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NULL RECURRENCE AND TRANSIENCE OF RANDOM DIFFERENCE EQUATIONS IN THE CONTRACTIVE CASE

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Abstract

Given a sequence $(M_k, Q_k)_{k\geq 1}$ of independent and identically distributed random vectors with nonnegative components, we consider the recursive Markov chain $(X_n)_{n\geq 0}$, defined by the random difference equation $X_n = M_n X_{n-1} + Q_n$ for $n \geq 1$, where X_0 is independent of $(M_k, Q_k)_{k\geq 1}$. Criteria for the null recurrence/transience are provided in the situation where $(X_n)_{n\geq 0}$ is contractive in the sense that $M_1 \cdots M_n \to 0$ almost surely, yet occasional large values of the Q_n overcompensate the contractive behavior so that positive recurrence fails to hold. We also investigate the attractor set of $(X_n)_{n\geq 0}$ under the sole assumption that this chain is locally contractive and recurrent.

Keywords: Attractor set; null recurrence; perpetuity; random difference equation; transience

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1. Introduction

Let $(M_n, Q_n)_{n\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) $\mathbb{R}_+^2$-valued random vectors with common law μ and generic copy (M, Q), where $\mathbb{R}_+ := [0, \infty)$. Further, let X_0 be a nonnegative random variable which is independent of $(M_n, Q_n)_{n\geq 1}$. Then the sequence $(X_n)_{n\geq 0}$, recursively defined by the random difference equation (RDE)

$$X_n := M_n X_{n-1} + Q_n, \qquad n \geq 1, \tag{1.1}$$

forms a temporally homogeneous Markov chain with transition kernel P given by

$$Pf(x) = \int f(mx + q)\, \mathrm{d}\mu(m, q)$$

for bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$. The operator P is Feller because it maps bounded continuous f to functions of the same type. To underline the role of the starting point we occasionally write X_n^x when $X_0 = x$ almost surely (a.s.). Since M, Q, and X_0 are nonnegative, $(X_n)_{n\geq 0}$ has state space $\mathbb{R}_+$.

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The sequence $(X_n)_{n\geq 0}$ may also be viewed as a *forward iterated function system*, i.e.

$$X_n \;=\; \Psi_n(X_{n-1}) = \Psi_n \circ \cdots \circ \Psi_1(X_0), \qquad n \geq 1,$$

where $\Psi_n(t) := Q_n + M_n t$ for $n \geq 1$ and $\circ$ denotes composition, and, thus, opposed to its closely related counterpart of *backward iterations*

$$\widehat{X}_0 := X_0 \quad \text{and} \quad \widehat{X}_n := \Psi_1 \circ \cdots \circ \Psi_n(X_0), \qquad n \geq 1.$$

The relation is established by the obvious fact that X_n has the same law as $\widehat{X}_n$ for each n, regardless of the law of X_0.

Set

$$\Pi_0 := 1 \quad \text{and} \quad \Pi_n := M_1 M_2 \cdots M_n, \qquad n \geq 1.$$

Assuming that

$$\mathbb{P}(M = 0) = 0 \quad \text{and} \quad \mathbb{P}(Q = 0) < 1, \tag{1.2}$$

and

$$\mathbb{P}(Mr + Q = r) < 1 \quad \text{for all } r \geq 0, \tag{1.3}$$

Goldie and Maller [12, Theorem 2.1] showed (actually, these authors did not assume that M and Q are nonnegative) that the series $\sum_{k\geq 1} \Pi_{k-1} Q_k$, called *perpetuity*, is a.s. convergent provided that

$$\lim_{n\to\infty} \Pi_n = 0 \quad \text{a.s.} \quad \text{and} \quad I_Q := \int_{(1,\infty)} J_-(x)\mathbb{P}(\log Q \in \mathrm{d}x) < \infty, \tag{1.4}$$

where

$$J_-(y) := \frac{y}{\mathbb{E}(y \wedge \log_- M)}, \qquad y > 0$$

and $\log_- x \;=\; -\min(\log x, 0)$. Equivalently, the Markov chain $(X_n)_{n\geq 0}$ is then positive recurrent with unique invariant distribution given by the law of the perpetuity. It is also well known as to what happens in the 'trivial cases' when at least one of the conditions in (1.2) and (1.3) fails [12, Theorem 3.1].

(i) If $\mathbb{P}(M = 0) > 0$ then $\tau := \inf\{k \geq 1 \colon M_k = 0\}$ is a.s. finite, and the perpetuity trivially converges to the a.s. finite random variable $\sum_{k=1}^{\tau} \Pi_{k-1} Q_k$, its law being the unique invariant distribution of $(X_n)_{n\geq 0}$.

(ii) If $\mathbb{P}(Q = 0) = 1$ then $\sum_{k\geq 1} \Pi_{k-1} Q_k = 0$ a.s.

(iii) If $\mathbb{P}(Q + Mr = r) = 1$ for some $r \geq 0$ and $\mathbb{P}(M = 0) = 0$, then either δ_r, the Dirac measure at r, is the unique invariant distribution of $(X_n)_{n\geq 1}$ or every distribution is invariant.

For further information on RDE and perpetuities, we refer the reader to [8] and [15].

If (1.2), (1.3),

$$\lim_{n\to\infty} \Pi_n = 0 \quad \text{a.s.}, \quad \text{and} \quad I_Q = \infty \tag{1.5}$$

hold, which are assumptions in most of our results hereafter (with the exception of Section 7) and particularly satisfied if

$$-\infty \leq \mathbb{E}\log M < 0 \quad \text{and} \quad \mathbb{E}\log_+ Q = \infty, \tag{1.6}$$

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where $\log_+ x = \max(\log x, 0)$, then the aforestated result [12, Theorem 2.1] implies that $(X_n)_{n\geq 0}$ must be either null recurrent or transient. Our purpose is to provide conditions for each of these alternatives and also to investigate the path behavior of $(X_n)_{n\geq 0}$. We refer to (1.5) as the *divergent contractive case* because, on the one hand, $\Pi_n \to 0$ a.s. still renders $\Psi_n \circ \cdots \circ \Psi_1$ to be contractions for sufficiently large n, while, on the other hand, $I_Q = \infty$ entails that occasional large values of the Q_n overcompensate this contractive behavior in such a way that positive recurrence no longer holds. As a consequence, $\sum_{k\geq 1} \Pi_{k-1} Q_k = \infty$ a.s. and so the backward iterations $\widehat{X}_n = \Pi_n X_0 + \sum_{k=1}^n \Pi_{k-1} Q_k$ diverge to ∞ a.s. regardless of whether the chain $(X_n)_{n\geq 0}$ is null recurrent or transient. The question of which alternative occurs relies on a delicate interplay between the Π_n and the Q_n.

Our main results (Theorems 3.1 and 3.2), for simplicity here confined to the situation when (1.2), (1.3), (1.6) hold and $s := \lim_{t\to\infty} t\,\mathbb{P}(\log Q > t)$ exists, assert that $(X_n)_{n\geq 0}$ is null recurrent if $s < -\mathbb{E}\log M$ and transient if $s > -\mathbb{E}\log M$. For deterministic $M \in (0, 1)$, i.e. autoregressive sequences $(X_n)_{n\geq 0}$, this result can be traced back to Kellerer [17, Theorem 3.1] and was later also proved by Zeevi and Glynn [24, Theorem 1], though under a further extra assumption, namely that Q has log-Cauchy tails with scale parameter s, i.e.

$$\mathbb{P}(\log(1 + Q) > t) = \frac{1}{1 + st} \quad \text{for all } t > 0.$$

On the other hand, they could show null recurrence of $(X_n)_{n\geq 0}$ even in the boundary case $s = -\log M$. Kellerer's result will be of some relevance here because we will take advantage of it in combination with a stochastic comparison technique (see Section 4, in particular Proposition 4.1). Finally, we mention work by Bauernschubert [2], Buraczewski and Iksanov [7], Pakes [19], and, most recently, Zerner [25] on the divergent contractive case, yet only Zerner [25] studied the recurrence problem for $(X_n)_{n\geq 0}$ and is, in fact, close to our work. We will therefore comment on the connections in more detail in Remark 3.1.

In the critical case $\mathbb{E}\log M = 0$, not studied here, when $\limsup_{n\to\infty} \Pi_n = \infty$ a.s. and, thus, noncontraction holds, sufficient conditions for the null recurrence of $(X_n)_{n\geq 0}$ and the existence of an essentially unique invariant Radon measure ν were given by Babillot *et al.* [1], namely,

$$\mathbb{E}|\log M|^{2+\delta} < \infty \quad \text{and} \quad \mathbb{E}(\log_+ Q)^{2+\delta} < \infty \quad \text{for some } \delta > 0.$$

For other aspects such as the tail behavior of ν or the convergence of $\widehat{X}_n$ after suitable normalization; see [4], [6], [13], [14], [16], and [21].

The paper is organized as follows. In Section 2 we review known results about general locally contractive Markov chains which form the theoretical basis of the present work. Our main results are stated in Section 3 and proved in Sections 4, 5, and 6. In Section 7 we investigate the attractor set of the Markov chain $(X_n)_{n\geq 0}$ under the sole assumption that $(X_n)_{n\geq 0}$ is locally contractive and recurrent.

2. Theoretical background

We start by giving some useful necessary and sufficient conditions for the transience and recurrence of the sequence $(X_n)_{n\geq 0}$. The following definition plays a fundamental role in the critical case $\mathbb{E}\log M = 0$; see [1], [3]–[6], and [20]. A general Markov chain $(X_n)_{n\geq 0}$, possibly taking values of both signs, is called *locally contractive* if, for any compact set K and all $x, y \in \mathbb{R}$,

$$\lim_{n\to\infty} |X_n^x - X_n^y|\,\mathbf{1}_{\{X_n^x \in K\}} = 0 \quad \text{a.s.}$$

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For the chain $(X_n)_{n\geq 0}$ to be studied here, we observe that, under (1.5),

$$|X_n^x - X_n^y| = \Pi_n |x - y| \to 0 \quad \text{a.s. as } n \to \infty$$

for all $x, y \in \mathbb{R}$. This means that $(X_n)_{n\geq 0}$ is contractive and, hence, locally contractive. Yet, it may hold that

$$\mathbb{P}\Big(\lim_{n\to\infty} |X_n^x - x| = \infty\Big) = 1$$

for any $x \in \mathbb{R}$ in which case the chain is called *transient*. We quote the following result from [20, Lemma 2.2].

Lemma 2.1. *If $(X_n)_{n\geq 0}$ is locally contractive then the following dichotomy holds: either*

$$\mathbb{P}\Big(\lim_{n\to\infty} |X_n^x - x| = \infty\Big) = 0 \quad \textit{for all } x \in \mathbb{R}$$

or

$$\mathbb{P}\Big(\lim_{n\to\infty} |X_n^x - x| = \infty\Big) = 1 \quad \textit{for all } x \in \mathbb{R}. \tag{2.1}$$

The lemma states that either $(X_n)_{n\geq 0}$ is transient or visits a large interval infinitely often (i.o.). The Markov chain $(X_n)_{n\geq 0}$ is called *recurrent* if there exists a nonempty closed set $L \subset \mathbb{R}$ such that $\mathbb{P}(X_n^x \in U \text{ i.o.}) = 1$ for every $x \in L$ and every open set U that intersects L. Plainly, recurrence is a local property of the path of $(X_n)_{n\geq 0}$.

The next lemma can be found in [3, Theorem 3.8] and [20, Theorem 2.13].

Lemma 2.2. *If $(X_n)_{n\geq 0}$ is locally contractive and recurrent then there exists a unique (up to a multiplicative constant) invariant locally finite measure ν.*

The Markov chain $(X_n)_{n\geq 0}$ is called *positive recurrent* if $\nu(L) < \infty$ and *null recurrent*, otherwise.

Our third lemma was stated as Proposition 1.3 of [3]. Since this report has never been published, we present a short proof.

Lemma 2.3. *Let $(X_n)_{n\geq 0}$ be a locally contractive Markov chain and U an open subset of $\mathbb{R}$. Then $\mathbb{P}(X_n^x \in U \text{ i.o.}) < 1$ for some $x \in \mathbb{R}$ implies that $\sum_{n\geq 0} \mathbb{P}(X_n^y \in K) < \infty$ for all $y \in \mathbb{R}$ and all compact $K \subset U$.*

Proof. Take x such that $\mathbb{P}(X_n^x \in U \text{ i.o.}) < 1$. Then there exists $n_1 \in \mathbb{N}$ such that

$$\mathbb{P}(X_n^x \notin U \text{ for all } n \geq n_1) > 0.$$

Now fix an arbitrary $y \in \mathbb{R}$ and a compact $K \subset U$. Defining the compact set $K_y := K \cup \{y\}$, the local contractivity implies that, for some $n_2 \in \mathbb{N}$,

$$\mathbb{P}(X_n^z \notin K \text{ for all } n \geq n_2 \text{ and some } z \in K_y) =: \delta > 0. \tag{2.2}$$

For $z \in K_y$, consider the sequence of stopping times

$$T_0^z = 0 \quad \text{and} \quad T_n^z = \inf\{k > T_{n-1}^z \colon X_k^z \in K\} \quad \text{for } n \geq 1.$$

Then (2.2) implies that $\mathbb{P}(T_{n_2}^z < \infty) \leq 1 - \delta$ for each $z \in K_y$. Consequently,

$$\mathbb{P}(T_{nn_2}^y < \infty) \leq (1 - \delta)\mathbb{P}(T_{(n-1)n_2}^y < \infty) \leq (1 - \delta)^n \quad \text{for all } n \geq 1$$

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and, thus,

$$\begin{aligned}\sum_{n\geq 0}\mathbb{P}(X_n^y \in K) &= \mathbb{E}\bigg(\sum_{n\geq 0}\mathbf{1}_{\{X_n^y\in K\}}\bigg)\\ &\leq \sum_{n\geq 0}\mathbb{P}(T_n^y < \infty)\\ &\leq \sum_{n\geq 0} n_2\mathbb{P}(T_{nn_2}^y < \infty)\\ &\leq \frac{n_2}{\delta}\\ &< \infty.\end{aligned}$$

□

A combination of Lemmas 2.1 and 2.3 provide us with the following proposition.

Proposition 2.1. *For a locally contractive Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}$, the following assertions are equivalent.*

(i) *The chain is transient.*

(ii) $\lim_{n\to\infty}|X_n^x| = \infty$ *a.s. for all* $x \in \mathbb{R}$.

(iii) $\mathbb{P}(X_n^x \in U \text{ i.o.}) < 1$ *for any bounded open* $U \subset \mathbb{R}$ *and some/all* $x \in \mathbb{R}$.

(iv) $\sum_{n\geq 0}\mathbb{P}(X_n^x \in K) < \infty$ *for any compact* $K \subset \mathbb{R}$ *and some/all* $x \in \mathbb{R}$.

Proof. The equivalence of (i), (ii), and (iii) is obvious. By Lemma 2.3, (iii) entails (iv), while the Borel–Cantelli lemma yields the converse. □

Now we consider the case when (2.1) is satisfied. For any ω, we define $L^x(\omega)$ to be the set of accumulation points of $(X_n^x(\omega))_{n\geq 0}$, i.e.

$$L^x(\omega) := \bigcap_{m\geq 1}\overline{\{X_n^x(\omega)\colon n \geq m\}},$$

where $\overline{C}$ denotes the closure of a set C. It is known (see [3] and [20]) that $L^x(\omega)$ does not depend on x and ω. In fact, there exists a deterministic set $L \subset \mathbb{R}$ (called the *attractor set* or *limit set*) such that

$$\mathbb{P}(L^x(\cdot) = L \text{ for all } x \in \mathbb{R}) = 1.$$

Proposition 2.2. *For a locally contractive Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{R}$, the following assertions are equivalent.*

(i) *The chain is recurrent.*

(ii) $\liminf_{n\to\infty}|X_n^x - x| < \infty$ *a.s. for all* $x \in \mathbb{R}$.

(iii) $\liminf_{n\to\infty}|X_n| < \infty$ *a.s.*

(iv) $\sum_{n\geq 0}\mathbb{P}(X_n^x \in K) = \infty$ *for a nonempty compact set* K *and some/all* $x \in \mathbb{R}$.

Proof. In view of the contrapositive Proposition 2.1, we must only verify for '(i)⇒(iv)' that the sum in (iv) is indeed infinite for some compact $K \neq \emptyset$ and *all* $x \in \mathbb{R}$. Without loss of generality, let $K = [-2b, 2b]$ for some $b > 0$ and $y \in \mathbb{R}$ such that, by (i), $\sum_{n\geq 0}\mathbf{1}_{(|X_n^y|\leq b)} = \infty$ a.s. and, thus, $\sum_{n\geq 0}\mathbb{P}(|X_n^y| \leq b) = \infty$. Local contractivity implies

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that $\sigma_x := \sup\{n \geq 0\colon |X_n^x - X_n^y| > b, |X_n^y| \leq b\}$ is a.s. finite for *all* $x \in \mathbb{R}$. Consequently, X_n^x hits $[-2b, 2b]$ whenever X_n^y hits $[-b, b]$ for $n > \sigma_x$, in particular, $\sum_{n\geq 0} \mathbf{1}_{(|X_n^x|\leq 2b)} = \infty$ a.s. and, thus, $\sum_{n\geq 0} \mathbb{P}(|X_n^x| \leq 2b) = \infty$ for all $x \in \mathbb{R}$. □

3. Results

In order to formulate the main result, we need

$$s_* := \liminf_{t\to\infty} t\mathbb{P}(\log Q > t), \qquad s^* := \limsup_{t\to\infty} t\mathbb{P}(\log Q > t) \tag{3.1}$$

for which $0 \leq s_* \leq s^* \leq \infty$ holds. In some places, the condition

$$\lim_{t\to\infty} t\mathbb{P}(\log Q > t) = s \ \in \ [0, \infty] \tag{3.2}$$

will be used. Finally, set $\mathfrak{m}^{\pm} := \mathbb{E}\log_{\pm} M$ and, if $\mathfrak{m}^+ \wedge \mathfrak{m}^- < \infty$,

$$\mathfrak{m} := \mathbb{E}\log M = \mathfrak{m}^+ - \mathfrak{m}^-,$$

which is then in $[-\infty, 0)$ by our standing assumption $\Pi_n \to 0$ a.s.

Theorem 3.1. *Let* $\mathfrak{m} \in [-\infty, 0)$ *and (1.2), (1.3), (1.5) be valid. Then the following assertions hold:*

(i) $(X_n)_{n\geq 0}$ *is null recurrent if* $s^* < -\mathfrak{m}$*;*

(ii) $(X_n)_{n\geq 0}$ *is transient if* $s_* > -\mathfrak{m}$ *(thus,* $\mathfrak{m} > -\infty$*).*

Remark 3.1. Zerner [25] studied the recurrence/transience of $(X_n)_{n\geq 0}$ defined by (1.1) in the more general setting when M is a nonnegative $d \times d$ random matrix and Q an $\mathbb{R}_+^d$-valued random vector. A specialization of his Theorem 5 to the one-dimensional case $d = 1$ reads as follows. Suppose that

$$M \in [a, b] \quad \text{a.s. for some } 0 < a < b < \infty \tag{3.3}$$

and that either $\lim_{t\to\infty} t^{\beta}\mathbb{P}(\log Q > t) = 0$ for some $\beta \in (\frac{2}{3}, 1)$, or $s_* > -\mathfrak{m}$. Let $y \in (0, \infty)$ be such that $\mathbb{P}(Q \leq y) > 0$. Then $(X_n)_{n\geq 0}$ is recurrent if and only if

$$\sum_{n\geq 0}\prod_{k=0}^{n} \mathbb{P}(Q \leq y\mathrm{e}^{-k\mathfrak{m}}) = \infty. \tag{3.4}$$

It is not difficult to verify that (3.4) holds if $s^* < -\mathfrak{m}$ and that it fails if $s_* > -\mathfrak{m}$. Therefore, Zerner's result contains our Theorem 3.1 under the additional assumption (3.3).

Remark 3.2. If $\log M$ and $\log Q$ are both integrable and $D(x) := \log X_1^x - \log X_0^x$, then

$$\mathbb{E}D(x) = \mathbb{E}\log(M + x^{-1}Q) \to 0 \quad \text{as } x \to \infty$$

shows that $(\log X_n)_{n\geq 0}$ forms a Markov chain with asymptotic drift zero. Such chains were studied at length by Denisov *et al.* [10]. They also provided conditions for recurrence and transience in terms of truncated moments of $D(x)$, see their Corollaries 2.11 and 2.16, but these appear to be more complicated and more restrictive than ours.

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Remark 3.3. Here is a comment on the boundary case $s = -\mathfrak{m}$ not covered by Theorem 3.1. Assuming $M = \mathrm{e}^{\mathfrak{m}}$ a.s., it can be shown that the null recurrence/transience of $(X_n)_{n\geq 0}$ is equivalent to the divergence/convergence of the series

$$\sum_{n\geq 1} \mathbb{P}\Big(\max_{1\leq k\leq n} e^{\mathfrak{m}(k-1)} Q_k \leq \mathrm{e}^x\Big) = \sum_{n\geq 1} \prod_{k=0}^{n-1} F(x-\mathfrak{m}k) \quad \text{for some/all } x \geq 0,$$

where $F(y) := \mathbb{P}(\log Q \leq y)$. Indeed, assuming $X_0 = 0$, the transience assertion follows when using $\mathbb{P}(X_n \leq \mathrm{e}^x) \leq \mathbb{P}(\max_{1\leq k\leq n} \mathrm{e}^{\mathfrak{m}(k-1)} Q_k \leq \mathrm{e}^x)$, while the null recurrence claim is shown by a thorough inspection and adjustment of the proof of Theorem 3.1(i). Using Kummer's test as stated in [23], we then further conclude that $(X_n)_{n\geq 0}$ is null recurrent if and only if there exist positive $p_1, p_2, \ldots$ such that $F(-\mathfrak{m}k) \geq p_k/p_{k+1}$ and $\sum_{k\geq 1}(1/p_k) = \infty$. For applications, the following sufficient condition, which is a consequence of Bertrand's test [22, p. 408], may be more convenient. If

$$1 - F(x) = \frac{-\mathfrak{m}}{x} + \frac{f(x)}{x \log x}$$

then $(X_n)_{n\geq 0}$ is null recurrent if $\limsup_{x\to\infty} f(x) < -\mathfrak{m}$, and transient if $\liminf_{x\to\infty} f(x) > -\mathfrak{m}$.

If $\mathfrak{m}^- = \mathfrak{m}^+ = \infty$ and $s^* < \infty$ then $(X_n)_{n\geq 0}$ is *always* null recurrent, as we confirm in the next theorem. Its proof will be based on finding an appropriate subsequence of $(X_n)_{n\geq 0}$ which satisfies the assumptions of Theorem 3.1(i).

Theorem 3.2. *Let* $\mathfrak{m}^+ = \mathfrak{m}^- = \infty$, $s^* < \infty$, *and (1.2), (1.3), (1.5) be valid. Then* $(X_n)_{n\geq 0}$ *is null recurrent.*

The two theorems are proved in Section 6 after some preparatory work in Sections 4 and 5.

Remark 3.4. It is worthwhile to point out that the assumptions of the previous theorem impose some constraint on the tails of $\log_+ M$. Namely, given these assumptions, the negative divergence of the random walk $S_n := \log \Pi_n$, $n \geq 0$, that is, $S_n \to -\infty$ a.s., entails

$$I_M = \int_{(1,\infty)} J_-(x)\mathbb{P}(\log M \in \mathrm{d}x) < \infty,$$

by Erickson's theorem [11, Theorem 2]. But this, in combination with $I_Q = \infty$ and $s^* < \infty$, further implies, by stochastic comparison, that

$$r_* := \liminf_{t\to\infty} t\mathbb{P}(\log M > t) = 0.$$

Indeed, if the latter failed to hold, i.e. $r_* > 0$, then

$$\mathbb{P}(\log M > t) \geq \frac{r_*}{2t} \geq \frac{r_*}{4s^*}\mathbb{P}(\log Q > t)$$

for all sufficiently large t, say $t \geq t_0$, which, in turn, would entail the contradiction

$$I_M - J_-(0+) \geq \frac{r_*}{4s^*} \int_{(t_0,\infty)} J'_-(x)\mathbb{P}(\log Q > x)\,\mathrm{d}x = \infty.$$

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4. The cases $M \le \gamma$ and $M \ge \gamma$: two comparison lemmas and Kellerer's result

In this section we collect some useful results for the cases when $M \le \gamma$ or $M \ge \gamma$ a.s. for a constant $\gamma \in (0, 1)$, in particular, Kellerer's unpublished recurrence result [17] for this situation; see Proposition 4.1 below. Whenever given i.i.d. nonnegative $Q_1, Q_2, \ldots$ with generic copy Q, let $(X_n(\gamma))_{n\ge 0}$ be defined by

$$X_n(\gamma) = \gamma X_{n-1}(\gamma) + Q_n, \qquad n \ge 1,$$

where $X_0(\gamma)$ is independent of $(Q_n)_{n\ge 1}$. We start with two comparison lemmas which treat two RDE with identical $M \le \gamma$ but different Q.

Lemma 4.1. *Let $(M_n, Q_n, Q'_n)_{n\ge 1}$ be a sequence of i.i.d. random vectors with nonnegative components and generic copy (M, Q, Q') such that $M \le \gamma$ a.s. for some $\gamma \in (0, 1)$ and*

$$\mathbb{P}(Q' > t) \ge \mathbb{P}(Q > t) \tag{4.1}$$

for some $t_0 \ge 0$ and all $t \ge t_0$. Define

$$X_n := M_n X_{n-1} + Q_n \quad \text{and} \quad X'_n := M_n X'_{n-1} + Q'_n$$

for $n \ge 1$, where X'_0 is independent of X_0 and $(M_k, Q_k)_{k\ge 1}$. Then

$$(X_n)_{n\ge 0} \text{ transient} \quad \Longrightarrow \quad (X'_n)_{n\ge 0} \text{ transient}$$

or, equivalently,

$$(X'_n)_{n\ge 0} \text{ recurrent} \quad \Longrightarrow \quad (X_n)_{n\ge 0} \text{ recurrent}.$$

Proof. The tail condition (4.1) ensures that we may choose a coupling (Q, Q') such that $Q' \ge Q - t_0$ a.s. Then, with $(Q_n, Q'_n)_{n\ge 1}$ being i.i.d. copies of (Q, Q'), it follows that

$$\begin{aligned} X'_n - X_n &= M_n(X'_{n-1} - X_{n-1}) + Q'_n - Q_n \\ &\ge M_n(X'_{n-1} - X_{n-1}) - t_0 \\ &\vdots \\ &\ge \Bigl(\prod_{k=1}^{n} M_k\Bigr)(X'_0 - X_0) - t_0 \sum_{k=0}^{n-1} \gamma^k \quad \text{a.s.} \end{aligned}$$

and, thereby,

$$\liminf_{n\to\infty} (X'_n - X_n) \ge -\frac{t_0}{1-\gamma} \quad \text{a.s.},$$

which obviously proves the asserted implication. □

Lemma 4.2. *Replace condition (4.1) in Lemma 4.1 with*

$$Q' = \mathbf{1}_{\{Q>\beta\}}\, Q \quad \text{for some } \beta > 0,$$

thus, $\mathbb{P}(Q' = 0) = \mathbb{P}(Q \le \beta)$. Then

$$(X'_n)_{n\ge 0} \text{ recurrent} \quad \Longleftrightarrow \quad (X_n)_{n\ge 0} \text{ recurrent}.$$

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Proof. Here it suffices to point out that

$$\begin{aligned}|X'_n - X_n| &= |M_n(X'_{n-1} - X_{n-1}) + Q'_n - Q_n| \\ &\le M_n|X'_{n-1} - X_{n-1}| + \mathbf{1}_{\{Q_n \le \beta\}}\, Q_n \\ &\vdots \\ &\le \left(\prod_{k=1}^{n} M_k\right)(X'_0 - X_0) + \beta \sum_{k=0}^{n-1} \gamma^k \\ &\le \gamma^n (X'_0 - X_0) + \frac{\beta}{1-\gamma} \quad \text{a.s. for all } n \ge 1. \end{aligned}$$

□

The announced result by Kellerer including its proof (with some minor modifications), taken from his unpublished technical report [17, Theorem 3.1], is given next.

Proposition 4.1. *Let* $0 < \gamma < 1$. *Then the following assertions hold:*

(i) $(X_n)_{n\ge 0}$ *is transient if* $M \ge \gamma$ *a.s. and*

$$s_* = \liminf_{t\to\infty} t\mathbb{P}(\log Q > t) > \log\left(\frac{1}{\gamma}\right);$$

(ii) $(X_n)_{n\ge 0}$ *is null recurrent if* $M \le \gamma$ *a.s. and*

$$s^* = \limsup_{t\to\infty} t\mathbb{P}(\log Q > t) < \log\left(\frac{1}{\gamma}\right).$$

Proof. It is enough to consider (in both parts) the case when $M = \gamma$ a.s. and, thus, the Markov chain $(X_n(\gamma))_{n\ge 0}$ as defined above. We may further assume that $X_0(\gamma) = 0$ and set $\theta := \log(1/\gamma)$.

Transience. It suffices to show that $\sum_{n\ge 1} \mathbb{P}(\widehat{X}_n(\gamma) \le \mathrm{e}^t) < \infty$ for all $t \ge 0$. Fixing t and any $\varepsilon > 0$ with $(1+\varepsilon)\theta < s_*$, pick $m \in \mathbb{N}$ so large that

$$\inf_{k\ge m+1} k\theta\mathbb{P}(\log Q > t + k\theta) \ge (1+\varepsilon)\theta.$$

Then we infer that, for all $n > m$,

$$\begin{aligned}\mathbb{P}(\widehat{X}_n(\gamma) \le \mathrm{e}^t) &= \mathbb{P}\left(\sum_{k=1}^{n} \gamma^{k-1} Q_k \le \mathrm{e}^t\right) \\ &\le \mathbb{P}(\log Q_k \le t + (k-1)\theta,\ 1 \le k \le n) \\ &\le \prod_{k=m+1}^{n} (1 - \mathbb{P}(\log Q > t + k\theta)) \\ &\le \prod_{k=m+1}^{n} \left(1 - \frac{1+\varepsilon}{k}\right) \\ &\le \prod_{k=m+1}^{n} \left(1 - \frac{1}{k}\right)^{1+\varepsilon} \\ &= \left(\frac{m}{n}\right)^{1+\varepsilon}, \end{aligned}$$

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where $(1-x)^{1+\varepsilon} \geq 1-(1+\varepsilon)x$ for all $x \in [0,1]$ has been utilized for the last inequality. Consequently, $\sum_{n\geq 1} \mathbb{P}(\widehat{X}_n(\gamma) \leq \mathrm{e}^t) < \infty$, and the transience of $(X_n(\gamma))_{n\geq 0}$ follows by Proposition 2.1.

Null recurrence. By Lemma 4.2, we may assume without loss of generality that, for some sufficiently small $\varepsilon > 0$, $\delta := \mathbb{P}(Q=0) \geq \gamma^{\varepsilon}$ and

$$\sup_{t\geq 1} t\mathbb{P}(\log Q > t) \leq (1-\varepsilon)\theta.$$

Also set $m_n := \theta^{-1}(m+\log n)$ for integer $m \geq 1$ so large that

$$g(x,n) := (x-1)\theta - \log n \geq 1 \vee (1-\varepsilon)\theta \quad \text{for all } x \in (m_n, \infty).$$

Note that $\delta^{m_n} \geq (\mathrm{e}^m n)^{-\varepsilon}$. For all $n \geq 1$ so large that $g(n,n) > \theta$, we then infer that

$$\begin{aligned}
\mathbb{P}(\widehat{X}_n(\gamma) \leq 1) &\geq \mathbb{P}\Big(\max_{1\leq k\leq n} \gamma^{k-1} Q_k \leq \frac{1}{n}\Big) \\
&\geq \mathbb{P}(Q=0)^{m_n} \prod_{m_n+1\leq k\leq n} \mathbb{P}(\log Q \leq g(k,n)) \\
&\geq \delta^{m_n} \prod_{m_n+1\leq k\leq n} \Big(1 - \frac{(1-\varepsilon)\theta}{g(k,n)}\Big) \\
&\geq (\mathrm{e}^m n)^{-\varepsilon} \prod_{k=m}^{n} \Big(1 - \frac{1-\varepsilon}{k}\Big) \\
&\geq (\mathrm{e}^m n)^{-\varepsilon} \prod_{k=2}^{n} \Big(1 - \frac{1}{k}\Big)^{1-\varepsilon} \\
&= \frac{\mathrm{e}^{-m\varepsilon}}{n}.
\end{aligned}$$

Here $(1-x)^{1-\varepsilon} \leq 1-(1-\varepsilon)x$ for all $x \in [0,1]$ has been utilized for the last inequality. Hence, $\sum_{n\geq 1} \mathbb{P}(\widehat{X}_n(\gamma) \leq 1) = \infty$, giving the recurrence of $(X_n(\gamma))_{n\geq 0}$ by Proposition 2.2. This completes the proof. □

Given a Markov chain $(Z_n)_{n\geq 0}$, a sequence $(\sigma_n)_{n\geq 0}$ is called a *renewal stopping sequence* for this chain if the following conditions hold:

(R1) $\sigma_0 = 0$ and the $\tau_n := \sigma_n - \sigma_{n-1}$ are i.i.d. for $n \geq 1$;

(R2) there exists a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ such that $(Z_n)_{n\geq 0}$ is Markov-adapted and each σ_n is a stopping time with respect to $\mathcal{F}$.

We define

$$S_n := \log \Pi_n = \sum_{k=1}^{n} \log M_k \quad \text{for } n \geq 0$$

and recall that, by our standing assumption, $(S_n)_{n\geq 0}$ is a negative divergent random walk ($S_n \to -\infty$ a.s.). For $c \in \mathbb{R}$, let $(\sigma_n^{>}(c))_{n\geq 0}$ and $(\sigma_n^{<}(c))_{n\geq 0}$ denote the possibly defective

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renewal sequences of ascending and descending ladder epochs associated with the random walk $(S_n + cn)_{n\geq 0}$, in particular,

$$\sigma^{>}(c) = \sigma_1^{>}(c) := \inf\{n \geq 1 \colon S_n + cn > 0\} = \inf\{n \geq 1 \colon \Pi_n > \mathrm{e}^{-cn}\},$$
$$\sigma^{<}(c) = \sigma_1^{<}(c) := \inf\{n \geq 1 \colon S_n + cn < 0\} = \inf\{n \geq 1 \colon \Pi_n < \mathrm{e}^{-cn}\}.$$

Plainly, these are renewal stopping sequences for $(X_n)_{n\geq 0}$ whenever nondefective.

Lemma 4.3. *Let $c \geq 0$ and $\gamma = \mathrm{e}^{-c}$.*

(i) *If c is such that $\sigma^{<}(c) < \infty$ a.s. then, with $(\sigma_n)_{n\geq 0} := (\sigma_n^{<}(c))_{n\geq 0}$,*

$$X_{\sigma_n} \leq X_{\sigma_n}(\gamma) \quad \text{and} \quad \widehat{X}_{\sigma_n} \leq \widehat{Y}_n \quad \text{a.s. for all } n \geq 0,$$

where $X_0 = X_0(\gamma) = \widehat{Y}_0$ and

$$\widehat{Y}_n := \sum_{k=1}^{n} \gamma^{\sigma_{k-1}} Q_k^* \quad \text{with } Q_n^* := \sum_{k=1}^{\sigma_n - \sigma_{n-1}} \frac{\Pi_{\sigma_{n-1}+k-1}}{\Pi_{\sigma_{n-1}}} Q_k$$

for $n \geq 1$ denotes the sequence of backward iterations pertaining to the recursive Markov chain $Y_n = \gamma^{\sigma_n - \sigma_{n-1}} Y_{n-1} + Q_n^*$.

(ii) *If c is such that $\sigma^{>}(c) < \infty$ a.s. then, with $(\sigma_n)_{n\geq 0} := (\sigma_n^{>}(c))_{n\geq 0}$,*

$$X_{\sigma_n} \geq X_{\sigma_n}(\gamma) \quad \text{and} \quad \widehat{X}_{\sigma_n} \geq \widehat{Y}_n \quad \text{a.s. for all } n \geq 0,$$

where $X_0 = X_0(\gamma) = \widehat{Y}_0$ and $\widehat{Y}_n$ is defined as in (i) for the σ_n given here.

Plainly, one can take $c \in (0, -\mathfrak{m})$ in (i) and $c \in (-\mathfrak{m}, \infty)$ in (ii) if $-\infty < \mathfrak{m} < 0$.

Proof of Lemma 4.3. (i) Suppose that the $\sigma_n^{<}(c)$ are a.s. finite. To prove our claim for X_{σ_n}, we use induction over n. Since $\sigma_0 = 0$, we have $X_{\sigma_0} = X_{\sigma_0}(\gamma)$. For the inductive step, suppose that $X_{\sigma_{n-1}} \leq X_{\sigma_{n-1}}(\gamma)$ for some $n \geq 1$. Observe that, with $\tau_n = \sigma_n - \sigma_{n-1}$,

$$\begin{aligned} M_{\sigma_{n-1}+k+1} \cdots M_{\sigma_n} &= \frac{\Pi_{\sigma_n}}{\Pi_{\sigma_{n-1}+k}} \\ &= \exp((S_{\sigma_n} + c\sigma_n) - (S_{\sigma_{n-1}+k} + c(\sigma_{n-1} + k)) - c(\tau_n - k)) \\ &\leq \gamma^{\tau_n - k} \quad \text{for all } 0 \leq k \leq \tau_n. \end{aligned} \tag{4.2}$$

Using this and the inductive hypothesis, we obtain

$$\begin{aligned} X_{\sigma_n} &= \frac{\Pi_{\sigma_n}}{\Pi_{\sigma_{n-1}}} X_{\sigma_{n-1}} + \sum_{k=1}^{\tau_n} \frac{\Pi_{\sigma_n}}{\Pi_{\sigma_{n-1}+k}} Q_{\sigma_{n-1}+k} \\ &\leq \gamma^{\tau_n} X_{\sigma_{n-1}}(\gamma) + \sum_{k=1}^{\tau_n} \gamma^{\tau_n - k} Q_{\sigma_{n-1}+k} \\ &= X_{\sigma_n}(\gamma) \quad \text{a.s.}, \end{aligned}$$

as asserted. Regarding the backward iteration $\widehat{X}_{\sigma_n}$, we find more directly that

$$\widehat{X}_{\sigma_n} = \sum_{k=1}^{n} \Pi_{\sigma_{k-1}} \sum_{j=1}^{\tau_k} \frac{\Pi_{\sigma_{k-1}+j-1}}{\Pi_{\sigma_{k-1}}} Q_j \leq \sum_{k=1}^{n} \gamma^{\sigma_{k-1}} \sum_{j=1}^{\tau_k} \frac{\Pi_{\sigma_{k-1}+j-1}}{\Pi_{\sigma_{k-1}}} Q_j = \widehat{Y}_n \quad \text{a.s.}$$

for each $n \geq 1$.

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(ii) If c is such that the $\sigma_n^{>}(c)$ are a.s. finite then (4.2) becomes

$$M_{\sigma_{n-1}+k+1}\cdots M_{\sigma_n} \geq \gamma^{\tau_n-k} \quad \text{for all } n\in\mathbb{N} \text{ and } 0\leq k\leq \tau_n.$$

Now it is easily seen that the inductive argument in (i) remains valid when reversing inequality signs and the same holds for $\widehat{X}_{\sigma_n}$. □

5. Tail lemmas

In order to prove our results, we need to verify that the tail condition (3.2) is preserved under stopping times with finite mean. To be more precise, let σ be any such stopping time for $(M_k, Q_k)_{k\geq 1}$ and consider

$$\widehat{X}_\sigma = \sum_{k=1}^{\sigma} \Pi_{k-1} Q_k.$$

Obviously,

$$\max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k \leq \widehat{X}_\sigma \leq \sigma \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k. \tag{5.1}$$

Lemma 5.1. *Assuming (1.2), (1.3), and* $\mathtt{m}<0$*, condition (3.2) entails*

$$\lim_{t\to\infty} t\mathbb{P}(\log \widehat{X}_\sigma > t) = s\mathbb{E}\sigma,$$

where the right-hand side is equal to 0 *if* $s=0$ *and* ∞ *if* $s=\infty$.

Proof. It suffices to prove

$$\lim_{t\to\infty} t\mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > t\Big) = s\mathbb{E}\sigma$$

since (5.1) in combination with $\mathbb{E}\sigma<\infty$ entails

$$\begin{aligned}\mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > t\Big) &\leq \mathbb{P}(\log \widehat{X}_\sigma > t)\\ &\leq \mathbb{P}(\log\sigma > \varepsilon t) + \mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > (1-\varepsilon)t\Big)\\ &= o(t^{-1}) + \mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > (1-\varepsilon)t\Big) \quad \text{for all } \varepsilon>0.\end{aligned}$$

(i) We first prove that

$$\limsup_{t\to\infty} t\mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > t\Big) \leq s\mathbb{E}\sigma, \tag{5.2}$$

which is nontrivial only when assuming $s\in[0,\infty)$. Set $\eta_n := \log Q_n$ for $n\in\mathbb{N}$. For any $\varepsilon\in(0,1)$, we then have

$$\begin{aligned}\mathbb{P}\Big(\log \max_{1\leq k\leq\sigma} \Pi_{k-1}Q_k > t\Big) &= \mathbb{P}\Big(\max_{1\leq k\leq\sigma} (S_{k-1}+\eta_k) > t\Big)\\ &\leq \mathbb{P}\Big(\max_{0\leq k\leq\sigma} S_k > \varepsilon t\Big) + \mathbb{P}\Big(\max_{1\leq k\leq\sigma} \eta_k > (1-\varepsilon)t\Big)\\ &= I_1(t) + I_2(t).\end{aligned}$$

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Regarding $I_1(t)$, note that

$$\max_{0\le k\le\sigma} S_k \le \sum_{k=1}^{\sigma} \log_+ M_k.$$

Since $\mathfrak{m} \in [-\infty, 0)$ entails $\mathbb{E}\log_+ M < \infty$, we conclude with the help of Wald's identity

$$\mathbb{E}\Big(\max_{0\le k\le\sigma} S_k\Big) \le \mathbb{E}\Big(\sum_{k=1}^{\sigma} \log_+ M_k\Big) = \mathbb{E}\sigma\, \mathbb{E}\log_+ M < \infty.$$

As a consequence, $\lim_{t\to\infty} t I_1(t) = 0$.

Turning to $I_2(t)$, we obtain

$$\begin{aligned} t I_2(t) &\le t\mathbb{E}\sum_{k=1}^{\sigma} \mathbf{1}_{\{\eta_k > (1-\varepsilon)t\}} \\ &= t\mathbb{E}\sum_{k\ge 1} \mathbf{1}_{\{\eta_k > (1-\varepsilon)t,\, \sigma \ge k\}} \\ &= t\mathbb{P}(\eta_1 > (1-\varepsilon)t) \sum_{k\ge 1} \mathbb{P}(\sigma \ge k) \\ &= t\mathbb{E}\sigma\, \mathbb{P}(\eta_1 > (1-\varepsilon)t) \\ &< \infty \end{aligned}$$

and, thereupon,

$$\limsup_{t\to\infty} t\,(I_1(t) + I_2(t)) = \limsup_{t\to\infty} t I_2(t) \le \frac{s\mathbb{E}\sigma}{1-\varepsilon}.$$

Hence, (5.2) follows upon letting ε tend to 0.

(ii) It remains to prove the inequality

$$\liminf_{t\to\infty} t\mathbb{P}\Big(\log \max_{1\le k\le\sigma} \Pi_{k-1} Q_k > t\Big) \ge s\mathbb{E}\sigma, \tag{5.3}$$

which is nontrivial only when assuming $s \in (0, \infty]$. To this end, observe that

$$\log \max_{1\le k\le\sigma} \Pi_{k-1} Q_k = \max_{1\le k\le\sigma} (S_{k-1} + \eta_k) \ge \max_{1\le k\le\sigma\wedge\tau(c)} \eta_k - c \quad \text{for any } c > 0,$$

where $\tau(c) := \inf\{n \ge 1 \colon S_n < -c\}$. Since, furthermore,

$$\begin{aligned} \mathbb{P}\Big(\max_{1\le k\le\sigma\wedge\tau(c)} \eta_k > t\Big) &= \mathbb{E}\Big(\sum_{k=1}^{\sigma\wedge\tau(c)} \mathbf{1}_{\{\max_{1\le j\le k-1}\eta_j \le t,\, \eta_k > t\}}\Big) \\ &= \sum_{k\ge 1} \mathbb{P}\Big(\max_{1\le j\le k-1} \eta_j \le t, \eta_k > t, \sigma\wedge\tau(c) \ge k\Big) \\ &= \mathbb{P}(\eta_1 > t) \sum_{k\ge 1} \mathbb{P}\Big(\max_{1\le j\le k-1} \eta_j \le t, \sigma\wedge\tau(c) \ge k\Big), \end{aligned}$$

we find

$$\begin{aligned} t\mathbb{P}\Big(\log \max_{1\le k\le\sigma} \Pi_{k-1} Q_k > t\Big) &\ge t\mathbb{P}(\eta_1 > t + c) \sum_{k\ge 1} \mathbb{P}\Big(\max_{1\le j\le k-1} \eta_j \le t, \sigma\wedge\tau(c) \ge k\Big) \\ &\to s\mathbb{E}(\sigma\wedge\tau(c)) \quad \text{as } t\to\infty, \end{aligned}$$

and this implies (5.3) upon letting c tend to ∞ for $\sigma\wedge\tau(c) \uparrow \sigma$. □

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By combining the previous result with a simple stochastic majorization argument, we obtain the following extension.

Lemma 5.2. *Let s_* and s^* be as defined in (3.1). Then*

$$\limsup_{t\to\infty} t\mathbb{P}(\log \widehat{X}_\sigma > t) \le s^*\mathbb{E}\sigma, \tag{5.4}$$

$$\liminf_{t\to\infty} t\mathbb{P}(\log \widehat{X}_\sigma > t) \ge s_*\mathbb{E}\sigma. \tag{5.5}$$

Proof. For (5.4), we may assume that $s^* < \infty$. Recall the notation $F(t) = \mathbb{P}(\log Q \le t)$ and set $\overline{F} := 1 - F$. Then define the new distribution function G by

$$\overline{G}(t) := \mathbf{1}_{(-\infty,0]}(t) + \left(\overline{F}(t) \vee \frac{s}{s+t}\right)\mathbf{1}_{(0,\infty)}(t)$$

for some arbitrary $s > s^*$ (we can even choose $s = s^*$ unless $s^* = 0$). Since $\overline{G} \ge \overline{F}$, we may construct (on a possibly enlarged probability space) random variables Q', Q'_1, Q'_2, ... such that (M, Q, Q'), (M_1, Q_1, Q'_1), (M_2, Q_2, Q'_2), ... are i.i.d., the distribution function of $\log Q'$ is G, and $Q' \ge Q$, thus,

$$\widehat{X}'_\sigma := \sum_{k=1}^{\sigma} \Pi_{k-1} Q'_k \ge \widehat{X}_\sigma.$$

On the other hand, $\overline{G}(t) = \mathbb{P}(\log Q' > t)$ satisfies the tail condition (3.2), whence, by an appeal to Lemma 5.1,

$$\limsup_{t\to\infty} t\mathbb{P}(\log \widehat{X}_\sigma > t) \le \lim_{t\to\infty} t\mathbb{P}(\log \widehat{X}'_\sigma > t) = s\mathbb{E}\sigma.$$

This proves (5.4) since $s - s^*$ can be chosen arbitrarily small.

Assertion (5.5) for $s > 0$ is proved in a similar manner. Indeed, pick any $s \in (0, s_*)$ (or even s_* itself unless $s_* = \infty$) and define

$$\overline{G}(t) := \left(\overline{F}(t) \wedge \frac{s}{s+t}\right)\mathbf{1}_{[0,\infty)}(t),$$

which obviously satisfies $\overline{G} \le \overline{F}$. In the previous notation, we now have $Q' \le Q$ and, thus, $\widehat{X}'_\sigma \le \widehat{X}_\sigma$. Since again $\overline{G}(t) = \mathbb{P}(\log Q' > t)$ satisfies the tail condition (3.2), we easily arrive at the desired conclusion by another appeal to Lemma 5.1. □

Our last tail lemma will be crucial for the proof of Theorem 3.2. Given any $0 < \gamma < 1$, recall that $X_0(\gamma) = X_0$ and

$$X_n(\gamma) = \gamma X_{n-1}(\gamma) + Q_n \quad \text{for } n \ge 1.$$

Let σ be any integrable stopping time for $(X_n)_{n\ge 0}$ and note that

$$X_\sigma(\gamma) = \gamma^\sigma X_0 + Q(\gamma),$$

where

$$Q(\gamma) := \sum_{k=1}^{\sigma} \gamma^{\sigma-k} Q_k.$$

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More generally, if $(\sigma_n)_{n\geq 0}$ denotes a renewal stopping sequence for $(X_n)_{n\geq 0}$ with $\sigma = \sigma_1$, then

$$X_{\sigma_n}(\gamma) = \gamma^{\sigma_n - \sigma_{n-1}} X_{\sigma_{n-1}}(\gamma) + Q_n(\gamma) \quad \text{for } n \geq 1$$

with i.i.d. $(\gamma^{\sigma_n - \sigma_{n-1}}, Q_n(\gamma))_{n\geq 1}$ and $Q_{\sigma_1}(\gamma) = Q(\gamma)$.

Lemma 5.3. *Let $\gamma \in (0,1)$ and σ, $Q(\gamma)$ be as just introduced. If Q satisfies condition (3.2) then*

$$\lim_{t\to\infty} t\mathbb{P}(\log Q(\gamma) > t) = s\mathbb{E}\sigma,$$

where the right-hand side is equal to 0 if $s = 0$ and ∞ if $s = \infty$. More generally, with s_*, s^* as defined in (3.1), it always holds that

$$s_*\mathbb{E}\sigma \leq \liminf_{t\to\infty} t\mathbb{P}(\log Q(\gamma) > t) \leq \limsup_{t\to\infty} t\mathbb{P}(\log Q(\gamma) > t) \leq s^*\mathbb{E}\sigma.$$

Proof. Embarking on the obvious inequality (compare (5.1))

$$\max_{1\leq k\leq \sigma} \gamma^{\sigma-k} Q_k \leq Q(\gamma) \leq \sigma \max_{1\leq k\leq \sigma} \gamma^{\sigma-k} Q_k,$$

the arguments are essentially the same and even slightly simpler than those given for the proofs of Lemmas 5.1 and 5.2. We therefore omit further details. □

6. Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1. (i) *Null recurrence.* We retain the notation of the previous sections, in particular, $S_n = \log \Pi_n$ for $n \geq 0$ and $\eta_n = \log Q_n$ for $n \geq 1$. For an arbitrary $c > 0$, let $(\sigma_n)_{n\geq 0}$ be the integrable renewal stopping sequence with

$$\sigma = \sigma_1 := \inf\{n \geq 1 \colon S_n < -c\}.$$

Then

$$(M_n^*, Q_n^*) := \left(\frac{\Pi_{\sigma_n}}{\Pi_{\sigma_{n-1}}}, \sum_{k=\sigma_{n-1}+1}^{\sigma_n} \frac{\Pi_{k-1}}{\Pi_{\sigma_{n-1}}} Q_k\right), \qquad n \geq 1,$$

are independent copies of $(\Pi_\sigma, \sum_{k=1}^{\sigma} \Pi_{k-1} Q_k)$. Set

$$\Pi_0^* := 1 \quad \text{and} \quad \Pi_n^* := \prod_{k=1}^{n} M_k^* \quad \text{for } n \geq 1.$$

By Lemma 5.2,

$$\limsup_{t\to\infty} t\mathbb{P}(\log Q_1^* > t) \leq s^*\mathbb{E}\sigma.$$

As already pointed out in the introduction, the validity of (1.2), (1.3), and (1.5) implies that $(X_n)_{n\geq 0}$ cannot be positive recurrent. We will always assume that $X_0 = \widehat{X}_0 = 0$ hereafter. By Proposition 2.2, the null recurrence of $(X_n)_{n\geq 0}$ follows if we can show that

$$\sum_{n\geq 1} \mathbb{P}(X_n \leq t) = \sum_{n\geq 1} \mathbb{P}(\widehat{X}_n \leq t) = \infty \quad \text{for some } t > 0 \tag{6.1}$$

or, a fortiori,

$$\sum_{n\geq 1} \mathbb{P}(\widehat{X}_{\sigma_n} \leq t) = \infty. \tag{6.2}$$

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We note that $\widehat{X}_{\sigma_n} = \sum_{k=1}^{\sigma_n} \Pi_{k-1} Q_k = \sum_{k=1}^{n} \Pi_{k-1}^* Q_k^*$ and pick an arbitrary nondecreasing sequence $0 = a_0 \le a_1 \le \cdots$ such that

$$a := \sum_{n\ge 0} \mathrm{e}^{-a_n} < \infty.$$

Fix any $z > 0$ so large that

$$\mathbb{P}\left(Q_1^* \le \frac{z}{a}\right) > 0.$$

Using $M_n^* < 1$ for all $n \ge 1$, we then infer that

$$\mathbb{P}\left(\max_{1\le k\le n} \mathrm{e}^{a_{k-1}} \Pi_{k-1}^* Q_k^* \le \frac{t}{a}\right) \ge \mathbb{P}\left(\max_{1\le k\le n} \Pi_{k-1}^* Q_k^* \le \frac{t}{a}\right) \ge \mathbb{P}\left(Q_1^* \le \frac{t}{a}\right)^n > 0.$$

Furthermore,

$$\mathbb{P}(\widehat{X}_{\sigma_n} \le t) = \mathbb{P}\left(\sum_{k=1}^{n} \Pi_{k-1}^* Q_k^* \le t\right) \ge \mathbb{P}\left(\max_{1\le k\le n} \mathrm{e}^{a_{k-1}} \Pi_{k-1}^* Q_k^* \le \frac{t}{a}\right),$$

since $\max_{1\le k\le n} \mathrm{e}^{a_{k-1}} \Pi_{k-1}^* Q_k^* \le t/a$ implies that

$$\sum_{k=1}^{n} \Pi_{k-1}^* Q_k^* \le \frac{t}{a} \sum_{k=1}^{n} \mathrm{e}^{-a_{k-1}} \le t.$$

Consequently,

$$\sum_{n\ge 1} \mathbb{P}\left(\max_{1\le k\le n} \mathrm{e}^{a_{k-1}} \Pi_{k-1}^* Q_k^* \le \frac{t}{a}\right) = \infty \tag{6.3}$$

implies (6.2), and, thus, (6.1).

By the choice of σ_n, we have $\log \Pi_k^* \le -ck$ a.s. Setting $x = \log t - \log a$, we have, with $a_k = o(k)$ as $k \to \infty$ (choose, e.g. $a_k = 2\log(1+k)$),

$$\begin{aligned}\sum_{n\ge 1} \mathbb{P}\left(\max_{1\le k\le n} \mathrm{e}^{a_{k-1}} \Pi_{k-1}^* Q_k^* \le \frac{t}{a}\right) &\ge \sum_{n\ge 1} \mathbb{P}\Big(\max_{1\le k\le n} (-c(k-1) + a_{k-1} + \log Q_k^*) \le x\Big) \\ &= \sum_{n\ge 1} \prod_{k=1}^{n} \mathbb{P}(\log Q_1^* \le x - a_{k-1} + c(k-1)).\end{aligned}$$

Defining b_n as the nth summand in the previous sum and writing $\sigma = \sigma(c)$ to show the dependence on c, Lemma 5.2 provides us with

$$\liminf_{n\to\infty} n\left(\frac{b_{n+1}}{b_n} - 1\right) \ge -s^* \frac{\mathbb{E}\sigma(c)}{c}.$$

Hence, Raabe's test entails (6.3) if we can fix $c > 0$ such that

$$s^* \frac{\mathbb{E}\sigma(c)}{c} < 1. \tag{6.4}$$

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Plainly, the latter holds for any $c > 0$ if $s^* = 0$. But if $s^* \in (0, \infty)$ then use the elementary renewal theorem to infer that (also in the $\mathfrak{m} = -\infty$ case)

$$\lim_{c\to\infty} s^* \frac{\mathbb{E}\sigma(c)}{c} = \frac{s^*}{-\mathfrak{m}} < 1.$$

Hence, (6.4) follows by our assumption $s^* < -\mathfrak{m}$.

(ii) *Transience.* By Proposition 2.1, it must be shown that

$$\sum_{n\geq 0} \mathbb{P}(\widehat{X}_n \leq t) < \infty \quad \text{for any } t > 0.$$

We point out first that it suffices to show

$$\sum_{n\geq 0} \mathbb{P}(\widehat{X}_{\sigma_n} \leq t) < \infty \tag{6.5}$$

for some integrable renewal stopping sequence $(\sigma_n)_{n\geq 0}$. Namely, since $(\widehat{X}_n)_{n\geq 0}$ is nondecreasing, it follows that

$$\sum_{n\geq 0} \mathbb{P}(\widehat{X}_n \leq t) = \sum_{n\geq 0} \mathbb{E}\Bigg(\sum_{k=\sigma_n}^{\sigma_{n+1}-1} \mathbf{1}_{\{\widehat{X}_k \leq t\}} \Bigg) \leq \sum_{n\geq 0} \mathbb{E}(\sigma_{n+1} - \sigma_n)\, \mathbf{1}_{\{\widehat{X}_{\sigma_n} \leq t\}} = \mathbb{E}\sigma \sum_{n\geq 0} \mathbb{P}(\widehat{X}_{\sigma_n} \leq t),$$

where we have used the fact that $\sigma_{n+1} - \sigma_n$ is independent of $\widehat{X}_{\sigma_n}$ for each $n \geq 0$.

Choosing $(\sigma_n)_{n\geq 0} = (\sigma_n^{>}(c))_{n\geq 0}$ as defined before Lemma 4.3 for an arbitrary $c \in (-\mathfrak{m}, s_*)$, part (ii) of this lemma provides us with

$$\widehat{X}_{\sigma_n} \geq \widehat{Y}_n = \sum_{k=1}^{n} \gamma^{\sigma_{k-1}} Q_k^* \quad \text{a.s. for all } n \geq 0, \tag{6.6}$$

where the Q_n^* are formally defined as in (i) for the σ_n given here and the $\widehat{Y}_n$ are the backward iterations of the Markov chain defined by the RDE

$$Y_n = \gamma^{\sigma_n - \sigma_{n-1}} Y_{n-1} + Q_n^*, \qquad n \geq 1.$$

By Lemma 5.2,

$$\liminf_{t\to\infty} t\mathbb{P}(\log Q^* > t) \geq s_* \mathbb{E}\sigma.$$

Let $(Q_n')_{n\geq 1}$ be a further sequence of i.i.d. random variables with generic copy Q', independent of all other occurring random variables and such that

$$\lim_{t\to\infty} t\mathbb{P}(\log Q' > t) =: s \in (c, s_*). \tag{6.7}$$

Set $\gamma := \mathrm{e}^{-c}$. Then Kellerer's result (Proposition 4.1) implies the transience of the Markov chain $X_n'(\gamma) = \gamma X_{n-1}'(\gamma) + Q_n'$, $n \geq 1$, and, thus, also of the subchain $(X_{\sigma_n}'(\gamma))_{n\geq 0}$. Observe that $\widehat{X}_{\sigma_n}'(\gamma) = \sum_{k=1}^{n} \gamma^{\sigma_{k-1}} \widehat{Q}_k$ with

$$\widehat{Q}_n = \sum_{k=\sigma_{n-1}+1}^{\sigma_n} \gamma^{k-\sigma_{n-1}-1} Q_k' \quad \text{for } n \geq 1$$

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and that, by (6.7) and Lemma 5.1,

$$\lim_{t\to\infty} t\mathbb{P}(\log \widehat{Q} > t) = s\mathbb{E}\sigma,$$

thus, $\mathbb{P}(Q^* > t) \geq \mathbb{P}(\widehat{Q} > t)$ for all sufficiently large t. We now infer by invoking our comparison Lemma 4.1 that the transience of $(X'_{\sigma_n}(\gamma))_{n\geq 0}$ entails the transience of $(Y_n)_{n\geq 0}$ given above and, thus,

$$\sum_{n\geq 0} \mathbb{P}(\widehat{Y}_n \leq t) < \infty \quad \text{for all } t > 0.$$

Finally, use (6.6) to arrive at (6.5). This completes the proof of part (ii). □

Proof of Theorem 3.2. Fix $c > s^*$ and set, as before, $\gamma = \mathrm{e}^{-c}$. Since $S_n = \log \Pi_n \to -\infty$ a.s. and $\mathfrak{m}^+ = \mathfrak{m}^- = \infty$, we have

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{S_n + an}{n} = -\infty \quad \text{a.s. for all } a \in \mathbb{R}$$

owing to Kesten's trichotomy (see, e.g. [18, p. 3]) and, hence, in particular, $S_n + cn \to -\infty$ a.s. As a consequence, the sequence $(\sigma_n)_{n\geq 0} = (\sigma_n^<(c))_{n\geq 0}$ as defined before Lemma 4.3 is an integrable renewal stopping sequence for $(X_n)_{n\geq 0}$. Part (i) of this lemma implies that

$$X_{\sigma_n} \leq X_{\sigma_n}(\gamma) = \gamma^{\sigma_n - \sigma_{n-1}} X_{\sigma_{n-1}}(\gamma) + Q_n(\gamma) \quad \text{a.s. for all } n \geq 0,$$

where $Q_n(\gamma) = \sum_{k=\sigma_{n-1}+1}^{\sigma_n} \gamma^{\sigma_n - k} Q_k$ for $n \geq 1$. Hence, it is enough to prove the null recurrence of $(X_{\sigma_n}(\gamma))_{n\geq 0}$. To this end, note first that $\mathfrak{m}(\gamma) := \mathbb{E}\log \gamma^{\sigma_1} = -c\,\mathbb{E}\sigma_1 \in (-\infty, 0)$. Moreover, Lemma 5.3 provides us with

$$\limsup_{t\to\infty} t\mathbb{P}(\log Q_1(\gamma) > t) \leq s^*\mathbb{E}\sigma_1 < c\mathbb{E}\sigma_1 = -\mathfrak{m}(\gamma),$$

and so the null recurrence of $(X_{\sigma_n}(\gamma))_{n\geq 0}$ follows from Theorem 3.1. □

7. On the structure of the attractor set

The purpose of this section is to investigate the structure of the attractor set L for the Markov chain $(X_n)_{n\geq 0}$ defined by (1.1). Unlike before, we assume hereafter that $(X_n)_{n\geq 0}$ is locally contractive and recurrent, the latter being an inevitable assumption for $L \neq \emptyset$. To exclude the 'trivial case' (as explained in the introduction) we assume that $\mathbb{P}(M = 0) = 0$. Recall from the paragraph preceding Proposition 2.2 that L consists of all accumulation points of $(X_n^x(\omega))_{n\geq 0}$ which turns out to be the same for all $x \in \mathbb{R}_+$ and $\mathbb{P}$-almost all ω. As already mentioned in the introduction, $(X_n)_{n\geq 0}$ possesses a unique invariant distribution, say ν, if (1.3) and (1.4) hold. The attractor set then coincides with the support of ν. In the positive recurrent case, the structure of L was analyzed in [8]. According to Theorem 2.5.5 of [8], L necessarily equals a half-line $[a, \infty)$ for some $a \geq 0$ if it is unbounded. If L is bounded, no general results concerning local properties of L are known. It may equally well be a fractal (for instance, a Cantor set) or an interval. Below we consider both the positive and null recurrent cases. The latter case is implied by hypotheses of Theorem 3.1(i), but also holds when $\mathbb{E}\log M = 0$ (see [1] and [3] for more details).

For $(m, q) \in \mathbb{R}_+^2$, let g be the affine transformation of $\mathbb{R}$ defined by

$$g(x) = mx + q, \qquad x \in \mathbb{R}.$$

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We will write $g = (m, q)$, thereby identifying g with (m, q). The affine transformations constitute a group $\mathrm{Aff}(\mathbb{R})$ with identity $(1, 0)$ and multiplication defined by

$$g_1 g_2 = (m_1, q_1)\,(m_2, q_2) = (m_1 m_2, q_1 + m_1 q_2)$$

for $g_i = (m_i, q_i)$, $i = 1, 2$. The inverse of $g = (m, q)$ is given by $g^{-1} = (m^{-1}, -m^{-1}q)$.

Assuming $m \neq 1$, let $x_0 = x_0(g) = q/(1 - m)$ be the unique fixed point of g, that is, the unique solution to the equation $g(x) = x$. Then

$$g(x) = m(x - x_0) + x_0, \qquad x \in \mathbb{R}$$

and, similarly,

$$g^n(x) = m^n x + q_n = m^n(x - x_0) + x_0, \qquad x \in \mathbb{R},\ n \geq 1, \tag{7.1}$$

where $q_n = \sum_{i=0}^{n-1} m^i q$. Equation (7.1) tells us that, modulo x_0, the action of g is either contractive or expanding depending on whether $m < 1$ or $m > 1$, respectively.

We interpret μ, the distribution of (M, Q), as a probability measure on $\mathrm{Aff}(\mathbb{R})$ hereafter and let $\operatorname{supp}\mu$ denote its support. Consider the subsemigroup T of $\mathrm{Aff}(\mathbb{R})$ generated by $\operatorname{supp}\mu$, i.e.

$$T := \{g_1 \cdots g_n \colon g_i \in \operatorname{supp}\mu,\ i = 1, \ldots, n, n \geq 1\},$$

and let $\overline{T}$ be its closure. A set $S \subset \mathbb{R}$ is said to be $\overline{T}$*-invariant* if, for every $g \in \overline{T}$ and $x \in S$, $g(x) = mx + q \in S$. The following result was stated in a slightly different setting as Proposition 2.5.3 of [8] and can be proved by the same arguments after minor changes.

Lemma 7.1. *Let $(X_n)_{n\geq 0}$ be locally contractive and recurrent. Then $L = \overline{S}_0$, where*

$$S_0 := \{(1 - m)^{-1}q \colon g = (m, q) \in T,\ m < 1\}.$$

Moreover, L is equal to the smallest $\overline{T}$-invariant subset of $\mathbb{R}$.

For positive recurrent $(X_n)_{n\geq 0}$, we have already pointed out that L, if unbounded, must be a half-line $[a, \infty)$ for some $a \geq 0$. The subsequent theorem provides the extension of this fact to any locally contractive and recurrent $(X_n)_{n\geq 0}$.

Theorem 7.1. *Let $(X_n)_{n\geq 0}$ be locally contractive and recurrent with unbounded attractor set L. If $\mathbb{P}(M = 0) = 0$ then $L = [a, \infty)$ for some $a \geq 0$.*

Proof. By Lemma 7.1, the set L is uniquely determined by $\operatorname{supp}\mu$ and does not depend on the values $\mu(A)$ for any particular sets A. Consequently, any modification of μ with the same support leaves L invariant. We will use this observation and define a tilting $\widetilde{\mu}$ of μ of the form

$$\widetilde{\mu}(\mathrm{d}m, \mathrm{d}q) = f(m)h(q)\mu(\mathrm{d}m, \mathrm{d}q)$$

for suitable positive functions f, h such that, if (M, Q) has law $\widetilde{\mu}$ then the corresponding Markov chain $(\widetilde{X}_n)_{n\geq 0}$ is positive recurrent with unique invariant distribution $\widetilde{\nu}$. We thus conclude $\operatorname{supp}\widetilde{\nu} = L$ and, thereupon, the claim $L = [a, \infty)$ if L is unbounded.

Set

$$f(m) := \begin{cases} \dfrac{c_0}{|\log m|} & \text{if } 0 < m < 1/\mathrm{e}, \\ c_0 & \text{if } 1/\mathrm{e} \leq m < 1, \\ c_0 c_1 & \text{if } 1 \leq m < \mathrm{e}, \\ \dfrac{c_0 c_1}{\log m} & \text{if } m \geq \mathrm{e}, \end{cases} \quad \text{and} \quad h(q) := \begin{cases} c_2 & \text{if } 0 \leq q < \mathrm{e}, \\ \dfrac{c_2}{\log q} & \text{if } q \geq \mathrm{e}, \end{cases}$$

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and fix $c_0, c_1, c_2 > 0$ such that

$$\int f(m)h(q)\mu(\mathrm{d}m, \mathrm{d}q) = 1.$$

Observe that, if $\mathbb{P}_{\widetilde{\mu}}$ is such that (M, Q) has law $\widetilde{\mu}$ under this probability measure, then

$$\mathbb{E}_{\widetilde{\mu}} \log M = c_0 \Bigg[- \int_{(0,1]\times\mathbb{R}_+} (1 \wedge |\log m|) h(q)\mu(\mathrm{d}m, \mathrm{d}q) + c_1 \int_{(1,\infty)\times\mathbb{R}_+} (1 \wedge \log m) h(q)\mu(\mathrm{d}m, \mathrm{d}q) \Bigg],$$

and from this it is readily seen that we can specify c_0, c_1 further so as to have

$$\mathbb{E}_{\widetilde{\mu}} \log M < 0.$$

Regarding $\mathbb{E}_{\widetilde{\mu}} \log^+ Q$, we find that

$$\mathbb{E}_{\widetilde{\mu}} \log^+ Q = c_2 \Bigg[\int_{\mathbb{R}_+\times[1,e]} \log q f(m)\mu(\mathrm{d}m, \mathrm{d}q) + \int_{\mathbb{R}_+\times(e,\infty)} f(m)\mu(\mathrm{d}m, \mathrm{d}q) \Bigg] < \infty.$$

Hence, if (M, Q) has law $\widetilde{\mu}$ then the corresponding Markov chain $(\widetilde{X}_n)_{n\geq 0}$ defined by (1.1) is indeed positive recurrent. This completes the proof of the theorem. □

In the next lemma we provide some conditions on μ that are easily checked and sufficient for L to be unbounded.

Lemma 7.2. *If $\mathbb{P}(M = 0) = 0$ then each of the following conditions on the law of (M, Q) implies that L is unbounded:*

(C1) *the law of Q has unbounded support;*

(C2) $\mathbb{P}(M > 1) > 0$ *and* $\mathbb{P}(Q = 0) < 1$.

Proof. Assume first (C1), put $\beta := \sup\{x : x \in L\}$ and recall from Lemma 7.1 that L is invariant under the action of $\operatorname{supp}\mu$, i.e. if $(m, q) \in \operatorname{supp}\mu$ and $x \in L$ then $mx + q \in L$. In particular,

$$m\beta + q \leq \beta \quad \text{for any } (m, q) \in \operatorname{supp}\mu.$$

Hence, if $\beta > 0$, we have $\beta \geq m\beta + q \geq q$ and conclude that $\beta = \infty$, because q can be chosen arbitrarily large.

Assuming now (C2), pick $g = (m, q) \in \operatorname{supp}\mu$ such that $m > 1$. Note that $x = x_0(g) = q/(1 - m)$, the unique fixed point of g, is negative or zero because $m > 1$. Since, under our hypothesis, the attractor set consists of at least two points, one can choose some positive $y \in L$. Using (7.1), we then infer that

$$g^n(y) = m^n(y - x) + x \to \infty \quad \text{as } n \to \infty,$$

which completes the proof. □

The assumptions of Lemma 7.2 are not optimal. Even if $\mathbb{P}(M < 1) = 1$ and the support of the distribution of Q is bounded, the attractor set may be unbounded, as demonstrated by the next lemma.

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Lemma 7.3. *Assume that* $\mathbb{P}(M < 1) = 1$. *Then the attractor set* L *is bounded if and only if the set*

$$S_1 = \{x_0 = x_0(g) \colon g \in \operatorname{supp} \mu\}$$

is bounded or, equivalently, $Q/(1 - M)$ *is a.s. bounded.*

Proof. Assuming that S_1 is bounded, denote by a and b its infimum and supremum, respectively. Since the closed interval $[a, b]$ is obviously $\overline{T}$-invariant, it must contain L by Lemma 7.1, which implies that L is bounded.

If S_1 is unbounded then $S_1 \subset S_0$ implies that S_0 and, thus, also $L = \overline{S}_0$ is unbounded by another appeal to Lemma 7.1. □

Finally, we turn to the case when the attractor set L is bounded. As already mentioned, the local structure of L cannot generally be described precisely. If μ is supported by $(a, 0)$ and $(a, 1 - a)$ for some $0 < a < \frac{1}{2}$, then $L \subset [0, 1]$ is equal to the Cantor set obtained by initially removing $(a, 1 - a)$ from $[0, 1]$ and successive self-similar repetitions of this action for the remaining intervals (see also [9, Remark 7]). So the Cantor ternary set is obtained if $a = \frac{1}{3}$. On the other hand, we have the following result.

Lemma 7.4. *For* $\alpha, \beta < 1$ *with* $\alpha + \beta \geq 1$, *suppose that* $(\alpha, q_\alpha), (\beta, q_\beta) \in \operatorname{supp} \mu$ *and, further,* $x_\alpha := q_\alpha/(1 - \alpha) \leq q_\beta/(1 - \beta) =: x_\beta$. *Then the interval* $[x_\alpha, x_\beta]$ *is contained in* L.

Proof. Without loss of generality, we assume that $x_\alpha = 0$ and $x_\beta = 1$ so that the points in $\operatorname{supp} \mu$ are $f_\alpha := (\alpha, 0)$ and $f_\beta := (\beta, 1 - \beta)$ rather than (α, q_α), (β, q_β) and $[0, 1] \subset L$ must be verified.

Pick any $x \in (0, 1)$. Let U be the subsemigroup of $\operatorname{Aff}(\mathbb{R})$ generated by f_α and f_β. To prove that $x \in L$, it is sufficient by Lemma 7.1 to find a sequence $(g_n)_{n \geq 1}$ in U such that x is an accumulation point of $(g_n(0))_{n \geq 1}$.

We construct this sequence inductively. Observe first that $\alpha + \beta \geq 1$ implies

$$x \in (0, 1) \subset [0, \alpha] \cup [1 - \beta, 1].$$

If x is an element of $[0, \alpha]$, take $g_1 = f_\alpha$, otherwise take $g_1 = f_\beta$. In both cases,

$$x \in [g_1(0), g_1(1)] \quad \text{and} \quad |g_1(1) - g_1(0)| \leq \alpha \vee \beta.$$

Assume that we have found $g_n = (a_n, b_n)$ such that

$$x \in [g_n(0), g_n(1)] \quad \text{and} \quad |g_n(1) - g_n(0)| = a_n \leq (\alpha \vee \beta)^n.$$

Using again $\alpha + \beta \geq 1$, we have

$$\begin{aligned} x \in [g_n(0), g_n(1)] = [b_n, a_n + b_n] &\subset [b_n, \alpha a_n + b_n] \cup [(1 - \beta)a_n + b_n, a_n + b_n] \\ &= [g_n f_\alpha(0), g_n f_\alpha(1)] \cup [g_n f_\beta(0), g_n f_\beta(1)]. \end{aligned}$$

Thus, x must belong to one of these intervals. If $x \in [g_n f_\alpha(0), g_n f_\alpha(1)]$, set $g_{n+1} = g_n f_\alpha$, otherwise, set $g_{n+1} = g_n f_\beta$. In both cases,

$$x \in [g_{n+1}(0), g_{n+1}(1)] \quad \text{and} \quad |g_{n+1}(1) - g_{n+1}(0)| = a_n(\alpha \vee \beta) \leq (\alpha \vee \beta)^{n+1}.$$

Hence, x is indeed an accumulation point of the sequence $(g_n(0))_{n \geq 1}$ and, therefore, an element of L. □

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