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Fine limits of superharmonic functions

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1. *Introduction*

Let O denote the origin of $\mathbb{R}^n$ (where $n \geqslant 2$), let $B(X, r)$ denote the open ball in $\mathbb{R}^n$ of centre X and radius r, and define $\phi_n : (0, +\infty) \to \mathbb{R}$ by $\phi_n(t) = t^{2-n}$ if $n \geqslant 3$ and $\phi_2(t) = \log(1/t)$. A sequence (X_k) in $B(O, 1)$ is said to *converge regularly to* O if $X_k \to O$ and there is a constant $\kappa > 1$ such that $\phi_n(|X_{k+1}|) < \kappa \phi_n(|X_k|)$ for each $k \in \mathbb{N}$. If $n \geqslant 3$, this is clearly equivalent to saying that there is a constant $\kappa' \in (0, 1)$ such that $|X_{k+1}| > \kappa' |X_k|$ for each k (cf. [**18**], p. 149).

If u is a superharmonic function on a neighbourhood of O, then u has (by definition of the fine topology) a fine limit at O equal to $u(O)$. Also (see [**11**], I·XI·4(*c*)) $u(X)/\phi_n(|X|)$ has fine limit at O equal to $\mu(\{O\})$, where μ is the Riesz measure associated with u. Let $0 < \alpha < 1$. In this paper we show that the function $u(X)(\phi_n(|X|))^{-\alpha}$ has fine limit zero at O if there is a sequence (X_k), converging regularly to O, for which the corresponding sequence $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_1. The converse is false, but the result is nevertheless sharp.

Next we consider what can be said if l_1 is replaced by l_p, where $p \geqslant 1$. In this case we have

$$u(X)(\phi_n(|X|))^{-\alpha} \to 0 \quad (X \to O; X \notin E),$$

where E has a certain geometrical covering property depending on the value of p. In particular, if $1 \leqslant p < (n-1)/(n-2)$, then there is a set $F \subset \partial B(O, 1)$, of Hausdorff dimension $(n-2)p$, such that

$$u(rY)(\phi_n(r))^{-\alpha} \to 0 \quad (r \to 0+; Y \in \partial B(O, 1) \backslash F).$$

Finally we consider boundary analogues of the above results for a positive superharmonic function u on the half-space

$$D = \{X = (x_1, \ldots, x_n) : x_n > 0\}.$$

Let $w(X) = |X|/x_n$ for $X \in D$. A sequence (X_k) in D is said to *converge regularly to infinity* if $|X_k| \to \infty$ and there exists a constant $\kappa > 1$ such that $|X_{k+1}| < \kappa |X_k|$ for each $k \in \mathbb{N}$. A particular case of Theorem 3 below asserts that the function uw (and hence u) has minimal fine limit zero at infinity if $(u(X_k)\, w(X_k))$ is in l_1 for some sequence (X_k) converging regularly to infinity. It follows that, if the latter condition is true, then there is a polar subset F of $\sigma = \partial B(O, 1) \cap D$ such that $u(rY) \to 0$ as $r \to \infty$, for any $Y \in \sigma \backslash F$. This improves a result of Beardon [**4**]. We also consider what can be said if $(u(X_k)\, w(X_k))$ belongs to l_p.

Some of the results in this paper can be regarded as fine limit counterparts to known results on semi-fine limits of superharmonic functions obtained by Lelong-Ferrand [**18**], Rippon [**22**], and Mizuta [**20**, **21**]. We mention also recent work of

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Armitage and Goldstein [2] on the characterization of thin sets by the existence of a Newtonian potential with certain growth properties.

2. *Internal limits*

Theorem 1 below is concerned with the fine topology on $\mathbb{R}^n$, the general theory of which is described in Brelot [6], part 1 and Doob [11], I.XI. In particular we recall (see [6], theorem IX·7 or [11], I·XI·4) that a set E is *thin at* O if and only if there is a superharmonic function u (with associated measure μ) on a neighbourhood N of O such that $u(X) \geqslant \phi_n(|X|)$ for $X \in E \cap N$, and $\mu(\{O\}) = 0$. Further, a function f on a neighbourhood of O is said to have *fine limit* l at O if there is a set $E \subset D$, thin at O, for which $f(X) \to l$ as $X \to O$, where $X \notin E$.

THEOREM 1. *Let u be a superharmonic function on a neighbourhood of O and let $0 < \alpha < 1$. If there is a sequence (X_k) converging regularly to O for which the function $(u(X_k)\,(\phi_n(|X_k|))^{-\alpha})$ belongs to l_1, then the function $u(X)\,(\phi_n(|X|))^{-\alpha}$ has fine limit zero at O.*

The converse to Theorem 1 is false, as can be seen by considering the function

$$u(X) = \begin{cases} (\phi_n(|X|))^{\alpha}/\log(1/|X|) & (n \geqslant 3) \\ (\phi_2(|X|))^{\alpha}/\log(\log(1/|X|)) & (n = 2). \end{cases}$$

In each case the function u is easily checked to be superharmonic on a neighbourhood of the origin, and $u(X)\,(\phi_n(|X|))^{-\alpha}$ has limit (and hence fine limit) zero at O. Yet clearly there is no sequence (X_k), converging regularly to O, for which the corresponding sequence $(u(X_k)\,(\phi_n(|X_k|))^{-\alpha})$ belongs to l_1. Nevertheless, as is demonstrated in Section 4·5, Theorem 1 is sharp.

COROLLARY 1. *Under the hypotheses of Theorem 1 there is a G_δ polar set $F \subset \partial B(O, 1)$ such that*

$$u(rY)\,(\phi_n(r))^{-\alpha} \to 0 \quad (r \to 0+\,;\, Y \in \partial B(O, 1) \backslash F).$$

COROLLARY 2. *Let $n = 2$. Under the hypotheses of Theorem 1 there is an open set $F \subset (0, 1)$ such that*

$$\sum_{k=1}^{\infty} \frac{k}{\log(1/\lambda_k)} < +\infty, \tag{1}$$

where λ_k is the Lebesgue measure of $F \cap (2^{-k-1}, 2^{-k})$, and such that

$$u(rY)\,(\phi_2(r))^{-\alpha} \to 0 \quad (r \to 0+\,;\, r \notin F)$$

uniformly in $Y \in \partial B(O, 1)$.

The proofs of Theorem 1 and its corollaries may be found in Section 4. In connection with Corollary 1, we note that Mizuta [19] has studied the radial limit behaviour of Riesz potentials of non-negative functions f satisfying a condition of the form $\int |X|^{\beta} f(X)^p dX < \infty$, where $p > 1$.

We next mention a covering property of thin sets in $\mathbb{R}^n$. Let $h : [0, +\infty) \to [0, +\infty)$ be a continuous increasing (wide sense) function such that $h(0) = 0$.

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THEOREM A. *Let $E \subset B(O,1)$ be thin at O and h be such that*

$$\int_0^1 h(t)\, t^{1-n}\, dt < +\infty. \tag{2}$$

Then there is a sequence of balls $(B(Z_k, r_k))$, where $r_k < \frac{1}{2}|Z_k|$, such that

$$E \subset \bigcup_{k=1}^{\infty} B(Z_k, r_k) \quad \text{and} \quad \sum_{k=1}^{\infty} h(r_k/|Z_k|) < +\infty.$$

Theorem A appears not to have been explicitly stated before, but its proof requires only self-evident modification to an argument of Essén and Jackson ([**13**], theorem 5) concerning minimally thin sets in a half-space. When $n = 2$ a suitable choice of $h(t)$ is $(\max\{\log(1/t), 1\})^{-\beta}$, where $\beta > 1$ (cf. [**12**], theorem 7); when $n \geqslant 3$ we can take $h(t) = t^{\beta}$ for any $\beta > n-2$.

The interesting point about Theorem A in the present context is that it suggests a way of studying superharmonic functions u such that $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_p for some (X_k). For each $k = 0, 1, \ldots$ we define

$$A_k = \{X : 2^k < \phi_n(|X|) \leqslant 2^{k+1}\}.$$

THEOREM 2. *Let u be a superharmonic function on a neighbourhood of O, let $0 < \alpha < 1$ and $p \geqslant 1$, and suppose there is a sequence (X_k) converging regularly to O for which $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_p. If h satisfies (2), then there is a family of balls $(B(Z_{j,k}, r_{j,k}))_{j,k}$, where $B(Z_{j,k}, r_{j,k}) \subset A_{k-1} \cup A_k \cup A_{k+1}$ for each j, such that*

$$\sum_k \Big(\sum_j h(r_{j,k}/|Z_{j,k}|)\Big)^p < +\infty,$$

and

$$u(X)(\phi_n(|X|))^{-\alpha} \to 0 \quad \Big(X \to O;\, X \notin \bigcup_{j,k} B(Z_{j,k}, r_{j,k})\Big).$$

The *Hausdorff h-content* of a set F is defined by

$$M_h(F) = \inf \sum_j h(\rho_j),$$

where the infimum is taken over all coverings of F by a countable number of open balls of radii ρ_j. Properties of M_h can be found in [**8**], chapter II and [**9**], chapter VI; in particular, M_h is a general capacity (see [**9**], VI·19). If $h(t) = t^{\beta}$ for some $\beta > 0$, then M_h is called *β-dimensional Hausdorff content*. The *Hausdorff dimension* of a set F is the infimum of all such β for which $M_h(F) = 0$.

Recalling that a bounded polar subset of $\mathbb{R}^n$ has Hausdorff dimension $n-2$ (see [**8**], IV.1), we have the following analogue of Corollary 1.

COROLLARY 3. *Under the hypotheses of Theorem 2 with $1 \leqslant p < (n-1)/(n-2)$, there is a G_{δ} set $F \subset \partial B(O,1)$, of Hausdorff dimension $(n-2)p$, such that*

$$u(rY)(\phi_n(r))^{-\alpha} \to 0 \quad (r \to 0+\,;\, Y \in \partial B(O,1) \backslash F).$$

When $n = 2$, as will become apparent from the proof of Corollary 3, the above set F has zero Hausdorff content with respect to the measure function $h(t) = (\max\{\log(1/t), 1\})^{-\beta}$, for any $\beta > p$. The proofs of Theorem 2 and its corollary may be found in Section 5.

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3. *Boundary limits*

Boundary analogues of the results in Section 2 will now be described. Theorem 3 below is concerned with the minimal fine topology on the half-space D. (For the general theory of the minimal fine topology on the Martin compactification of a Green space, see [**6**], part 2 or [**11**], 1·XII.) In particular, we recall that a set $E \subset D$ is said to be *minimally thin at infinity* if there is a positive superharmonic function v on D such that $v(X) \geqslant x_n$ for $X \in E$, and $v(X)$ does not majorize x_n on D. Further, a function f on D is said to have *minimal fine limit* l at infinity if there is a set $E \subset D$, minimally thin at infinity, for which

$$f(X) \to l \quad (|X| \to \infty ; X \in D \backslash E).$$

Aikawa [1] has introduced a more general notion of thinness at infinity which can be formulated as follows. Let $0 \leqslant a \leqslant 1$. A set $E \subset D$ is *a-minimally thin at infinity* if and only if there is a positive superharmonic function v on D such that $v(X) \geqslant x_n^a |X|^{1-a}$ for $X \in E$ and such that $v(X)$ does not majorize x_n on D (cf. [**1**], theorem 3·1). Thus the case $a = 1$ gives the usual idea of minimal thinness at infinity. The case $a = 0$ corresponds to what Essén and Jackson (in [**14**], p. 249) call a *rarefied* set. (The term *rarefied* is used by Lelong-Ferrand [**18**] to mean something different.)

THEOREM 3. *Let u be a positive superharmonic function on D and let $-1 < \gamma < n-1$. If $(u(X_k)\, w(X_k) |X_k|^\gamma)$ belongs to l_1 for some sequence (X_k) converging regularly to infinity, then for each $a \in [0, 1]$ there is a set $E(a)$, a-minimally thin at infinity, such that*

$$u(X)\,(w(X))^a |X|^\gamma \to 0 \quad (|X| \to \infty ; X \in D \backslash E(a)).$$

In particular, the function $u(X)\, w(X) |X|^\gamma$ has minimal fine limit zero at infinity.

The converse to Theorem 3 is false. To see this, we note that, if r_γ is chosen to be a sufficiently large positive number, then the function

$$v(X) = \begin{cases} (w(X) |X|^\gamma \log |X|)^{-1} & (|X| > r_\gamma) \\ x_n\, r_\gamma^{-\gamma-1} / (\log r_\gamma) & (|X| \leqslant r_\gamma) \end{cases} \tag{3}$$

is positive and superharmonic on D. Clearly $v(X)\, w(X) |X|^\gamma$, and hence $v(X)\,(w(X))^a |X|^\gamma$, has limit zero as $|X| \to \infty$, yet there is no sequence (X_k), converging regularly to infinity, for which $(v(X_k)\, w(X_k) |X_k|^\gamma)$ belongs to l_1. Nevertheless it is readily checked that Theorem 3 is sharp in the same sense as is Theorem 1. In Section 6·3 we also point out that (in contrast to the case of minimal semi-fine limits) it is possible for a positive superharmonic function u, but not uw, to have minimal fine limit zero at infinity.

Beardon ([**4**], theorem 4·4) has obtained a related result for negative subharmonic functions on a two-dimensional half-strip, which can be reformulated for the half-space as in the following theorem. If (X_k) converges regularly to infinity and the sequence $(w(X_k))$ is bounded above (i.e. (X_k) approaches infinity in a non-tangential fashion), then we call (X_k) *admissible* (cf. [**4**], p. 127).

THEOREM B. *Let $0 < p \leqslant 1$ and u be a positive superharmonic function on the set $\mathbb{R} \times (0, +\infty)$. If there is an admissible sequence (X_k) such that $\sum (u(X_k))^p < +\infty$, then there is a set $F \subset \sigma$ of p-dimensional Hausdorff content zero such that $u(rY) \to 0$ as $r \to \infty$ for each $Y \in \sigma \backslash F$.*

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In view of the comment preceding Corollary 3, the following (with $n = 2$ and $\gamma = 0$) is an improvement of Theorem B.

COROLLARY 4. *Let u be a positive superharmonic function on D and let $-1 < \gamma < n-1$. If $(u(X_k)\, w(X_k)|X_k|^\gamma)$ belongs to l_1 for some sequence (X_k) converging regularly to infinity, then there is a G_δ polar set $F \subset \sigma$ such that $r^\gamma u(rY) \to 0$ as $r \to \infty$ for any $Y \in \sigma \backslash F$.*

A measurable subset F of the interval $(1, +\infty)$ is said to be of *finite logarithmic length* if $\int_F t^{-1}\, dt < +\infty$.

COROLLARY 5. *Let u be a positive superharmonic function on $\mathbb{R} \times (0, +\infty)$ and let $-1 < \gamma < n-1$. If $(u(X_k)\, w(X_k)|X_k|^\gamma)$ belongs to l_1 for some sequence (X_k) converging regularly to infinity, then there is an open set $F \subset (1, +\infty)$ of finite logarithmic length such that*

$$r^\gamma u(rY) \to 0 \quad (r \to \infty\,;\, r \notin F)$$

uniformly in $Y \in \sigma$.

There is a natural boundary analogue of Theorem 2. This can be proved by combining ideas from Section 5·1 and [**15**], section 5, but both the statement and proof are rather technical. For the sake of brevity we give only the following analogue of Corollary 3.

THEOREM 4. *Let u be a positive superharmonic function on D, let $-1 < \gamma < n-1$ and $1 \leqslant p < (n-1)/(n-2)$. If $(u(X_k)\, w(X_k)|X_k|^\gamma)$ belongs to l_p for some sequence (X_k) converging regularly to infinity, then there is a set $F \subset \sigma$, of Hausdorff dimension $(n-2)\, p$, such that*

$$r^\gamma u(rY) \to 0 \quad (r \to \infty\,;\, Y \in \sigma \backslash F).$$

When $n = 2$, the range of p allowed in Theorem 4 is $[1, +\infty)$, and the conclusion is always that there is a set $F \subset \sigma$ of Hausdorff dimension zero with the stated property. Thus Theorem 4 gives a further improvement of Theorem B.

Theorem 3 and its corollaries are proved in Section 6. The proof of Theorem 4 is indicated in Section 7.

4. *Proof of Theorem 1 and Corollaries*

4·1. In proving Theorem 1, we can assume that u is defined on $B(O, 1)$. By the Riesz decomposition, u can be represented in $B(O, \frac{1}{2})$ as the sum of a harmonic function H and a (Newtonian or logarithmic) potential. In fact, it is clearly sufficient to prove Theorem 1 and its corollaries when $H \equiv 0$, in which case

$$u(X) = \int_{B(O, \frac{1}{2})} \phi_n(|X - Y|)\, d\mu(Y) \quad (X \in B(O, \tfrac{1}{2})), \tag{4}$$

where μ is a finite measure on $B(O, \frac{1}{2})$. The following two lemmas will be used in the proof of Theorem 1.

4·2. LEMMA 1. *Let $0 < \alpha < 1$. There is a sequence (X_k) converging regularly to O for which $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_1 if and only if*

$$\int_{B(O, \frac{1}{2})} (\phi_n(|X|))^{1-\alpha}\, d\mu(X) < +\infty. \tag{5}$$

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To see this, first suppose that the above sequence is in l_1. There exists $\kappa > 1$ such that $\phi_n(|X_{k+1}|) < \kappa\phi_n(|X_k|)$ for each $k \in \mathbb{N}$. Let

$$B_k = \{X : \phi_n(|X|) > \kappa^k\} \quad (k \in \mathbb{N}).$$

There is no loss of generality in assuming that $X_k \in B_k \backslash B_{k+1}$ for all sufficiently large k. If $Y \in B_k$, then $\phi_n(|Y - X_k|) > C(n)\kappa^k$. (Here, and elsewhere, we use $C(a, b, c, \ldots)$ to denote a positive constant, depending at most on $a, b, c, \ldots$, not necessarily the same on any two occurrences.) Hence, from (4),

$$u(X_k)(\phi_n(|X_k|))^{-\alpha} \geqslant C(n, \kappa, \alpha)\kappa^{k(1-\alpha)}\mu(B_k), \tag{6}$$

and (5) now follows.

Conversely, suppose (5) holds, let $\kappa > 1$, and let $\mathcal{M}(u; r)$ denote the mean of u over the sphere $\partial B(O, r)$. We recall that

$$\mathcal{M}(\phi_n(|Y - \cdot|); r) = \min\{\phi_n(r), \phi_n(|Y|)\}$$

(see, for example, [11], I·1·5). It follows from (4) and a change in order of integration that

$$\mathcal{M}(u; \phi_n^{-1}(\kappa^{k+1})) - \mathcal{M}(u; \phi_n^{-1}(\kappa^k)) \leqslant \kappa^{k+1}\mu(B_k),$$

whence

$$\sum_{k=1}^{\infty} \kappa^{-(k+1)\alpha}\{\mathcal{M}(u; \phi_n^{-1}(\kappa^{k+1})) - \mathcal{M}(u; \phi_n^{-1}(\kappa^k))\} \leqslant \sum_{k=1}^{\infty} \kappa^{(k+1)(1-\alpha)}\mu(B_k).$$

Thus, from (5),

$$\sum_{k=1}^{\infty} \kappa^{-k\alpha}\mathcal{M}(u; \phi_n^{-1}(\kappa^k)) < +\infty,$$

and so there is a sequence (X_k) such that $\phi_n(|X_k|) = \kappa^k$ and $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_1.

4·3. With μ as in Section 4·1, let

$$A(X) = \{Y : \tfrac{1}{2} < \phi_n(|X|)/\phi_n(|Y|) < 2\}$$

and

$$u_0(X) = \int_{A(X)} \phi_n(|X - Y|)\, d\mu(Y) \quad (X \in B(O, \tfrac{1}{4})).$$

LEMMA 2. *Let $0 < \alpha < 1$. If there is a sequence (X_k) converging regularly to O for which $u(X_k)(\phi_n(|X_k|))^{-\alpha} \to 0$, then*

$$(u(X) - u_0(X))(\phi_n(|X|))^{-\alpha} \to 0 \quad (X \to O).$$

When $n \geqslant 3$ this lemma is essentially contained in [20], lemma 1. We give below the proof in the case $n = 2$ for the sake of completeness. From (6) and the hypothesis of the lemma it is clear that

$$(\log(1/r))^{1-\alpha}\mu(B(O, r)) \to 0 \quad (r \to 0+). \tag{7}$$

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Using integration by parts and (7) we have

$$\limsup_{r\to 0+}(\log(1/r))^{-\alpha}\int_{\{\frac{1}{2}>|Y|\geqslant r\}}\log(1/|Y|)\,d\mu(Y)$$

$$=\limsup_{r\to 0+}(\log(1/r))^{-\alpha}\int_r^{r_0}\mu(B(O,\rho))\,\rho^{-1}\,d\rho \quad (0<r_0\leqslant\tfrac{1}{2})$$

$$=0. \tag{8}$$

If $|X|<\frac{1}{4}$ then, from (4),

$$u(X)-u_0(X)=\int_{B(O,\frac{1}{2})\setminus A(X)}\log(1/|X-Y|)\,d\mu(Y)$$

$$\leqslant\log(4/(3|X|))\,\mu(\overline{B(O,|X|^2)})+\int_{\{\frac{1}{2}>|Y|\geqslant\sqrt{(|X|)}\}}\log(2/|Y|)\,d\mu(Y).$$

The lemma now follows from (7) and (8).

4·4. The proof of Theorem 1 will now be completed. Let μ_k be the measure defined on Borel sets F by

$$\mu_k(F)=\mu(F\cap(A_{k-1}\cup A_k\cup A_{k+1}))\quad(k\in\mathbb{N}),$$

where A_k is as defined in Section 2, let

$$u_k(X)=\int\phi_n(|X-Y|)\,d\mu_k(Y)\quad(X\in\mathbb{R}^n),$$

and let $\|\mu_k\|=\mu_k(\mathbb{R}^n)$. It follows from Lemma 1 that

$$\sum_{k=1}^{\infty}2^{k(1-\alpha)}\|\mu_k\|<+\infty.$$

Now let (a_k) be a sequence of positive numbers such that

$$(\|\mu_k\|+2^{-k})/a_k\to 0\quad\text{and}\quad\sum_{k=1}^{\infty}2^{k(1-\alpha)}a_k<+\infty, \tag{9}$$

let ν be the finite measure defined by

$$\nu=\sum_{k=1}^{\infty}2^{k(1-\alpha)}a_k(\|\mu_k\|+2^{-k})^{-1}\mu_k,$$

and let

$$v(X)=\int\phi_n(|X-Y|)\,d\nu(Y).$$

Further, let

$$E_k=\{X:2^k<\phi_n(|X|)<2^{k+2}\quad\text{and}\quad u_k(X)\,(\phi_n(|X|))^{-\alpha}>(\|\mu_k\|+2^{-k})/a_k\} \tag{10}$$

and $E=\bigcup_k E_k$. (We note, for future reference, that each E_k is open.) If $X\in E_k$, then by (10) and the definition of ν

$$v(X)\geqslant 2^{k(1-\alpha)}a_k(\|\mu_k\|+2^{-k})^{-1}u_k(X)\geqslant 4^{\alpha-1}\phi_n(|X|).$$

Since $\nu(\{O\})=0$, the set E is thin at O.

It remains to show that

$$u(X)\,(\phi_n(|X|))^{-\alpha}\to 0\quad(X\to O;X\notin E).$$

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In fact, if $X \in A_k \backslash E$, it follows from (9), (10) and Lemma 2 that (as $k \to \infty$)

$$u(X)(\phi_n(|X|))^{-\alpha} \leqslant (u(X)-u_0(X))(\phi_n(|X|))^{-\alpha} + u_k(X)(\phi_n(|X|))^{-\alpha}$$
$$\leqslant o(1) + (\|\mu_k\| + 2^{-k})/a_k = o(1). \tag{11}$$

Theorem 1 is now established.

4·5. Let (a_k) be a decreasing sequence of positive numbers with the property that $\sup_k(a_k/a_{k+1}) < +\infty$. Theorem 1 is sharp in the sense that, if $(a_k) \notin l_1$, then there is a positive superharmonic function u on a neighbourhood of O and a sequence (X_k) converging regularly to O such that $u(X_k)(\phi_n(|X_k|))^{-\alpha} \leqslant a_k$ and yet $u(X)(\phi_n(|X|))^{-\alpha}$ fails to have fine limit zero at O.

We will show this for $n \geqslant 3$, the modifications required for the case $n = 2$ being minor. Let $c > (\sup_k(a_k/a_{k+1}))^{2/\alpha}$, let

$$X_k = (0, \ldots, 0, \phi_n^{-1}(c^k)), \quad b_k = c^{k(\alpha-1)}a_k, \tag{12}$$

and

$$u(X) = \sum_{k=1}^{\infty} b_k \phi_n(|X + X_k|) \quad (X \in \mathbb{R}^n).$$

Since $\sum b_k$ converges, u is superharmonic in $\mathbb{R}^n$. Also (X_k) converges regularly to O and (using (12))

$$u(X_k) \leqslant \sum_{j=1}^{k-1} b_j c^j + c^k \sum_{j=k}^{\infty} b_j$$
$$\leqslant c^{k\alpha}a_k \sum_{j=1}^{k-1} c^{(-\alpha/2)(k-j)} + c^k a_k \sum_{j=k}^{\infty} c^{j(\alpha-1)}$$
$$\leqslant c^{k\alpha}a_k\{c^{-\alpha/2}(1-c^{-\alpha/2})^{-1} + (1-c^{\alpha-1})^{-1}\},$$

whence

$$u(X_k)(\phi_n(|X_k|))^{-\alpha} \leqslant C(c, \alpha)\, a_k.$$

However, $u(X)(\phi_n(|X|))^{-\alpha} \geqslant c^{-\alpha}$ whenever $X \in \bigcup_k E_k$, where

$$E_k = \{X : c^k < \phi_n(|X|) \leqslant c^{k+1} \quad \text{and} \quad c^{-\alpha k} b_k \phi_n(|X + X_k|) \geqslant 1\}.$$

Also, the Newtonian capacity of E_k is at least

$$2^{2-n} \min\{c^{-\alpha k} b_k, c^{-k}(1 - c^{1/(2-n)})^{n-2}\}.$$

It now follows from Wiener's criterion ([**6**], theorem IX, 10) that, if $(a_k) \notin l_1$, then $\bigcup_k E_k$ is not thin at O, so u does not have fine limit zero at O.

4·6. To prove Corollary 1, let E and E_k be as in Section 4·4 and define E_k^* to be the radial projection of E_k onto $\partial B(O, 1)$. Then, since each E_k is open, the set

$$F = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k^* \right)$$

is a G_δ subset of $\partial B(O, 1)$. Further, since E is thin at O, a result of Deny [**10**] asserts that F is polar. If $Y \in \partial B(O, 1) \backslash F$, it follows from (9), (10) and (11) that

$$\limsup_{r \to 0+} u(rY)(\phi_n(r))^{-\alpha} \leqslant \limsup_{k \to \infty} (\|\mu_k\| + 2^{-k})/a_k = 0.$$

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Corollary 1 is now established.

4·7. In Corollary 2 we are dealing only with the case $n = 2$. By Theorem 1 there is an open set E, thin at O, for which

$$u(X)(\phi_2(|X|))^{-\alpha} \to 0 \quad (X \to O; X \notin E).$$

(We can choose E to be open because of the assertion following (10).) Let

$$F = \{r \in (0,1) : \partial B(O,r) \cap E \neq \varnothing\}.$$

Then F is open and

$$u(rY)(\phi_2(r))^{-\alpha} \to 0 \quad (r \to 0+; r \notin F)$$

uniformly in $Y \in \partial B(O,1)$. Further, it follows from Wiener's criterion that (1) holds (cf. [5], p. 336 for details). Corollary 2 is now established.

5. *Proof of Theorem 2 and Corollary*

5·1. In what follows we adapt ideas from [13, 15]. We will prove the theorem for $n \geqslant 3$, and then indicate briefly how to modify the argument to deal with the two-dimensional case. Again we can assume that u has the form (4).

Let κ, μ, B_k, μ_k, and u_k be as in Section 4. Since the sequence $(u(X_k)(\phi_n(|X_k|))^{-\alpha})$ belongs to l_p, so does $(\kappa^{k(1-\alpha)}\mu(B_k))$ by (6), whence

$$\sum_{k=1}^{\infty} 2^{k(1-\alpha)p}\|\mu_k\|^p < +\infty.$$

We redefine (a_k) as a sequence of positive numbers such that

$$(\|\mu_k\| + 2^{-k})/a_k \to 0 \quad \text{and} \quad \sum_{k=1}^{\infty} 2^{k(1-\alpha)p} a_k^p < +\infty, \tag{13}$$

and put $E = \bigcup_k E_k$, where

$$E_k = \{X \in A_k : u_k(X)(\phi_n(|X|))^{-\alpha} > (\|\mu_k\| + 2^{-k})/a_k\}. \tag{14}$$

We also let u_k^* be the Newtonian potential due to the measure μ_k^* defined on Borel sets F by

$$\mu_k^*(F) = 2^{(k+2)(1-\alpha)}\mu_k(\{2^{-(k+2)/(n-2)}X : X \in F\}),$$

and define

$$E_k^* = \{X : 2 \leqslant |X|^{n-2} < 4 \quad \text{and} \quad u_k^*(X)(\phi_n(|X|))^{-\alpha} > (\|\mu_k\| + 2^{-k})/a_k\}. \tag{15}$$

We now fix k and choose an arbitrary compact subset J of E_k^*. By a result of Frostman ([17], lemma 5·4) there is a measure ν (with corresponding Newtonian potential v) supported by J such that $\nu(B(X,r)) \leqslant h(r)$ for any ball $B(X,r)$, and such that

$$M_h(J) \leqslant C(n)\,\nu(J). \tag{16}$$

Since $J \subset B(O,4)$, it follows from (2) and integration by parts that, if $X \in B(O,4)$, then

$$\begin{aligned} v(X) &= \int_J \phi_n(|X-Y|)\,d\nu(Y) \\ &= (n-2)\int_0^8 \nu(B(X,t))\,t^{1-n}\,dt + \nu(B(X,8))\,\phi_n(8) \\ &\leqslant C(n,h). \end{aligned}$$

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Then, using (15), (16), and a change in order of integration,

$$\begin{aligned} M_h(J) &\leqslant C(n,\alpha)\, a_k(\|\mu_k\|+2^{-k})^{-1} \iint \phi_n(|X-Y|)\, d\mu_k^*(X)\, d\nu(Y) \\ &= C(n,\alpha)\, a_k(\|\mu_k\|+2^{-k})^{-1} \int v(X)\, d\mu_k^*(X) \\ &\leqslant C(n,\alpha,h)\, 2^{k(1-\alpha)} a_k. \end{aligned}$$

In view of the arbitrary nature of J, it follows that

$$M_h(E_k^*) \leqslant C(n,\alpha,h)\, 2^{k(1-\alpha)} a_k.$$

Thus there is a family of balls $(B(Z_{j,k}^*, r_{j,k}^*))_j$ such that

$$E_k^* \subset \bigcup_j B(Z_{j,k}^*, r_{j,k}^*)$$

and

$$(\sum_j h(r_{j,k}^*))^p \leqslant C(n,\alpha,h,p)\,(2^{k(1-\alpha)p} a_k^p + 2^{-k}). \tag{17}$$

Further, there exists k_0 such that, for $k \geqslant k_0$, each $B(Z_{j,k}^*, r_{j,k}^*)$ is contained in the set $\{X : 8 > |X|^{n-2} \geqslant 1\}$. Putting

$$Z_{j,k} = 2^{-(k+2)/(n-2)} Z_{j,k}^* \quad \text{and} \quad r_{j,k} = 2^{-(k+2)/(n-2)} r_{j,k}^*,$$

we have

$$E_k \subset \bigcup_j B(Z_{j,k}, r_{j,k}), \quad B(Z_{j,k}, r_{j,k}) \subset A_{k-1} \cup A_k \cup A_{k+1}.$$

Also, from (13), (17), the monotonicity of h, and the fact that $|Z_{j,k}^*| \geqslant 1$,

$$\sum_{k=k_0}^{\infty} (\sum_j h(r_{j,k}/|Z_{j,k}|))^p \leqslant \sum_{k=k_0}^{\infty} (\sum_j h(r_{j,k}^*))^p < +\infty. \tag{18}$$

Finally, if $X \in A_k \backslash E$, it follows from (13), (14) and Lemma 2 that

$$u(X)\,(\phi_n(|X|))^{-\alpha} = o(1) \quad (k \to \infty)$$

(cf. the final step of Section 4·4). Since $E \subset \bigcup_{j,k} B(Z_{j,k}, r_{j,k})$, Theorem 2 is proved.

The above argument can be adapted to the case $n = 2$ by using μ_k to denote the restriction of μ to $\{X : e^{-2(k+2)} \leqslant |X| < e^{-2(k-1)}\}$ and using the fact that, with this notation,

$$\sum_{j=1}^{\infty} 2^{j(1-\alpha)p} \left(\sum_{k=2^j}^{2^{j+1}-1} \|\mu_k\| \right)^p < +\infty.$$

5·2. We now prove Corollary 3. Let h satisfy (2), let $Z_{j,k}^*$ and $r_{j,k}^*$ be as in Section 5·1, let F_k denote the radial projection of $\bigcup_j B(Z_{j,k}^*, r_{j,k}^*)$ onto $\partial B(O,1)$ and put

$$F = \bigcap_{i=1}^{\infty} \left(\bigcup_{k=i}^{\infty} F_k \right).$$

Then F is a G_δ subset of $\partial B(O,1)$. Further, given $\epsilon > 0$, there exists i such that F has a countable covering by balls of radii $\rho_{j,k} \leqslant r_{j,k}^*$ satisfying

$$\sum_{k=i}^{\infty} \sum_j (h(\rho_{j,k}))^p \leqslant \sum_{k=i}^{\infty} (\sum_j h(r_{j,k}^*))^p < \epsilon,$$

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using (18) and the fact that each $B(Z^*_{j,k}, r^*_{j,k})$ is contained in $\{|X| \geqslant 1\}$. Hence F has zero Hausdorff content with respect to the measure function $(h(t))^p$. Taking $h(t) = t^{n-2}(\max\{\log(1/t), 1\})^{-2}$, so that $t^\beta/h(t) \to 0$ as $t \to 0+$ for $\beta > n-2$, the corollary is established.

Taking $h(t) = (\log(1/t))^{-1}(\log\{\log(1/t)\})^{-2}$ for small $t > 0$, the remark following Corollary 3 is also proved.

6. *Proof of Theorem 3 and Corollaries*

6·1. Before proving Theorem 3, we collect together some preliminary material. The Green function for the half-space D is defined by

$$G(X, Y) = \phi_n(|X-Y|) - \phi_n(|X-Y^*|) \quad (X, Y \in D),$$

where Y^* is the reflection of Y in ∂D. There is a one-to-one correspondence between positive superharmonic functions u on D and pairs (c, μ), where $c \in [0, +\infty)$ and μ is a measure on $\bar{D}$ satisfying

$$\int_{\bar{D}} (1+|X|)^{-n}\, d\mu(X) < +\infty.$$

This follows from the Riesz–Herglotz representation,

$$u(X) = cx_n + \int_{\bar{D}} K(Y, X)\, d\mu(Y), \tag{19}$$

where

$$K(Y, X) = \begin{cases} G(X, Y)/y_n & \text{if} \quad (X, Y) \in D \times D \\ 2c_n x_n |X-Y|^{-n} & \text{if} \quad (X, Y) \in D \times \partial D, \end{cases}$$

and $c_n = \max\{1, n-2\}$. We will use the well-known inequalities

$$2c_n x_n y_n |X-Y^*|^{-n} \leqslant G(X, Y) \leqslant 2c_n x_n y_n |X-Y|^{-n}, \tag{20}$$

which follow easily from the fact that

$$|X-Y^*|^2 = |X-Y|^2 + 4x_n y_n.$$

6·2. Let (c, μ) be the pair corresponding to a positive superharmonic function u on D. Aikawa [1], theorem 3·2 has shown that the conclusion of Theorem 3 holds if $c = 0$ and

$$\int_{\bar{D}} (1+|X|)^{1+\gamma-n}\, d\mu(X) < +\infty. \tag{21}$$

Thus Theorem 3 will follow from the *only if* part of the lemma below.

LEMMA 3. *Let u be a positive superharmonic function on D and let $-1 < \gamma < n-1$. Then $(u(X_k)\, w(X_k)\, |X_k|^\gamma)$ belongs to l_1 for some sequence (X_k) converging regularly to infinity if and only if $c = 0$ and (21) holds.*

To prove the *only if* part, let $\kappa > 1$ be such that $|X_{k+1}| < \kappa |X_k|$ for each $k \in \mathbb{N}$, and let $I_k = B(O, \kappa^k) \cap \bar{D}$. There is no loss of generality in assuming that $X_k \in I_k \backslash I_{k-1}$ for all sufficiently large $k \in \mathbb{N}$. If $Y \in I_k$, then $|X_k - Y^*| \leqslant 2\kappa^k$. Hence, from (19) and (20),

$$u(X_k)\, w(X_k)\, |X_k|^\gamma \geqslant C(n, \kappa)\, \kappa^{(1+\gamma-n)k}\, \mu(I_k), \tag{22}$$

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and
$$u(X_k)\,w(X_k)\,|X_k|^\gamma \geqslant c\kappa^{(1+\gamma)(k-1)}. \tag{23}$$

Combining the l_1 condition with (23), it is clear that $c = 0$. Combining the l_1 condition with (22), (21) is established.

Conversely, suppose $c = 0$ and (21) holds, and define

$$\mathcal{N}(u;r) = r^{-n-2}\int_{\partial B(O,r)\cap D} x_n^2\,u(X)\,w(X)\,d\sigma(X),$$

where σ denotes surface area measure. Noting that

$$\mathcal{N}(u;r) = C(n)\int_{\bar{D}} \min\{r^{-n}, |X|^{-n}\}\,d\mu(X)$$

(see, for example, [**16**], lemma 1), it follows from (21) that

$$\sum_{k=1}^{\infty} 2^{k(\gamma+1)}\mathcal{N}(u;2^k) < +\infty$$

(cf. the second paragraph of Section 4·2). Thus there is a sequence (X_k) in D such that $|X_k| = 2^k$ for each k and such that $(u(X_k)\,w(X_k)\,|X_k|^\gamma)$ belongs to l_1.

6·3. Let u be a positive superharmonic function on D. Then (as can be observed by comparing [**22**] with [**21**]) u has minimal semi-fine limit zero at infinity if and only if uw has. In the case of minimal fine limits, the *if* part of this assertion clearly remains true, but the converse is false as we now show.

Let $\gamma = 0$, let v be as in (3) and let $u = v^{\frac{1}{2}}$. Then u is a positive superharmonic function on D with limit (and hence minimal fine limit) zero at infinity. Since

$$\int_1^\infty t^{-2}[t/\log t]\,dt = +\infty,$$

it follows from theorem 2 of Brelot and Doob [**7**] that the set

$$E = \{(X', x_n)\in\mathbb{R}^{n-1}\times(0,+\infty) : x_n < |X'|/\log|X'|\}$$

is not minimally thin at infinity. But it is easy to see that $u(X)\,w(X) \geqslant 1$ on $E\backslash B(O,\rho)$, for some $\rho > 0$. Hence uw cannot have minimal fine limit zero at infinity.

6·4. Corollary 4 follows from the first assertion of Theorem 3 by an argument directly analogous to that of Corollary 1. In place of the cited result of Deny, we appeal to its analogue for minimally thin sets in D, which is due to Lelong-Ferrand [**18**], §8.

6·5. In Corollary 5 we are dealing only with $n = 2$. Let $E(0)$ be the rarefied set in Theorem 3 (taking $a = 0$). Azarin [**3**] (although he did not use this terminology) has proved a geometrical covering result for rarefied sets. This asserts that there is a sequence of open discs $(B(Z_k, r_k))$ such that $E(0)\subset\bigcup_k B(Z_k, r_k)$ and $\sum(r_k/|Z_k|)$ converges. Letting

$$F = \{r\in(1,+\infty) : \partial B(O,r)\cap B(Z_k, r_k) \neq \varnothing \text{ for some } k\in\mathbb{N}\},$$

it is easy to see that F is open and of finite logarithmic length. Further, $r^\gamma u(rY)\to 0$ uniformly in $Y\in\sigma$ as $r\to\infty$, where $r\notin F$.

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7. *Proof of Theorem* 4

The proof of Theorem 4 is modelled on the argument of Section 5, so we give below only an indication of the proof. Let (c, μ) be the pair corresponding to a positive superharmonic function u on D. From (22), (23) and the l_p hypothesis it follows that $c = 0$ and that

$$\sum_k (2^{(1+\gamma-n)k}\mu(\{X \in \bar{D} : 2^{k-1} \leqslant |X| < 2^k\}))^p < +\infty. \tag{24}$$

Arguing as in Lemma 2, it is sufficient to show that there is a set $F \subset \sigma$ of dimension $(n-2)p$ such that

$$r^\gamma u'(rY) \to 0 \quad (r \to \infty ; Y \in \sigma \backslash F), \tag{25}$$

where

$$u'(X) = \int_{\{Y \in \bar{D} : \frac{1}{2}|X| < |Y| < 2|X|\}} K(Y, X)\, d\mu(Y).$$

Now let $1 < b_1 < b_2$ and $K_i = \{X \in D : w(X) < b_i\}$ for $i = 1, 2$. It is sufficient to prove that there is a set $F \subset \sigma$, of dimension $(n-2)p$, such that (25) holds for $Y \in (K_1 \cap \sigma) \backslash F$. In fact, since

$$2^{(1+\gamma-n)k}\mu(\{X \in \bar{D} : 2^{k-1} \leqslant |X| < 2^k\}) \to 0 \quad (k \to \infty)$$

(see (24)), it follows from (20) that we need only establish

$$r^\gamma u''(rY) \to 0 \quad (r \to \infty ; Y \in (K_1 \cap \sigma) \backslash F),$$

where

$$u''(X) = \int_{\{Y \in \bar{D} \cap K_2 : \frac{1}{2}|X| < |Y| < 2|X|\}} K(Y, X)\, d\mu(Y).$$

The remaining argument is a routine modification of Sections 5·1, 5·2 and is left to the reader.

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