Eight formulae for the area of triangle *OIH*

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1. *Introduction*

Which is your favourite formula in triangle geometry? Mine is definitely the formula for the area of triangle *OIH*. It is well known that the perpendicular bisectors to the sides of any triangle are concurrent at a point O (centre of the circumcircle), that the angle bisectors to the vertex angles are concurrent at a point I (centre of the incircle), and that the altitudes are concurrent at a point H. If the triangle is not isosceles, then these three points are all different and uniquely determine a new triangle *OIH* (see Figure 1), whose area can be expressed in terms of the sides a, b, c of the original triangle. I derived such a formula 20 years ago, and later found out that it had been studied a century earlier.

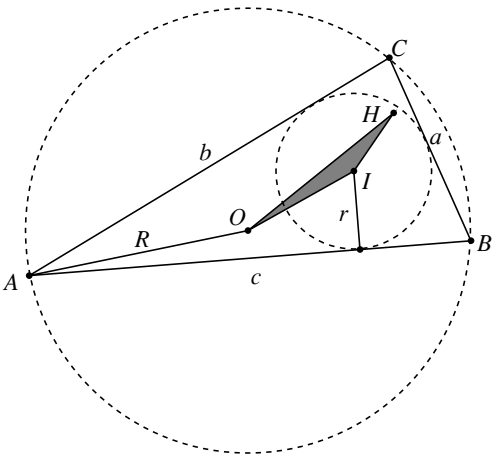


FIGURE 1: The triangle *OIH*

In 1890, R. Sondat posed a problem in [1] that was about deriving three different formulae for the area of triangle *OIH*. Solutions were given the next year by E. Lemoine. Sondat's second formula is my favourite triangle formula. It states that the area S of triangle *OIH* is given by

$$S = \frac{|a - b|\,|b - c|\,|c - a|}{8r} \tag{1}$$

where r is the inradius of the triangle, which, as is well known, can be expressed in terms of the triangle sides. The elegance and simplicity of formula (1) is striking considering the complexity of the problem it addresses. Writing it explicitly in terms of the sides gives

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$$S = \frac{|a-b||b-c||c-a|}{4}\sqrt{\frac{a+b+c}{(-a+b+c)(a-b+c)(a+b-c)}}.$$

Sondat did not use absolute values, but instead considered the case $a > b > c$. His first formula is

$$S = 2R^2 \sin\frac{|A-B|}{2} \sin\frac{|B-C|}{2} \sin\frac{|C-A|}{2} \tag{2}$$

and the third is

$$16S^2 = -s^4 + 2\left(2R^2 + 10Rr - r^2\right)s^2 - r(4R+r)^3. \tag{3}$$

Here s, R and r denote the semiperimeter, the circumradius and the inradius respectively of the original triangle ABC, with vertex angles A, B and C.

Lemoine used a determinant to derive formula (1). Other proofs of it have been given in [2, pp. 114-115] and [3], employing complex numbers and trigonometry respectively. Sondat's first formula (2) can be derived from (1) using

$$a - b = 4R \sin\frac{A-B}{2} \sin\frac{C}{2},$$

(which is derived from the extended law of sines) together with its cyclic versions, and the fact that

$$\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = \frac{\Delta}{4Rs},$$

where Δ is the area of the original triangle ABC. Sondat's third formula (3) is a direct consequence of the identity

$$(a-b)^2(b-c)^2(c-a)^2 = 4r^2\left(-s^4 + 2\left(2R^2 + 10Rr - r^2\right)s^2 - r(4R+r)^3\right),$$

which is proved using $a + b + c = 2s$, $ab + bc + ca = s^2 + r^2 + 4Rr$ and $abc = 4Rrs$, and performing some tedious algebraic manipulations best suited for a rainy day (preferably with a computer algebra system). All of this has already been published in mathematical magazines (though these formulae are rarely found in geometry books), and for those to whom these facts are unknown, filling in the blanks are rewarding exercises.

Before we derive some new formulae for the area of triangle OIH, let us make a few comments on the formulae we have reviewed. Obviously the area of triangle OIH is zero if, and only if, I lies on the line OH. This is the famous *Euler line*, so from (1) we get that the incentre of a triangle lies on the Euler line if, and only if, the triangle is isosceles (including equilateral triangles). In all other triangles it is impossible for the incentre to be on the Euler line. There are other well-known points on the Euler line, such as the centroid G and the centre N of the nine-point circle, see Figure 2.

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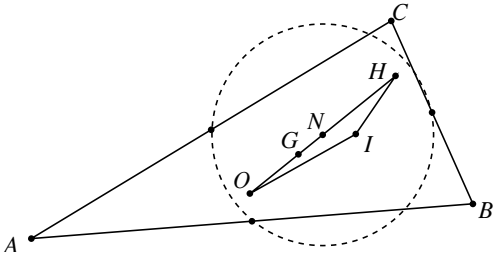


FIGURE 2: Four points on the Euler line

Since I is at a certain distance from the Euler line in any given triangle, the area of other triangles like HIG, OIG, OIN and GIN will be proportional to the distances HG, OG, ON and GN along the Euler line. It is well known that $HG = \frac{2}{3}OH$, $OG = \frac{1}{3}OH$, $ON = \frac{1}{2}OH$ and $GN = \frac{1}{6}OH$ [4, p. 27]. Thus we have, for example, that the area of triangles HIG and OIG are

$$S_{HIG} = \tfrac{2}{3}S = \frac{|a-b||b-c||c-a|}{12r}$$

and

$$S_{OIG} = \tfrac{1}{3}S = \frac{|a-b||b-c||c-a|}{24r}.$$

Sondat's third formula (3) is closely connected to the problem of finding a necessary and sufficient condition on three positive numbers s, R and r (often called the fundamental invariants) to determine a triangle with these as the semiperimeter, circumradius and inradius respectively. Using $S^2 \geqslant 0$ and solving for s^2, we get the *fundamental triangle inequality*

$$2R^2 + 10Rr - r^2 - 2(R-2r)d \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)d,$$

where $d = \sqrt{R^2 - 2Rr}$. This has also been called Blundon's inequality since it was studied by Blundon in [5]. Just over a year later, he proposed a problem in [6], which was to derive Sondat's second formula (1).

In the rest of this paper we shall derive five more formulae for the area of triangle OIH, in terms of other quantities, that we have not been able to find in the geometrical literature. We will use the same notations for points, lengths and areas in all sections.

2. *The area of OIH in terms of the exradii*

It is well known that every triangle has three excircles each of which is tangent to one side of the triangle and the extensions of the other two sides. Their radii, which are called the exradii, are given by [4, p. 18]

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$$r_a = \frac{\Delta}{s-a}, \qquad r_b = \frac{\Delta}{s-b}, \qquad r_c = \frac{\Delta}{s-c},$$

where Δ is the area of the original triangle. From these formulae, we get

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{a+b+c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

so the inradius can be expressed in terms of the exradii as

$$r = \frac{r_a r_b r_c}{r_a r_b + r_b r_c + r_c r_a}. \tag{4}$$

Hence with the help of Heron's formula,

$$r_a r_b r_c = \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \Delta s = \frac{\Delta^2}{r}.$$

Solving for Δ yields

$$\Delta = \frac{r_a r_b r_c}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}.$$

But we also get

$$\frac{1}{r_a} + \frac{1}{r_b} = \frac{c}{\Delta},$$

whence

$$c = \frac{r_a + r_b}{r_a r_b}\Delta = \frac{r_a + r_b}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}\, r_c.$$

Together with its cyclic versions, we have

$$|a-b| = \frac{|r_a - r_b|}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}\, r_c,$$

$$|b-c| = \frac{|r_b - r_c|}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}\, r_a,$$

$$|c-a| = \frac{|r_c - r_a|}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}\, r_b.$$

Inserting these three expressions and (4) into Sondat's second formula (1), we get the following formula

$$S = \frac{|r_a - r_b|\,|r_b - r_c|\,|r_c - r_a|}{8\sqrt{r_a r_b + r_b r_c + r_c r_a}},$$

where the area of triangle OIH is given in terms of the three exradii.

3. *The area of OIH in terms of the tangent lengths*

Let the distances from the vertices A, B, C of a triangle to the points where the incircle is tangent to the sides a, b, c be x, y, z

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respectively (the tangent lengths, see Figure 3). As we shall see next, it is possible to express the area of triangle OIH in terms of the tangent lengths. It is well known and easy to prove that $x = s - a$, $y = s - b$ and $z = s - c$ [4, p. 16]. From these formulae we get directly

$$|a - b| = |x - y|, \qquad |b - c| = |y - z|, \qquad |c - a| = |z - x|.$$

Now we need to express the inradius in terms of x, y, z.

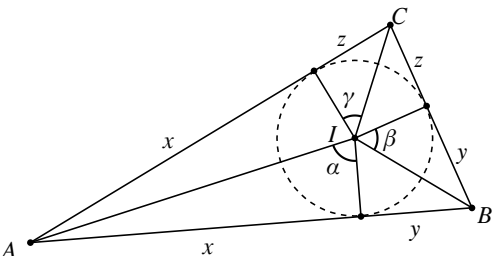


FIGURE 3: The tangent lengths

Let α be the angle between IA and the inradius perpendicular to side AB, and similar for β and γ (see Figure 3). We get that $\alpha + \beta = \pi - \gamma$, and taking the tangent of both sides gives

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = -\tan\gamma.$$

Thus

$$\frac{\frac{x}{r} + \frac{y}{r}}{1 - \frac{xy}{r^2}} = -\frac{z}{r}$$

and solving for r, we get

$$r = \sqrt{\frac{xyz}{x + y + z}}.$$

(Note that using $\Delta = rs$, this is quickly turned into a derivation of Heron's formula.) Sondat's second formula (1) now yields

$$S = \frac{|x - y||y - z||z - x|}{8}\sqrt{\frac{x + y + z}{xyz}}$$

so we have expressed the area of triangle OIH in terms of the tangent lengths.

4. *The area of OIH in terms of the altitudes*

If h_a, h_b, h_c are the altitudes to sides a, b, c respectively in a triangle, then $a = \frac{2\Delta}{h_a}$ and cyclic versions, so we get directly

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$$|a - b| = 2\Delta\left|\frac{1}{h_a} - \frac{1}{h_b}\right| = \frac{2\Delta}{h_a h_b}|h_a - h_b|.$$

Thus

$$|a - b||b - c||c - a| = \frac{8\Delta^3}{(h_a h_b h_c)^2}|h_a - h_b||h_b - h_c||h_c - h_a|. \quad (5)$$

We also have

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{2s}{2\Delta} = \frac{1}{r},$$

so the inradius is given by

$$r = \frac{h_a h_b h_c}{h_a h_b + h_b h_c + h_c h_a}. \quad (6)$$

Now we need the area Δ of the original triangle expressed in terms of the altitudes. According to Heron's formula,

$$\begin{aligned}\Delta^2 &= (a + b + c)(-a + b + c)(a - b + c)(a + b - c)\\ &= (2\Delta)^4\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)\end{aligned}$$

so

$$\Delta = \frac{1}{\sqrt{\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(-\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} - \frac{1}{h_b} + \frac{1}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_b} - \frac{1}{h_c}\right)}}. \quad (7)$$

A similar version of (7) was derived in [7].

Inserting (5) and (6) into Sondat's second formula (1), we get

$$S = \frac{|h_a - h_b||h_b - h_c||h_c - h_a|(h_a h_b + h_b h_c + h_c h_a)}{(h_a h_b h_c)^3}\Delta^3,$$

where Δ is given by (7), and we have a formula for the area of *OIH* in terms of the altitudes.

5. *The area of OIH in terms of three subtriangle inradii*

If we draw tangents to the incircle, parallel to the sides of a triangle, then we get three smaller triangles at the vertices that are similar to the original triangle (see Figure 4). Call the inradii of these triangles r_1, r_2, r_3. By similarity,

$$\frac{r_3}{r} = \frac{h_c - 2r}{h_c}$$

and we get

$$r_3 = r - \frac{2r^2}{h_c} = r - \frac{r^2 c}{\Delta}.$$

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Using this and its cyclic versions yields

$$r_1 + r_2 + r_3 = 3r - (a + b + c)\frac{r^2}{\Delta} = 3r - 2s \times \frac{r}{s} = r, \quad (8)$$

a formula whose derivation was given as a problem at [8].

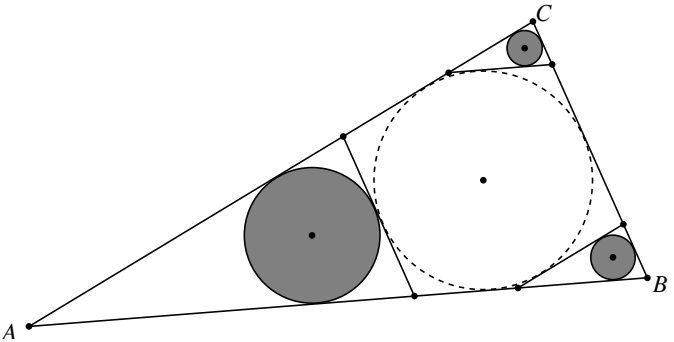


FIGURE 4: The three subtriangles and their incircles

We also have

$$c = \frac{\Delta}{r^2}(r - r_3) = \frac{\Delta}{r^2}(r_1 + r_2)$$

and the cyclic versions

$$b = \frac{\Delta}{r^2}(r_1 + r_3), \qquad a = \frac{\Delta}{r^2}(r_2 + r_3)$$

so

$$|a - b||b - c||c - a| = \left(\frac{\Delta}{r^2}\right)^3 |r_1 - r_2||r_2 - r_3||r_3 - r_1|. \quad (9)$$

Further we have

$$a + b + c = \frac{\Delta}{r^2} \times 2(r_1 + r_2 + r_3),$$

$$-a + b + c = \frac{\Delta}{r^2} \times 2r_1,$$

$$a - b + c = \frac{\Delta}{r^2} \times 2r_2,$$

$$a + b - c = \frac{\Delta}{r^2} \times 2r_3.$$

Using Heron's formula, the original triangle has an area satisfying

$$16\Delta^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$

$$= \frac{16\Delta^4}{r^8} r_1 r_2 r_3 (r_1 + r_2 + r_3),$$

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whence

$$\Delta^2 = \frac{r^8}{r_1 r_2 r_3 (r_1 + r_2 + r_3)} = \frac{(r_1 + r_2 + r_3)^7}{r_1 r_2 r_3}.$$

From this formula we get

$$\frac{\Delta^3}{r^7} = \Delta \times \frac{\Delta^2}{r^7} = \sqrt{\frac{(r_1 + r_2 + r_3)^7}{r_1 r_2 r_3}} \times \frac{1}{r_1 r_2 r_3} = \sqrt{\frac{(r_1 + r_2 + r_3)^7}{(r_1 r_2 r_3)^3}}.$$

Using the last expression and (9) in Sondat's second formula (1), we get that the area of triangle OIH is given by

$$S = \frac{|r_1 - r_2|\,|r_2 - r_3|\,|r_3 - r_1|}{8} \sqrt{\frac{(r_1 + r_2 + r_3)^7}{(r_1 r_2 r_3)^3}},$$

which is in terms of the three subtriangle inradii.

6. *The area of OIH in terms of three subtriangle areas*

Let T_1, T_2, T_3 be the areas of the three subtriangles whose inradii we used in the previous section. We shall express the area of triangle OIH in terms of their areas. By similarity, we have

$$\left(\frac{r_1}{r}\right)^2 = \frac{T_1}{\Delta}$$

so

$$r_1 = r\sqrt{\frac{T_1}{\Delta}} = k\sqrt{T_1} \tag{10}$$

where $k = \dfrac{r}{\sqrt{\Delta}}$. Using cyclic versions yields

$$r_1 + r_2 + r_3 = \frac{r}{\sqrt{\Delta}}\left(\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}\right)$$

and, since $r = r_1 + r_2 + r_3$ according to (8), we get

$$\sqrt{\Delta} = \sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}. \tag{11}$$

To derive this lovely formula was a problem in [9].

Even though we did not note it explicitly in the previous section, it is easy to see from the formulae we derived there together with (10) that the area of triangle OIH is given by

$$S = \frac{|r_1 - r_2|\,|r_2 - r_3|\,|r_3 - r_1|}{8 r_1 r_2 r_3}\Delta$$

$$= \frac{k^3\left|\sqrt{T_1} - \sqrt{T_2}\right|\left|\sqrt{T_2} - \sqrt{T_3}\right|\left|\sqrt{T_3} - \sqrt{T_1}\right|}{8k^3\sqrt{T_1 T_2 T_3}}\Delta.$$

Inserting the expression for the area of the original triangle from (11), we get

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$$S = \frac{\left|\sqrt{T_1} - \sqrt{T_2}\right|\left|\sqrt{T_2} - \sqrt{T_3}\right|\left|\sqrt{T_3} - \sqrt{T_1}\right|\left(\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}\right)^2}{8\sqrt{T_1T_2T_3}},$$

which is the formula for the area of triangle *OIH* expressed in terms of the three subtriangle areas.

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