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A metric minimal PI cascade with $2^{\mathfrak{c}}$ minimal ideals

ELI GLASNER† and YAIR GLASNER‡

† *Department of Mathematics, Tel Aviv University, Tel Aviv, Israel*
(e-mail: glasner@math.tau.ac.il)
‡ *Department of Mathematics, Ben-Gurion University of the Negev, Beõer Sheva, Israel*
(e-mail: yairgl@math.bgu.ac.il)

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Abstract. We first improve an old result of McMahon and show that a metric minimal flow whose enveloping semigroup contains less than $2^{\mathfrak{c}}$ (where $\mathfrak{c} = 2^{\aleph_0}$) minimal left ideals is proximal isometric (PI). Then we show the existence of various minimal PI-flows with many minimal left ideals, as follows. For the acting group $G = \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$, we construct a metric minimal PI G-flow with $\mathfrak{c}$ minimal left ideals. We then use this example and results established in Glasner and Weiss. [On the construction of minimal skew-products. *Israel J. Math.* **34** (1979), 321–336] to construct a metric minimal PI cascade (X, T) with $\mathfrak{c}$ minimal left ideals. We go on to construct an example of a minimal PI-flow $(Y, \mathcal{G})$ on a compact manifold Y and a suitable path-wise connected group $\mathcal{G}$ of a homeomorphism of Y, such that the flow $(Y, \mathcal{G})$ is PI and has $2^{\mathfrak{c}}$ minimal left ideals. Finally, we use this latter example and a theorem of Dirbák to construct a cascade (X, T) that is PI (of order three) and has $2^{\mathfrak{c}}$ minimal left ideals. Thus this final result shows that, even for cascades, the converse of the implication 'less than $2^{\mathfrak{c}}$ minimal left ideals implies PI', fails.

Key words: enveloping semigroup, PI-flow, minimal left ideals
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1. *Introduction*

For a compact metric space X, let $\mathrm{Homeo}(X)$ denote the Polish group of self homeomorphisms of X equipped with the compact open topology. In this work, a *G-flow* (X, G) is a pair consisting of a compact (usually metrizable) space X and a continuous homomorphism of the topological group G into $\mathrm{Homeo}(X)$. We usually write $(g, x) \mapsto gx$ for the action of G on X that is defined via this representation. A flow $(X, \mathbb{Z})$, with the group of integers $\mathbb{Z}$ as the acting group, is called a *cascade* and is usually denoted as (X, T), where T is the homeomorphism that corresponds to $1 \in \mathbb{Z}$.

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For a flow (X, G), let $E(X, G)$ denote its enveloping semigroup. Recall that $E(X, G)$ is defined as the closure, in the compact space X^X, of the collection of homeomorphisms that is the image of the homomorphism from G into Homeo(X) that defines the dynamical system (X, G). $E(X, G)$ has a structure of a compact right topological semigroup and it is also a G-dynamical system. By a theorem of Ellis, it always has minimal left ideals, which coincide with its minimal subsystems. We let mi(X, G) denote the cardinality of the collection of minimal left ideals in $E(X, G)$. For the definition of proximal-isometric (PI)-flows and for more details on enveloping semigroups and the structure theory of minimal flows, we refer the reader to [**2**, **7**, **16**] (see also §2 below).

The main result of [**6**], actually stated as the title of that paper, is that (for any acting group G) a metric minimal flow whose enveloping semigroup contains finitely many minimal left ideals is PI. This result was later greatly improved by McMahon [**14**], who showed that a metric minimal flow with mi$(X, G) < 2^{\aleph_1}$ is PI. In the first section of this note, following a suggestion of Akin, we improve McMahon's result and show that a metric minimal flow whose enveloping semigroup contains less than $2^{\mathfrak{c}}$ (where $\mathfrak{c} = 2^{\aleph_0}$) minimal left ideals is PI.

At the end of [**6**], the first named author claimed that a certain metric minimal PI G-flow (X, G), with $G = \mathrm{SL}(2, \mathbb{R})$, has an enveloping semigroup $E(X, G)$ with $\mathfrak{c}$ minimal left ideals. Unfortunately, the argument given in [**6**, example on p. 91] does not at all show this. Instead, it actually shows that each minimal left ideal in $E(X, G)$ contains $\mathfrak{c}$ idempotents. In §3, we construct an example of a metric minimal PI-flow $(\Omega, \mathbf{G})$ (of order two), with $\mathbf{G} = G^{\mathbb{N}} = \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$ and with the property that mi$(\Omega, \mathbf{G}) = \mathfrak{c}$. Then, in §4, we use this example and results established in [**10**] to construct a metric minimal PI cascade (X, T) (of order three) with mi$(X, T) = \mathfrak{c}$. In §5, we construct a minimal PI-flow (X, G) (of order two) on the Cantor set X, with a suitable acting group G and with mi$(X, G) = 2^{\mathfrak{c}}$. Finally, in §6, we construct a minimal PI cascade (of order three) (X, T) with mi$(X, T) = 2^{\mathfrak{c}}$. Thus this final result shows that, even for cascades, the converse of the implication, mi$(X, T) < 2^{\mathfrak{c}} \Rightarrow PI$, does not hold. The Morse minimal set (X, T), which is PI of order three, has mi$(X, T) = 2$ (see [**11**, Theorem 3.1] and [**15**]). In Remark 4.4 at the end of §4 we indicate how to construct, given $n \in \mathbb{N}$, a minimal PI cascade with 2^n minimal left ideals. It seems that it should not be too hard to construct minimal PI cascades with mi$(X, T) = n$ for any $n = 1, 2, \ldots, \aleph_0$. However, we leave this question open.

2. *Non PI-flows have $2^{\mathfrak{c}}$ minimal left ideals*

The way in which we will usually control the number of minimal left ideals in an enveloping semigroup E is by computing the number of minimal idempotents in E that are equivalent to a given minimal idempotent u. We recall the definitions and basis facts concerning these notions (see, e.g., [**7**, Ch. 1, §2] or [**8**, Ch. 1]).

THEOREM 2.1.

(1) *A subset M of E is a minimal left ideal of the semigroup E if and only if it is a minimal subsystem of (E, G). In particular, a minimal left ideal is closed. Minimal left ideals M in E exist and for each such ideal the set of idempotents in M, denoted by $J = J(M)$, is non-empty. We say that an idempotent in E is a* minimal idempotent *if it belongs to some minimal left ideal.*

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(2) *Let M be a minimal left ideal and let J be its set of idempotents.*
   (a) *For $v \in J$ and $p \in M$, $pv = p$.*
   (b) *For each $v \in J$, $vM = \{vp : p \in M\}$ is a subgroup of M with identity element v. For every $w \in J$ the map $p \mapsto wp$ is a group isomorphism of vM onto wM.*
   (c) *$\{vM : v \in J\}$ is a partition of M. Thus if $p \in M$, then there exists a unique $v \in J$ such that $p \in vM$; we denote by p^{-1} the inverse of p in vM.*

(3) *Let K, L and M be minimal left ideals of E. Let v be an idempotent in M. Then there exists a unique idempotent v' in L such that $vv' = v'$ and $v'v = v$. (We write $v \sim v'$ and say that v' is* equivalent *to v.) If $v'' \in K$ is equivalent to v', then $v'' \sim v$. The map $p \mapsto pv'$ of M to L is an isomorphism of G-systems.*

Now to PI-flows. We say that a minimal flow (X, G) is a *strictly PI-flow* if there is an ordinal η (which is countable when X is metrizable) and a family of flows $\{(W_\iota, w_\iota)\}_{\iota \le \eta}$ such that (i) W_0 is the trivial flow, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ that is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \le \eta$ the flow W_ν is the inverse limit of the flows $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that (X, G) is a *PI-flow* if there exists a strictly PI-flow $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

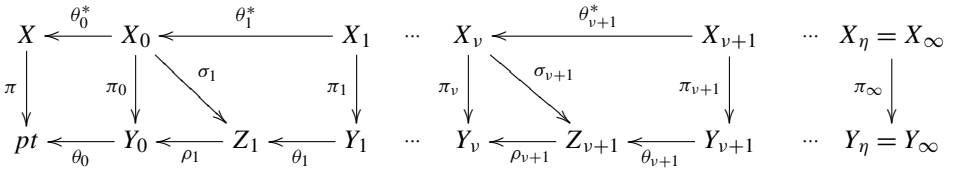
We say that the extension $(X, G) \xrightarrow{\pi} (Y, G)$ is an RIC (*relatively incontractible*) *extension* if, for every $n \in \mathbb{N}$, the minimal points are dense in the relation

$$R_\pi^{(n)} = \{(x_1, x_2, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j)\ \forall 1 \le i, j \le n\}.$$

It is easy to see that an RIC extension satisfies the *generalized Bronstein condition* as defined in [**16**, p. 813], with $\pi_i = \pi$, for all $1 \le i \le n$. Also, an RIC extension is necessarily an open map. Clearly, every distal extension is RIC, so, in particular, every distal extension is open.

The structure theorem for the general minimal flow is proved in [**4**, **13**] (see also [**16**]) and asserts that every minimal flow admits a canonically defined proximal extension which is a weakly mixing RIC extension of a strictly PI-flow. Both the Furstenberg and the Veech–Ellis structure theorems (for minimal distal and point-distal flows, respectively) are corollaries of this general structure theorem. For more details on the structure theorem see [**2**, **7**, **16**].

THEOREM 2.2. (Structure theorem for minimal flows) *Given a minimal flow (X, G), there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-tower),*



where, for each $\nu \le \eta$, π_ν is RIC, ρ_ν is isometric, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of X_ι, Y_ι, π_ι etc. for $\iota < \nu$. Thus X_∞ is a proximal extension of X and an RIC weakly

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mixing extension of the strictly PI-flow Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI-flow.

The weak mixing of all orders of the extension π_∞ means that for every $n \in \mathbb{N}$ the relation

$$R^{(n)}_{\pi_\infty} = \{(x_1, x_2, \ldots, x_n) \in X^n_\infty : \pi_\infty(x_i) = \pi_\infty(x_j)\ \forall 1 \le i,\ j \le n\}$$

is topologically transitive. This augmented version of the structure theorem follows from Theorem 2.6.2 in Veech's review paper [**16**]. It can also be proved by a slight modification of E. Glasner's proof of the structure theorem as it appears in [**2**, Theorem 27, Ch. 14 (p. 219)].

Now let (X, G) be a minimal metric flow and let $\pi : (X, G) \to (Y, G)$ a homomorphism. Let Ξ denote the Cantor set. We choose a sequence $\mathcal{U}_n$ of clopen partitions of Ξ so that $\mathcal{U}_{n+1} \prec \mathcal{U}_n$ and so that $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ generates the topology on Ξ. We let $k_n = \operatorname{card} \mathcal{U}_n$.

Let R^Ξ_π be the compact subset of X^Ξ defined by the condition

$$R^\Xi_\pi = \{r \in X^\Xi : \pi(r(\xi_1)) = \pi(r(\xi_2))\ \forall \xi_1, \xi_2 \in \Xi\}.$$

That is, the set of r such that $\pi \circ r : \Xi \to Y$ is a constant function. We denote by CR^Ξ_π the subset of R^Ξ_π consisting of continuous maps, and by UCR^Ξ_π the topological space whose underlying set is CR^Ξ_π but equipped with the topology of uniform convergence: i.e., the topology induced by the metric

$$D(r_1, r_2) = \sup_{\xi\in\Xi} d(r_1(\xi), r_2(\xi)).$$

It is easy to see that D is a complete metric. For each $n \in \mathbb{N}$, we will identify the space $R^{k_n}_\pi$ with the subspace of UCR^Ξ_π consisting of functions that are constant on each cell of $\mathcal{U}_n$. Clearly, $\bigcup_{n\in\mathbb{N}} R^{k_n}_\pi$ is a dense subset of both spaces UCR^Ξ_π and R^Ξ_π. In particular, it follows that UCR^Ξ_π is a Polish space.

Let $\iota : UCR^\Xi_\pi \to CR^\Xi_\pi \subset R^\Xi_\pi$ denote the natural map. Clearly, ι is a continuous injection. We observe that the image space CR^Ξ_π is dense in R^Ξ_π. Of course all of these spaces are G-invariant; however, whereas the spaces $R^{k_n}_\pi$ and R^Ξ_π are indeed flows, in the sense that their phase spaces are compact, the space UCR^Ξ_π is merely a Polish space and thus (UCR^Ξ_π, G) is a *Polish dynamical system*. We now have the following crucial proposition whose proof mimics that of [**1**, Proposition 4.10].

PROPOSITION 2.3. *Let (X, G) be a minimal metric flow and let $(X, G) \xrightarrow{\pi} (Y, G)$ be a homomorphism. Suppose that π is a weakly mixing extension of all (finite) orders. Then the (non-metrizable) flow (R^Ξ_π, G) is topologically transitive and has a dense set of transitive points.*

Proof. We will first show that the Polish dynamical system (UCR^Ξ_π, G) is topologically transitive. So let V_1, V_2 be two non-empty open subsets of UCR^Ξ_π. There are $n_1, n_2 \in \mathbb{N}$ with $V_i \cap R^{k_{n_i}}_\pi \neq \emptyset$, $i = 1, 2$. Then, with $n = \max(n_1, n_2)$ we have $V_i \cap R^{k_n}_\pi \neq \emptyset$, $i = 1, 2$. Since, by assumption, the flow $(R^{k_n}_\pi, G)$ is topologically transitive, there is some $g \in G$ with

$$\emptyset \neq g(V_1 \cap R^{k_n}_\pi) \cap (V_2 \cap R^{k_n}_\pi) \subset gV_1 \cap V_2.$$

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Thus the Polish system (UCR_π^Ξ, G) is indeed topologically transitive, and being Polish, this fact implies that it has a dense G_δ subset of transitive points. Now, under ι, this set is pushed onto a dense set of transitive points in the flow (R_π^Ξ, G) and our proof is complete. □

We are now ready to state and prove the main result of this section, Theorem 2.4 below. Our proof of this theorem follows the main ideas of McMahon's proof in [**14**]. The tools that enable us to improve his result are the augmented form of the structure theorem, Theorem 2.2 and Proposition 2.3.

THEOREM 2.4. *Let (X, G) be a minimal metric flow that is not PI. Then $E(X, G)$ has $2^\mathfrak{c}$ minimal left ideals.*

Proof. We start by recalling the following proposition from [**14**].

PROPOSITION 2.5. *If the extension $\pi : (X, G) \to (Y, G)$ is proximal, then π induces a one-to-one correspondence between the collections of left minimal ideals in $E(X, G)$ and $E(Y, G)$.*

Proof. Let π_* be the induced semigroup homomorphism from $E(X, G)$ onto $E(Y, G)$. Suppose I_1, I_2 are two distinct (and hence disjoint) minimal ideals in $E(X, G)$ with $I = \pi_*(I_1) = \pi_*(I_2)$. Let $u_1 \in I_1$ and $u_2 \in I_2$ be equivalent idempotents, so that $u_1u_2 = u_2$ and $u_2u_1 = u_1$. Then, as $\pi_*(u_1)$ and $\pi_*(u_2)$ belong to the same minimal left ideal I in $E(Y, G)$, we have $\pi_*(u_1) = \pi_*(u_2u_1) = \pi_*(u_2)\pi_*(u_1) = \pi_*(u_2)$. So, for every $x \in X$,

$$\pi(u_1x) = \pi_*(u_1)\pi(x) = \pi_*(u_2)\pi(x) = \pi(u_2x).$$

Thus the points u_1x and u_2x are proximal. But, since we have both $u_1(u_1x) = u_1x$ and $u_1(u_2x) = u_2x$, these points are also distal, and hence $u_1x = u_2x$. This means that $u_1 = u_2 \in I_1 \cap I_2$, which is a contradiction. □

Now, in the PI diagram constructed for (X, G) in the structure Theorem 2.2, the extension $X_\infty \to X$ is a proximal extension. Hence, in view of Proposition 2.5, in counting the minimal left ideals we can replace X by X_∞. So, from now on, we assume that $X = X_\infty$ and that the extension $X = X_\infty \to Y = Y_\infty$ is nontrivial, RIC and weakly mixing of all orders.

Applying Proposition 2.3, we conclude that the flow (R_π^Ξ, G) has a dense set of transitive points. Recall that the enveloping semigroup $E(X, G)$ can be identified with the enveloping semigroup $E(R_\pi^\Xi, G)$ (in fact, also with $E(X^\Xi, G)$) by letting $p \in E(X, G)$ act on X^Ξ coordinatewise.

Let $r \in R_\pi^\Xi$ be a fixed transitive point. Let y be the unique point in Y such that $r(\Xi) \subset \pi^{-1}(y)$. We also fix a left minimal ideal I in $E(X, G)$. Next observe that the extension π, being non-trivial and RIC, cannot be proximal. It follows that there is a minimal idempotent $u \in I$ with card $(u\pi^{-1}(y)) > 1$. We fix two distinct points x and x' in $u\pi^{-1}(y)$.

For each non-empty subset $A \subset \Xi$, let $r_A \in R_\pi^\Xi$ be defined by

$$r_A(\xi) = \begin{cases} x, & \xi \in A, \\ x', & \xi \in A^c. \end{cases}$$

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Since r is a transitive point, there is an element $q_A \in E(X, G)$ with $q_A r = r_A$. Let $I_A = Iq_A$. This is a minimal left ideal in $E(X, G)$. Let u_A be the unique minimal idempotent in I_A that is equivalent to u. Then, for some element $s_A \in I_A$, we have $u_A = s_A q_A$. Suppose A, A' are two distinct non-empty subsets of Ξ. Then $u_A r(\xi) = s_A q_A r(\xi) = s_A r_A(\xi)$, which equals $s_A x$ or $s_A x'$ according to whether $\xi \in A$ or $\xi \in A^c$. Similarly, $u_{A'} r(\xi) = s_{A'} q_{A'} r(\xi) = s_{A'} r_{A'}(\xi)$, which equals $s_{A'} x$ or $s_{A'} x'$ according to whether $\xi \in A'$ or $\xi \in A'^c$. Since x and x' form a pair of distinct points in uX ($ux = x$, $ux' = x'$), we conclude that $s_A x \neq s_A x'$ and $s_{A'} x \neq s_{A'} x'$. (E.g., if $s_A x = s_A x'$, then also $x = u_A x = u_A s_A^{-1} s_A x = u_A s_A^{-1} s_A x' = u_A x' = x'$, which is a contradiction.) It follows that $u_A r \neq u'_A r$, and hence $u_A \neq u_{A'}$. Thus $I_A \neq I_{A'}$ and the map $A \mapsto I_A$ is one-to-one. This shows that $E(X, G)$ has at least $2^{\mathfrak{c}}$ minimal ideals. By cardinality arguments, $\mathrm{mi}(X, G) \leq 2^{\mathfrak{c}}$ and the proof of Theorem 2.4 is complete. □

3. *A metric minimal PI-flow with* $\mathfrak{c}$ *minimal left ideals*

Let $X \cong S^1$ and $Y \cong \mathbb{P}^1$ be as in [**6**], namely, the spaces of rays emanating from the origin and lines through the origin of $\mathbb{R}^2$, respectively. Let $G = \mathrm{SL}_2(\mathbb{R})$ and we consider (X, G) and (Y, G) as G-flows. Then the map $\pi : X \to Y$, which sends a ray into the unique line in which it is contained, is a homomorphism of minimal G-flows. It is easy to see that Y is proximal and that the extension π is a group extension with fiber group $\{\rho, \mathrm{id}\} \cong \mathbb{Z}_2$, where $\rho : X \to X$ is the map that sends a ray onto its antipodal ray. Thus the G-flow (X, G) is, in particular, a PI-flow.

As was shown in [**6**, Example on p. 91] each minimal left ideal in $E(X, G)$ contains $\mathfrak{c}$ idempotents, called *minimal idempotents*. Any minimal idempotent u corresponds to a partition of the circle S^1 (naturally identified with X) into two arcs, say, J_1 and J_2 with common (antipodal) end points x_1 and x_2, which are fixed by u, and are such that $u(J_1) = \{x_1\}$, $u(J_2) = \{x_2\}$. For example, the figure below describes two minimal equivalent idempotents u and u' (and thus belonging to two different minimal left ideals), which correspond to the arcs with end points N, S.

In fact, it is not hard to see that $E(X, G)$ contains exactly two minimal left ideals. For the proof of the following lemma, see Appendix A below.

LEMMA 3.1. *Let $E = E(X, G)$ be the enveloping semigroup of (X, G). For any line $y \in Y$, with corresponding pair of antipodal rays x_1, x_2 in X, let u_y and u'_y be the maps from X into itself that send the corresponding arcs onto the endpoints x_1, x_2 in the counter-clockwise direction and in the clockwise direction, respectively.*

(1) *G embeds topologically into E.*
(2) *For each $y \in Y$, the maps u_y, u'_y are idempotents in E.*
(3) *The sets $I_+ = \{u_y : y \in Y\}$ and $I_- = \{u'_y : y \in Y\}$ are the two minimal left ideals in E.*
(4) *The set $\{e\} \cup I_+ \cup I_-$ comprises all the idempotents in E.*

Next we will describe a construction of a flow $(\Omega, \mathbf{G})$, for a suitable group $\mathbf{G}$, which is metric, minimal, PI and whose enveloping semigroup has $\mathfrak{c}$ minimal left ideals.

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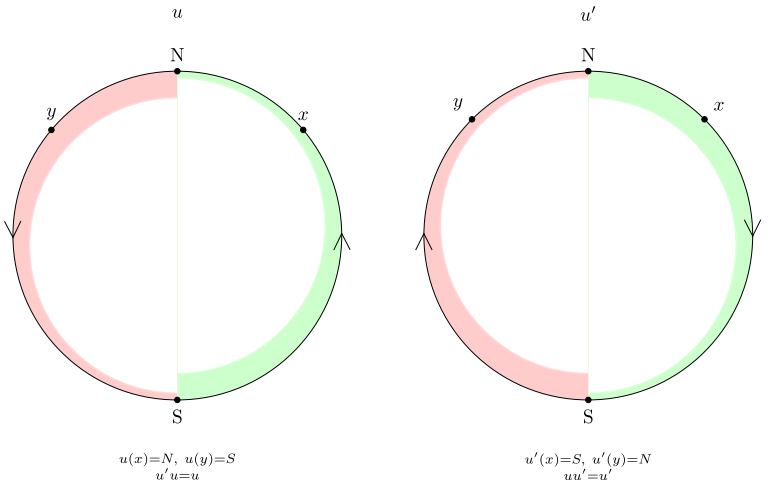


FIGURE 1. The minimal idempotents u and u'.

We let $\Omega = X^{\mathbb{N}}$, where X is as above. The group $\mathbf{G}$ is the product group $\mathbf{G} = G^{\mathbb{N}} = \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$. The action of $\mathbf{G}$ on Ω is coordinatewise. For each $n \in \mathbb{N}$, the projection $P_n : \Omega \to X$ is a homomorphism of the flow $(\Omega, \mathbf{G})$ onto the flow (X, G).

As $\mathbf{G}$ acts coordinatewise, we can achieve, for every sequence $\theta \in \{1, -1\}^{\mathbb{N}}$, a minimal idempotent $\mathbf{u}_\theta \in E(\Omega, \mathbf{G})$ such that $P_n(\mathbf{u}_\theta)$ is either u or u', as in the example above, according to whether $\theta(n) = 1$ or -1.

Ranging over the elements $\theta \in \{1, -1\}^{\mathbb{N}}$, we obtain, in this way, a collection of $\mathfrak{c}$ pairwise equivalent distinct minimal idempotents, which, in turn, belong to $\mathfrak{c}$ distinct minimal left ideals.

Finally, it is not hard to see that $(\Omega, \mathbf{G})$ is minimal and PI. In fact, the natural map $\Pi = (\pi \circ P_1, \pi \circ P_2, \dots) : \Omega \to \Omega_0 = Y^{\mathbb{N}}$ is a group extension, with compact fiber group $K \cong \mathbb{Z}_2^{\mathbb{N}}$, and the factor flow $(\Omega_0, \mathbf{G})$ is proximal.

4. *A cascade with* $\mathfrak{c}$ *minimal left ideals*

We will next apply the construction of skew-product minimal flows from [**10**] and the above example to obtain a minimal PI cascade with $\mathfrak{c}$ minimal left ideals.

In the following, we use the notation of [**10**]. In particular, (Z, σ) is a fixed minimal metric cascade and $(Y, \mathcal{G})$ is a minimal $\mathcal{G}$-flow with $\mathcal{G}$ a path-wise connected topological group. In addition, we assume that $\pi : (Y, \mathcal{G}) \to (Y_0, \mathcal{G})$ is a group extension with compact fiber group K. That is, there is a compact subgroup $K \subset \mathrm{Homeo}(Y)$, the group of homeomorphisms of the compact metric space Y, such that $gky = kgy$ for every $y \in Y$, $g \in \mathcal{G}$ and $k \in K$, and is such that $\pi(y_1) = \pi(y_2)$ if and only if there is a $k \in K$ with $y_2 = ky_1$. Note that any homeomorphism h of Y that commutes with K defines a unique homeomorphism $\tilde{h}$ of Y_0.

As in [**10**], on the product space $X = Z \times Y$, we let $\mathcal{H}_{\mathcal{G}}(X)$ be the subgroup of $\mathrm{Homeo}(X)$ that consists of homeomorphisms $g : X \to X$ of the form $g(z, y) = (z, g_z(y))$,

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where $z \mapsto g_z \in \mathcal{G}$ is a continuous map from Z into $\mathcal{G}$. We then let

$$\mathcal{S}_{\mathcal{G}}(\sigma) = \{g^{-1} \circ \sigma \circ g : g \in \mathcal{H}_{\mathcal{G}}(X)\}$$

(here σ is identified with $\sigma \times \mathrm{id}$, where id is the identity map on Y, so that $g(z, y) = (\sigma z, g_{\sigma z}^{-1} g_z(y))$). We set $X_0 = Z \times Y_0$ and define the map $\pi_X : X \to X_0$ as $\pi_X(z, y) = (z, \pi(y))$.

We note that every homeomorphism $T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$ (where the closure is taken in the Polish group $\mathcal{H}(X)$ equipped with its uniform convergence topology) commutes with the K-action on the Y coordinate of X. Thus, every such T defines also a corresponding homeomorphism $\tilde{T}$ in $\mathcal{H}(X_0)$. The proof of the following lemma is straightforward and we omit it.

LEMMA 4.1. *For every $T \in \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$, each element p in the enveloping semigroup $E(X, T)$ has the form*

$$p(z, y) = (\pi_*(p)z, p_z y),$$

where $\pi_*(p)$ is the image of p in $E(Z, \sigma)$ and p_z is an element of $E(Y, \mathcal{G})$. Moreover, p is an idempotent if and only if $\pi_*(p)$ and all the p_z, $z \in Z$, are idempotents.

We will assume that the action of $\mathcal{G}$ on Y_0 has the following property $(*)$.

> For every pair of points y_1, y_2 in Y_0, there exist neighborhoods U and V of y_1 and y_2, respectively, such that, for every $\epsilon > 0$, there exists $h \in \mathcal{G}$ with $\mathrm{diam}(h(V \cup U)) < \epsilon$.

PROPOSITION 4.2. *Under the above assumptions, there exists a dense G_δ subset $\mathcal{R} \subset \overline{\mathcal{S}_{\mathcal{G}}(\sigma)}$ such that, for every $T \in \mathcal{R}$:*

(1) *the cascade (X, T) is minimal;*
(2) *the extension $\pi_X : (X, T) \to (X_0, \tilde{T})$ is a K-extension; and*
(3) *the projection $P : (X_0, \tilde{T}) \to (Z, \sigma)$ is a proximal extension.*

In particular, for $T \in \mathcal{R}$, the minimal cascade (X, T) is PI.

Proof. Part (1) is just [**10**, Theorem 1]. Part (2) is clear. For part (3), apply the proof of [**10**, Theorem 3] with the following modification. In the definition of the sets $E_{U,V,\epsilon}$, we consider only non-empty open subsets U and V of Y that are K-invariant, and the diameter is taken with respect to the pseudometric on X, which is obtained by lifting the X_0 metric via π_X. Thus, for this kind of open set and with respect to this pseudometric, the conditions required in the proof of [**10**, Theorem 3] are satisfied by our assumption on the $\mathcal{G}$-action on Y_0. □

THEOREM 4.3. *There exists a metric minimal PI cascade (of order three) whose enveloping semigroup has $\mathfrak{c}$ minimal left ideals.*

Proof. We are going to apply Proposition 4.2 with the following input. (Z, σ) is a fixed metric minimal equicontinuous cascade (e.g., one can take (Z, σ) to be an irrational rotation on the unit circle in $\mathbb{C}$). The role of Y and Y_0 will be played by Ω and Ω_0, respectively, with $\mathcal{G} = \mathbf{G} = \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$, as described in §3. Clearly, property $(*)$ holds for

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the action of $\mathbf{G}$ on Ω_0. Therefore, Proposition 4.2 applies and we pick any T in the residual set $\mathcal{R}$ provided by this proposition. We also fix an arbitrary point $z_0 \in Z$.

Let now u be any minimal idempotent in $E(X, T)$. By the equicontinuity of (Z, σ), u projects to Z as the identity. By Lemma 4.1, for any $\omega \in \Omega$, $u(z_0, \omega) = (uz_0, u_{z_0}\omega) = (z_0, u_{z_0}\omega)$, where u_{z_0} is an idempotent in $E(\Omega, \mathbf{G})$. Next observe that an element p of $E(\Omega, \mathcal{G})$ is completely determined by its projections $P_n(p)$, $n \in \mathbb{N}$, in $E(S^1, \mathrm{SL}_2(\mathbb{R}))$. Of course, for each $n \in \mathbb{N}$, $P_n(u_{z_0})$ is an idempotent in $E(X, G)$ and hence, by Lemma 3.1(4), either $P_n(u_{z_0}) = u_{P_n(\omega)}$ or $P_n(u_{z_0}) = u'_{P_n(\omega)}$ (by Lemma 3.1(1), $P_n(u_{z_0}) \neq e$). One more observation: the reflection ρ (which sends a ray onto its antipodal ray) satisfies $\rho(u_{z_0}) = u'_{z_0}$ and it follows that, under the action of $K \cong \{\rho, \mathrm{id}\}^{\mathbb{N}}$, the orbit Ku, restricted to the fiber over z_0, namely, Ku_{z_0}, exhausts all the possible $\mathbf{u}_\theta$, $\theta \in \{1, -1\}^{\mathbb{N}}$.

We finally conclude, as in the discussion in §3, that $E(X, T)$ indeed contains $\mathfrak{c}$ minimal left ideals. □

Remark 4.4. For any $n \in \mathbb{N}$, the same method, in which in §3 we set $\Omega = X^n$ instead of $X^{\mathbb{N}}$, will produce, a minimal cascade with 2^n minimal left ideals.

5. *A PI-flow on the Cantor set with $2^{\mathfrak{c}}$ minimal left ideals*

Let Z denote the Cantor set. Let $G_0 \subset \mathrm{Homeo}(Z)$ be a finitely generated subgroup such that the corresponding flow (Z, G_0) is *totally minimal* in the sense that, for every two ordered finite subsets of distinct points, $\{z_1, z_2, \ldots, z_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ of X, there is a sequence of elements $\gamma_j \in G_0$ with $\lim_{j\to\infty} \gamma_j(z_i) = w_i$, for $i = 1, 2, \ldots, n$. Actually a less stringent condition will suffice: for every two ordered finite subsets of distinct points, $\{z_1, z_2, \ldots, z_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ of X and every pair of disjoint non-empty open sets $V_1, V_2 \subset X$, there is an element $g \in G_0$ with $\gamma z_i \in V_1$ and $\gamma w_i \in V_2$ for all $1 \leq i \leq n$.

Clearly then, the flow (Z, G_0) is, in particular, minimal and proximal. We note that the existence of G_0, as above, follows from a result of Kechris and Rosendal [**12**, Theorem 2.10], according to which the Polish group $\mathrm{Homeo}(Z)$ is topologically 2-generated: i.e., it contains a dense subgroup that is generated by two elements.

Let $\mathbb{Z}_2 = \{1, -1\}$ and set $X = Z \times \mathbb{Z}_2$. Choose a fixed non-empty clopen proper subset U of Z. Let $\phi : Z \to \mathbb{Z}_2$ be the function

$$\phi(z) = \begin{cases} -1, & z \in U, \\ 1, & z \in U^c, \end{cases}$$

and define a homeomorphism Φ of X by the formula

$$\Phi(z, \epsilon) = (z, \phi(z)\epsilon).$$

By abuse of notation, given $\gamma \in G_0$, we also denote by the same letter γ the homeomorphism $\gamma \times \mathrm{id}$ of X. Thus $\gamma(z, \epsilon) = (\gamma z, \epsilon)$. Finally, let G denote the subgroup of $\mathrm{Homeo}(X)$ generated by G_0 (acting on X) and Φ.

PROPOSITION 5.1. *The flow (X, G) is PI and its enveloping semigroup has $2^{\mathfrak{c}}$ minimal left ideals.*

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Proof. By construction, (X, G) is a group extension (with fiber group $\mathbb{Z}_2$) of a proximal flow. Thus it is a strictly PI-flow of order two. It is easy to check that it is minimal. Fix an arbitrary point $z_0 \in Z$ and let $\{V_n\}_{n\in\mathbb{N}}$ be a nested sequence of clopen neighborhoods of z_0 with $\bigcap V_n = \{z_0\}$.

Given any two non-empty disjoint finite sets $E, F \subset Z$ and an $n \in \mathbb{N}$, choose elements $\alpha = \alpha_{E,F,n}$ and $\beta = \beta_{E,F,n}$ in G_0 such that (i) $\alpha(E) \subset U$, (ii) $\alpha(F) \subset U^c$ and (iii) $\beta(\alpha(E) \cup \alpha(F)) \subset V_n$. Then form the product

$$\gamma_{E,F,n} = \beta\Phi\alpha.$$

Given an infinite subset $A \subset Z$ with an infinite complement $B = Z\backslash A$, we consider the directed set $\{(E, F, n) : E \subset A, F \subset B, n \in \mathbb{N}\}$, where $(E, F, n) \prec (E', F', n')$ if and only if $E \subset E'$, $F \subset F'$ and $n \le n'$. A moment's reflection will convince the reader that, for $A \subset Z$ such that $z_0 \notin A$, the net $\{\gamma_{E,F,n}\} \subset G$ converges to a minimal idempotent $u_A \in E(X, G)$ defined by the formula

$$u_A(z, \epsilon) = \begin{cases} (z_0, -\epsilon), & z \in A, \\ (z_0, \epsilon), & z \in B. \end{cases}$$

Clearly, $u_A u_{A'} = u_{A'}$ for any choice of sets $A, A' \subset Z$, as above, i.e., u_A and $u_{A'}$ are equivalent minimal idempotents, and it follows that $E(X, G)$ has at least $2^{\mathfrak{c}}$ minimal left ideals. By cardinality arguments, $\mathrm{mi}(X, G) \le 2^{\mathfrak{c}}$ and our proof is complete. □

6. *A cascade with $2^{\mathfrak{c}}$ minimal left ideals*

In this last section, we show how to construct a minimal PI cascade with $2^{\mathfrak{c}}$ minimal left ideals. The construction is similar to the constructions described in §§ 4 and 5.

Now, back to the context of §4. Let $Y_0 = S^2$, the 2-sphere in $\mathbb{R}^3$, and let $\mathcal{G}_0 = \mathrm{Homeo}_0(S^2)$ denote the connected component of the identity of the Polish group $\mathrm{Homeo}(S^2)$. We observe that $\mathcal{G}_0 = \mathrm{Homeo}_0(S^2)$ acts totally minimally and extremely proximally on S^2. Let $K = S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ be the circle group and set $Y = S^2 \times S^1 = Y_0 \times K$.

Choose two open sets $U_1, U_2 \subset Y_0$ with $\overline{U_1} \cap \overline{U_2} = \emptyset$. Let $\phi : Y_0 \to [0, 1]$ be a continuous function such that

$$\phi(y) = \begin{cases} 1, & y \in U_1, \\ 0, & y \in U_2. \end{cases}$$

Now define a one parameter family of homeomorphism Φ_t of Y by the formula

$$\Phi_t(y, \zeta) = (y, e^{i\pi t\phi(y)}\zeta).$$

We set $\Phi = \Phi_1$ so that $\Phi(y, \zeta) = (y, \zeta)$ for $\zeta \in U_2$ and $(y, -\zeta)$ for $\zeta \in U_1$. By abuse of notation, given $g \in \mathcal{G}_0$, we also denote by the same letter g the homeomorphism $g \times \mathrm{id}$ of $Y = Y_0 \times S^1$. Thus $g(y, \zeta) = (gy, \zeta)$. Denote by $\mathcal{G}$ the subgroup of $\mathrm{Homeo}(Y)$ generated by the following three path-wise connected subgroups: (i) $\mathcal{G}_0$ (acting on Y), (ii) $\{\Phi_t \mid t \in \mathbb{R}\}$ and (iii) $\mathrm{id} \times K$. Clearly, $\mathcal{G}$ is itself path-wise connected. Also one can easily check that K is central in $\mathcal{G}$.

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PROPOSITION 6.1. *The flow* $(Y, \mathcal{G})$ *is PI and its enveloping semigroup has* $2^{\mathfrak{c}}$ *minimal left ideals.*

Proof. By construction, $(Y, \mathcal{G})$ is a group extension (with fiber group S^1) of the extremely proximal flow $(Y_0, \mathcal{G}_0)$. Thus it is a strictly PI-flow of order two. It is clearly minimal. Fix an arbitrary point $y_0 \in Y_0$ and let $\{V_n\}_{n\in\mathbb{N}}$ be a nested sequence of clopen neighborhoods of y_0 with $\bigcap V_n = \{y_0\}$.

Given two non-empty disjoint finite sets $E, F \subset Y_0$ and $n \in \mathbb{N}$ and choose elements $\alpha = \alpha_{E,F,n}$ and $\beta = \beta_{E,F,n}$ in $\mathcal{G}_0$ such that (i) $\alpha(E) \subset U_1$, (ii) $\alpha(F) \subset U_2$ and (iii) $\beta(\alpha(E) \cup \alpha(F)) \subset V_n$. Then form the product

$$g_{E,F,n} = \beta\Phi\alpha.$$

Given an infinite subset $A \subset Y_0$ with an infinite complement $B = Y_0 \setminus A$, we consider the directed set $\{(E, F, n) : E \subset A,\ F \subset B \text{ finite},\ n \in \mathbb{N}\}$, where $(E, F, n) \prec (E', F', n')$ if and only if $E \subset E'$, $F \subset F'$ and $n \le n'$.

It is now easy to see that, for A with $y_0 \notin A$, the net $\{g_{E,F,n}\} \subset \mathcal{G}$ converges to the minimal idempotent $u_A \in E(Y, \mathcal{G})$ defined by the formula

$$u_A(y, \zeta) = \begin{cases} (y_0, -\zeta), & y \in A, \\ (y_0, \zeta), & y \in B. \end{cases}$$

(Clearly, u_A is an idempotent. To see that it is minimal, observe that for, $\gamma \in \mathcal{G}_0$, we have $\gamma u_A(y, \zeta) = (\gamma y_0, \pm\zeta)$.)

It is easy to see that $u_A u_{A'} = u_{A'}$ for any choice of sets A, $A' \subset Y_0$, as above, i.e., u_A and $u_{A'}$ are equivalent minimal idempotents. It follows that $E(Y, \mathcal{G})$ has at least $2^{\mathfrak{c}}$ minimal left ideals. By cardinality arguments, $\mathrm{mi}(Y, \mathcal{G}) \le 2^{\mathfrak{c}}$ and our proof is complete. □

In [**3**], Dirbák extended some of the results of [**10**] and, among other things, proved the next theorem which we reformulate in terms that are convenient for our purpose (see [**3**, Theorem 6]).

Let (Z, σ) be an infinite minimal cascade on a compact metric space Z. Let $\mathcal{G}$ be a Polish group with a dense path-wise connected subgroup. Let $C(Z, \mathcal{G})$ denote the Polish space of continuous functions from Z into $\mathcal{G}$ (the *cocycles*) and let $B(Z, \mathcal{G})$ be the collection of *co-boundaries*, i.e.,

$$B(Z, \mathcal{G}) = \{\psi \in C(Z, \mathcal{G}) : \psi(z) = \phi(\sigma z)\phi(z)^{-1} \text{for some } \phi \in C(Z, \mathcal{G})\},$$

and let $C'(Z, \mathcal{G}) = \overline{B(Z, \mathcal{G})}$.

THEOREM 6.2. (Dirbák) *There exists a dense* G_δ *subset* $\mathcal{R} \subset C'(Z, \mathcal{G})$ *such that, for every cocycle* $\phi \in \mathcal{R}$*, the corresponding Polish dynamical system (skew-product) defined on* $Z \times \mathcal{G}$ *by the formula*

$$T_\phi(z, g) = (\sigma z, \phi(z)g)$$

is topologically transitive.

THEOREM 6.3. *There exists a metric minimal PI cascade (of order three) whose enveloping semigroup has* $2^{\mathfrak{c}}$ *minimal left ideals.*

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Proof. Again we are going to apply Proposition 4.2, and this time also Theorem 6.2, with the following input.

(Z, σ) is a fixed metric minimal equicontinuous cascade (e.g., one can take (Z, σ) to be an irrational rotation on the unit circle). As above, we let $Y_0 = S^2$, $Y = Y_0 \times S^1$, and consider the corresponding actions of $\mathcal{G}_0$ and $\mathcal{G}$. We set $X_0 = Z \times Y_0$, $X = Z \times Y_0 \times S^1$ and define the map $\pi_X : X \to X_0$ as $\pi_X(z, y) = (z, \pi(y))$, where $\pi : (Y, \mathcal{G}) \to (Y_0, \mathcal{G}_0)$. Clearly, property $(*)$ and extreme proximality hold for the action of $\mathcal{G}_0$ on Y_0. Therefore, Proposition 4.2 applies, as well as Theorem 6.2. We pick any $T = T_\phi$ in the intersection of the residual sets provided by these theorems. We also fix a point $z_0 \in Z$ such that the orbit of the point (z_0, e), where e is the identity element of $\mathcal{G}$, is dense in $Z \times \mathcal{G}$.

Given an infinite subset $A \subset Y_0$ with an infinite complement $B = Y_0 \setminus A$, we now consider the directed set $\{(E, F, n, \epsilon) : E \subset A, F \subset B, n \in \mathbb{N}, \epsilon > 0\}$, where $(E, F, n, \epsilon) \prec (E', F', n', \epsilon')$ if and only if $E \subset E'$, $F \subset F'$, $n \le n'$ and $\epsilon' < \epsilon$. For each (E, F, n, ϵ), we choose $\nu_{(E,F,n,\epsilon)} \in \mathbb{Z}$ such that (in $Z \times \mathcal{G}$)

$$d(T_\phi^{\nu_{(E,F,n,\epsilon)}}(z_0, e), (z_0, g_{(E,F,n)})) < \epsilon.$$

If we let $\phi_k(z) = \phi(\sigma^{k-1}z)\phi(\sigma^{k-2}z) \cdots \phi(\sigma z)\phi(z)$, then

$$T_\phi^{\nu_{(E,F,n,\epsilon)}}(z_0, e) = (\sigma^{\nu_{(E,F,n,\epsilon)}} z_0, \phi_{\nu_{(E,F,n,\epsilon)}}(z_0)).$$

It now follows that, along the directed set $\{(E, F, n, \epsilon)\}$, we have, in $X = Z \times Y$,

$$\lim T_\phi^{\nu_{(E,F,n,\epsilon)}}(z_0, y, \zeta) = (z_0, u_A(y, \zeta)), \quad \text{and in } E(Y, \mathcal{G}),$$
$$\lim \phi_{\nu_{(E,F,n,\epsilon)}}(z_0) = u_A,$$

where

$$u_A(y, \zeta) = \begin{cases} (y_0, -\zeta), & y \in A, \\ (y_0, \zeta), & y \in B = A^c. \end{cases}$$

Thus, denoting $q = \lim T_\phi^{\nu_{(E,F,n,\epsilon)}}$, the limit of the net $T_\phi^{\nu_{(E,F,n,\epsilon)}}$ in $E(X, T_\phi)$, we have

$$q(z_0, y, \zeta) = (z_0, u_A(y, \zeta)) = (z_0, y_0, \pm\zeta). \tag{1}$$

Now let v be any minimal idempotent in $E(X, T)$. Clearly, v restricts to Z as the identity. By Lemma 4.1, for any $y \in Y$, $v(z_0, y) = (vz_0, v_{z_0}y) = (z_0, v_{z_0}y)$, where v_{z_0} is an idempotent in $E(Y, \mathcal{G})$. We let $\pi_* v_{z_0} = \tilde{v}_{z_0}$ be the corresponding idempotent in $E(Y_0, \mathcal{G}_0)$. Because v is a minimal idempotent and because the extension $P : (X_0, \tilde{T}) \to (Z, \sigma)$ is proximal, it follows that $\tilde{v}_{z_0}$ is a minimal idempotent in $E(Y_0, \mathcal{G}_0)$: i.e., its range is a singleton, a one-point set (in $Y_0 = S^2$). We can further require, as we may, that this single point be y_0. We now have

$$v_{z_0}(y, \zeta) = (y_0, \xi(y)\zeta) \quad \text{for all } (y, \zeta) \in Y = Y_0 \times S^1, \tag{2}$$

for some function $\xi : Y_0 \to S^1$ and, as u_{z_0} is an idempotent, we must have $\xi(y_0) = 1$.

Next consider the element $vq \in E(X, T_\phi)$. It belongs to some minimal left ideal, say, $I \subset E(X, T_\phi)$ and (by (1) and (2))

$$\begin{aligned} vq(z_0, y, \zeta) &= v(z_0, y_0, \pm\zeta) = (z_0, y_0, \xi(y_0)(\pm\zeta)) \\ &= (z_0, y_0, \pm\zeta) = q(z_0, y, \zeta) = (z_0, u_A(y, \zeta)). \end{aligned}$$

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Denote by $(vq)^{-1}$ the inverse of vq, in the maximal subgroup of I which contains vq, and let $\hat{u}_A = vq(vq)^{-1} = (vq)^{-1}vq$. Then $\hat{u}_A$ is a minimal idempotent in I and

$$\hat{u}_A(z_0, y, \zeta) = (z_0, u_A(y, \zeta)).$$

Again it is easy to check that, restricted to the fiber of X over z_0, $\hat{u}_A\hat{u}_{A'} = \hat{u}_{A'}$ for any choice of sets $A, A' \subset Y_0$, as above. This implies that, for $A \neq A'$, the minimal idempotents $\hat{u}_A$ and $\hat{u}_{A'}$ belong to different minimal left ideals in $E(X, T)$. In fact, if they belong to the same minimal left ideal then, in $E(X, T)$, we have $\hat{u}_A\hat{u}_{A'} = \hat{u}_A$, which, in turn, will imply the same equality for the restriction on the fiber over z_0, leading to the equality $\hat{u}_A = \hat{u}_{A'}$ on this fiber, which is a contradiction. It follows that $E(X, T_\phi)$ has at least $2^\mathfrak{c}$ minimal left ideals. By cardinality arguments, $\mathrm{mi}(X, T_\phi) \leq 2^\mathfrak{c}$ and our proof is complete. □

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A. *Appendix. A proof of Lemma 3.1*

We first recall the following lemma of Furstenberg from [**5**, Lemma 2]. Let V be a finite-dimensional real linear space. $P(V)$ will denote the corresponding projective space. If $v \in V$, $\bar{v}$ will denote the corresponding point of $P(V)$; if W is a subspace of V, $\overline{W}$ will designate the corresponding linear subvariety of $P(V)$. Finite unions of linear subvarieties will be called *quasi-linear subvarieties*. As for all algebraic subvarieties, these satisfy the descending chain condition. This leads to the following lemma.

LEMMA A.1. *Let $\tau_n \in \mathrm{GL}(V)$ and let $\bar{\tau}_n$ denote the corresponding projective transformations. Assume that* $\det \tau_n = 1$ *and $\|\tau_n\| \to \infty$, where $\|\cdot\|$ is a suitable norm on the linear endomorphisms of V. There exists a transformation π of $P(V)$ whose range is a quasi-linear subvariety $\subsetneq P(V)$, and a sequence $\{n_k\}$ with $\bar{\tau}_{n_k}(x) \to \pi(x)$ for every $x \in P(V)$.*

In our case, where $\dim(V) = 2$, the transformation π is either of the form $\bar{g}$ with $g \in \mathrm{SL}_2(\mathbb{R})$, or it has the form $\pi = \pi_{x_0,x_1,x_2}$, where $\pi(x) = x_0$ for every $x \in \mathbb{P}^1 \setminus \{x_1\}$ and $\pi(x_1) = x_2$ (possibly with $x_0 = x_2$). Note that, in this case, the range of π consists of at most two points.

Proof of Lemma 3.1. (1) First, consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on $Y = \mathbb{P}^1$. If $\bar{g}y = y$ for $g \in \mathrm{PSL}_2(\mathbb{R})$ and every $y \in Y$, then $ge_i = \lambda_i e_i$, $i = 1, 2$, with $\lambda_1\lambda_2 = 1$, so that $g = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$. If $|\lambda_1| \neq |\lambda_2|$, then $g\binom{1}{1} = \binom{\lambda_1}{\lambda_2}$ and $\overline{\binom{1}{1}} \neq \overline{\binom{\lambda_1}{\lambda_2}}$, which contradicts our assumption. Thus $|\lambda_1| = |\lambda_2|$, and hence $g = \mathrm{id}$ in $\mathrm{PSL}_2(\mathbb{R})$. We conclude that the map $g \mapsto \bar{g}$, from $\mathrm{PSL}_2(\mathbb{R})$ to $E(Y, \mathrm{PSL}_2(\mathbb{R}))$ is a continuous injection.

Next assume that $\bar{g}_n y \to \bar{g}y$ for $g \in \mathrm{PSL}_2(\mathbb{R})$, a sequence $g_n \in \mathrm{PSL}_2(\mathbb{R})$ and every $y \in Y$. It follows from Furstenberg's lemma that the topological space $E(Y, \mathrm{PSL}_2(\mathbb{R}))$

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is Fréchet: i.e., its topology is determined by sequences, rather than nets. In other words, the dynamical system $(Y, \mathrm{PSL}_2(\mathbb{R}))$ is *tame*; see [**9**]. By Furstenberg's lemma (and its proof), we have pointwise convergence $g_n \to h$, with h a semi-linear map $\mathbb{R}^2 \to \mathbb{R}^2$. In particular, $\bar{g}y = \bar{h}y$ for every $y \in Y$. But, as observed above, if h is not a linear map, then $\bar{h}$ has a range that consists of at most two points; so it follows that h is an invertible linear map. Now, the equality $\bar{g} = \bar{h}$ implies that $g = h$ as elements of $\mathrm{PSL}_2(\mathbb{R})$. Thus the map $g \mapsto \bar{g}$ from $\mathrm{PSL}_2(\mathbb{R})$ to $E(Y, \mathrm{PSL}_2(\mathbb{R}))$ is a homeomorphism. Now, when dealing with X, the space of rays, instead of $Y = \mathbb{P}^1$ we have the canonical surjective map $E(X, \mathrm{SL}_2(\mathbb{R})) \to E(Y, \mathrm{PSL}_2(\mathbb{R}))$. Again the map $g \mapsto \tilde{g} \in E(X, \mathrm{SL}_2(\mathbb{R}))$ is clearly a continuous injection and, as we are now in a situation where rays are mapped onto rays, it follows easily that the assumption '$\tilde{g}_n x \to \tilde{g}x$ for $g \in \mathrm{SL}_2(\mathbb{R})$, a sequence $g_n \in \mathrm{SL}_2(\mathbb{R})$ and every $x \in X$' implies that $g_n \to g$ in $\mathrm{SL}_2(\mathbb{R})$. Thus $g \mapsto \tilde{g}$ is indeed a homeomorphism from $\mathrm{SL}_2(\mathbb{R})$ onto its image in $E(X, \mathrm{SL}_2(\mathbb{R}))$.

(2) This is easily checked.

(3)–(4) It follows from the discussion so far that an element p of $E(Y, \mathrm{PSL}_2(\mathbb{R}))$ is either of the form $p = \bar{g}$ for some $g \in \mathrm{PSL}_2(\mathbb{R})$ or it has the form $p = \bar{h}$ for some semi-linear map h. If, in addition, $p^2 = p$ is an idempotent, then either $p = \mathrm{id}$ or it has the form $\pi = \pi_{y_0}$, where $\pi(y) = y_0$ for every $y \in Y$. Now observe that there are exactly two possible liftings of p to minimal idempotents u, u' in $E(X, \mathrm{SL}_2(\mathbb{R}))$, as claimed. This is because $X \cong S^1$ is 1-dimensional; the situation is radically different when one goes to S^2 and $\mathbb{P}^2$ (see the proof of Theorem 6.3). This proves part (4) and part (3) follows as well. □

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