_0) \oplus \tilde{X},\ Y = \tilde{Y} \oplus R(A_0),$$

with $R(A_0)$ closed and

$$\dim\,(N(A_0)) = \dim\,(\tilde{Y}) < \infty;$$

and we suppose that $N(A_0) \subseteq X_n$ when n is sufficiently large and $\tilde{Y}$ can be chosen with $\tilde{Y} \subseteq Y_n$ for n sufficiently large.

A few words about these assumptions are in order. Assumption (A 3) means simply that $A(\lambda)$ has a first order Taylor expansion in λ, and even in the scalar case some such assumption is needed if one wishes to proceed by linearization. The second assumption, (A 4), is a transversality assumption, which, as was shown in (10), is equivalent to the following growth condition of (6).

(A 4′) There exists an $\epsilon > 0$ such that

$$\|A(\lambda)(x)\| \geqslant \epsilon\,|\lambda|\,\|x\|,$$

when λ is sufficiently small and $x \in X$.

(For a variation on (A 4′), see (2).)

Even when one does local bifurcation, some such transversality assumption is necessary. In fact, the transversality assumption in (7) for $k = 2$ is equivalent to (A 4). As we shall see in Section 3, and was shown in the finite dimensional case in (10), (A 4) is particularly suitable for clarifying the role of this transversality condition in obtaining a non-trivial invariation in $\pi_{k-1}(GL(\infty))$. In (A 5), the requirement that A_0 be Fredholm of index 0 seems to be essential for our analysis. However, the compatibility assumption on Γ is of a technical nature, and one should be able to weaken it. However, we impose it since for any of the general classes of A-proper mappings of which we are aware, when a mapping T is A-proper with respect to

$$\Gamma = \langle \{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\} \rangle$$

and W and V are finite-dimensional subspaces of X and Y, respectively, T is also A-proper with respect to a scheme $\Gamma' = \langle \{X_n'\}, \{P_n'\}, \{X_n'\}, \{Q_n'\} \rangle$, where $W \subseteq X_n'$ and $V \subseteq Y_n'$ for each n. Of course, when one is interested in using the A-properness to *construct* approximation solutions via the Galerkin equations the actual approximation scheme one uses will play a crucial role. In view of the topological tools we will use to prove our results, we do not feel our results contribute to the actual construction of solutions.

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THEOREM 2. *Suppose* $f: \bar{\mathcal{O}} \subseteq \mathbb{R}^k \times X \to Y$ *satisfies assumptions* (*A* 1), (*A* 3), (*A* 4) *and* (*A* 5). *Suppose that* $k \equiv 1, 2, 4$ *or* 8 (mod 8) *and define* C_k *by*

$$C_k = \begin{cases} k & \text{if} \quad k = 1, 2, 4 \text{ or } 8 \\ 16C_{k-8} & \text{if} \quad k > 8. \end{cases}$$

Let $m = \dim N(A_0)$. *Then* m *is an integral multiple of* C_k, *and if it is an odd integral multiple of* C_k *then there is global bifurcation at* (0, 0) *of the solutions of equation* (1·1) *from the trivial solutions.*

Global bifurcation results, when $k = 2$, have been obtained in recent years. In (5), letting $X = \mathbb{R}^n$, a global bifurcation result is obtained where Hopf bifurcation is considered for periodic solutions of autonomous differential equations. In (16) Hopf bifurcation is considered for autonomous functional differential equations, where $X = C([0, 1], Z)$, Z being a Banach space, and certain trajectory operators are shown to be A-proper with respect to an appropriate projection scheme. In both of these papers a condition on the parity of certain eigenvalue crossings plays the role of our condition (A 3) and (A 4). In (12) the case of a single complex parameter is considered, and global bifurcation is proven when the complex dimension of $N(A_0)$ is odd.

For general k, a global bifurcation result was proven for differentiable compact perturbations of the identity in (1), where the bifurcation condition was given in terms of the linearization $A(\lambda)$ generating a nontrivial element of $\pi_{k-1} GL_C(X)$, where $GL_C(X)$ denotes the nonsingular linear maps which are of the form $I - C$, C compact. Finally, in (3) a global bifurcation result for general nonlinear condensing perturbations of the identity is proven, which when things are suitably differentiable, yields a sufficient condition analogous to (A 3) and (A 4).

Some of the global bifurcation results for vector parameter problems have already found interesting applications (cf. (5), (4), (13) and (16)). The global bifurcation results proven here furnish a general setting in which to consider a wide variety of problems in differential equations.

2. We devote this section to proving Theorem 1. In the previous section we have already defined X, Y, Γ, $\mathcal{O}$, V, λ_0 and δ_0.

Letting $C = \{(\lambda, x) \in \mathcal{O} \mid |\lambda| = \lambda_0, \|x\| \leqslant \delta_0\}$ and $S^{k-1} = \{(\lambda, 0) \in \mathcal{O} \mid |\lambda| = \lambda_0\}$, we have $f(\lambda, x) \neq 0$ when $(\lambda, x) \in C \setminus S^{k-1}$. Define D_0 to be $\{(\lambda, 0) \mid |\lambda| < \lambda_0\}$ and

$$D_1 = \{(\lambda, 0) \mid |\lambda| > \lambda_0, \lambda \in V\},$$

and for $n \in N$, let $C_n = (\mathbb{R}^k \times X_n) \cap C$.

Let $(\mathbb{R}^k \times X)^+$ denote the space $\{\mathbb{R}^k \times X\} \cup \{\infty\}$, where complements of bounded subsets of $\mathbb{R}^k \times X$ form a neighbourhood basis of ∞, and $\mathbb{R}^k \times X$ has the given topology. For the proof of Theorem 1, all sets are considered subsets of $(\mathbb{R}^k \times X)^+$. In particular the closure of any unbounded set contains ∞.

First we prove a result on general topology. In (1) this was proved using the continuity property of Cech cohomology. Here we give a direct proof. We say two sets A, B of a topological space are separated by W if W is open and closed, $A \supset W$ and $B \cap W = \varnothing$.

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PROPOSITION 2.1. *Suppose $\{F_n\}$ is a family of closed subsets of a normal topological space, with $T = \cap F_n$. Assume for each neighbourhood U of T, there exists n such that $F_n \subset U$. Suppose A, B are closed subsets of T that cannot be separated in any F_n. Then A and B cannot be separated in T.*

Proof. Assume the contrary. Let W be open and closed in T with $A \subset W, B \cap W = \phi$. By normality, there exists an open V on the ambient space with $V \cap T = \bar{V} \cap T = W$ Let U be the union of the interiors of V and its complements; U is a neighbourhood of T. Thus there exists $F_n \subset U$ and $V \cap U$ is open and closed in F_n and separates A and B in F_n. This is the contradiction that proves the proposition.

Proof of Theorem 1. Recall the definition of Ω and Ω_0 in Definition 1·1. Let Ω_1 be the component of $(0, 0)$ in $\bar{\Omega} \cup \{(0, 0)\}$.

Now the A-properness of f implies that when f is restricted to closed bounded sets it is, in fact, proper (18). Consequently, $S \equiv \bar{\Omega} \cup \{(0, 0), \infty\}$, as a subset of $(\mathbb{R}^k \times X)^+$, is compact.

Now we claim that $\Omega_1 \cap K \neq \varnothing$, where $K = (\bar{\Omega} \cap \partial\mathcal{O}) \cup (\bar{\Omega} \cap \partial V) \cup \{\infty\}$, and to verify this claim we assume the contrary. Since S is compact and Hausdorff there exists a subset W of S which is both open and closed, contains Ω_1, and is disjoint from K ((14), theorem 47.II.2): i.e. Ω_1 and K are separated by W. But then it is clear that $W \cup D_0$ is both open and closed in

$$T = \{\{f^{-1}(0)\} \cup \{(\mathbb{R}^k \times X)^+ \backslash \mathcal{O}\}\} \backslash C,$$

$W \cup D_0$ contains D_0 and is disjoint from D_1. We now derive a contradiction to the existence of W.

We may, be relabelling if necessary, assume $\gamma_n \neq 0$ for all $n \in N$. For $n \in N$, let $\mathcal{O}_n = \mathcal{O} \cap (\mathbb{R}^k \times X_n)$ and let $f_n = Q_n f | \mathcal{O}_n$. Choose $\beta > 0$ such that if we let $J = \{t(\lambda, 0) + (1-t)(\lambda_0, x) \mid 0 \leqslant t \leqslant 1, \|x\| = \delta_0, |\lambda| = \lambda_0 \pm \beta, (\lambda, x) \in \mathcal{O}_n\}$ then $f(\lambda, x) \neq 0$ when $(\lambda, x) \in J \backslash (V \times \{0\})$.

Now define $\hat{f}_n : \mathcal{O}_n \to Y_n$:

$$\hat{f}_n = \begin{cases} f_n \text{ outside of } J_n \\ 0 \text{ on } \mathcal{O}_n \cap (V \times \{0\}), \\ \text{linear in } x \text{ inside } J_n. \end{cases}$$

We note that $\hat{f}_n = f_n$ on C_n:

We define

$$T_n = \{\{(\hat{f}_n)^{-1}(0)\} \cup \{\mathbb{R}^k \times X_n)^+ \backslash \mathcal{O}_n\}\} \backslash \{C_n\}.$$

It is proved in § 3 of (1) that D_0, D_1 cannot be separated in T_n, since $\gamma_n \neq 0$.

Let

$$F_n = T_n \cup T.$$

We claim

(i) For any neighbourhood $\mathcal{U}$ of T in $(\mathbb{R}^k \times X)^+ \backslash C$, there is an $n \in N$ such that $F_n \subseteq \mathcal{U}$,

(ii) Each F_n is closed and $\cap F_n = T$.

Once we have verified (i) and (ii) we see with $D_0 = A$, $D_1 = B$, that the conditions of Proposition 2·1 are satisfied, so D_0 and D_1 cannot be separated in T. This contradicts the existence of W.

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We will verify (i) by supposing the contrary. Then there exists a neighbourhood $\mathscr{U}$ of T in $(\mathbb{R}^k \times X)^+ \backslash C$, and a sequence $\{(\lambda_n, x_n)\}$, such that $(\lambda_n, x_n) \in T_n \backslash \mathscr{U}$, for each $n \in \mathbb{N}$. It follows that $\hat{f}_n(\lambda_n, x_n) = 0$, for each n, and moreover, $\{(\lambda_n, x_n)\}$ is bounded since $\infty \in \mathscr{U}$, and $\mathscr{U}$ is open.

Case 1. Suppose (λ_n, x_n) lies outside of J for infinitely many n (by relabelling we may suppose for all n). Then $f_n(\lambda_n, x_n) = 0$, for each n, and the A-properness implies that a subsequence of $\{(\lambda_n, x_n)\}$ converges to (λ_*, x_*) with $f(\lambda_*, x_*) = 0$. But

$$(\lambda_*, x_*) \notin S^{k-1},$$

since none of the $\{(\lambda_n, x_n)\}$ lies inside of J. Thus $(\lambda_*, x_*) \in T \subseteq \mathscr{U}$, which contradicts the choice of $(\lambda_n, x_n) \notin \mathscr{U}$, for each n.

Case 2. Suppose Case 1 does not occur. In this situation, (λ_n, x_n) lies on or inside of J_n, when n is sufficiently large. Now $x_n \neq 0$ since $(\lambda_n, x_n) \notin \mathscr{U} \supset T$, so by the linearity of $\hat{f}_n$, inside of J_n, we may choose $\beta_n \geqslant 0$ such that $(\lambda_n, \beta_n x_n) \in J_n$ and $f_n(\lambda_n, \beta_n x_n) = 0$. Finally, we invoke the A-properness of f to extract a subsequence of $\{(\lambda_n, \beta_n x_n)\}$ which converges to (λ_*, x_*) and $(\lambda_*, x_*) \in J, f(\lambda_*, x_*) = 0$. But

$$f^{-1}(0) \cap J = \{(\lambda, x) \mid |\lambda| = \lambda_0 \pm \beta, x = 0\},$$

so that $(\lambda_*, x_*) \in T$. This again gives a contradiction, and the proof of (i) is complete.

Now (ii) follows immediately from the continuity of f and f_n, and from the regularity of our spaces in conjunction with (i).

To conclude our proof we need the following connectedness result, which may be found in Kuratowski (14); if A is a compact connected metric space, with $p, q \in A$, then there is a connected component, B, of $A \backslash \{p, q\}$, whose closure in A contains both p and q. We apply this result with A being the quotient space obtained by collapsing K to a point in Ω_1, letting p be the collapsed point, and letting $q = (0, 0)$. Then, letting Ω_0 be the preimage, under the canonical map, of the connected component B, it is clear that Ω_0 is contained in Ω, $(0, 0) \in \overline{\Omega}_0$, and $\overline{\Omega}_0 \cap K \neq \varnothing$. This completes the proof.

3. We devote this section to the proof of Theorem 2. As we already noted, assumption (A 4) is equivalent to assumption (A 4′), so that $A_\lambda \equiv (\partial f(\lambda, x)/\partial x)_{x=0}$ is a bijection when λ is sufficiently small. Hence, the Inverse Function Theorem yields assumption (A 2). So it remains to verify that γ_n is nontrivial for infinitely many n. Choose λ_0 and c positive real numbers, and N_1 a positive integar, such that

$$\|Q_n A(\lambda)(x)\| \geqslant c\|x\|,$$

if $x \in X_n$, $n \geqslant N_1$ and $\lambda \in \mathbb{R}^k$ with $|\lambda| = \lambda_0$. By the Uniform Boundedness Principle, $\{Q_n\}$ is bounded, so choose M with $\|Q_n\| \leqslant M$, for each $n \in N$. By the continuous Fréchet differentiability of $f(\lambda, x)$ with respect to x, we may choose β_0 such that

$$\|f(\lambda, x) - A(\lambda)(x)\| \leqslant \frac{c}{2M} \quad \text{if} \quad |\lambda| = \lambda_0, \|x\| = \beta_0,$$

from which it follows that

$$tQ_n f(\lambda, x) + (1-t)Q_n A(\lambda)(x) \neq 0,$$

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when $0 \leqslant t \leqslant 1$, $\|x\| = \beta_0$, $|\lambda| = \lambda_0$, $x \in X_n$ and $n \geqslant N_1$. Thus the elements of π^s_{k-1} determined by

$$\lambda \to Q_n f(\lambda, x),\ |\lambda| = \lambda_0,\ \|x\| = \beta_0,\ x \in X_n,\ n \geqslant N_1,$$

and

$$\lambda \to Q_n A(\lambda)(x),\ |\lambda| = \lambda_0,\ \|x\| = \beta_0,\ x \in X_n,\ n \geqslant N_1$$

are the same. Now if we choose a linear bijection T_n of Y_n onto X_n, then the non-triviality of the element in π^s_{k-1} is unchanged if we compose with T_n. We will show that T_n can be chosen so that $\lambda \to T_n \circ Q_n \circ A(\lambda) \in L(X_n, X_n)$ is homotopic through non-singular mapping to a mapping of the form

$$\lambda \to \begin{bmatrix} I & 0 \\ 0 & \sum_{i=1}^{k} \lambda_i S_i \end{bmatrix},$$

where $X_n = M_n \oplus N(A_0)$, and $\sum_{i=1}^k \lambda_i S_i$ is nonsingular for each $(\lambda_1, \ldots, \lambda_k) \neq 0$. It was proven in (6) that the conditions we have imposed on $\dim N(A_0)$ and k guarantee that the above element is non-trivial in $GL(\infty)$, and that its image in π^s_{k-1} under the Whitehead J-homomorphism is also nontrivial. Thus γ_n is non-trivial for $n \geqslant N_1$.

The proof that the above deformation may be carried out will be in several steps.

LEMMA 3.1. *Let X and Y be Banach spaces, with $\Gamma = \langle \{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\} \rangle$ a complete scheme for mappings from X to Y. Let $A_0 \in \mathscr{L}(X, Y)$ be A-proper with respect to Γ and Fredholm of index 0. Suppose, moreover, that $N(A_0) \subseteq X_n$, for each n, and that there exists a subspace $\tilde{Y}$ of Y with $\tilde{Y} \oplus R(A_0) = Y$ and $\tilde{Y} \subseteq Y_n$, for each n. Let $\tilde{X}$ be any closed subspace of X with $\tilde{X} \oplus N(A_0) = X$. Then there exists a positive integer N such that:*

(i) $N(Q_n A_0|_{X_n}) = N(A_0)$ *when* $n \geqslant N$,

(ii) $Q_n A_0(X_n) \oplus \tilde{Y} = Y_n$ *when* $n \geqslant N$,

and

(iii) *there is a $\beta > 0$ such that*

$$\|Q_n A_0(x)\| \geqslant \beta \|x\| \quad \text{when} \quad x \in X_n \cap \tilde{X} \quad \text{and} \quad n \geqslant N.$$

Proof. (i) Suppose such an N does not exist. Then, by taking subsequences if necessary, we may choose $\{z_n\}$, such that $Q_n A_0(z_n) = 0$, $\|z_n\| = 1$ and $z_n \in \tilde{X} \cap X_n$ for each n. By the A-properness of A_0, we may, again by choosing subsequences if neccessary, assume $\{z_n\}$ converges to z_0 with $A_0(z_0) = 0$. But $\tilde{X}$ is closed, so $z_0 \in N(A_0) \cap \tilde{X}$ and $\|z_0\| = 1$, which is a contradiction. This proves (i).

(ii) We first prove that the sum is direct for sufficiently large n, by assuming the contrary and obtaining a contradiction. Thus, again by taking subsequences if necessary, we have sequences $\{y_n\}$ and $\{z_n\}$, with $0 = y_n + z_n$, $y_n \in \tilde{Y}$ and $y_n \in Q_n A_0(X_n)$, $z_n \neq 0$, for each n. Then

$$y_n + Q_n A_0(x_n) = 0 \quad \text{where} \quad x_n \in X_n \cap \tilde{X} \quad \text{and} \quad \|x_n\| = 1.$$

Consequently, $\{y_n\}$ is a bounded sequence in $\tilde{Y}$, and $\tilde{Y}$ is finite dimensional, so that, by taking subsequences if necessary, we may assume $\{y_n\} \to y_0 \in \tilde{Y}$. The A-properness

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of A_0 means that, again taking subsequences if necessary, we may assume $\{x_n\} \to x_0 \in \tilde{X}$, and $y_0 + A(x_0) = 0$. But $\tilde{Y} \cap R(A_0) = \{0\}$, so that $y_0 = A(x_0) = 0$. Since $\|x_0\| = 1$ and $x_n \in \tilde{X}$, with $N(A_0) \cap \tilde{X} = \{0\}$, we obtain a contradiction. Thus $Q_n A_0(X_n) \cap \tilde{Y} = \{0\}$, for sufficiently large n.

Now, by part (i), $N(Q_n A_0|_{X_n}) = N(A_0)$, when $n \geqslant N$, so that the codimension of $Q_n A_0(X_n)$ in Y_n is $\dim N(A_0)$, when $n \geqslant N$. But $\dim N(A_0) = \dim(\tilde{Y})$, so that

$$Y_n = Q_n A_0(X_n) \oplus \tilde{Y},$$

when n is sufficiently large.

(iii) Suppose (iii) does not hold. Then, as above, the A-properness of A_0 yields a contradiction.

PROPOSITION 3.1. *Suppose hypotheses* (A 3), (A 4) *and* (A 5) *hold for* $A(\lambda)$, *and letting* $A(0) = A_0$, *let* N *be as prescribed in the previous Lemma. Let* W_n *be the projection of* Y_n *on to* $Q_n A_0(X_n)$, *along the space* $\tilde{Y}$, *when* $n \geqslant N$. *Let* $L: \tilde{Y} \to N(A_0)$ *be any linear bijection, and define* $B_n \in \mathscr{L}(Y_n, X_n)$ *by*

$$B_n = L \circ (I - W_n) + (Q_n A(0)|_{X_n \cap \tilde{X}})^{-1} \circ W_n.$$

Then

$$B_n \circ Q_n A(\lambda)|_{X_n} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \lambda \begin{bmatrix} C^n_{1,1} & C^n_{1,2} \\ C^n_{2,1} & C^n_{2,2} \end{bmatrix} + S^n(\lambda),$$

where the matrix representation is respect to the decomposition of X_n *as* $N(A_0) \oplus (\tilde{X} \cap X_n)$ *and the following growth conditions hold.*

(i) $\|S^n(\lambda)\| = 0(|\lambda|)$ *as* $|\lambda| \to 0$,

(ii) *there is an* $\alpha > 0$ *such that*

$$\|\lambda C^n_{1,1}(x)\| \geqslant \alpha \|x\| \quad \text{when} \quad x \in N(A_0), \quad |\lambda| = 1$$

and n *is sufficiently large.*

Proof of (i). By the definition of $A(\lambda)$, and of B_n, and conclusion (iii) of Lemma 3·1, it clearly suffices to show that $\{W_n\}$ is bounded. Indeed, if $\{W_n\}$ is not bounded, then by choosing a subsequence if necessary, we can choose $\{y_n\}$ with $y_n \in Y_n$, and $\|W_n(y_n)\| = 1$, while $\{y_n\} \to 0$. Now, for each n there exists a unique $x_n \in X_n \cap \tilde{X}$ such that

$$W_n(y_n) = Q_n A_0(x_n),$$

and by (iii) of Lemma 3·1, $\{x_n\}$ is bounded. But $y_n - W_n(y_n) \in \tilde{Y}$, for each n, and $\tilde{Y}$ is finite dimensional. so that since $\{y_n\}$ converges, we may, by taking subsequences if necessary, suppose $\{W_n(y_n)\} \to y_0$, with $\|y_0\| = 1$. Again the A-properness of A_0 implies, taking subsequences, as usual, if necessary, that $\{x_n\} \to x_0$ and $y_0 = A(x_0)$, with $y_0 \in \tilde{Y}$, since $\{y_n - W_n(y_n)\} \subseteq \tilde{Y}$ and $\{y_n\} \to 0$. Since $R(A_0) \cap \tilde{Y} = \{0\}$, we get a contradiction.

Proof of (ii). Suppose that an $\alpha > 0$ does not exist such that the stated inequality holds when n is sufficiently large. Then, again taking subsequences if necessary, we obtain $\{\lambda_n\}$ and $\{x_n\}$ such that $\|x_n\| = |\lambda_n| = 1$ and $x_n \in N(A_0)$, for each n, and

$$\|\lambda_n C^n_{1,1}(x_n)\| \leqslant 1/n.$$

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Now $\lambda C^n_{1,1}(x) = [L \circ (I - W_n) \circ (\lambda A^1)](x)$, when $x \in N(A_0)$, and where λA^1 denotes $\sum_{i=1}^k \lambda_i A_i$. Thus, since L is bijective,

$$\{(I - W_n)\lambda_n A^1(x_n)\} \to 0.$$

Now $W_n \lambda_n A^1(x_n) = Q_n A_0(z_n)$, where $z_n \in X_n \cap \tilde{X}$, and since $\{x_n\}$, $\{\lambda_n\}$ and $\{W_n\}$ are bounded, it follows from (iii) of Lemma 3·1, that $\{z_n\}$ is also bounded. Since $\{x_n\} \subseteq N(A_0)$, we may suppose that $\{x_n\}$ converges to $x_0 \in N(A_0)$, and also that $\{\lambda_n\}$ converges. Thus we are in the position that, since

$$\begin{aligned}\{\lambda_n Q_n A_0(z_n)\} &= \{W_n \lambda_n A^1(x_n)\} \\ &= \{(W_n - I)\lambda_n A^1(x_n) + \lambda_n A^1(x_n)\}, \{Q_n A_0(z_n)\}\end{aligned}$$

converges. Hence, the A-properness of A_0 implies, taking subsequences if necessary, that $\{z_n\}$ converges to $z_0 \in X$. Supposing $\{\lambda_n\} \to \lambda_0$, we then have $\lambda_0 A^1(x_0) = A_0(z_0)$, with $\lambda_0 \neq 0$, $x_0 \neq 0$, and $A_0(x_0) = 0$. This directly contradicts (A 4).

Remark 3·1. Suppose W is any linear space and $W = W_1 \oplus W_2$. Let $S: W \to W$ be linear and let its matrix with respect to the above decomposition be

$$\begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix}.$$

Suppose $S_{2,2}: W_2 \to W_2$ is bijective. Then if $w = w_1 + w_2 \in W$, $S(w) = 0$ if and only if

$$S_{1,1}(w_1) - [S_{1,2} \circ (S_{2,2})^{-1} \circ S_{2,1}](w_1) \neq 0.$$

This simple observation is needed in the final proposition.

PROPOSITION 3·2. *Let $C^n_{i,j}$ and $S^n(\lambda)$ be as defined in the previous Proposition. Then there exists a positive integar N_1 and $\lambda_1 > 0$ such that*

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \lambda \begin{bmatrix} C^n_{1,1} & tC^n_{1,2} \\ tC^n_{2,1} & tC^n_{2,2} \end{bmatrix} + tS^n(\lambda)$$

is non-singular when $0 \leqslant t \leqslant 1$, $0 < |\lambda| < \lambda_1$, and $n \geqslant N_1$.

Proof. Letting

$$S^n(\lambda) = \begin{bmatrix} S^n_{1,1}(\lambda) & S^n_{1,2}(\lambda) \\ S^n_{2,1}(\lambda) & S^n_{2,2}(\lambda) \end{bmatrix}$$

it is clear that $\|S^n_{i,j}(\lambda)\| = o(|\lambda|)$, as $|\lambda| \to 0$ where $1 \leqslant i, j \leqslant 2$. Now we can choose $\lambda_1 > 0$ such that

$$I + \lambda t C^n_{2,2} + t S^n_{2,2}(\lambda): (\tilde{X} \cap X_n) \to (\tilde{X} \cap X_n)$$

is bijective when $0 \leqslant |\lambda| \leqslant \lambda_1$ $0 \leqslant t \leqslant 1$.

If the conclusion of the Proposition does not hold we may by choosing a subsequence if necessary, and by using Remark 3·1, choose $\{\lambda_n\} \subseteq \mathbb{R}^k \backslash \{0\}$, $\{t_n\} \subseteq [0, 1]$ and $\{x_n\} \subseteq N(A_0)$ with $\{\lambda_n\} \to 0$, and such that for each n, $\|x_n\| = 1$ and

$$\begin{aligned}[\lambda_n C^n_{1,1} + {}_{1n}S^n_{1,1}(\lambda_n)](x_n) = (t\lambda_n C^n_{1,2} + t_n S^n_{1,2}(\lambda_n))(I + t_n \lambda_n C^n_{2,2} + t_n S^n_{2,2}(\lambda_n))^{-1} \\ \times (t_n \lambda_n C^n_{2,1}(\lambda_n) + t_n S^n_{2,1}(\lambda_n))(x_n).\end{aligned}$$

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But if we use the growth conditions ensured by Proposition 3·1, and divide by $|\lambda_n|$ for each n, we clearly get a contradiction.

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