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Unsolved graph colouring problems

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Our book Graph Coloring Problems [85] *appeared in 1995. It contains descriptions of unsolved problems, organized into sixteen chapters. A large number of publications on graph colouring have appeared since then, and in particular around thirty of the* 211 *problems in that book have been solved. In this chapter we review some of our favourite problems that remain unsolved.*

1. Introduction

A main reason for the continued interest in the area of graph colouring is its wealth of interesting unsolved problems. Many of these are easy to state, but seemingly difficult to solve. However they are not impossible, as the literature in the field will testify.

The seven most striking results of the past twenty years are:

- the 5-list-colourability of planar graphs (dating back to V. G. Vizing in 1975 and to P. Erdős, A. L. Rubin and H. Taylor in 1979) by Thomassen [159]
- the confirmation by Robertson, Sanders, Seymour and Thomas [137] of the truth of the four-colour theorem (F. Guthrie and A. De Morgan (1852))
- the asymptotic solution by Reed [134] of the problem as to whether for $k \geq 9$ there are k-chromatic graphs without complete k-graphs and of maximum degree k (V. G. Vizing (1968) and O. V. Borodin and A. V. Kostochka (1977))

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- the proof by Chudnovsky, Robertson, Seymour and Thomas [39] of the strong perfect graph conjecture of C. Berge around 1960
- the proof by Thomassen [161] of the weak 3-flow conjecture of W. T. Tutte (1954) and F. Jaeger (1988)
- the solution by Kostochka and Yancey [111] to the problem of critical graphs with few edges (due to T. Gallai (1963) and O. Ore (1967))
- the description found by Borodin, Dvořák, Kostochka, Lidický and Yancey [24] of all 4-critical planar graphs with exactly four triangles (B. Grünbaum (1963), V. A. Aksenov (1974) and P. Erdős (1990)).

In addition to these major achievements there are many other important results – in fact, thirty-one of the original 211 problems from the lists in Jensen and Toft [85] were solved by 2013. The area continues to surprise, and if we had been asked in 1995 which problems would be solved first our guesses would have been totally wrong. Although graph colouring has its origins in the four-colour problem, the latter's solution does not diminish the interest. To cite W. T. Tutte from his 1978 paper [166], where he describes several difficult colouring problems:

The Four Colour Theorem is the tip of the iceberg, the thin edge of the wedge and the first cuckoo of spring.

The subsequent advances bear witness to this.

2. Complete graphs and chromatic numbers

Hadwiger's conjecture

As we saw in Chapter 4, the following conjecture was proposed by Hadwiger [74] in 1943.

Every k-chromatic graph can be transformed into a complete k-graph by successive contractions of edges and deletions of edges and vertices.

If a graph H can be obtained from a graph G by successive contractions and deletions, then H is a *minor* of G. Hadwiger's conjecture thus suggests that any graph with no complete k-graph as a minor is $(k-1)$-colourable. Hadwiger's idea was to create a new combinatorial classification of graphs, different from the classical topological classification via embeddability in surfaces.

A more striking formulation of Hadwiger's conjecture is this, where a class of graphs is *proper* if it is not the class of all graphs:

For any proper minor-closed class of graphs the maximum chromatic number of graphs in the class equals the number of vertices in the largest complete graph belonging to the class.

In particular, this suggests that all planar graphs are 4-colourable, because the complete graph K_4 is the largest complete planar graph. Thus Hadwiger's conjecture

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may be seen as a far-reaching generalization of the four-colour theorem. In 1937 Wagner [170] proved that the 4-colourability of all planar graphs can be extended to the 4-colourability of all graphs without K_5 as a minor, thereby proving that the case $k = 5$ of Hadwiger's conjecture is equivalent to the four-colour theorem.

For $k = 4$, Hadwiger's conjecture was proved by Hadwiger in his 1943 paper, and independently by Dirac [43]. By a deep theorem of Robertson, Seymour and Thomas [138], the case $k = 6$ is also equivalent to the four-colour theorem. Kawarabayashi and Toft [96] proved that a 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, and Kawarabayashi [90] subsequently proved the corresponding statement with $K_{4,4}$ replaced by $K_{3,5}$.

Perhaps the most interesting special unsolved case of Hadwiger's conjecture arises by assuming that G is a graph with independence number α at most 2, so G contains no three mutually non-adjacent vertices. In 1982 Duchet and Meyniel [46] observed in general that, if G is a graph with n vertices, then G contains as a minor a complete $\lceil n/(2\alpha - 1)\rceil$-graph. In particular, if $\alpha \leq 2$, then G contains a complete $\lceil \frac{1}{3}n\rceil$-graph as a minor.

The importance of this special case was first pointed out by W. Mader who noticed that, whereas the truth of Hadwiger's conjecture would imply that if $\alpha \leq 2$, then G contains a complete $\lceil \frac{1}{2}n\rceil$-graph as a minor, it seems very difficult to improve on the lower bound of $\lceil \frac{1}{3}n\rceil$ for the order of a largest complete minor. P. D. Seymour has asked for a proof of the existence of a number $\varepsilon > 0$ such that, if $\alpha \leq 2$, then G contains a complete $\lceil (\frac{1}{3} + \varepsilon)n\rceil$-graph as a minor. As far as we know, this problem is still open. Partial results have been obtained by Plummer, Stiebitz and Toft [132] and Pedersen and Toft [130]. Duchet and Meyniel's original bound of $\lceil n/(2\alpha - 1)\rceil$ was improved for $\alpha > 2$ by Kawarabayashi, Plummer and Toft [92] to $\lceil n/(2\alpha - \frac{3}{2})\rceil$, and further by Kawarabayashi and Song [94] to $\lceil n/(2\alpha - 2)\rceil$.

In an interesting paper, Kawarabayashi and Reed [93] showed that, for any fixed number $k \geq 2$, there exist a constant N and an $O(n^2)$ algorithm which, for any input graph G with at most n vertices, produces one of the following:

- a colouring of G with $k - 1$ colours
- a K_k-minor of G
- a minor H of G with at most N vertices, for which $\chi(H) \geq k$ and H has no K_k-minor.

In particular it follows that, for any fixed k, the corresponding case of Hadwiger's conjecture is a finite problem, in the sense that it is sufficient to check the finite set of graphs with order at most N to find or rule out a counter-example. However, for the case $k = 5$ this does not imply a positive answer to the finiteness part of the four-colour problem (see Section 3), because Kawarabayashi and Reed [93] used the four-colour theorem to prove the existence of the constant N above.

Borowiecki [29] asked about the extent to which Hadwiger's conjecture may be generalized to list-colourings. For $k \leq 4$, every graph with list-chromatic number k has K_k as a minor. Since there are planar graphs that are not list-4-colourable, as

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shown by Voigt [169], the same correspondence does not hold for $k > 4$. In general, Barát, Joret and Wood [13] proved that, for each $t \geq 1$, there exists a list-$(4t + 1)$-chromatic graph with no complete minor larger than K_{3t+1}.

The Erdős–Faber–Lovász problem

In 1972 Erdős, Faber and Lovász posed the following problem and conjectured that the answer is affirmative.

Let G be the union of k complete graphs, each with at most k vertices, and suppose that the intersection of any two of them is at most a single vertex. Can G be coloured with k colours?

It is surprising that this innocent sounding problem is still open. Some advances have been described by Romero and Sánchez-Arroyo [141].

A reformulation of the conjecture was studied by Klein and Margraf [103], who observed that any graph G is the line graph of a linear simple hypergraph H, where *linear* means that any two edges have at most one vertex in common, and *simple* means that no hyperedge is a multiple edge. The smallest value of $|V(H)|$ for such an H is called the *linear intersection number* $li(G)$ of G. They conjectured that $\chi(G) \leq li(G)$ holds for all graphs G, and proved that this is equivalent to the Erdős–Faber–Lovász conjecture. Moreover, Klein and Margraf proved for any graph G that their conjecture holds for at least one of G and $\overline{G}$, where $\overline{G}$ is the complement of G.

Another more direct and well-known equivalent version of the Erdős–Faber–Lovász conjecture is the statement that, for any simple linear hypergraph H, the chromatic index $\chi'(H)$ is bounded by the number of vertices in H.

Gyárfás's forbidden subgraph conjecture

Gyárfás [71] and Sumner [158] made the following conjecture. In its statement, the parameter $\omega(G)$ denotes the clique-number of G, the number of vertices in a largest complete subgraph of G.

For any forest F, there exists a function $f_F : \mathbb{N} \to \mathbb{N}$, such that $\chi(G) \leq f_F(\omega(G))$ if G does not contain F as an induced subgraph.

A family $\mathcal{G}$ of graphs is called *χ-bound* if there exists a function f for which

$$\chi(G) \leq f(\omega(G)), \quad \text{for all } G \in \mathcal{G}.$$

If $\mathcal{G}(H)$ denotes the family of all graphs that do not contain the graph H as an induced subgraph, then the above conjecture suggests that $\mathcal{G}(H)$ is χ-bound when H is a forest.

Erdős [48] proved that, for all $g > 0$ and all $k > 0$, there exist graphs of chromatic number k without cycles of length less than g. So, for a family $\mathcal{G}(H)$ to be χ-bound, the graph H has to be a forest.

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For a forest F, Kierstead and Penrice [99] and Sauer [144] proved that the family $\mathcal{G}(F)$ is χ-bound if and only if $\mathcal{G}(T)$ is χ-bound for each connected component T of F. The conjecture was proved for stars, paths and brooms by Gyárfás [72], where a *broom* (also called a *comet*) is obtained by identifying an end-vertex of a path with a vertex of a star, and for trees of radius 2 by Kierstead and Penrice [100].

We may reformulate the Gyárfás–Sumner conjecture as follows.

The graphs G in the family of graphs $\mathcal{G}(T, K)$ that do not contain a fixed tree T or a fixed complete graph K as induced subgraphs have bounded chromatic number.

A set of forbidden subgraphs $\{T, K\}$ with this property is called *heroic*. Chudnovsky and Seymour [40] characterized all finite heroic sets of graphs, provided that the Gyárfás–Sumner conjecture is true. Their characterization states that a set of graphs is heroic if and only if it contains a forest, the complement of a forest (such as a complete graph), a complete multipartite graph or the complement of a complete multipartite graph (also called a *clique partition graph*). They also considered a corresponding problem for tournaments, where a heroic set may consist of just one graph, and where a colouring of a tournament has colour classes consisting of transitive subtournaments.

The classes of graphs $\mathcal{G}(P_n)$, where P_n is a path on n vertices, have been investigated with respect to their colouring complexity by Broersma, Golovach, Paulusma and Song [33]. The existence of a polynomial algorithm for k-colouring graphs in $\mathcal{G}(P_n)$ seems unknown for $k = 3$ and $n \geq 7$, for $k = 4$ and $n = 6$ and 7, for $k = 5$ and $n = 6$ and 7, and for $k \geq 6$ and $n = 6$.

Double-critical graphs

The following question arises as a special case of the Erdős–Lovász Tihany conjecture from 1968 (see [85]).

Let G be a connected graph in which the removal of any two adjacent vertices together with all their incident edges results in a $(\chi(G) - 2)$-colourable graph. Must G be a complete graph?

A connected graph G for which $\chi(G - v - w) \leq \chi(G) - 2$ for any edge $vw \in E(G)$ is called *double-critical*.

The Erdős–Lovász Tihany conjecture is the following statement.

Let $\omega(G) < \chi(G) = s + t - 1$ $(s, t \geq 2)$. Then G contains two disjoint subgraphs G_1 and G_2 for which $\chi(G_1) \geq s$ and $\chi(G_2) \geq t$.

The case $s = 2$ is the question for double-critical graphs.

The Erdős–Lovász Tihany conjecture has been proved for quasi-line graphs (where the neighbourhood of each vertex is the complement of a bipartite graph), and also for graphs of independence number 2, by Balogh, Kostochka, Prince and Stiebitz [12]. Another relaxation of the conjecture has been obtained by Stiebitz [155], replacing

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the chromatic number by the so-called colouring number. The precise statement of the result is: let $\omega(G) < \chi(G) = s + t - 1$ $(s, t \geq 2)$. Then G contains two disjoint subgraphs G_1 and G_2 for which either $\chi(G_1) \geq s$ and $\mathrm{col}(G_2) \geq t$, or $\mathrm{col}(G_1) \geq s$ and $\chi(G_2) \geq t$.

The double-critical graph problem is open for $k \geq 6$. Instead of requiring that G be complete, one might relax the condition to G having a complete minor of the desired size. The cases $k = 6, 7$ and 8 have been obtained for this relaxation (see [91] and [4]).

An interesting equivalent version of the double-critical graph question was formulated by Stehlik [152] in terms of hypergraphs with the *Helly property* – that is, whenever some hyperedges intersect pairwise, then they all have a non-empty intersection. The *covering number* of a hypergraph is the smallest size of a set of vertices that intersects every hyperedge.

Let H be any hypergraph with the Helly property. If the covering number decreases by 2 whenever two disjoint hyperedges are removed from H, is it then true that H consists entirely of disjoint hyperedges?

Adaptive colouring

Let f be any integer-valued function defined on the edge-set of a graph G. An (improper) colouring of the vertices of G is called *adaptive to* f if, for every edge vw of G, the colour assigned to v or w differs from $f(vw)$. The *adaptive chromatic number* $\chi_{\mathrm{ad}}(G)$ is the least number k for which G allows, for each such function f, a colouring adaptive to f and using colours $1, 2, \ldots, k$. The following questions were posed by Hell and Zhu [78].

What is the value of the adaptive chromatic number $\chi_{\mathrm{ad}}(K_n)$ of the complete graph K_n with n vertices?

Is it true that K_n has the smallest adaptive chromatic number among all graphs with chromatic number n?

Is there a function F such that $\chi(G) \leq F(\chi_{\mathrm{ad}}(G))$ holds for all graphs G?

It is clear that $\chi_{\mathrm{ad}}(G) \leq \chi(G)$ is true for every graph, since a usual k-colouring is adaptive to all functions f as above. Hell and Zhu [78] proved a few other additional close ties and similarities between the two parameters:

- If G is connected, then $\chi_{\mathrm{ad}}(G) = 2$ if and only if $G - e$ is bipartite for some edge e of G.
- For any numbers $k, g \geq 3$, there exists a graph G without cycles of length less than g for which $\chi_{\mathrm{ad}}(G) = \chi(G) = k$.
- For each $k \geq 3$, it is an NP-hard problem to decide, with a graph G as the input, whether $\chi_{\mathrm{ad}}(G) \leq k$.

However, in contrast to Brooks's theorem [34] (see Chapter 2), the maximum adaptive chromatic number grows no faster than the square-root of the maximum

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degree. Hell and Zhu proved that $\chi_{\mathrm{ad}}(G) \leq \lceil \sqrt{e(2\Delta - 1)} \rceil$, where Δ is the maximum degree of G. This bound was improved by Molloy and Thron [125] to the best-possible $\chi_{\mathrm{ad}}(G) \leq (1 + \varepsilon)\sqrt{\Delta}$, for every $\varepsilon > 0$. In particular, this implies an upper bound $\chi_{\mathrm{ad}}(K_n) \leq \sqrt{n}$ for the complete graph K_n. A similar upper bound for $\chi_{\mathrm{ad}}(K_n)$ also follows from earlier results of Gyárfás and Erdős [73]. Robert Šámal and Alexander Kostochka independently proved the lower bound $\chi_{\mathrm{ad}}(K_n) \geq \sqrt{n/\ln n}$ for large enough n.

In an informative paper, Hell, Pan, Wong and Zhu [77] proved that the four-fold direct (or 'Cartesian') product K_n^4 of the complete graph K_n has adaptive chromatic number n. This again is in contrast to the behaviour of the chromatic number, which never increases when we form direct products (see also Problem 11.1 of [85]). They conjectured that in general, for any graph G, there exists a number d for which $\chi_{\mathrm{ad}}(G^d) = \chi(G)$.

If G is planar, then $\chi_{\mathrm{ad}}(G) \leq 4$; this follows from the four-colour theorem. It was also proved in a list-colouring version, without using the four-colour theorem, by Esperet, Montassier and Zhu [51], thereby answering a further question raised by Hell and Zhu [78].

The list-colouring version of the third question above was answered affirmatively by Molloy and Thron [124]. Adaptive list-colourings have also been studied by Kostochka and Zhu [112] and by Montassier, Raspaud and Zhu [126].

3. Graphs on surfaces

The four-colour problem

Interesting accounts have continued to appear with new views on different aspects of the four-colour problem, such as the books by Fritsch and Fritsch [58] and Wilson [172]. The following questions remain.

Is there a short proof of the four-colour theorem (that every planar graph is 4-colourable) which can be checked in detail by hand by a competent mathematician in at most two weeks?

Is there a short argument, independent of known proofs of the four-colour theorem, to demonstrate that the four-colour problem is a finite problem, in the sense that there exists a number N for which a smallest counter-example must have at most N vertices?

Is there a proof of the four-colour theorem, perhaps neither short nor easily checkable, which is not based on the method of finding an unavoidable set of reducible configurations?

Francis Guthrie's famous four-colour problem has a letter from Augustus De Morgan to W. R. Hamilton in 1852 as its earliest written source. It became the four-colour theorem 124 years later, when a computer-aided proof was obtained by Appel and Haken [10] and Appel, Haken and Koch [11]. One might have thought that

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the story would end here. But new and sometimes surprising facts are continually being discovered surrounding the early history, and also concerning the possibilities of finding proofs that are simpler according to specific criteria.

Recently McKay [120] discovered a printed publication of the four-colour problem which was earlier than the 1860 anonymous book review by De Morgan. A letter signed only by the initials 'F.G.' appeared in the *Athenæum* already in 1854 (see [60]), and read as follows:

Tinting maps. – In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two conterminous divisions ought to be tinted the same. Now, I have found by experience that four colours are necessary and sufficient for this purpose, – but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F.G.

The identity of the author is not known with certainty. Authors whose interests touched on the map-colour problem included Francis Guthrie's brother Frederick, and an illustrious explorer and mapmaker Francis Galton, both of whom shared the same initials. It appears likely that Francis or Frederick Guthrie was actually the author of the note.

In September 2004 Gonthier [67] and B. Werner completed a proof of the four-colour theorem, which they adapted from the proof of Robertson, Seymour, Sanders and Thomas [137]. It was written in a sufficiently basic formalization to make it checkable by the general mathematical theorem-proving software *Coq*, which had been developed during the 1980s and 1990s at several institutions in France. They reported that *Coq* was able to verify the correctness of this proof. (See also the article by Knight [105].)

The proof given by Robertson, Seymour, Sanders and Thomas is simpler than the original one by Appel and Haken [10], not only by exhibiting a much smaller unavoidable set of reducible configurations, but also by avoiding the use of 'ring-reduction' arguments first introduced by Bernhart [18]. A qualitatively even simpler proof was described by Steinberger [154] in 2010. Although Steinberger's proof involves a much larger unavoidable set of 2822 configurations, it avoids the use of more complicated *C*-reducibility techniques and applies only the simpler *D*-reducibility method; the existence of such a four-colour proof relying entirely on *D*-reducibility was first conjectured in 1975 by Stromquist [157] in his Ph.D. thesis.

The finiteness question for the four-colour problem may have been considered first by H. Heesch, who wrote a letter to G. Ringel in 1953 explaining the basic idea of step-wise extending an unavoidable set of configurations into a large enough finite such set, all members of which were reducible. According to Bigalke [19], Heesch told Ringel that it had occurred to him in 1935-36 that a proof of the finiteness of the four-colour problem might be obtained in this way. The idea, known as the *finitization of the four-colour problem* (see Wilson [172]), led Heesch in 1970 to

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propose a particular unavoidable set of 8900 configurations, so that a computational confirmation of the conjectured reducibility of all of them would produce a proof of the four-colour problem, after a large (but finite) amount of effort. The question about an upper bound N on the order of a smallest 5-chromatic counter-example differs from Heesch's question in the respect that, even if the existence of N were known and hence the proof of the four-colour theorem were reduced to a finite task, this does not immediately imply the truth of the four-colour theorem. (Compare this with the discussion in the previous section of the finiteness of Hadwiger's conjecture for graphs with a fixed chromatic number.)

Each proof of the four-colour theorem described above is based on an idea, implicitly used back in 1879 by A. B. Kempe. This is to establish by Euler's formula that at least one element of a collection of possible constituents is present in any given graph or map, and then to apply a 'reduction' to show that no such configuration appears in a smallest example that is not 4-colourable. It appears likely that such collections must be fairly large and that the actual calculations are infeasible without the aid of computers. Several such proofs are available by now, and it is a controversial issue among mathematicians (albeit to a much lesser extent than in the 1970s) as to whether computer-assisted proofs can be considered valid. Whether or not every correct proof of the four-colour theorem must be too long to be checked in every detail by a human, it would be interesting to see proofs based on different approaches, such as by algebraic means, as suggested by work of G. D. Birkhoff and (later) W. T. Tutte, or by using other powerful methods, such as advanced probabilistic methods.

Steinberg's 3-colour problem

As reported by Aksionov and Melnikov [3], the following question was asked by R. Steinberg in 1975.

Assume that no cycle in a planar graph has length 4 *or* 5. *Does it follow that the graph can be* 3*-coloured?*

Following a suggestion by P. Erdős in 1991, several authors have attempted to find the smallest possible value of k for which a planar graph must be 3-colourable if it contains no cycles of any length $4, 5, \ldots, k$. Abbott and Zhou [1] in 1991 were the first to prove the existence of such a number k, and they gave $k \leq 11$ as a bound. In 1995 the bound was improved to $k \leq 9$ by Sanders and Zhao [143], and then to $k \leq 8$ by Salavatipour [142] in 2002, and to $k \leq 7$ by Borodin, Glebov, Raspaud and Salavatipour [27] in 2005. In 2009 Borodin, Glebov, Montassier and Raspaud [26] proved the currently best result, that if a planar graph contains neither 5-cycles nor 7-cycles, nor any pair of 3-cycles that share an edge, then it can be 3-coloured.

A stronger conjecture by Borodin, Glebov, Jensen and Raspaud [25] suggests that if a planar graph contains no 3-cycle that shares an edge with a 4-cycle or a 5-cycle, then it can be 3-coloured; this suggests a common strengthening of

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Steinberg's problem and the theorem of Grötzsch [69] that every planar graph with no 3-cycle is 3-colourable. They proved that if a planar graph contains no 3-cycle which shares an edge with a cycle of length at most 9, then it can be 3-coloured.

Several related theorems and conjectures are found in the informative survey by Borodin [23] on the general theme of planar graph colouring.

Zhao [175] extended Erdős's suggestion to higher surfaces, proving that for any fixed surface S there is a constant k such that, if a graph without any cycles of length $4, 5, \ldots, k$ is embeddable in S, then it can be 3-coloured.

Ringel's earth–moon problem

What is the largest chromatic number f_2 that any union of two planar graphs can have?

The equivalent dual problem, asked in 1959 by Ringel [135], deals with a map of 'countries', each consisting of one connected region on the earth and one connected region on the moon. The aim is to colour the regions, using as few colours as possible, so that the two parts of each country receive the same colour, and so that any two regions sharing a common border receive different colours.

Ringel [135] noted the bounds $8 \leq f_2 \leq 12$, where the lower bound was improved to 9 by T. Sulanke in 1974, as first reported by Gardner [64]. An infinite class of critical 9-chromatic graphs of thickness 2 was constructed by Boutin, Gethner and Sulanke [30]. In general, the *thickness* of a graph G is the smallest number t for which there are t planar graphs whose union is G. For the maximum chromatic number f_t of graphs of thickness t, the known bounds are $6t - 2 \leq f_t \leq 6t$, for $t \geq 3$.

Motivated by the difficulty of computing both the thickness (proved NP-hard by Mansfield [117]) and the chromatic number of a given graph, Albertson, Boutin and Gethner [6], [7] compiled catalogues of known bounds for both parameters, as well as the fractional chromatic number, when restricted to special classes of graphs.

Several extensions of Ringel's earth–moon problem to higher surfaces were discussed by Jackson and Ringel [82].

Barnette's conjecture

Every simple planar 3-colourable graph allows a partitioning of its vertex-set into the vertex-sets of two induced forests.

The original conjecture, posed by Barnette [14], was stated dually in terms of the existence of a Hamiltonian cycle in any 3-connected bipartite cubic planar graph.

An early partial result in 1975 by Goodey [68] implies the truth of the conjecture for maximal planar graphs in which each vertex has degree 4 or 6. This was extended by Feder and Subi [54], who proved that if a maximal planar graph G allows a 3-colouring in which all vertices of degree larger than 6 receive the same colour, then the conjecture is true for G.

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Holton, Manvel and McKay [81] proved the conjecture in its original formulation for every 3-connected planar bipartite cubic graph with at most 64 vertices. Among other results, Aldred, Bau, Holton and McKay [8] showed that every 3-connected planar cubic graph with at most 176 vertices and with all faces of size at most 6 is Hamiltonian, independently of the bipartiteness condition. The bound 176 was improved to 250 by Brinkmann, McKay and von Nathusius [31].

Kelmans [97] proved that Barnette's conjecture is equivalent to the following interesting stronger statement: for any two edges e_1 and e_2 that belong to the boundary of a common face in a 3-connected bipartite cubic planar graph G, there exists a Hamiltonian cycle in G containing e_1 but not e_2.

The square of planar graphs

If a graph is planar with maximum degree 3, *can its vertices be coloured with seven colours, so that all vertices at distance* 1 *or* 2 *receive different colours?*

This question was asked in 1977 by Wegner [171], who observed that seven colours is best possible, by subdividing a non-triangular edge of a triangular prism, and who proved that eight colours would be sufficient.

Various authors have mentioned a positive solution to Wegner's problem by C. Thomassen, obtained in 2001. However, according to the survey paper published in 2008 by Kramer and Kramer [113], the problem is still considered open. It seems open as to whether six colours are always sufficient when we restrict attention to the special case of 3-connected graphs.

An interesting variation applies to planar graphs of arbitrary maximum degree. It requires, for some natural number d, that vertices are coloured differently when their distance is precisely equal to d. There is no upper bound on the number of colours needed when d is even, but the situation appears much more complicated when d is odd. For $d = 1$, we of course retrieve the four-colour problem. For $d = 3$ Nešetřil and Ossona de Mendez [127] have proved the first known (and very large) upper bound on the number of colours that are sufficient in a colouring, and they have presented an example for which six colours are needed for $d = 3$; this may be the best lower bound at present. In fact Nešetřil and Ossona de Mendez proved upper bounds for all odd $d \geq 3$; their bounds depend on d and are extremely large, even for $d = 3$.

The game-chromatic number and Brams's problem

In April 1981 Martin Gardner publicised the following problem as one of several short problems in his column in *Scientific American*.

Let G be a graph and let k be a positive integer, and suppose that the following two-person game is played on G. Alice and Bob alternate turns, with Alice having the first move. Both players know G and k from the beginning. A move consists in choosing a vertex v from G that has not yet been assigned a colour and assigning to v a colour

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from $1, 2, \ldots, k$ *distinct from the colours assigned previously (by either player) to the vertices adjacent to v. The game stops with Alice the winner if G has been coloured after* $|V(G)|$ *moves. Bob is the winner if, at some point of the game, there is a vertex that cannot be assigned a colour – that is, the vertex is already adjacent to vertices coloured with all k colours.*

The game-chromatic number $\chi_g(G)$ *is the least number k for which Alice has a winning strategy. If Alice has a winning strategy for G using k colours, does she also then have a winning strategy with* $k+1$ *colours? What is the least number k for which Alice has a winning strategy for all planar graphs?*

The part of this problem that involves planar graphs is due to Steven Brams, a professor of politics at New York University. Brams is a game theory expert who invented the game for planar maps, thinking about possible alternative proofs of the four-colour theorem. The problem came up in correspondence with Martin Gardner [63] in late 1980. In a later letter to Gardner in October 1980 Lloyd Shapley showed that $k \geq 6$ (with five colours for the map of the dodecahedron: Bob wins if he always colours the opposite face of Alice's last choice with the same colour), and Robert High improved the bound to 7. Brams and Gardner contemplated that there might exist no least k for planar graphs.

As far as we know, a proof that there is a least number k for which Alice has a winning strategy for all planar graphs was first given by Kierstead and Trotter [101] in 1993. In a more recent paper, Bartnicki, Grytczuk, Kierstead and Zhu [15] gave 8 and 17 as the best-known lower and upper bounds for the least k for which Alice has a winning strategy for planar graphs, with a proof that $k = 18$ suffices.

For graphs in general, the delightful paper [15] contains the following intriguing remark:

It is not clear what influence increasing the number of colours has on the colouring game. Suppose that Alice wins with k colours on a graph G. Then it would seem trivial that she should also win on G with $k+1$ *colours. But can you prove it? As yet, nobody has been able to do so!*

For more on graphs and games, see Chapter 14.

Albertson's 4-colour problem

Let S be a surface. Is there a constant M_S *such that, if G is embeddable on S, then* $G - X$ *is 4-colourable for some subset* $X \subseteq V(G)$ *with at most* M_S *vertices? In particular, does this hold for the torus T with* $M_T = 3$?

These questions were asked by Albertson [5] in 1981. Not much recent progress has been achieved on either of them.

DeVos, Kawarabayashi and Mohar [42] proved the similar general statement for 5-list-colourability: there is a constant k such that, for every graph G embedded in S, there exists a vertex-set U with at most k vertices for which $G-U$ is 5-list-colourable.

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The similar statement with 4-list-colourability instead of 5-list-colourability is false, since there exist non-4-list-colourable planar graphs that may be drawn in any number of copies on any surface, as proved by Voigt [169] and Mirzakhani [121].

The similar question for 3-colourability of triangle-free graphs on S, which by Grötzsch's theorem [69] has a positive answer when S is the plane, was answered affirmatively by Kawarabayashi and Thomassen [95].

We are not aware of similar results concerning the 4-list-colourability of triangle-free graphs, or the 3-list-colourability of graphs with girth at least 5. Both of these results hold for the plane; the former follows easily from Euler's formula and the latter was proved by Thomassen [160].

4. Degrees and colourings

Edge-disjoint placements

Let G_1 and G_2 be graphs with the same number n of vertices, and suppose that their maximum degrees $\Delta(G_1)$ and $\Delta(G_2)$ satisfy

$$(\Delta(G_1)+1)(\Delta(G_2)+1) \leq n+1.$$

Does it follow that there exists a graph on n vertices which is an edge-disjoint union of a copy of G_1 and a copy of G_2?

This question was asked independently by Bollobás and Eldridge [20], [21] and by Catlin [35], [36]. A positive answer would extend considerably the classical colouring theorem of Hajnal and Szemerédi [75], which states that if G has maximum degree Δ and exactly $k(\Delta+1)$ vertices, then there exists a colouring of G with $\Delta+1$ colours in which each colour class has size k. A short proof of this theorem was given by Kierstead and Kostochka [98].

The case when one of the maximum degrees is at most 2 was solved affirmatively by Aigner and Brandt [2]. So if a graph on n vertices has minimum degree at least $\frac{1}{3}(2n-1)$, then it contains as a subgraph every graph on n vertices of maximum degree at most 2. For maximum degree 3 and sufficiently large n a positive solution was given by Csaba, Shokoufandeh and Szemerédi [41].

Kaul, Kostochka and Yu [89] proved that the conclusion holds under the extra assumption that $\min\{\Delta(G_1), \Delta(G_2)\} \geq 300$, and the stronger condition

$$(\Delta(G_1)+1)(\Delta(G_2)+1) \leq \tfrac{3}{5}n+1.$$

For graphs with large maximum degree their result improves a theorem of Sauer and Spencer [145] stating that the conclusion follows for all G_1 and G_2 satisfying

$$\Delta(G_1)\Delta(G_2) < \tfrac{1}{2}n.$$

A generalization of the problem to hypergraph packing was studied by Rödl, Ruciński and Taraz [140].

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Reed's ω-Δ-χ conjecture

If G has maximum degree Δ and contains no complete subgraph with more than ω vertices, then

$$\chi(G) \leq \left\lceil \tfrac{1}{2}(\Delta + 1) + \tfrac{1}{2}\omega \right\rceil.$$

This conjecture was posed by Reed [133] in 1998 and quickly became popular. The idea behind it is to combine the obvious lower and upper bounds

$$\omega \leq \chi(G) \leq \Delta + 1$$

into an improved upper bound obtained as a convex combination of them. For $\omega \geq \Delta$ the conjectured bound is trivial, and for $\omega = \Delta - 1$ the truth follows from Brooks's theorem [34].

In the special case $\omega = \Delta = 2$ we must round up the average of ω and $\Delta + 1$, due to odd cycles. Classical constructions by Gallai [61] of 4-critical graphs with $\Delta = 4$ and $\omega = 2$, and graphs obtained as joins of these with complete graphs, show in general that either rounding up is necessary, or the weight of the $\Delta + 1$ term must be increased in the convex combination.

If ω is as small as possible – that is, in the non-trivial case when G is triangle-free and $\omega = 2$ – the best possible bound by A. Johansson (unpublished, see [123]) is

$$\chi(G) \leq \frac{c \cdot \Delta}{\log \Delta},$$

where $c > 0$ is a constant, and shows that the conjecture is true when Δ is large enough.

Reed [133] also made a second conjecture for $\Delta \geq 3$:

$$\chi(G) \leq \tfrac{2}{3}(\Delta + 1) + \tfrac{1}{3}\omega.$$

By giving more weight to the trivial upper bound $\Delta + 1$ in the convex combination, rounding is avoided. As one of the first positive results, Reed proved that there exists a constant a, with $0 < a < 1$, for which

$$\chi(G) \leq (1 - a)(\Delta + 1) + a\omega, \quad \text{for } \Delta \geq 3.$$

The following stronger 'local version' of Reed's conjecture was conjectured by King [102] in his Ph.D. thesis in 2009. For a vertex v of G, let $\omega(v)$ denote the size of a largest complete subgraph of G that contains v. King's conjecture is

$$\chi(G) \leq \max_{v \in V(G)} \left\{ \left\lceil \tfrac{1}{2}(d_G(v) + 1 + \omega(v)) \right\rceil \right\}.$$

Chudnovsky, King, Plumettaz and Seymour [38] proved this conjecture for *quasi-line graphs* – graphs with the property that the neighbourhood of each vertex induces the complement of a bipartite graph. It is easy to see that all line graphs are quasi-line

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graphs. Edwards and King [47] have suggested an even stronger 'superlocal version' of Reed's conjecture.

The total colouring conjecture

Let G be a simple graph with maximum degree Δ. Then it is possible to colour the edges and vertices of G with $\Delta + 2$ colours in such a way that no two adjacent or incident elements in $V(G) \cup E(G)$ receive the same colour.

There has been some controversy about the priority to this conjecture, as described by Soifer [150]. In 1964 the conjecture was proposed by V. G. Vizing in graph theory seminars at the Academy of Sciences in Akademgorodok (outside Novosibirsk), organized by A. A. Zykov. It was independently proposed by M. Behzad at the same time, and its first written appearence is in Behzad's Ph.D. thesis [17] at Michigan State University in 1965. The controversy arising from this has been addressed by Stiebitz, Scheide, Toft and Favrholdt [156].

In 1968 Vizing [167] published a paper containing a wealth of problems, including the *total graph colouring conjecture* for multigraphs, which suggests that $\Delta(G) + \mu(G) + 1$ colours suffice in a simultaneous colouring of the vertices and edges of a multigraph G with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. A natural stronger conjecture is that $\chi'(G) + 1$ colours suffice for all multigraphs. Unfortunately, Vizing [167] used the term 'chromatic index' to denote the least number of colours needed in a simultaneous colouring of vertices and edges! This number is now commonly called the *total chromatic number*, denoted by χ'', and a simultaneous colouring of the vertices and edges of a graph is called a *total colouring*, following Behzad.

A further strengthening when $\chi' \geq \Delta + 3$ was proposed by Goldberg [66]:

A graph G with $\chi'(G) \geq \Delta(G) + 3$ satisfies $\chi''(G) = \chi'(G)$.

We observe that Goldberg's conjecture says that any graph G with $\chi'(G) \geq \Delta(G) + 2$ satisfies $\chi'(G) = \mathcal{W}(G)$, where $\mathcal{W}(G)$ is the *density* of G (see Section 5). Goldberg [66] noticed that any critical graph G with $\chi'(G) = \mathcal{W}(G)$ satisfies $\chi''(G) = \chi'(G)$: we first take a $(k-1)$-edge-colouring of $G - e$, where e is an edge of G and $k = \chi'(G)$. He then proved (see Stiebitz, Scheide, Toft and Favrholdt [156]) that none of the $k - 1$ colours is missing at two distinct vertices of G. So we can colour each vertex v of G, using a colour that is missing at v, and we can colour e with colour k. This results in a total colouring of G with k colours.

We may also ask, as an analogue of Goldberg's conjecture, whether every graph satisfies $\chi''(G) \leq \max\{\Delta(G) + 2, \mathcal{W}(G)\}$?

It is folklore that any graph G satisfies $\chi''(G) \leq \chi'^{\ell}(G) + 2$, where $\chi'^{\ell}(G)$ is the list-chromatic index of G (see Section 5). In 1997 Borodin, Kostochka and Woodall [28] used list-colouring arguments to show that $\chi''(G) \leq \frac{3}{2}\Delta + 1$ for graphs G with

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maximum degree $\Delta \geq 2$. One year later, Molloy and Reed [122] used probabilistic arguments to show that $\chi'' \leq \Delta + 10^{26}$ for simple graphs.

Clearly, there is a list variant for the total chromatic number, and Borodin, Kostochka and Woodall [28] conjectured that the total list-chromatic number equals the total chromatic number.

Critical graphs with many edges or high degrees

Let $k \geq 4$ be an integer. A graph is *k-critical* if it has chromatic number k and every proper subgraph is $(k-1)$-colourable.

If $F_k(n)$ is the largest number of edges of a k-critical graph on n vertices, does

$$\lim_{n\to\infty} F_k(n)/n^2$$

exist? Is $F_6(n) = \frac{1}{4}n^2 + n$, for $n \equiv 2 \pmod 4$?

If $\delta_k(n)$ is the largest minimum degree of a k-critical graph on n vertices, does there exist a constant $c > 0$ for which $\delta_4(n) \geq cn$? What are the orders of magnitude of $\delta_4(n)$ and $\delta_5(n)$?

Critical graphs were first defined and used by G. A. Dirac, when he studied for his Ph.D. degree at the University of London around 1950. In 1949 he met Paul Erdős, whose first reaction was to ask about the maximum number of edges in these minimal graphs. Dirac observed that the join of two odd cycles of equal length is a 6-critical graph with many edges: no better examples have since been found. For $|V(G)| = n$, the number of edges is $\frac{1}{4}n^2 + n$. Jensen and Toft [85] conjectured, however, that this is not the best possible value.

For $k = 4$, it is known that

$$\liminf_{n\to\infty} F_4(n)/n^2 \geq \tfrac{1}{16},$$

and for $k = 5$ the corresponding value is $\frac{4}{31}$, as proved by Toft [162]. Jensen [84] observed the general upper bound

$$F_k(n) \leq \frac{k-2}{2(k-1)}n^2, \quad \text{for all } n \text{ and } k \text{ with } n > k \geq 4.$$

For the largest minimum degree in critical graphs the above examples of Dirac show that $\delta_6(n) \geq \frac{1}{2}n$ for $n \equiv 2 \pmod 4$. Simonovits [149] and Toft [163] proved that $\delta_4(n) \geq cn^{1/3}$. This may be the best order of magnitude, but it seems difficult to prove this. As for the order of magnitude of $\delta_5(n)$, there is no reasonable guess.

Erdős [50] asked specifically about the existence of regular 4-critical graphs. The only 3-regular example is K_4, by Brooks's theorem [34]. On the other hand, Gallai [61] constructed an infinite family of 4-regular examples. An infinite family

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of 5-regular 4-critical graphs was constructed by Jensen [84]. Examples of regular 4-critical graphs of even degrees 6 to 16 were found by Dobrynin, Melnikov and Pyatkin [44], [45]. We are not aware of any examples of 4-critical graphs that are regular of odd degree 7 or more.

Probability colouring

Assume that $(p_1, p_2, \ldots, p_k)$ is a probability distribution on the set $\{1, 2, \ldots, k\}$ – that is, $p_i \geq 0$ for $1 \leq i \leq k$, and $\sum_{i=1}^{k} p_i = 1$. Let the vertices of G be coloured randomly and independently so that each vertex of G receives colour i with probability p_i, for $i = 1, 2, \ldots, k$. For $m \geq 0$, let P_G^m be the probability of at most m monochromatic edges. As usual, Δ denotes the maximum degree.

Is P_G^m maximized by the uniform distribution $p_i = 1/k$ when $k = k(G, m)$ is large enough – say, whenever $k \geq f(\Delta, m)$ for some function f?

Is there a structural characterization of those graphs G for which, for each $k > 0$, the probability P_G^0 of a proper colouring is maximized by the uniform distribution?

If, for all k and m, the probability P_G^m is maximized by the uniform distribution, then G is called *P-uniform*.

Is there a a structural characterization of P-uniform graphs?

We note that

$$P_G^m = \sum_f \prod_{v \in V(G)} p_{f(v)},$$

where the sum is over all mappings $f : V(G) \to \{1, 2, \ldots, k\}$ for which there are at most m edges xy with $f(x) = f(y)$. Permutations of the colours do not change this probability, and so P_G^m may be expressed as a symmetric polynomial in $p_1, p_2, \ldots, p_k$.

We also note in particular that

$$P_G^0 = \chi(G, k) k^{-|V(G)|} \quad \text{when} \quad (p_1, p_2, \ldots, p_k) = (1/k, 1/k, \ldots, 1/k),$$

where $\chi(G, k)$ is the chromatic polynomial of G.

Fadnavis [52] and her Ph.D. advisor P. Diaconis asked the first question above, and she proved that the answer is affirmative for $m = 0$ if the number of colours satisfies $k > 6.3 \times 10^5 \Delta^4$. For $G = K_{1,n}$ and $m = 0$ it suffices that $k \geq n + 1$. Diaconis had conjectured that the answer to the second question above is affirmative when G is claw-free – that is, when $K_{1,3}$ is not an induced subgraph. This conjecture was also proved by Fadnavis.

The graph $K_{1,4}$ is not P-uniform, as witnessed by $m = 0$ and $k = 2$. Fadnavis [52] exhibited a claw-free graph on 19 vertices which is not P-uniform (for values $m = 30$ and $k = 2$), but proved that all complete graphs and cycles are P-uniform.

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5. Edge-colourings

In this and the following section we no longer restrict our attention to simple graphs. Accordingly the term 'graph' is used more generally to denote a multigraph – that is, two vertices may be joined by more than one edge, but loops do not occur.

Goldberg's conjecture

The *density* $\mathcal{W}(G)$ of a graph G is defined by

$$\mathcal{W}(G) = \max\{\left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil \ : \ H \subseteq G, |V(H)| \geq 2\}.$$

The following conjecture was proposed independently by Goldberg [65] and Seymour [148] in the 1970s. A detailed account of its history is given in Stiebitz, Scheide, Toft and Favrholdt [156].

Every graph G satisfies $\chi'(G) \leq \max\{\Delta(G)+1, \mathcal{W}(G)\}$.

This conjecture can be stated in different equivalent ways, as follows.

Every graph G satisfies $\chi'(G) \in \{\Delta(G), \Delta(G)+1, \mathcal{W}(G)\}$.

Every graph G satisfies, for each $r \in \mathbb{N}$,

$$\chi'(G) \leq \max\{\frac{(2r+3)\Delta(G)+2r}{2r+2}, \mathcal{W}_1(G), \mathcal{W}_2(G), \ldots, \mathcal{W}_r(G)\},$$

where $\mathcal{W}_r(G) = \max\{\lceil |E(H)|/r \rceil \ : \ H \subseteq G, |V(H)| = 2r+1\}$ (Gupta [70]).

Let G be a χ'-critical graph (that is, $\chi'(H) < \chi'(G)$ for every proper subgraph H of G), let $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, let e be an edge of G, and let φ be an edge-colouring of $G-e$ with k colours. Then no colour is missing at two distinct vertices of G with respect to φ. (Andersen [9]).

Every χ'-critical graph G with $\chi'(G) \geq \Delta(G)+2$ has odd order and satisfies

$$2|E(G)| = (\chi'(G)-1)(|V(G)|-1)+2$$

(Stiebitz, Scheide, Toft and Favrholdt [156]).

The conjecture of Goldberg and Seymour is one of the most important conjectures in edge-colouring theory. It generalizes Vizing's theorem in a strong way, by showing that there are only three possible values for the chromatic index of an arbitrary graph. While it appears unknown whether the density $\mathcal{W}(G)$ can be computed in polynomial time, the parameter $\kappa(G) = \max\{\Delta(G), \mathcal{W}(G)\}$ can be determined in polynomial time: this follows from Edmonds's matching polytope theory and the theory of anti-blocking polyhedra (see Schrijver [146]). So Goldberg's conjecture supports the following conjecture of Hochbaum, Nishizeki and Shmoys [79]:

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There is a polynomial-time algorithm that colours the edges of any graph G with at most $\chi'(G) + 1$ colours.

Results related to Goldberg's conjecture were discussed by Stiebitz, Scheide, Toft and Favrholdt [156]. In particular, Goldberg's conjecture was confirmed for graphs with at most 15 vertices, and also for graphs with maximum degree at most 15. Gupta's version of Goldberg's conjecture is true for $1 \leq r \leq 6$. Furthermore, it is known that every graph G satisfies

$$\chi'(G) \leq \max\{\Delta(G) + \sqrt{(\Delta(G) - 1)/2}, \mathcal{W}(G)\}.$$

Whether Goldberg's conjecture holds for all planar graphs is unknown.

The Berge–Fulkerson conjecture

This conjecture is attributed to C. Berge (see Seymour [147]), but it first appeared in print in Fulkerson [59]:

Every bridgeless cubic graph G contains a family of six perfect matchings such that each edge of G is contained in precisely two of them.

An equivalent form of it is that $\chi'(2G) = 6$ for every bridgeless cubic graph. The conjecture was extended by Seymour [148] to the generalized Fulkerson conjecture:

Every r-graph G satisfies $\chi'(2G) = 2r$.

Here, an *r-graph* is an r-regular graph G such that, for each $X \subseteq V(G)$ with an odd number of vertices, there are at least r edges in G with precisely one endpoint in X. Since the fractional chromatic index of every r-graph G is $\chi'^*(G) = r$, there are integers t for which $\chi'(tG) = tr$ (see Stiebitz, Scheide, Toft and Favrholdt [156]). Seymour's conjecture suggests that in general $t = 2$ is one of these integers.

The following seemingly weaker version of the Berge–Fulkerson conjecture was proposed by Berge (personal communication from F. Jaeger in 1994). It is easy to see that this conjecture is implied by the Berge–Fulkerson conjecture, and Mazzuoccolo [119] proved that the two conjectures are actually equivalent.

Every bridgeless cubic graph G contains a family of five perfect matchings such that each edge of G is contained in at least one of them.

We note that Frink's proof [57] of Petersen's theorem [131] tells us that each edge of a cubic bridgeless graph is contained in a perfect matching (see also König's book [110]). On the other hand, it is not known whether Berge's conjecture holds, even with 'five' replaced by any other fixed number. For a positive integer k, define

$$m_k(G) = \max\left\{|\bigcup_{i=1}^{k} M_i|/|E(G)| \ : \ M_1, M_2, \ldots, M_k \text{ are perfect matchings of } G\right\}.$$

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Clearly, $m_1(G) = \frac{1}{3}$ and Berge's conjecture suggests that $m_5(G) = 1$. For the Petersen graph P, we have

$$m_2(P) = \tfrac{3}{5},\ m_3(P) = \tfrac{4}{5},\ m_4(P) = \tfrac{14}{15},\ m_5(P) = 1,$$

and Patel [129] made the following conjecture:

Every bridgeless cubic graph G satisfies $m_k(G) \geq m_k(P)$, for $1 \leq k \leq 5$.

Patel proved that this conjecture is implied by the Berge–Fulkerson conjecture, and thus, by the result of Mazzuoccolo, the two conjectures are equivalent. Lower bounds on $m_k(G)$ for $2 \leq k \leq 5$ were established by Kaiser, Král' and Norine [87]: in particular, they verified the case $k = 2$ of the conjecture. The paper by Fouquet and Vanherpe [56] provides some partial results related to the perfect matching index of bridgeless cubic graphs – that is, the minimum number of perfect matchings that cover all edges.

For the Petersen graph P, only twelve edges can be covered with three perfect matchings, but thirteen edges can be covered with three matchings. Rizzi [136] proved that if G is a simple graph with $\Delta(G) \leq 3$, then there are three matchings M_1, M_2 and M_3 for which $|M_1 \cup M_2 \cup M_3| \geq \frac{6}{7}|E(G)|$. The *max edge t-colouring problem* is to colour the largest possible number of edges of a graph G in a proper way using t colours; the complexity of this problem was studied by Feige, Ofek and Wieder [55].

There is another attractive formulation of the Berge–Fulkerson conjecture in terms of even graphs – that is, graphs in which all vertices have even degree. That the following conjecture is equivalent to the Berge–Fulkerson conjecture was proved by Jaeger [83]:

Every bridgeless graph G contains a family of six even subgraphs such that each edge of G is contained in exactly four of them.

Hao, Niu, Wang, Zhang and Zhang [76] proved that a cubic graph G satisfies $\chi'(2G) = 6$ if and only if there are two edge disjoint matchings M_1 and M_2 such that $\chi'(G - M_i) \leq 3$ for $i = 1, 2$ and $M_1 \cup M_2$ is the edge-set of a cycle. They used this to verify the Berge–Fulkerson conjecture for certain families of snarks.

The relation between the Berge–Fulkerson conjecture and shortest cycle cover problems has been investigated by Fan and Raspaud [53]. They also drew attention to the following implication of the Berge–Fulkerson conjecture which has become known as the Fan–Raspaud conjecture:

Every bridgeless cubic graph contains three perfect matchings with empty intersection.

If a cubic graph has a family of six perfect matchings covering each edge twice, then the intersection of any three of them must be empty. On the other hand, even

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the existence of an integer $k \geq 3$ for which every bridgeless cubic graph contains k perfect matchings with empty intersection is not clear.

The list-chromatic index

The following list-colouring conjecture, suggested independently by various researchers (including V. G. Vizing, M. O. Albertson, K. Collins, A. Tucker, R. P. Gupta, B. Bollobás and A. J. Harris) is the most popular problem in edge-colouring theory. It first appeared in print in a paper by Bollobás and Harris [22].

Let G be a graph with $\chi'(G) = k$, and let each edge of G be assigned a list of k colours. Then it is possible to colour the edges of G so that each edge receives a colour from its list.

The study of list-colouring problems for graphs was initiated in the 1970s by Vizing [168], and independently by Erdős, Rubin and Taylor [49]. The *list-chromatic index* $\chi'^{\ell}(G)$ of a graph G is the least number k for which, whenever we assign a list of k colours to each edge of G, there is a proper colouring of the edges of G for which each edge receives a colour from its list. By definition, every graph G satisfies $\chi'(G) \leq \chi'^{\ell}(G)$ and the list-colouring conjecture suggests that equality holds. In contrast, the gap between the chromatic number and the list-chromatic number can be arbitrarily large, even for bipartite graphs, as observed by Vizing [168] and Erdős, Rubin and Taylor [49].

One of the most spectacular results about edge-colourings from the 1990s is the verification by Galvin [62] of the list-colouring conjecture for the class of bipartite graphs – that is, the proof that $\chi'(G) = \chi'^{\ell}(G)$ for every bipartite graph G. This particular case of the list-colouring conjecture is due to J. Dinitz (see Chetwynd and Häggkvist [37]).

Borodin, Kostochka and Woodall [28] obtained an attractive generalization of Galvin's landmark theorem, allowing lists of different lengths, where an edge e with ends u and v has a list of length $\max\{d_G(u), d_G(v)\}$. Based on this they proved that

$$\chi'^{\ell}(G) \leq \left\lfloor \tfrac{3}{2}\Delta(G) \right\rfloor$$

for all graphs G. The list-colouring conjecture has been verified for the class of simple graphs, at least asymptotically, by Kahn [86], who proved that every simple graph G satisfies $\chi'^{\ell}(G) = \Delta(G) + o(\Delta(G))$. The following conjecture, posed by Bojan Mohar, deals with list-colourings of χ'-critical class 2 graphs – that is, simple graphs G for which $\chi'(G) = \Delta(G)+1$ and $\chi'(H) < \chi'(G)$, whenever H is a proper subgraph of G.

Suppose that G is a χ'-critical graph of class 2, and for each edge of G there is a list of $\Delta(G)$ colours. Then the edges of G can be properly coloured so that each edge receives a colour from its list unless all the lists are equal to each other.

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6. Flow problems

A *k-flow* in a graph (also called a 'nowhere-zero k-flow' for emphasis) is an assignment to each edge of a direction and of a flow value in $\{1, 2, \ldots, k-1\}$, so that the net flow entering and leaving each vertex is 0. The concept of a k-flow was introduced by Tutte [164], who showed that a plane graph allows a k-flow if and only if its dual graph is vertex-k-colourable. A comprehensive survey of k-flows written by Lovász [115] is available on-line (see also Chapter 9).

Tutte's 5-flow conjecture

The following conjecture was posed by Tutte [164] in 1954:

Every graph without bridges allows a 5*-flow.*

Kochol [108] proved that if the 5-flow conjecture is false, then a smallest counter-example is a cyclically 6-edge-connected cubic graph – that is, a 3-regular graph for which, whenever fewer than six edges are deleted, the remaining graph has at most one component that contains a cycle.

A bound on the possible order of a counter-example was obtained by Steffen [151], who proved Tutte's 5-flow conjecture for all graphs with fewer than 44 vertices. As a consequence, the conjecture is true for all graphs embeddable on a surface with Euler characteristic at least -3. A result by Kochol [109] shows that the girth of a smallest counter-example is at least 9. Since the order of a $(3, 9)$-cage is known (see [32]), this improves Steffen's lower bound to 58 vertices, and the bound on the Euler characteristic to -10.

An interesting equivalent reformulation of the 5-flow conjecture, stated in terms of a parity condition, is due to Matamala and Zamora [118]. Let F be a subgraph of a graph G. Then F is called a $(1, 2)$-*factor* of G if each vertex of G is a vertex of F of degree 1 or 2. An edge of G is called F-*balanced* if it is either an edge of F or its ends have the same degree in F. If every cycle of G contains an even number of F-balanced edges, then F is called an *even subgraph*. Matamala and Zamora [118] proved that a cubic graph without bridges allows a 5-flow if and only if it contains an even $(1, 2)$-factor.

As a common weakening of Tutte's 5-flow and 4-flow conjectures (see below), Kochol [106] observed that if a cubic graph has no bridge and does not contain the Petersen graph as a minor, then it allows a 5-flow.

Hochstättler [80] related Tutte's k-flow problems to Hadwiger's conjecture by extending both problems to the more general setting of regular matroids (see, for example, Oxley [128]).

Tutte's 4-flow conjecture

The following conjecture, formally a strengthening of the four-colour theorem, was made by Tutte [165] in 1966.

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Every graph without a bridge and without a subdivision of the Petersen graph as a subgraph allows a 4*-flow.*

When restricted to cubic graphs, the existence of a 4-flow becomes equivalent to the 3-colourability of edges. The truth of the conjecture in this special case was proved by Robertson, Seymour and Thomas [139], with a proof extending the proof by Robertson, Sanders, Seymour and Thomas [137] of the four-colour theorem.

A 4-flow in a graph is equivalent to a covering of its edges by two subgraphs, each of which is an edge-disjoint union of cycles. Thus, many statements about 4-flows have equivalent reformulations in terms of cycle covers. The excellent monograph by Zhang [173] presents several additional connections between Tutte's k-flow conjectures and other theorems and conjectures on flows and cycle covers in graphs. Such connections were also explored by Kaiser and Škrekovski [88].

Extending these ideas from graphs to regular matroids, Lai, Li and Poon [114] showed (using the four-colour theorem) that if a regular matroid without a co-loop (a singleton circuit in the dual matroid) does not allow a 4-flow, then it contains a minor that is either K_5 or the matroid dual of K_5. In particular, this proves the conjecture of Jensen and Toft [85] that every graph without a bridge and without a K_5-minor has a 4-flow.

Tutte's 3-flow conjecture

According to a survey by Steinberg [153] on the three-colour problem, the following 3-flow conjecture was made by Tutte in 1972. Its restriction to the planar case is dually equivalent to Grötzsch's theorem [69].

Every graph without a bridge and without an edge-cut of size 3 *allows a* 3*-flow.*

A weaker conjecture by Jaeger [83], called 'the weak 3-flow conjecture', states that there exists a number λ for which every graph of edge-connectivity at least λ allows a 3-flow. This conjecture has been proved by Thomassen [161] for $\lambda = 8$, and improved to $\lambda = 6$ by Lovász, Thomassen, Wu and Zhang [116]. Tutte's conjecture corresponds to $\lambda = 4$ in the weak 3-flow conjecture. However, Kochol [107] proved that Tutte's conjecture already follows from the case $\lambda = 5$ in the weak 3-flow conjecture.

Similar to a k-flow in a graph, except with non-integral flow values, a *circular r-flow* for a real number $r \geq 2$ is defined as an assignment to each edge of a direction and of a flow value in the interval $[1, r-1]$, for which the net flow entering and leaving each vertex is 0. According to Zhang [174], L. A. Goddyn and P. D. Seymour have conjectured that, for each $\varepsilon > 0$, there exists a number λ for which every graph of edge-connectivity at least λ allows a circular $(2+\varepsilon)$-flow. Klostermeyer and Zhang [104] proved Goddyn and Seymour's conjecture for planar graphs. The general theorem on circular flows by Thomassen [161] implies, for each positive natural number k, that if a graph has large enough edge-connectivity, then it allows a circular flow in which all flow values are either 1 or $1 + 1/k$, thus proving the conjecture also for the general case (personal communication from C. Thomassen).

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7. Concluding remarks

As indicated in this chapter, even though many important graph-colouring problems have been solved, there is a large variety of interesting unsolved problems still around. We made a selection, mostly guided by our own taste, and mostly from our book [85] in updated form. As for historical remarks about the problems, the original book is usually a more comprehensive source.

The state of the art for edge-colourings is described in Stiebitz, Scheide, Toft and Favrholdt [156]. This book also contains a chapter with twenty attractive edge-colouring conjectures. Here we have used formulations from [156], as well as from [85].

Probabilistic methods play a fast-growing role in the development of combinatorial mathematics. These we have to some degree neglected. For those wishing insight into this side of colouring theory we strongly recommend the excellent book by Molloy and Reed [123], as well as the well-written book by Beck [16], which is a rich source of how the probabilistic method, in a game-theoretic setting, may be used to obtain exact results by 'derandomization'.

Graph colouring theory flourishes as never before, and we hope and expect to see solutions and partial solutions to several of the above problems in the near future. The area continues to show surprising developments, and it is hard to predict which problems will find solutions. Tutte's iceberg is still there, with many beautiful hidden secrets waiting for discovery just beneath the surface.

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