Ergod. Th. & Dynam. Sys. (2006), **26**, 1623–1651
doi:10.1017/S0143385706000277
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Printed in the United Kingdom

Furstenberg transformations on irrational rotation algebras

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(*Received* 15 *April* 2005 *and accepted in revised form* 13 *January* 2006)

Abstract. We introduce a general class of automorphisms of rotation algebras, the non-commutative Furstenberg transformations. We prove that fully irrational non-commutative Furstenberg transformations have the tracial Rokhlin property, which is a strong form of outerness. We conclude that crossed products by these automorphisms have stable rank one, real rank zero, and order on projections determined by traces (Blackadar's Second Fundamental Comparability Question). We also prove that several classes of simple quotients of the C^*-algebras of discrete subgroups of five-dimensional nilpotent Lie groups, considered by Milnes and Walters, are crossed products of simple C^*-algebras (C^*-algebras of minimal ordinary Furstenberg transformations) by automorphisms which have the tracial Rokhlin property. It follows that these algebras also have stable rank one, real rank zero, and order on projections determined by traces.

0. *Introduction*

Furstenberg introduced in [**8**] a family of homeomorphisms of $S^1 \times S^1$, now called Furstenberg transformations. They have the form

$$h_{\gamma,d,f}(\zeta_1, \zeta_2) = (e^{2\pi i\gamma}\zeta_1, \exp(2\pi i f(\zeta_1))\zeta_1^d \zeta_2),$$

with fixed $\gamma \in \mathbb{R}$, $d \in \mathbb{Z}$, and $f : S^1 \to \mathbb{R}$ continuous. For $\gamma \notin \mathbb{Q}$ and $d \neq 0$, Furstenberg proved that $h_{\gamma,d,f}$ is minimal. These homeomorphisms, and higher dimensional analogs (which also appear in [**8**]), have attracted significant interest in operator algebras (see, for example, [**30, 10, 15, 42, 43**]) and in dynamics (see, for example, [**9**] and [**45**]).

For any $\theta \in \mathbb{R}$, the formula for the automorphism $f \mapsto f \circ h_{\gamma,d,f}$ of $C(S^1 \times S^1)$ also defines an automorphism of the rotation algebra A_θ. Taking the generators of A_θ

https://doi.org/10.1017/S0143385706000277 Published online by Cambridge University Press

to be unitaries u and v satisfying $vu = e^{2\pi i\theta}uv$, we obtain an automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ such that

$$\alpha_{\theta,\gamma,d,f}(u) = e^{2\pi i\gamma}u \quad \text{and} \quad \alpha_{\theta,\gamma,d,f}(v) = \exp(2\pi i f(u))u^d v.$$

We call it a *Furstenberg transformation* on A_θ. When $\theta \notin \mathbb{Q}$, it is the most general automorphism α of A_θ for which $\alpha(u)$ is a scalar multiple of u (Proposition 1.6). Several special cases of these have been considered previously (see the discussion after Proposition 1.6), but, as far as we know, not the general case.

In this paper, we prove that when $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$, then $\alpha_{\theta,\gamma,d,f}$ has the tracial Rokhlin property introduced in [**29**]. In particular, all non-trivial powers of $\alpha_{\theta,\gamma,d,f}$ are outer in a strong sense. Combining this result with that of [**29**], we see that $C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has stable rank one and real rank zero, and that the order on projections over this algebra is determined by traces (Blackadar's Second Fundamental Comparability Question).

Other cases, in which $1, \theta, \gamma$ are linearly dependent over $\mathbb{Q}$, or even in which θ, γ, or both are rational, are also interesting, but the analysis becomes more difficult. Sometimes the automorphism has the tracial Rokhlin property (see Theorem 3.7), sometimes the automorphism does not have the tracial Rokhlin property but the crossed product is simple anyway (see Example 3.8), and sometimes the crossed product is not simple (such as when θ and γ are both rational). It seems to be an interesting question to understand the structure of the crossed products in these cases, but we leave it for future work.

The method of proof applies to other examples as well. In a series of papers [**25–28, 49**], Milnes and Walters have studied the simple quotients of the C^*-algebras of certain discrete subgroups of nilpotent Lie groups of dimension up to five. These algebras are a kind of generalization of the irrational rotation algebras, which occur when the Lie group is the three-dimensional Heisenberg group. Among the algebras in these papers, the least is known about the algebras $A_\theta^{5,3}$ and $A_\theta^{5,6}$ of [**26**] and the algebras $A_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \varepsilon)$ of [**28**]. We show that each of these is the crossed product of a simple C^*-algebra (the C^*-algebra of an ordinary minimal Furstenberg transformation on $S^1 \times S^1$) by an automorphism with the tracial Rokhlin property. Again, it follows that these algebras have stable rank one and real rank zero, and that the order on projections over them is determined by traces.

This paper is organized as follows. In §1, we define Furstenberg transformations on rotation algebras, and prove some basic properties. Among other things, we compute the ordered K-theory of the crossed product when $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$, and we identify several cases in which one can prove by elementary means that crossed products of two different rotation algebras by two different Furstenberg transformations are isomorphic.

In §2, we derive an easy-to-verify criterion for the tracial Rokhlin property. Let A be a simple unital C^*-algebra with tracial rank zero in the sense of [**18, 19**], and with a unique tracial state. Then an automorphism of A has the tracial Rokhlin property if and only if the corresponding automorphism of the type II_1 factor obtained from the tracial state via the Gelfand–Naimark–Segal construction is aperiodic. This is a kind of generalization of the equivalence of conditions (i) and (iii) in Theorem 2.1 of [**13**], and we essentially

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follow the proof there. In §3, we apply this result to $\alpha_{\theta,\gamma,d,f}$ in the case that $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$, and also in the case $\theta \notin \mathbb{Q}$, $\gamma = 0$, $d \neq 0$, and f constant. Section 4 contains the results on simple quotients of discrete subgroups of five-dimensional nilpotent Lie groups.

1. *Furstenberg transformations on rotation algebras*

For $\theta \in \mathbb{R}$ let A_θ be the usual rotation algebra, generated by unitaries u and v satisfying $vu = e^{2\pi i\theta}uv$.

Definition 1.1. Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f\colon S^1 \to \mathbb{R}$ be a continuous function. The *Furstenberg transformation* on A_θ determined by (θ, γ, d, f) is the automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ such that

$$\alpha_{\theta,\gamma,d,f}(u) = e^{2\pi i\gamma}u \quad \text{and} \quad \alpha_{\theta,\gamma,d,f}(v) = \exp(2\pi i f(u))u^d v.$$

We check below that $\alpha_{\theta,\gamma,d,f}$ is an automorphism. It might be called a non-commutative Furstenberg transformation. Note that the parameter θ does not appear in the formulas; its only role is to specify the algebra on which the automorphism acts.

Special cases of these have been studied previously. See the discussion after Proposition 1.6.

LEMMA 1.2. *Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Then there is a unique automorphism given by the formulas in Definition 1.1.*

Proof. One checks easily that

$$u_0 = e^{2\pi i\gamma}u \quad \text{and} \quad v_0 = \exp(2\pi i f(u))u^d v$$

are unitaries which satisfy the commutation relation $v_0u_0 = e^{2\pi i\theta}u_0v_0$. Therefore there is a homomorphism $\alpha_{\theta,\gamma,d,f}$ which sends u to u_0 and v to v_0. Uniqueness is obvious. To prove that it is an automorphism, we exhibit an inverse. With $g(\zeta) = -f(e^{-2\pi i\gamma}\zeta) + d\gamma$ for $\zeta \in S^1$, one can check that $\alpha_{\theta,-\gamma,-d,g}$ is a two-sided inverse for $\alpha_{\theta,\gamma,d,f}$. □

We are interested in the crossed products $C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ by these automorphisms. There is some immediate redundancy, which we exhibit in the next three lemmas.

LEMMA 1.3. *Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Then*

$$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f}) \cong C^*(\mathbb{Z}, A_\gamma, \alpha_{\gamma,\theta,-d,-f}).$$

Proof. Let w be the canonical implementing unitary of the crossed product

$$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f}).$$

Then $C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ is the universal C^*-algebra generated by three unitaries u, v, w, subject to the relations

$$vu = e^{2\pi i\theta}uv, \quad wuw^* = e^{2\pi i\gamma}u, \quad \text{and} \quad wvw^* = \exp(2\pi i f(u))u^d v.$$

We take adjoints in the third relation, exchange the first two, and rearrange algebraically, to get

$$wu = e^{2\pi i\gamma}uw, \quad vuv^* = e^{2\pi i\theta}u, \quad \text{and} \quad vwv^* = \exp(-2\pi i f(u))u^{-d}w.$$

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The universal C^*-algebra generated by unitaries satisfying these relations is

$$C^*(\mathbb{Z}, A_\gamma, \alpha_{\gamma,\theta,-d,-f}),$$

with v playing the role of the implementing unitary of the crossed product. □

LEMMA 1.4. *Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Let $k, l \in \mathbb{Z}$. Set $g(\zeta) = f(e^{2\pi i l\theta}\zeta) + (ld - k)\theta$. Then*

$$\mathrm{Ad}(u^k v^l) \circ \alpha_{\theta,\gamma,d,f} = \alpha_{\theta,\gamma+l\theta,d,g}.$$

Moreover, the crossed products by $\alpha_{\theta,\gamma,d,f}$ and $\alpha_{\theta,\gamma+l\theta,d,g}$ are isomorphic.

Proof. The first statement is a computation. The second is then a special case of the isomorphism $C^*(\mathbb{Z}, A, \mathrm{Ad}(u) \circ \alpha) \cong C^*(\mathbb{Z}, A, \alpha)$ for any unitary u. Lacking a suitable reference for this isomorphism, we outline the proof here. The actions of $\mathbb{Z}$ generated by α and $\mathrm{Ad}(u) \circ \alpha$ are exterior equivalent, in the sense of 8.11.3 of [**34**], via the cocycle defined by $u_n = u\alpha(u) \cdots \alpha^{n-1}(u)$ for $n \geq 0$ and $u_{-n} = \alpha^{-n}(u_n^*)$ for $n > 0$. The crossed products are then isomorphic as in the proof of Theorem 2.8.3(5) of [**35**]. □

LEMMA 1.5. *Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z} \setminus \{0\}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Let $r \in \mathbb{R}$, and set $g(\zeta) = f(e^{2\pi i r/d}\zeta) + r$. Then the automorphism β of A_θ, determined by $\beta(u) = \exp(2\pi i r/d)u$ and $\beta(v) = v$, satisfies*

$$\beta \circ \alpha_{\theta,\gamma,d,f} \circ \beta^{-1} = \alpha_{\theta,\gamma,d,g}.$$

Moreover, the crossed products by $\alpha_{\theta,\gamma,d,f}$ and $\alpha_{\theta,\gamma,d,g}$ are isomorphic.

Proof. The first statement is a computation. The second statement is immediate from the first. □

In this lemma, one does not get anything new by taking $\beta(v) = \exp(2\pi i s)v$.

As we now show, when $\theta \in \mathbb{R} \setminus \mathbb{Q}$, automorphisms of the form $\alpha_{\theta,\gamma,d,f}$ are the most general automorphisms of A_θ which send u to a scalar multiple of u.

PROPOSITION 1.6. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $\gamma \in \mathbb{R}$. Let $\alpha \in \mathrm{Aut}(A_\theta)$ be an automorphism such that $\alpha(u) = e^{2\pi i\gamma}u$. Then there exist $d \in \mathbb{Z}$ and a continuous function $f : S^1 \to \mathbb{R}$ such that $\alpha = \alpha_{\theta,\gamma,d,f}$.*

Proof. We have

$$u\alpha(v)u^* = [e^{2\pi i\gamma}u]\alpha(v)[e^{2\pi i\gamma}u]^* = \alpha(uvu^*) = e^{-2\pi i\theta}\alpha(v).$$

Therefore u commutes with $\alpha(v)v^*$. Since θ is irrational, $C^*(u)$ is a maximal commutative subalgebra of A_θ by Corollary 3.3.3 on p. 79 of [**48**]. (See the previous page of this reference for the notation.) So there is a continuous function $g : S^1 \to S^1$ such that $\alpha(v)v^* = g(u)$. We may write $g(\zeta) = \exp(2\pi i f(\zeta))\zeta^d$ for suitable $d \in \mathbb{Z}$ (the degree of g) and a suitable continuous function $f : S^1 \to \mathbb{R}$. Then $\alpha(v) = \exp(2\pi i f(u))u^d v$, whence $\alpha = \alpha_{\theta,\gamma,d,f}$. □

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When $\theta = 0$, $d \neq 0$, and $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, then $\alpha_{\theta,\gamma,d,f}$ is a minimal Furstenberg transformation on the torus, as first introduced in §2 of [**8**]. Crossed products by these have been well studied, sometimes only in special cases such as f constant or $d = 1$, and sometimes in considerably greater generality, such as in higher dimensions. See, for example, §4 of [**30**], [**45**], [**15**], Example 4.9 of [**36**], and [**43**]. When f is smooth, this homeomorphism is uniquely ergodic by Theorem 2.1 of [**8**], and it follows from [**24**] (also see the survey article [**23**]) that the crossed product is classifiable. More recently, classifiability has been proved whenever f is uniquely ergodic [**22**]. In particular, the crossed product does not depend on the choice of f, and when $d = 1$ it is an AT algebra. For arbitrary continuous f, Theorem 2 (in §4) of [**9**] shows that the Furstenberg transformation need not be uniquely ergodic. (See the discussion in Example 3.8 below.)

The case $d = 1$ and f equal to the constant function 0 has been studied by Packer [**31, 32**]. She classified the crossed products up to isomorphism and Morita equivalence, and computed the ordered K-theory. In the case that at least one of θ and γ is not rational, she also proved that the projections satisfy cancellation.

The case $\gamma = 0, d = 1$, and $\theta \in \mathbb{R} \setminus \mathbb{Q}$ has been considered by Kodaka and Osaka [**16**]. They have shown that $\alpha_{\theta,0,1,f}$ is outer. Lemma 1.3 shows that in this case the crossed product is isomorphic to the transformation group C^*-algebra of a minimal Furstenberg transformation on the ordinary torus. Using Lemmas 1.4 and 1.3, one can show that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ but $1, \theta, \gamma$ are linearly dependent over $\mathbb{Q}$, then

$$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f}) \cong C^*(\mathbb{Z}, A_{\theta_0}, \alpha_{\theta_0,\gamma_0,d_0,f_0})$$

with $\theta_0 \in \mathbb{Q}$, that is, the crossed product is isomorphic to the crossed product of a Furstenberg transformation on a rational rotation algebra. Although we will not consider such crossed products in this paper, it is plausible that one can relate them to the transformation group C^*-algebras of ordinary Furstenberg transformations using Morita equivalence.

We compute the K-theory of the crossed product by a Furstenberg transformation on A_θ when θ and γ are not both rational. The computation of the groups is valid without this restriction. The statement about the range of a tracial state should be as well, but we do not investigate the rational case further here.

LEMMA 1.7. *Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Let $B = C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$. Suppose θ and γ are not both rational and $d \neq 0$. Then*

$$K_0(B) \cong \mathbb{Z}^3 \quad \text{and} \quad K_1(B) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/d\mathbb{Z}.$$

Moreover, if τ is any tracial state on B, then $\tau_*(K_0(B)) = \mathbb{Z} + \theta\mathbb{Z} + \gamma\mathbb{Z}$.

Proof. By Lemma 1.3, without loss of generality $\theta \notin \mathbb{Q}$. Let $\alpha = \alpha_{\theta,\gamma,d,f}$. Without loss of generality $0 < \theta < 1$. Let $\iota : A_\theta \to B$ be the inclusion map. The composition $\tau \circ \iota$ is the unique tracial state on A_θ, and we denote it again by τ.

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We use the Pimsner–Voiculescu exact sequence [**39**], which here takes the following form.

$$\begin{array}{ccccc} K_0(A_\theta) & \xrightarrow{\mathrm{id}-\alpha_*^{-1}} & K_0(A_\theta) & \xrightarrow{\iota_*} & K_0(B) \\ {\scriptstyle\exp}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(B) & \xleftarrow[\iota_*]{} & K_1(A_\theta) & \xleftarrow[\mathrm{id}-\alpha_*^{-1}]{} & K_1(A_\theta) \end{array}$$

Recall that $K_0(A_\theta) \cong \mathbb{Z}^2$ with generators $[1]$ and $[p]$, where $p \in A_\theta$ is a projection with $\tau(p) = \theta$. Also, recall that $K_1(A_\theta) \cong \mathbb{Z}^2$ with generators $[u]$ and $[v]$, the classes of the standard unitary generators. The induced map α_* on $K_0(A_\theta)$ is the identity, since $K_0(A_\theta)$ has no non-trivial automorphisms preserving order and $[1]$ when $\theta \notin \mathbb{Q}$. We have $[e^{2\pi i\gamma}u] = [u]$ and $[\exp(2\pi i f(u))u^d v] = d[u] + [v]$, so α_* on $K_1(A_\theta)$ has the matrix

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

Therefore the upper left horizontal map in the sequence above is zero and the lower right horizontal map is

$$\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}.$$

The computation of the groups follows easily.

It remains to determine the range of τ_*. If $\gamma = 0$ then the result on the range of the tracial state follows from Lemma 1.3 and Example 4.9 of [**36**]. If γ is rational, we may use Lemma 1.4 to replace γ by $\gamma + \theta$. Accordingly, we may assume $\gamma \notin \mathbb{Q}$. We may further assume $0 < \gamma < 1$.

Let $u_0 \in C(S^1)$ be the standard generating unitary, and let $\varphi\colon C(S^1) \to A_\theta$ be the homomorphism determined by $\varphi(u_0) = u$. Then φ is equivariant for the automorphisms $u_0 \to e^{2\pi i\gamma}u_0$ of $C(S^1)$ and α of A_θ. Therefore there is a homomorphism $\psi\colon A_\gamma \to B$ of the crossed products. Taking suitable portions of the Pimsner–Voiculescu exact sequences for these crossed products, we get the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^0(S^1) & \xrightarrow{(\iota_0)_*} & K_0(A_\gamma) & \xrightarrow{\partial_0} & K^1(S^1) & \longrightarrow & 0 \\ & & {\scriptstyle\varphi_*}\downarrow & & {\scriptstyle\psi_*}\downarrow & & {\scriptstyle\varphi_*}\downarrow & & \\ 0 & \longrightarrow & K_0(A_\theta) & \xrightarrow{\iota_*} & K_0(B) & \xrightarrow{\partial} & K_1(A_\theta) & \xrightarrow{\mathrm{id}-\alpha_*^{-1}} & K_1(A_\theta) \end{array}$$

As for A_θ, the composition $\tau \circ \psi$ is the unique tracial state on A_γ, and we write simply τ. Further, $K_0(A_\gamma) \cong \mathbb{Z}^2$ with generators $[1]$ and $[q]$, where $q \in A_\gamma$ is a projection with $\tau(q) = \gamma$. Moreover, $\partial_0([q]) = [u_0]$.

We claim that $K_0(B)$ is generated by $[1]$, $\iota_*([p])$, and $\psi_*([q])$. Since these have trace 1, θ, and γ, the result will follow. Let $\omega \in K_0(B)$. Then

$$\partial(\omega) \in \mathrm{Ker}(\mathrm{id} - \alpha_*^{-1}) = \mathbb{Z} \cdot [u].$$

Choose $m \in \mathbb{Z}$ such that $\partial(\omega) = m[u]$. Then

$$\partial(m\psi_*([q])) = m \cdot \varphi_* \circ \partial_0([q]) = m\varphi_*([u_0]) = m[u] = \partial(\omega).$$

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Therefore $\partial(m\psi_*([q]) - \omega) = 0$, so there are $k, l \in \mathbb{Z}$ such that $m\psi_*([q]) - \omega = \iota_*(k[1] + l[p])$. Thus $\omega = k[1] + l\iota_*([p]) + m\psi_*([q])$, as desired. □

To further analyze the crossed products by the Furstenberg transformations $\alpha_{\theta,\gamma,d,f}$ with $1, \theta, \gamma$ linearly independent over $\mathbb{Q}$, we need to show that these automorphisms have the tracial Rokhlin property. In the next section, we give a general method for proving this property.

2. *The tracial Rokhlin property in terms of trace norms*

We recall the definition of the tracial Rokhlin property (Definition 1.1 of [**29**]).

Definition 2.1. Let A be a stably finite simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every non-zero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$;
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$;
(3) with $e = \sum_{j=0}^{n} e_j$, the projection $1 - e$ is Murray–von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

The definition requires no condition on $\alpha(e_n)$.

The main result of this section is a criterion for the tracial Rokhlin property for an automorphism α of a simple unital C^*-algebra A with tracial rank zero and unique tracial state, namely that α should induce an aperiodic automorphism of the type II_1 factor obtained as the weak operator closure of A in the Gelfand–Naimark–Segal representation associated with τ. It is a kind of generalization of the equivalence of conditions (i) and (iii) of Theorem 2.1 of [**13**]. We follow the outline in §4 of [**13**], but we give the details omitted there. Some of the lemmas here are of independent interest, such as the approximate supremum of finitely many projections in a C^*-algebra with real rank zero (Lemma 2.3). In Theorem 2.14, we also show, under the same hypotheses as above, that the tracial Rokhlin property is equivalent to an analog of the Rokhlin property in which operator norm estimates are replaced by trace norm estimates.

Since the following computation will be used several times, we state it separately.

LEMMA 2.2. *Let A be a C^*-algebra, and let $a \in A$ satisfy $0 \leq a \leq 1$. Then $\|ab - b\| \leq [2\|abb^* - bb^*\|]^{1/2}$ for every $b \in A$.*

Proof. We have

$$\|ab - b\|^2 \leq \|abb^* - bb^*\| \cdot \|a^*\| + \|abb^* - bb^*\| \leq 2\|abb^* - bb^*\|.$$

Take square roots. □

The following lemma is an approximate version for C^*-algebras with real rank zero of the projection onto the closed span of finitely many subspaces. We get the correct size control on the projection, but it only approximately does the correct thing on most of the subspaces.

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LEMMA 2.3. *Let A be a C*-algebra with real rank zero. Let $q_0, q_1, \ldots, q_n \in A$ be projections, and let $\varepsilon > 0$. Then there exists a projection $e \in A$ such that:*
(1) $q_0 \leq e$;
(2) $\|eq_k - q_k\| < \varepsilon$ *for* $1 \leq k \leq n$;
(3) $e \precsim q_0 \oplus q_1 \oplus \cdots \oplus q_n$ *in* $M_{n+1}(A)$.

Proof. We first prove this when $n = 1$. Set $p = q_0$ and $q = q_1$. Since A has real rank zero, there is a projection $r \in \overline{(1-p)qAq(1-p)}$ such that

$$\|r(1-p)q(1-p) - (1-p)q(1-p)\| < \tfrac{1}{2}\varepsilon^2.$$

Then $\|r(1-p)q - (1-p)q\| < \varepsilon$ by Lemma 2.2. Set $e = p + r$. Clearly $e \geq p$, which is part (1). We have

$$\begin{aligned}\|eq - q\| &= \|epq + e(1-p)q - q\| = \|pq + r(1-p)q - q\| \\ &= \|r(1-p)q - (1-p)q\| < \varepsilon,\end{aligned}$$

which is part (2). Finally, it follows from Lemma 4.1 of [**29**] that r is Murray–von Neumann equivalent to a projection in $\overline{q(1-p)A(1-p)q} \subset qAq$. So $e \precsim p \oplus q$, which is part (3).

The general case is proved by induction. If the statement holds for n, and $q_0, q_1, \ldots, q_{n+1} \in A$ are projections, then we find e_0 for $q_0, q_1, \ldots, q_n \in A$ by the induction hypothesis and e for the projections e_0 and q_{n+1} by the case $n = 1$. This projection satisfies the conclusion of the lemma. □

COROLLARY 2.4. *Let A be a C*-algebra with real rank zero. Let $q_0, q_1, \ldots, q_n \in A$ be projections, and let $\varepsilon > 0$. Then there exists a projection $e \in A$ such that:*
(1) $e \leq q_0$;
(2) $\|q_k e - e\| < \varepsilon$ *for* $1 \leq k \leq n$;
(3) $[e] \geq [1] - \sum_{k=0}^{n}([1] - [q_k])$ *in* $K_0(A)$.

Proof. Apply Lemma 2.3 with the same value of ε, and with $1 - q_0, 1 - q_1, \ldots, 1 - q_n$ in place of $q_0, q_1, \ldots, q_n$. Call the resulting projection f. Then take $e = 1 - f$. □

The next several lemmas will be used to control the traces of various projections that we construct.

LEMMA 2.5. *Let $\varepsilon > 0$. Then there is a continuous function $g\colon [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$ such that whenever A is a C*-algebra with real rank zero, and $a \in A$ satisfies $\|a\| \leq 1$, then there exists a projection $e \in A$ such that $g(aa^*)e = e$ and $\|ea - a\| < \varepsilon$.*

Proof. Set $\varepsilon_0 = \frac{1}{18}\varepsilon^2$. Define continuous functions $g, h\colon [0,1] \to [0,1]$ by

$$g(t) = \begin{cases} \varepsilon_0^{-1} t, & 0 \leq t \leq \varepsilon_0, \\ 1, & \varepsilon_0 \leq t \leq 1, \end{cases}$$

and

$$h(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon_0, \\ \varepsilon_0^{-1}(t - \varepsilon_0), & \varepsilon_0 \leq t \leq 2\varepsilon_0, \\ 1, & 2\varepsilon_0 \leq t \leq 1. \end{cases}$$

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Then $\|h(aa^*)aa^* - aa^*\| < 2\varepsilon_0$. Therefore Lemma 2.2 gives

$$\|h(aa^*)a - a\| < \sqrt{2\varepsilon_0} = \tfrac{1}{3}\varepsilon.$$

Since A has real rank zero, there is a projection e in the hereditary subalgebra B of A generated by $h(aa^*)$ such that $\|eh(aa^*) - h(aa^*)\| < \frac{1}{3}\varepsilon$. Now

$$\begin{aligned}\|ea - a\| &\leq \|e\| \cdot \|a - h(aa^*)a\| + \|eh(aa^*) - h(aa^*)\| \cdot \|a\| + \|h(aa^*)a - a\| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.\end{aligned}$$

Also $g(aa^*)e = e$ because $g(aa^*)b = b$ for all $b \in B$. □

LEMMA 2.6. *Let $g\colon [0,1] \to [0,1]$ be a continuous function such that $g(0) = 0$ and $g(1) = 1$, and let $\varepsilon > 0$. Then the following hold.*

(1) *There exists $\delta > 0$ such that, whenever ω is a state on a C*-algebra A and $a \in A$ satisfies $0 \leq a \leq 1$ and $\omega(a) < \delta$, then $\omega(g(a)) < \varepsilon$.*

(2) *There exists $\delta > 0$ such that, whenever ω is a state on a C*-algebra A and $a \in A$ satisfies $0 \leq a \leq 1$ and $\omega(a) > 1 - \delta$, then $\omega(g(a)) > 1 - \varepsilon$.*

Proof. We prove part (1). Choose $\delta_0 > 0$ such that $g(t) < \frac{1}{2}\varepsilon$ for all $t \in [0, \delta_0]$. Set $\delta = \frac{1}{2}\varepsilon\delta_0$. Let a and ω be as in the hypotheses of part (1).

By considering the homomorphism $\varphi\colon C([0,1]) \to A$ which sends the function $h(t) = t$ to a, and the state $\omega \circ \varphi$, we reduce to the case $A = C([0,1])$ and $a = h$. Then there is a probability measure μ on $[0,1]$ such that $\omega(f) = \int_0^1 f\,d\mu$ for all $f \in C([0,1])$. The assumption $\omega(h) < \delta$ becomes $\int_0^1 t\,d\mu(t) < \delta$. From $t \geq \delta_0\chi_{[\delta_0,1]}(t)$ we get $\delta_0\mu([\delta_0,1]) < \delta$, whence $\mu([\delta_0,1]) < \frac{1}{2}\varepsilon$. Since $g \leq 1$, we now get $\int_{\delta_0}^1 g\,d\mu < \frac{1}{2}\varepsilon$. Also

$$\int_0^{\delta_0} g\,d\mu < \frac{\varepsilon}{2}\mu([0,\delta_0]) \leq \frac{\varepsilon}{2}.$$

Putting these together gives $\omega(g(a)) = \int_0^1 g\,d\mu < \varepsilon$.

For part (2), apply part (1) to the function $h(t) = 1 - g(1-t)$. If $\omega(a) > 1 - \delta$ then $\omega(1-a) < \delta$, so $\omega(h(1-a)) < \varepsilon$, whence $\omega(g(a)) = 1 - \omega(h(1-a)) > 1 - \varepsilon$. □

LEMMA 2.7. *Let $g\colon [0,1] \to [0,1]$ be a continuous function such that $g(0) = g(1) = 0$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, whenever ω is a state on a C*-algebra A and $a \in A$ satisfies $0 \leq a \leq 1$ and $\omega(a - a^2) < \delta$, then $\omega(g(a)) < \varepsilon$.*

Proof. Choose $\delta_0 > 0$ such that $\delta_0 < \frac{1}{2}$ and $g(t) < \frac{1}{3}\varepsilon$ for all $t \in [0, \delta_0] \cup [1 - \delta_0, 1]$. Set $\delta = \frac{1}{6}\varepsilon\delta_0$. Let a and ω be as in the hypotheses. As in the proof of Lemma 2.6, we may assume that $A = C([0,1])$, that $a(t) = t$ for all t, and that there is a probability measure μ on $[0,1]$ such that $\omega(f) = \int_0^1 f\,d\mu$ for all $f \in C([0,1])$. For $t \in [\delta_0, 1 - \delta_0]$ we have $t - t^2 \geq \delta_0 - \delta_0^2 > \frac{1}{2}\delta_0$. From $\int_0^1 (t - t^2)\,d\mu(t) < \frac{1}{6}\varepsilon\delta_0$ we therefore get $\mu([\delta_0, 1 - \delta_0]) < \frac{1}{3}\varepsilon$. So

$$\begin{aligned}\omega(g(a)) &= \int_0^{\delta_0} g\,d\mu + \int_{\delta_0}^{1-\delta_0} g\,d\mu + \int_{1-\delta_0}^1 g\,d\mu \\ &\leq \frac{\varepsilon}{3}\mu([0,\delta_0]) + \mu([\delta_0, 1-\delta_0]) + \frac{\varepsilon}{3}\mu([1-\delta_0, 1]) < \varepsilon.\end{aligned}$$

This completes the proof. □

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We will make extensive use of the L^2-norm (or seminorm) associated with a tracial state τ of a C^*-algebra A, given by $\|a\|_{2,\tau} = \tau(a^*a)^{1/2}$. See the discussion before Lemma V.2.20 of [**46**] for more on this seminorm in the von Neumann algebra context. All the properties we need are immediate from its identification with the seminorm in which one completes A to obtain the Hilbert space H_τ for the Gelfand–Naimark–Segal representation associated with τ, and from the relation $\tau(ba) = \tau(ab)$ for $a, b \in A$. In particular, we always have $\|abc\|_{2,\tau} \leq \|a\| \cdot \|b\|_{2,\tau} \cdot \|c\|$ for $a, b, c \in A$.

We prove here L^2 analogs for C^*-algebras with real rank zero of various norm approximation results. These are probably well known in von Neumann algebras. However, as we see here, all that is really needed is real rank zero.

LEMMA 2.8. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that the following holds. Let A be a unital C^*-algebra with real rank zero, let $T \subset T(A)$, and let $p, f_1, f_2, \ldots, f_n \in A$ be projections such that $\|pf_k\|_{2,\tau} < \delta$ for $1 \leq k \leq n$ and all $\tau \in T$. Then there exists a projection $e \in A$ such that:*

(1) *$e \leq p$;*
(2) *$\|ef_k\| < \varepsilon$ for $1 \leq k \leq n$;*
(3) *$\tau(e) > \tau(p) - \varepsilon$ for all $\tau \in T$.*

Proof. Choose g as in Lemma 2.5 for $\frac{1}{3}\varepsilon$ in place of ε. Apply Lemma 2.6(1) with this g and with $n^{-1}\varepsilon$ in place of ε, and let δ_0 be the resulting positive number. Set $\delta = \delta_0^{1/2}$.

Let the C^*-algebra A, the subset $T \subset T(A)$, and the projections $p, f_1, f_2, \ldots, f_n \in A$ be as in the hypotheses, with this choice of δ. By the choice of g, for each k there is a projection q_k such that $g(pf_kp)q_k = q_k$ and $\|q_kpf_k - pf_k\| < \frac{1}{3}\varepsilon$. Apply Lemma 2.3 in the C^*-algebra pAp to find a projection $q \leq p$ such that $\|qq_k - q_k\| < \frac{1}{3}\varepsilon$ for $1 \leq k \leq n$ and $q \precsim q_1 \oplus q_2 \oplus \cdots \oplus q_n$ in $M_n(pAp)$. Then set $e = p - q$.

That $e \leq p$ is clear. To prove part (2), we estimate, using $q \leq p$ at the second step,

$$\begin{aligned}\|ef_k\| &= \|pf_k - qf_k\| = \|pf_k - qpf_k\| \\ &\leq \|pf_k - q_kpf_k\| + \|q_k - qq_k\| \cdot \|pf_k\| + \|q\| \cdot \|q_kpf_k - pf_k\| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.\end{aligned}$$

For part (3), for $\tau \in T$ we use the choice of δ_0 and the hypothesis $\tau(pf_kp) = \|pf_k\|_{2,\tau}^2 < \delta^2 = \delta_0$ to conclude that $\tau(g(pf_kp)) < n^{-1}\varepsilon$. So $g(pf_kp)q_k = q_k$ gives

$$\tau(q_k) = \tau(g(pf_kp)^{1/2}q_kg(pf_kp)^{1/2}) \leq \tau(g(pf_kp)) < n^{-1}\varepsilon.$$

Therefore $\tau(q) \leq \sum_{k=1}^n \tau(q_k) < \varepsilon$, whence $\tau(e) = \tau(p) - \tau(q) > \tau(p) - \varepsilon$. □

LEMMA 2.9. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that whenever A is a C^*-algebra with real rank zero, $T \subset T(A)$, $r \in A$ is a projection, and $p_1, p_2, \ldots, p_n \in rAr$ are projections with $\|p_jp_k\|_{2,\tau} < \delta$ for $j \neq k$ and $\tau \in T$, then there exist mutually orthogonal projections $q_1, q_2, \ldots, q_n \in rAr$ such that $\|q_k - p_k\|_{2,\tau} < \varepsilon$ for all k and all $\tau \in T$.*

Proof. We prove this by induction on n. The statement is trivial for $n = 1$. Suppose therefore that the statement is known for a particular value of n, and let $\varepsilon > 0$.

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Choose δ_0 for the statement with n projections and with $\frac{1}{4}\varepsilon$ in place of ε. Set $\delta_1 = \min(\frac{1}{3}\delta_0, \frac{1}{2}\varepsilon)$. Choose $\delta_2 > 0$ so small that whenever f, g are projections in a C^*-algebra B which satisfy $\|fg\| < \delta_2$, then there is a projection $h \in B$ such that $fh = 0$ and $\|g - h\| < \delta_1$. We also require $\delta_2 \leq \varepsilon^2$. Choose $\delta > 0$ as in Lemma 2.8 for the given value of n and with δ_2 in place of ε. We also require $\delta \leq \frac{1}{3}\delta_0$.

Now let A, T, and r, $p_1, p_2, \ldots, p_{n+1} \in A$ be as in the hypotheses, with this value of δ. Use the choice of δ following Lemma 2.8 to find a projection $q_{n+1} \in A$ with $q_{n+1} \leq p_{n+1}$ such that $\|q_{n+1} p_k\| < \delta_2$ for $1 \leq k \leq n$ and $\tau(p_{n+1} - q_{n+1}) < \delta_2$ for all $\tau \in T$. Then also $\|[q_{n+1} + (1 - r)]p_k\| < \delta_2$ for $1 \leq k \leq n$. Use the choice of δ_2 to find projections $e_1, e_2, \ldots, e_n \in A$ which are orthogonal to $q_{n+1} + (1 - r)$ and satisfy $\|e_k - p_k\| < \delta_1$ for $1 \leq k \leq n$.

For $j \neq k$ we now have

$$\|e_j e_k\|_{2,\tau} \leq \|e_j - p_j\| \cdot \|e_k\|_{2,\tau} + \|p_j\|_{2,\tau} \cdot \|e_k - p_k\| + \|p_j p_k\|_{2,\tau} < 2\delta_1 + \delta \leq \delta_0.$$

Apply the induction hypothesis with $e_1, e_2, \ldots, e_n$ in place of $p_1, p_2, \ldots, p_n$ and with $r - q_{n+1}$ in place of r, obtaining mutually orthogonal projections $q_1, q_2, \ldots, q_n \leq r - q_{n+1}$ with $\|q_k - e_k\|_{2,\tau} < \frac{1}{4}\varepsilon$ for all k and all $\tau \in T$. Then for $1 \leq k \leq n$ and $\tau \in T$ we have

$$\|q_k - p_k\|_{2,\tau} \leq \|q_k - e_k\|_{2,\tau} + \|e_k - p_k\| < \tfrac{1}{4}\varepsilon + \delta_1 \leq \varepsilon.$$

Since also $\|q_{n+1} - p_{n+1}\|^2 < \delta_2 \leq \varepsilon^2$ (using $q_{n+1} \leq p_{n+1}$), this completes the proof. □

The next two lemmas are special cases of Lemma 2.12, but are used in its proof. In the first, we could presumably, with more care, also require $\|pq - q\| < \varepsilon$.

LEMMA 2.10. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that, whenever A is a unital C^*-algebra with real rank zero, $T \subset T(A)$, and $(e_{j,k})_{1 \leq j,k \leq n}$ is a system of matrix units for a copy of M_n in A with identity e, and whenever $p \in A$ is a projection with $p \leq e$ such that $\|[p, e_{j,k}]\|_{2,\tau} < \delta$ for $1 \leq j, k \leq n$ and all $\tau \in T$, then there is a projection $q \in A$ with $q \leq e$ that commutes with all $e_{j,k}$ and such that $\|q - p\|_{2,\tau} < \varepsilon$ for all $\tau \in T$.*

Proof. Use Lemma 5.1 of [**29**] to choose a continuous function $g\colon [0, 1] \to [0, 1]$ such that $g(0) = 0$, such that $g(1) = 1$, and such that, whenever A is a C^*-algebra with real rank zero and $a \in A$ satisfies $0 \leq a \leq 1$, then there is a projection $f \in \overline{aAa}$ such that

$$\|af - f\| < \frac{\varepsilon}{6n} \quad \text{and} \quad \|fg(a) - g(a)\| < \frac{\varepsilon}{6n}.$$

Apply Lemma 2.7 to the function $h(t) = (t - g(t))^2$ with $\frac{1}{144}n^{-2}\varepsilon^2$ in place of ε. Let $\delta_0 > 0$ be the resulting number. Set

$$\delta = \min\left(\sqrt{2\delta_0}, \frac{\varepsilon}{2n^2}\right).$$

Now let A, T, $(e_{j,k})_{1 \leq j,k \leq n}$, e, and p be as in the statement of the lemma. Then, using the trace property several times at the last step,

$$\begin{aligned} 2\delta_0 \geq \delta^2 &> \tau([p, e_{1,1}]^*[p, e_{1,1}]) \\ &= \tau(e_{1,1} p e_{1,1} + p e_{1,1} p - e_{1,1} p e_{1,1} p - p e_{1,1} p e_{1,1}) \\ &= 2\tau(e_{1,1} p e_{1,1} - (e_{1,1} p e_{1,1})^2). \end{aligned}$$

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With $a = e_{1,1}pe_{1,1}$, the choice of δ_0 gives $\tau((g(a)-a)^2) < \frac{1}{144}n^{-2}\varepsilon^2$ for all $\tau \in T$. The choice of g then gives a projection $f \in \overline{aAa} \subset e_{1,1}Ae_{1,1}$ such that

$$\|af - f\| < \frac{\varepsilon}{6n} \quad \text{and} \quad \|fg(a) - g(a)\| < \frac{\varepsilon}{6n}.$$

We estimate

$$\begin{aligned}\|f - a\|_{2,\tau} &\leq \|f - fa\| + \|f\| \cdot \|a - g(a)\|_{2,\tau} + \|fg(a) - g(a)\| + \|g(a) - a\|_{2,\tau} \\ &< \frac{\varepsilon}{6n} + \left(\frac{\varepsilon^2}{144n^2}\right)^{1/2} + \frac{\varepsilon}{6n} + \left(\frac{\varepsilon^2}{144n^2}\right)^{1/2} = \frac{\varepsilon}{2n}.\end{aligned}$$

Now set $q = \sum_{k=1}^{n} e_{k,1}fe_{1,k}$. Clearly q commutes with all $e_{j,k}$ and $q \leq e = \sum_{k=1}^{n} e_{k,k}$. We estimate $\|p - q\|_{2,\tau}$, in several steps. First, for $j \neq k$ we have

$$\|e_{j,j}pe_{k,k}\|_{2,\tau} \leq \|[e_{j,j}, p]\|_{2,\tau}\|e_{k,k}\| < \delta \leq \frac{\varepsilon}{2n^2}.$$

Second, we get

$$\begin{aligned}\|e_{k,k}pe_{k,k} - e_{k,1}fe_{1,k}\|_{2,\tau} &\leq \|e_{k,k}pe_{k,k} - e_{k,1}pe_{1,k}\|_{2,\tau} + \|e_{1,1}pe_{1,1} - f\|_{2,\tau} \\ &\leq \|e_{k,k}\| \cdot \|[p, e_{k,1}]\|_{2,\tau} \cdot \|e_{1,k}\| + \|f - a\|_{2,\tau} \\ &< \delta + \frac{\varepsilon}{2n} \leq \frac{\varepsilon}{2n^2} + \frac{\varepsilon}{2n}.\end{aligned}$$

Adding things up, we get

$$\begin{aligned}\|p - q\|_{2,\tau} &\leq \sum_{k=1}^{n} \|e_{k,k}pe_{k,k} - e_{k,1}fe_{1,k}\|_{2,\tau} + \sum_{j \neq k} \|e_{j,j}pe_{k,k}\|_{2,\tau} \\ &< n\left(\frac{\varepsilon}{2n^2} + \frac{\varepsilon}{2n}\right) + (n^2 - n)\left(\frac{\varepsilon}{2n^2}\right) = \varepsilon.\end{aligned}$$

This completes the proof. □

LEMMA 2.11. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that, whenever A is a unital C^*-algebra with real rank zero, $T \subset T(A)$, and $e_1, e_2, \ldots, e_n \in A$ are projections such that $\sum_{k=1}^{n} e_k = 1$, and whenever $p \in A$ is a projection such that $\|[p, e_k]\|_{2,\tau} < \delta$ for $1 \leq k \leq n$ and all $\tau \in T$, then there is a projection $q \in A$ which commutes with all e_k and such that $\|q - p\|_{2,\tau} < \varepsilon$ for all $\tau \in T$.*

Proof. The proof is essentially the same as that of Lemma 2.10. Let g, δ_0, and δ be as there. With A, T, $e_1, e_2, \ldots, e_n$, and p as in the hypotheses, for $1 \leq k \leq n$ let $a_k = e_k pe_k$, and use the method of the first part of the proof of Lemma 2.10 to find projections $f_k \in e_k Ae_k$ such that $\|f_k - a_k\|_{2,\tau} < \frac{1}{2}n^{-1}\varepsilon$. Set $q = \sum_{k=1}^{n} f_k$. Using the method of the second part of the proof of Lemma 2.10, slightly simplified, one gets

$$\|q - p\|_{2,\tau} < n\left(\frac{\varepsilon}{2n}\right) + (n^2 - n)\left(\frac{\varepsilon}{2n^2}\right) < \varepsilon,$$

as desired. □

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LEMMA 2.12. *For every $\varepsilon > 0$ and every finite dimensional C^*-algebra E, there is $\delta > 0$ such that, whenever A is a unital C^*-algebra with real rank zero, $T \subset T(A)$, S is a system of matrix units for E, and $\varphi\colon E \to A$ is a unital homomorphism, and whenever $p \in A$ is a projection such that $\|[p, \varphi(x)]\|_{2,\tau} < \delta$ for all $x \in S$ and $\tau \in T$, then there is a projection $q \in A$ which commutes with $\varphi(E)$ and such that $\|q - p\|_{2,\tau} < \varepsilon$ for all $\tau \in T$.*

Proof. Write $E = \bigoplus_{k=1}^{n} M_{r(k)}$. Apply Lemma 2.10 with $\frac{1}{2}n^{-1}\varepsilon$ in place of ε and with the values $r(1), r(2), \ldots, r(n)$ in place of n. Let $\delta_0 > 0$ be the minimum of ε and the n resulting values of δ. Apply Lemma 2.11 with $\frac{1}{3}\delta_0$ in place of ε and with the given value of n, and let $\delta_1 > 0$ be the resulting number. Set $R = \max(r(1), r(2), \ldots, r(n))$, and set $\delta = \min(R^{-1}\delta_1, \frac{1}{3}\delta_0)$.

Now let A, T, E, S, and p be as in the statement of the lemma. Let $e_1, e_2, \ldots, e_n$ be the minimal central projections of E, with e_k being the identity of $M_{r(k)}$. Since e_k is a sum of $r(k) \leq R$ elements of S, we get $\|[p, \varphi(e_k)]\|_{2,\tau} < R\delta \leq \delta_1$ for $1 \leq k \leq n$ and for all $\tau \in T$. Therefore there is a projection $f \in A$ which commutes with all $\varphi(e_k)$ and such that $\|f - p\|_{2,\tau} < \frac{1}{3}\delta_0$ for all $\tau \in T$. Set $f_k = f\varphi(e_k)$. For $s \in S$ a matrix unit for the kth summand $M_{r(k)}$, we have $[\varphi(s), f_k] = [\varphi(s), f]$, so

$$\|[\varphi(s), f_k]\|_{2,\tau} \leq 2\|f - p\|_{2,\tau}\|\varphi(s)\| + \|[\varphi(s), p]\|_{2,\tau} < \tfrac{2}{3}\delta_0 + \tfrac{1}{3}\delta_0 = \delta_0.$$

Applying Lemma 2.10 to the set of all such $\varphi(s)$ with k fixed, and with e_k in place of e, we obtain a projection $q_k \leq e_k$ that commutes with all such $\varphi(s)$ and satisfies $\|q_k - f_k\|_{2,\tau} < \frac{1}{2}n^{-1}\varepsilon$. Set $q = \sum_{k=1}^{n} q_k$, which is a projection that commutes with all elements of E and satisfies

$$\|q - f\|_{2,\tau} \leq \sum_{k=1}^{n} \|q_k - f_k\|_{2,\tau} < n(\tfrac{1}{2}n^{-1}\varepsilon) = \tfrac{1}{2}\varepsilon.$$

So $\|q - p\|_{2,\tau} < \frac{1}{2}\varepsilon + \frac{1}{3}\delta_0 \leq \varepsilon$. □

LEMMA 2.13. *Let A be a simple separable unital C^*-algebra with tracial rank zero. Let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a discrete group. Let $F \subset \Gamma$ and $S \subset A$ be finite sets, and let $\varepsilon > 0$. Then there exist projections $q, q_0 \in A$, unital finite dimensional subalgebras $E \subset qAq$ and $E_0 \subset q_0Aq_0$, and automorphisms $\varphi_\gamma \in \mathrm{Aut}(A)$ for $\gamma \in F \cup \{1\}$, such that:*

(1) *$\varphi_1 = \mathrm{id}_A$ and $\|\varphi_\gamma - \alpha_\gamma\| < \varepsilon$ for all $\gamma \in F$;*
(2) *for every $\gamma \in F \cup \{1\}$ and $x \in E$, we have $q_0\varphi_\gamma(x) = \varphi_\gamma(x)q_0$ and $q_0\varphi_\gamma(x)q_0 \in E_0$;*
(3) *for every $a \in S$, we have $\|qa - aq\| < \varepsilon$ and $\mathrm{dist}(qaq, E) < \varepsilon$;*
(4) *$\tau(1 - q), \tau(1 - q_0) < \varepsilon$ for all $\tau \in T(A)$.*

Proof. Since A has tracial rank zero, there is a projection $q \in A$ and a unital finite dimensional subalgebra $E \subset qAq$, such that $\tau(1 - q) < \varepsilon$ for all $\tau \in T(A)$, and such that for every $a \in S$ we have $\|qa - aq\| < \varepsilon$ and $\mathrm{dist}(qaq, E) < \varepsilon$. With this choice, condition (3) clearly holds, as does the part of condition (4) which applies to q.

Use Lemma 2.5.10 of [**17**] to find $\delta > 0$ such that, whenever B and C are unital subalgebras of a unital C^*-algebra A such that $\dim(B) \leq \dim(E) + 1$ and such that B has a system of matrix units each of which has distance less than δ from C, then there is a unitary $u \in A$ with $\|u - 1\| < \frac{1}{4}\varepsilon$ such that $u^*Bu \subset C$.

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Let T_0 be a system of matrix units for $E + \mathbb{C}(1-q)$, and set $T = \bigcup_{\gamma \in F \cup \{1\}} \alpha_\gamma(T_0)$. Using again the fact that A has tracial rank zero, find a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that $\tau(1-p) < \varepsilon$ for all $\tau \in T(A)$, and such that for every $a \in T$, we have $\|pa - ap\| < \frac{1}{3}\delta$ and $\operatorname{dist}(pap, D) < \frac{1}{3}\delta$. Set $B = D + (1-p)A(1-p)$, and observe that if $a \in T$ then

$$\begin{aligned}\operatorname{dist}(a, B) &\leq \|a - [pap + (1-p)a(1-p)]\| + \operatorname{dist}(pap, D)\\ &\leq 2\|[p, a]\| + \operatorname{dist}(pap, D) < \delta.\end{aligned}$$

Using the choice of δ on the system of matrix units $\alpha_\gamma(T_0)$ for the subalgebra $\alpha_\gamma(E + \mathbb{C}(1-q))$, choose for each $\gamma \in F \cup \{1\}$ a unitary $u_\gamma \in A$ with $\|u_\gamma - 1\| < \frac{1}{4}\varepsilon$ such that

$$u_\gamma^* \alpha_\gamma(E + \mathbb{C}(1-q))u_\gamma \subset D + (1-p)A(1-p).$$

Now set $q_0 = u_1 p u_1^*$ and $E_0 = u_1 D u_1^*$. This gives

$$E + \mathbb{C}(1-q) \subset E_0 + (1-q_0)A(1-q_0).$$

Further define $\varphi_\gamma(x) = [u_1 u_\gamma^*]\alpha_\gamma(x)[u_1 u_\gamma^*]^*$ for $x \in A$. Then $\varphi_1 = \mathrm{id}_A$ and

$$\|\varphi_\gamma - \alpha_\gamma\| \leq 2\|u_1 - 1\| + 2\|u_\gamma - 1\| < \varepsilon.$$

Thus condition (1) holds. Condition (2) follows from

$$u_1[D + (1-p)A(1-p)]u_1^* = E_0 + (1-q_0)A(1-q_0),$$

and the remaining part of condition (4) holds by construction. □

In the following theorem, condition (3) can also be weakened to require merely $\sum_{k=0}^n \tau(e_k) > 1 - \varepsilon$ for every $\tau \in T(A)$.

THEOREM 2.14. *Let A be a simple separable unital C^*-algebra with tracial rank zero. Let $\alpha \in \mathrm{Aut}(A)$. Then α generates an action of $\mathbb{Z}$ with the tracial Rokhlin property if and only if for every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every finite subset $S \subset A$ there exist orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:*

(1) $\|\alpha(e_k) - e_{k+1}\|_{2,\tau} < \varepsilon$ *for* $0 \leq k \leq n-1$ *and all* $\tau \in T(A)$*;*
(2) $\|[e_k, a]\|_{2,\tau} < \varepsilon$ *for* $0 \leq k \leq n$*, all* $a \in S$*, and all* $\tau \in T(A)$*;*
(3) $\sum_{k=0}^n e_k = 1$.

Proof. Suppose α generates an action of $\mathbb{Z}$ with the tracial Rokhlin property. Let ε, S, and n be given. We may assume that $\|a\| \leq 1$ for all $a \in S$. Since A has tracial rank zero, Corollary 5.7 and Theorems 5.8 and 6.8 of [**19**] imply that the order on projections over A is determined by traces. (The definition is recalled in Definition 3.4 below.) We apply Lemma 1.4 of [**29**] with $\min(\frac{1}{3}\varepsilon, \frac{1}{9}\varepsilon^2)$ in place of ε. Call the resulting projections $f_0, f_1, \ldots, f_n$. Set $f = \sum_{j=0}^n f_j$, set $e_0 = f_0 + 1 - f$, and set $e_j = f_j$ for $1 \leq j \leq n$. Since $\|e_0 - f_0\|_{2,\tau} = \tau(1-f)^{1/2} < \frac{1}{3}\varepsilon$ and $\|a\|_{2,\tau} \leq \|a\|$ for all $\tau \in T(A)$, it is easy to show that $e_0, e_1, \ldots, e_n$ satisfy the conditions of the theorem.

Now we prove the other direction. Let $S \subset A$ be a finite set, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Without loss of generality $\varepsilon < 1$. We construct projections $e_0, e_1, \ldots, e_n$ such that $\|\alpha(e_k) - e_{k+1}\| < \varepsilon$ for $0 \leq k \leq n-1$, such that $\|[e_k, a]\| < \varepsilon$ for $a \in S$ and $0 \leq k \leq n$,

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and such that $1 - \sum_{k=0}^{n} \tau(e_k) < \varepsilon$ for all $\tau \in T(A)$. By Lemma 1.4 of [**29**], this is sufficient.

Choose $\varepsilon_0 > 0$ with $\varepsilon_0 < \frac{1}{3}\varepsilon(n+1)^{-1}$ and so small that whenever $p_0, p_1, \ldots, p_n$ are projections in a C^*-algebra B which satisfy $\|p_j p_k\| < 4\varepsilon_0$ for $j \neq k$, then there are orthogonal projections $e_0, e_1, \ldots, e_n \in B$ such that $e_0 = p_0$ and $\|e_k - p_k\| < \frac{1}{3}\varepsilon$ for $1 \leq k \leq n$.

Apply Lemma 2.8 with ε_0 in place of ε and with n as given, and let $\varepsilon_1 > 0$ be the resulting value of δ. We also require $\varepsilon_1 \leq \varepsilon_0$. Then set

$$\varepsilon_2 = \min\left(1, \frac{\varepsilon_1^2}{16}, \frac{\varepsilon_1}{2(n+3)}, \frac{\varepsilon}{18}\right).$$

Set

$$T = \bigcup_{k=0}^{n} \alpha^{-k}(S).$$

Apply Lemma 2.13 with ε_2 in place of ε, with T in place of S, and with $\{-n, -n+1, \ldots, 1, 0\}$ in place of F. We obtain projections $q, q_0 \in A$, unital finite dimensional subalgebras $E \subset qAq$ and $E_0 \subset q_0 A q_0$, and automorphisms $\psi_k = \varphi_{-k} \in \mathrm{Aut}(A)$ for $-n \leq k \leq 0$ as there. Apply Lemma 2.12 with ε_2 in place of ε and $E_0 + \mathbb{C}(1 - q_0)$ in place of E, obtaining $\delta > 0$.

Apply the hypothesis with n as given, with δ in place of ε, and with a system of matrix units for $E_0 + \mathbb{C}(1 - q_0)$ in place of S, getting projections $p_0, p_1, \ldots, p_n$. Let $B_0 = A \cap [E_0 + \mathbb{C}(1 - q_0)]'$, the subalgebra of A consisting of all elements which commute with everything in $E_0 + \mathbb{C}(1 - q_0)$. Apply the choice of δ using Lemma 2.12 to p_0, obtaining a projection $f \in B_0$ which satisfies $\|f - p_0\|_{2,\tau} < \varepsilon_2$ for all $\tau \in T(A)$. Since q_0 is in the center of $E_0 + \mathbb{C}(1-q_0)$, the element $f_0 = q_0 f$ is also a projection in B_0. Since $q_0 \psi_k(E) q_0 \subset E_0$, it follows that f_0 commutes with all elements of $q_0 \psi_k(E) q_0$ and hence with all elements of $\psi_k(E)$. Therefore $f_k = \psi_k^{-1}(f_0)$ commutes with all elements of E, including q. So f_k also commutes with $1 - q$. We estimate $\|f_0 f_k\|_{2,\tau}$ for $\tau \in T(A)$. To start, $\|f - p_0\|_{2,\tau} < \varepsilon_2$ and

$$\|f_0 - f\|_{2,\tau} = \tau((1-q_0)f)^{1/2} \leq \tau(1-q_0)^{1/2} < \varepsilon_2^{1/2} \leq \tfrac{1}{4}\varepsilon_1,$$

so

$$\|f_0 - p_0\|_{2,\tau} \leq \|f_0 - f\|_{2,\tau} + \|f - p_0\|_{2,\tau} < \tfrac{1}{4}\varepsilon_1 + \varepsilon_2.$$

Now for $1 \leq k \leq n$ we get

$$\begin{aligned} \|f_k - \alpha^k(f_0)\| &= \|\psi_k^{-1}(f_0) - \alpha^k(f_0)\| \leq \|\psi_k^{-1} - \alpha^k\| \\ &\leq \|\psi_k^{-1}\| \cdot \|\alpha^{-k} - \psi_k\| \cdot \|\alpha^k\| < \varepsilon_2. \end{aligned}$$

So

$$\begin{aligned} \|f_k - p_k\|_{2,\tau} &\leq \|f_k - \alpha^k(f_0)\|_{2,\tau} + \|\alpha^k(f_0) - \alpha^k(p_0)\|_{2,\tau} + \|\alpha^k(p_0) - p_k\|_{2,\tau} \\ &\leq \|f_k - \alpha^k(f_0)\| + \|f_0 - p_0\|_{2,\tau\circ\alpha^k} + \sum_{j=0}^{k-1} \|\alpha(p_j) - p_{j+1}\|_{2,\tau\circ\alpha^j} \\ &< \varepsilon_2 + (\tfrac{1}{4}\varepsilon_1 + \varepsilon_2) + n\varepsilon_2 = \tfrac{1}{4}\varepsilon_1 + (n+2)\varepsilon_2. \end{aligned}$$

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Therefore,

$$\|f_0 f_k\|_{2,\tau} = \|f_0 f_k - p_0 p_k\|_{2,\tau} \le \|f_0 - p_0\|_{2,\tau} \cdot \|f_k\| + \|p_0\| \cdot \|f_k - p_k\|_{2,\tau}$$
$$< (\tfrac{1}{4}\varepsilon_1 + \varepsilon_2) + (\tfrac{1}{4}\varepsilon_1 + (n+2)\varepsilon_2) = \tfrac{1}{2}\varepsilon_1 + (n+3)\varepsilon_2 \le \varepsilon_1.$$

We saw that $f_0, f_1, \ldots, f_n \in B = A \cap [E + \mathbb{C}(1-q)]'$. This algebra has real rank zero because $E + \mathbb{C}(1-q)$ is finite dimensional. Therefore Lemma 2.8, applied to B with $\{\tau|_B : \tau \in T(A)\}$ in place of T, and the choice of ε_1, provides a projection $g \in A \cap [E + \mathbb{C}(1-q)]'$ such that $g \le f_0$, such that $\|gf_k\| < \varepsilon_0$ for $1 \le k \le n$, and such that $\tau(g) > \tau(f_0) - \varepsilon_0$ for all $\tau \in T(A)$.

Now use Corollary 2.4 to find a projection $e_0 \in A \cap [E + \mathbb{C}(1-q)]'$ such that $e_0 \le q$, such that $\|ge_0 - e_0\| < \varepsilon_0$, and such that $[e_0] \ge 1 - (1 - [q]) - (1 - [g])$ in $K_0(A \cap [E + \mathbb{C}(1-q)]')$. The last inequality implies that

$$\tau(e_0) \ge \tau(g) - \tau(1-q) > \tau(f_0) - \varepsilon_0 - \varepsilon_2 \ge \tau(f_0) - 2\varepsilon_0$$

for all $\tau \in T(A)$.

We now show that $e_0, \alpha(e_0), \ldots, \alpha^n(e_0)$ are approximately orthogonal. It suffices to estimate $\|e_0\alpha^k(e_0)\|$ for $1 \le k \le n$. We first use $\psi_k^{-1}(g) \le \psi_k^{-1}(f_0) = f_k$ to estimate $\|\psi_k^{-1}(g)g\| \le \|f_k g\| < \varepsilon_0$. Then

$$\|\psi_k^{-1}(e_0)e_0\| \le 2\|e_0 - ge_0\| + \|\psi_k^{-1}(e_0)\| \cdot \|\psi_k^{-1}(g)g\| \cdot \|e_0\| < 2\varepsilon_0 + \varepsilon_0 = 3\varepsilon_0.$$

So

$$\|e_0\alpha^k(e_0)\| \le \|\alpha^{-k} - \psi_k\| + \|\psi_k^{-1}(e_0)e_0\| < \varepsilon_2 + 3\varepsilon_0 \le 4\varepsilon_0.$$

Now use the choice of ε_0 to find orthogonal projections

$$e_1, \ldots, e_n \in A \cap [E + \mathbb{C}(1-q)]',$$

all orthogonal to e_0, such that $\|e_k - \alpha^k(e_0)\| < \frac{1}{3}\varepsilon$ for $1 \le k \le n$. We prove that $e_0, e_1, \ldots, e_n$ satisfy the estimates asked for at the beginning of the proof of this direction.

The first estimate is easy:

$$\|\alpha(e_k) - e_{k+1}\| \le \|e_k - \alpha^k(e_0)\| + \|\alpha^{k+1}(e_0) - e_{k+1}\| < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon < \varepsilon.$$

For the second, we start by estimating $\|[e_0, a]\|$ for $a \in T$. By construction, there is $b_0 \in E$ such that $\|qaq - b_0\| < \varepsilon_2$. Then $b = b_0 + (1-q)a(1-q)$ satisfies

$$\|a - b\| \le \|qaq - b_0\| + 2\|[q, a]\| < 3\varepsilon_2.$$

Now $e_0 \le q$ and e_0 commutes with all elements of E, so e_0 commutes with b. Therefore $\|[e_0, a]\| < 2 \cdot 3\varepsilon_2$. If now $a \in S$ and $0 \le k \le n$, then $\alpha^{-k}(a) \in T$, so

$$\|[\alpha^k(e_0), a]\| = \|[e_0, \alpha^{-k}(a)]\| < 6\varepsilon_2,$$

and

$$\|[e_k, a]\| \le 2\|e_k - \alpha^k(e_0)\| + \|[\alpha^k(e_0), a]\| < \tfrac{2}{3}\varepsilon + 6\varepsilon_2 \le \varepsilon,$$

as desired.

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Finally, we estimate $1 - \sum_{k=0}^{n} \tau(e_k)$ for $\tau \in T(A)$. We saw above that $\tau(e_0) > \tau(f_0) - 2\varepsilon_0$ for all $\tau \in T(A)$. Thus also $\tau(\alpha^k(e_0)) > \tau(\alpha^k(f_0)) - 2\varepsilon_0$ for $\tau \in T(A)$ and $0 \leq k \leq n$. From $\|e_k - \alpha^k(e_0)\| < \frac{1}{3}\varepsilon < \frac{1}{3}$ we get $\tau(e_k) = \tau(\alpha^k(e_0))$, and from $\|f_k - \alpha^k(f_0)\| < \varepsilon_2 \leq 1$ we get $\tau(f_k) = \tau(\alpha^k(f_0))$. So $\tau(e_k) > \tau(f_k) - 2\varepsilon_0$ for $\tau \in T(A)$ and $0 \leq k \leq n$.

Next, using a previous estimate at the second step,

$$|\tau(f_k) - \tau(p_k)| \leq \|f_k - p_k\|_{2,\tau} < \tfrac{1}{4}\varepsilon_1 + (n+2)\varepsilon_2 \leq \varepsilon_1,$$

so $\tau(f_k) > \tau(p_k) - \varepsilon_1$. Thus $\tau(e_k) > \tau(p_k) - \varepsilon_1 - 2\varepsilon_0 \geq \tau(p_k) - 3\varepsilon_0$. Summing over k and using $\sum_{k=0}^{n} \tau(p_k) = 1$, we get $\sum_{k=0}^{n} \tau(e_k) > 1 - 3(n+1)\varepsilon_0 \geq 1 - \varepsilon$, as desired.

This completes the proof of Theorem 2.14. □

LEMMA 2.15. *Let H be a Hilbert space, and let $A \subset L(H)$ be a unital C*-subalgebra which has real rank zero. Let $p \in A''$ be a projection. Then p is a strong operator limit of projections in A.*

Proof. The proof of Lemma 4.6 of [**12**] (which gives this conclusion for the weak operator closure of an AF algebra in the Gelfand–Naimark–Segal representation from a tracial state) applies here with no essential change. □

The following lemma is known, and is a special case of results in [**3**]. However, a direct proof is much easier, and we have not found one in the literature.

LEMMA 2.16. *Let A be a simple separable unital C*-algebra with tracial rank zero, and suppose that A has a unique tracial state τ. Let $\pi_\tau : A \to L(H_\tau)$ be the Gelfand–Naimark–Segal representation associated with τ. Then $\pi_\tau(A)''$ is a hyperfinite factor.*

Proof. Set $N = \pi_\tau(A)''$, and regard A as a subalgebra of N. That N is a factor is well known. For hyperfiniteness, we verify Theorem III.7.3(iii) of [**5**]. The first part is trivial. For the second part (approximation of finite sets in trace norm by finite dimensional subalgebras), let $t_1, t_2, \ldots, t_n \in N$. Without loss of generality $\|t_j\| \leq 1$ for all j. Let $\varepsilon > 0$. The sets

$$\{x \in N : \|x\| \leq 1 \text{ and } \|x - t_j\|_{2,\tau} < \tfrac{1}{2}\varepsilon\}$$

are open in the *-strong operator topology on the closed unit ball of N because τ is normal (Proposition V.2.5 of [**46**]). So the Kaplansky Density Theorem provides $b_1, b_2, \ldots, b_n \in A$ with $\|b_j\| \leq 1$ and $\|b_j - t_j\|_{2,\tau} < \frac{1}{2}\varepsilon$ for all j. Use tracial rank zero to find a projection $p \in A$, a finite dimensional unital subalgebra $E \subset pAp$, and $c_1, c_2, \ldots, c_n \in E$ such that $\tau(1-p) < \frac{1}{36}\varepsilon^2$, such that $\|pb_j - b_j p\| < \frac{1}{6}\varepsilon$ for all j, and such that $\|pb_j p - c_j\| < \frac{1}{6}\varepsilon$ for all j. Now

$$\|b_j - pb_j p\|_{2,\tau} \leq \|b_j - pb_j\|_{2,\tau} + \|p\| \cdot \|b_j - b_j p\|_{2,\tau} \leq 2\|1-p\|_{2,\tau} < \tfrac{1}{3}\varepsilon,$$

so

$$\|c_j - t_j\|_{2,\tau} \leq \|c_j - pb_j p\| + \|pb_j p - b_j\|_{2,\tau} + \|b_j - t_j\|_{2,\tau} < \tfrac{1}{6}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof. □

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THEOREM 2.17. *Let A be a simple separable unital C^*-algebra with tracial rank zero, and suppose that A has a unique tracial state τ. Let $\pi_\tau : A \to L(H_\tau)$ be the Gelfand–Naimark–Segal representation associated with τ. Let $\alpha \in \mathrm{Aut}(A)$. Then α generates an action of A with the tracial Rokhlin property if and only if for every $n > 0$ the automorphism of $\pi_\tau(A)''$ induced by α^n is outer.*

Proof. Assume the automorphism of $\pi_\tau(A)''$ induced by α^n is outer for every $n > 0$. We verify the hypotheses of Theorem 2.14. Thus let $\varepsilon > 0$, let $n \in \mathbb{N}$, and let $S \subset A$ be a finite subset. Without loss of generality $\|a\| \leq 1$ for all $a \in S$. Set $\varepsilon_0 = (4n+5)^{-1}\varepsilon$. Choose $\delta > 0$ as in Lemma 2.9 with n as given and for ε_0 in place of ε.

We regard A as a subalgebra of $N = \pi_\tau(A)''$, we let τ also denote the extension of the tracial state to $\pi_\tau(A)''$, and we let $\overline{\alpha}$ denote the extension of the automorphism to $\pi_\tau(A)''$. The algebra N is hyperfinite by Lemma 2.16.

Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, and let N_ω be the central sequence algebra, as defined before Theorem XIV.4.6 of [**47**]. (We use [**47**] as our standard reference, but almost everything is also in [**4**].) By Lemma XIV.4.5, Theorems XIV.4.6 and XIV.4.18 of [**47**], this algebra is a type II_1 factor, whose unique tracial state τ_ω sends the image of a strongly ω-central sequence $(a_l)_{l\in\mathbb{N}}$ in N_ω to $\lim_{l\to\omega}\tau(a_l)$. Let $\overline{\alpha}_\omega$ be the induced automorphism of N_ω. Theorem XIV.4.16 and Lemma XVII.2.2 of [**47**] show that $\overline{\alpha}_\omega^n$ is properly outer for all $n \neq 0$. Apply Definition XVII.1.5 and Theorem XVII.1.6 of [**47**] to find projections $f_0, f_1, \ldots, f_n \in N_\omega$ such that $\sum_{k=0}^n f_k = 1$ and $\|\overline{\alpha}_\omega(f_k) - f_{k+1}\|_{2,\tau_\omega} < \varepsilon_0$. By Theorem XIV.4.6(v) of [**47**], we can represent each f_k by a sequence $(f_{k,l})_{l\in\mathbb{N}}$ in $l^\infty(N)$ such that each $f_{k,l}$ is a projection and $\sum_{k=0}^n f_{k,l} = 1$ for every l. For $0 \leq k \leq n$ and $a \in S$ we have

$$\lim_{l\to\omega} \tau([a, f_{k,l}]^*[a, f_{k,l}]) = \tau_\omega([a, f_k]^*[a, f_k]) = 0,$$

so there is a neighborhood U of ω in $\beta\mathbb{N}$ such that $l \in U$ implies $\|[a, f_{k,l}]\|_{2,\tau} < \varepsilon_0$ for $a \in S$ and $0 \leq k \leq n$. Similarly, for $l \in \mathbb{N}$ sufficiently close to ω we have $\|\overline{\alpha}(f_{k,l}) - f_{k+1,l}\|_{2,\tau} < \varepsilon_0$ for $0 \leq k \leq n-1$.

Choose $l_0 \in \mathbb{N}$ for which the above estimates hold. For $0 \leq k \leq n$ use Lemma 2.15 to find a projection $g_k \in A$ such that $\|g_k - f_{k,l_0}\|_{2,\tau} < \min(\frac{1}{2}\delta, \varepsilon_0)$. Then $\|g_j g_k\|_{2,\tau} < \delta$ for $j \neq k$, so the choice of δ using Lemma 2.9 provides mutually orthogonal projections $e_1, \ldots, e_n \in A$ such that $\|e_k - g_k\|_{2,\tau} < \varepsilon_0$, and therefore $\|e_k - f_{k,l_0}\|_{2,\tau} < 2\varepsilon_0$. Set $e_0 = 1 - \sum_{j=1}^n e_j$. Since $\sum_{j=0}^n f_{j,l_0} = 1$, we get

$$\|e_0 - f_{0,l_0}\|_{2,\tau} \leq \sum_{j=1}^n \|e_j - f_{j,l_0}\|_{2,\tau} < 2n\varepsilon_0.$$

Now

$$\begin{aligned}\|\alpha(e_0) - e_1\|_{2,\tau} &\leq \|\overline{\alpha}(e_0 - f_{0,l_0})\|_{2,\tau} + \|\overline{\alpha}(f_{0,l_0}) - f_{1,l_0}\|_{2,\tau} + \|f_{1,l_0} - e_1\|_{2,\tau} \\ &< 2n\varepsilon_0 + \varepsilon_0 + 2\varepsilon_0 < \varepsilon.\end{aligned}$$

Similarly $\|\alpha(e_j) - e_{j+1}\|_{2,\tau} < 5\varepsilon_0 < \varepsilon$ for $1 \leq j \leq n-1$. Finally, if $a \in S$ then

$$\|[a, e_0]\|_{2,\tau} \leq 2\|e_0 - f_{0,l_0}\|_{2,\tau} \cdot \|a\| + \|[a, f_{0,l_0}]\|_{2,\tau} < 2 \cdot 2n\varepsilon_0 + \varepsilon_0 < \varepsilon,$$

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and similarly for $j \neq 0$ we have $\|[a, e_j]\|_{2,\tau} < 2 \cdot 2\varepsilon_0 + \varepsilon_0 < \varepsilon$. This completes the proof that outerness in the trace representation implies the tracial Rokhlin property.

The converse follows from Proposition 2.3 of [**13**] and Corollary 4.6 of [**29**]. □

We can now give a version of Kishimoto's result, Theorem 2.1 of [**13**], giving conditions for the Rokhlin property on a simple unital AT algebra with real rank zero and a unique tracial state.

THEOREM 2.18. *Let A be a simple separable unital C^*-algebra with tracial rank zero, and suppose that A has a unique tracial state τ. Let $\pi_\tau : A \to L(H_\tau)$ be the Gelfand–Naimark–Segal representation associated with τ. Let $\alpha \in \mathrm{Aut}(A)$. Then the following conditions are equivalent:*

(1) *α has the tracial Rokhlin property;*
(2) *the automorphism of $\pi_\tau(A)''$ induced by α^n is outer for every $n \neq 0$, that is, α^n is not weakly inner in π_τ for any $n \neq 0$;*
(3) *$C^*(\mathbb{Z}, A, \alpha)$ has a unique tracial state;*
(4) *$C^*(\mathbb{Z}, A, \alpha)$ has real rank zero.*

Proof. The equivalence of (1) and (2) is Theorem 2.17. The implication from (1) to (4) is Theorem 4.5 of [**29**]. The implication from (3) to (2) is Proposition 2.3 of [**13**].

We prove that (4) implies (3). Assume that (4) holds. Proposition 2.2 of [**13**] implies that the restriction map $T(C^*(\mathbb{Z}, A, \alpha)) \to T(A)$ is a bijection to the set $T(A)^\alpha$ of α-invariant tracial states of A. Since $T(A)^\alpha = \{\tau\}$, it follows that $C^*(\mathbb{Z}, A, \alpha)$ has a unique tracial state. □

3. *The tracial Rokhlin property for non-commutative Furstenberg transformations*

In this section, we prove that the automorphism $\alpha_{\theta,\gamma,d,f}$ of Definition 1.1 has the tracial Rokhlin property when $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$. We do the same for the case $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\gamma = 0$, and f constant. Thus, the results of [**29**] apply to the crossed products by these automorphisms.

A question that remains unanswered is whether these automorphisms actually have the Rokhlin property. An argument in §6 of [**11**] and an argument in [**12**] proceed by first proving the approximate Rokhlin property, and then showing that, for the algebra and automorphism in question, the approximate Rokhlin property implies the Rokhlin property. Apparently, though, those arguments depend on approximate innerness. Our automorphisms are not approximately inner because they are non-trivial on K_1.

If A is a simple C^*-algebra with unique tracial state τ, and $\alpha \in \mathrm{Aut}(A)$, we write $\overline{\alpha}$ for the automorphism of $\pi_\tau(A)''$ determined by α. We also use the trace norm $\|\cdot\|_{2,\tau}$ described before Lemma 2.8.

LEMMA 3.1. *Let A be a separable unital C^*-algebra with a faithful tracial state τ. Let $(y_i)_{i\in I}$ be a family of unitaries such that $\tau(y_i^* y_j) = 0$ for $i \neq j$ and whose linear span is dense in A. Let $\pi : A \to L(H)$ be the Gelfand–Naimark–Segal representation associated with τ. We identify A with its image in $\pi(A)''$. Then every $a \in \pi(A)''$ has a unique representation as $a = \sum_{i\in I} \lambda_i y_i$, with convergence in $\|\cdot\|_{2,\tau}$ and whose coefficients satisfy $\sum_{i\in I} |\lambda_i|^2 = \|a\|_{2,\tau}^2$. If a is unitary then $\sum_{i\in I} |\lambda_i|^2 = 1$.*

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Proof. Let $\xi \in H$ be the standard cyclic vector for π. Then one immediately checks that $a \mapsto a\xi$ is an isometric linear map from $\pi(A)''$ with $\|\cdot\|_{2,\tau}$ to H, and that $(y_i\xi)_{i\in I}$ is an orthonormal basis for H. The coefficients λ_i are determined by $a\xi = \sum_{i\in I} \lambda_i y_i \xi$. □

LEMMA 3.2. *Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$ and let $\gamma \in \mathbb{R}$. Let $\alpha \in \mathrm{Aut}(A_\theta)$ satisfy $\alpha(u) = \exp(2\pi i\gamma)u$. If $\gamma \notin \mathbb{Z} + \theta\mathbb{Z}$ then $\overline{\alpha}$ is outer.*

Proof. Let τ be the unique tracial state on A_θ, and identify A_θ with its image in $\pi_\tau(A_\theta)''$.

Suppose $\overline{\alpha}$ is inner, so $\overline{\alpha} = \mathrm{Ad}(w)$ for some unitary $w \in \pi_\tau(A_\theta)''$. We apply Lemma 3.1 with $I = \mathbb{Z}\times\mathbb{Z}$ and $y_{m,n} = u^m v^n$ for $m, n \in \mathbb{Z}$, and write

$$w = \sum_{m,n\in\mathbb{Z}} \lambda_{m,n} u^m v^n$$

with convergence in $\|\cdot\|_{2,\tau}$. Then, also with convergence in $\|\cdot\|_{2,\tau}$,

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} e^{2\pi i n\theta}\lambda_{m,n}u^{m+1}v^n &= \sum_{m,n\in\mathbb{Z}} \lambda_{m,n}u^m v^n u \\ &= wu = e^{2\pi i\gamma}uw = \sum_{m,n\in\mathbb{Z}} e^{2\pi i\gamma}\lambda_{m,n}u^{m+1}v^n.\end{aligned}$$

We have $\lambda_{m,n} \neq 0$ for some $m, n \in \mathbb{Z}$, and uniqueness of the series representation then implies $\exp(2\pi i n\theta) = \exp(2\pi i\gamma)$. □

THEOREM 3.3. *Let $\theta, \gamma \in \mathbb{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$. Let $d \in \mathbb{Z}$. Then the automorphism $\alpha = \alpha_{\theta,\gamma,d,f} \in \mathrm{Aut}(A_\theta)$, of Definition 1.1, has the tracial Rokhlin property.*

Proof. It follows from Theorem 4 and Remark 6 of [**6**] and Proposition 2.6 of [**18**] that A_θ has tracial rank zero. Also, A_θ has a unique tracial state τ. Lemma 3.2 implies that $\overline{\alpha}^k$ is outer on $\pi_\tau(A_\theta)''$ for all $k \neq 0$. The theorem is therefore immediate from Theorem 2.17. □

Before stating the consequences, we recall a definition.

Definition 3.4. Let A be a unital C^*-algebra. We say that the *order on projections over A is determined by traces* if, whenever $n \in \mathbb{N}$ and $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \precsim q$.

This is Blackadar's Second Fundamental Comparability Question for $M_\infty(A)$. See 1.3.1 in [**1**].

COROLLARY 3.5. *Let $\theta, \gamma \in \mathbb{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$. Let $d \in \mathbb{Z}$. Let $\alpha_{\theta,\gamma,d,f} \in \mathrm{Aut}(A_\theta)$ be as in Definition 1.1. Then:*

(1) *$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ is simple;*
(2) *$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has a unique tracial state;*
(3) *$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has real rank zero;*
(4) *$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ has stable rank one;*
(5) *the order on projections over $C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ is determined by traces;*
(6) *$C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,d,f})$ satisfies the local approximation property of Popa* [**40**] *(is a Popa algebra in the sense of Definition 1.2 of* [**3**]*).*

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Proof. Part (1) follows from Theorem 3.3 and Corollary 1.14 of [**29**]. It is well known that A_θ has real rank zero (Theorem 1.5 of [**2**]; Remark 6 in §5 of [**6**]) and stable rank one (see [**41**]), and that the order on projections over A_θ is determined by traces (Corollary 2.5 of [**44**]). (These also all follow from tracial rank zero; see Theorem 3.4 of [**18**] and Theorems 5.8 and 6.8 of [**19**].) Therefore part (2) follows from Theorem 3.3 and Corollary 4.6 of [**29**], part (3) follows from Theorem 3.3 and Theorem 4.5 of [**29**], part (4) follows from Theorem 3.3 and Theorem 5.3 of [**29**], part (5) follows from Theorem 3.3 and Theorem 3.5 of [**29**], and part (6) follows from Theorem 3.3 and Corollary 4.7 of [**29**]. □

It is worth pointing out that one can verify the tracial Rokhlin property for the automorphisms above using methods related to those of [**38**] (also see the long unpublished preprint [**37**]), without considering the automorphism in the trace representation or using tracial rank zero. The basic idea is to find approximately central embeddings of irrational rotation algebras, which are equivariant for an action on the embedded algebra which rotates one of the standard generators. In the case at hand, the rotation of the generator is irrational; in [**38**], the group is finite cyclic, and the rotation of the generator is rational. This was our original approach. It has the advantage that it can be used in situations in which the algebra is not known to have tracial rank zero. But it requires more work than the proof of Theorem 3.3, and the difficulty increases rapidly with the complexity of the algebra. For example, we also originally proved Proposition 4.1 this way, but that proof is much harder than for Theorem 3.3.

We will show that the same methods give the tracial Rokhlin property for one other kind of Furstenberg transformation on an irrational rotation algebra.

LEMMA 3.6. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \in \mathrm{Aut}(A_\theta)$ satisfy $\alpha(u) = u$ and $\alpha(v) = \lambda u^d v$ with $\lambda \in S^1$ and $d \in \mathbb{Z} \setminus \{0\}$. Then $\overline{\alpha}$ is outer.*

Proof. Let τ, $A_\theta \subset \pi_\tau(A_\theta)''$, $\overline{\alpha} = \mathrm{Ad}(w)$, and $w = \sum_{m,n\in\mathbb{Z}} \lambda_{m,n} u^m v^n$ be as in the proof of Lemma 3.2. Then, with convergence in $\|\cdot\|_{2,\tau}$,

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i n\theta} \lambda_{m,n} u^{m+1} v^n = \sum_{m,n\in\mathbb{Z}} \lambda_{m,n} u^m v^n u$$
$$= wu = uw = \sum_{m,n\in\mathbb{Z}} \lambda_{m,n} u^{m+1} v^n.$$

Since $\theta \notin \mathbb{Q}$, uniqueness of the series representation implies $\lambda_{m,n} = 0$ for $n \neq 0$.

Set $\beta_m = \lambda_{m,0}$. Then, again with convergence in $\|\cdot\|_{2,\tau}$,

$$\sum_{m\in\mathbb{Z}} \beta_m u^m v = wv = \lambda u^d v w = \sum_{m\in\mathbb{Z}} \lambda \beta_m u^d v u^m = \sum_{m\in\mathbb{Z}} \lambda e^{2\pi i m\theta} \beta_m u^{m+d} v.$$

Uniqueness of the series representation implies $|\beta_{m+d}| = |\beta_m|$ for all $m \in \mathbb{Z}$. Since $\sum_{n=-\infty}^{\infty} |\beta_n|^2 = 1$ and $d \neq 0$, this is a contradiction. □

In fact, if $\overline{\alpha}_{\theta,0,d,f}$ is inner, then one can show that there are numbers $\beta_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ with $\sum_{n=-\infty}^{\infty} |\beta_n|^2 = 1$ such that

$$e^{2\pi i f(\zeta)} \zeta^d = \sum_{m,n\in\mathbb{Z}} \beta_m \overline{\beta_n} e^{-2\pi i n\theta} \zeta^{m-n}$$

for all $\zeta \in S^1$.

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THEOREM 3.7. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $\rho \in \mathbb{R}$, and let $d \in \mathbb{Z} \setminus \{0\}$. Then the automorphism $\alpha = \alpha_{\theta,0,d,\rho} \in \mathrm{Aut}(A_\theta)$ of Definition 1.1 (with ρ regarded as a constant function) has the tracial Rokhlin property.*

Proof. The proof is the same as for Theorem 3.3. □

We then get all the properties listed in Corollary 3.5 for the crossed products by these automorphisms as well. By Lemma 1.3, the crossed product is also the C^*-algebra of an ordinary smooth minimal Furstenberg transformation on the torus, and in this case it is in fact known that the C^*-algebra has tracial rank zero. (See [**24**].)

Theorem 3.7 can also be proved by methods related to those of [**38**], but with considerably more difficulty. See the comment after Corollary 3.5.

Example 3.8. We show that Theorem 3.3 and Corollary 3.5(2) can fail if $1, \theta, \gamma$ are not linearly independent over $\mathbb{Q}$, even if the crossed product is simple, $d = 1$, and θ and γ are both irrational.

By Theorem 2 (in §4) of [**9**] and the preceding discussion, there is an irrational number $\theta \in \mathbb{R}$ (called α there) and a continuous function $g\colon S^1 \to \mathbb{R}$ (which in the notation of [**9**] is $g(e^{2\pi it}) = f(t) - t$) such that the function from S^1 to S^1 given by $\zeta \mapsto e^{2\pi i g(\zeta)}\zeta$ is a coboundary with respect to the action of rotation by $e^{2\pi i\theta}$ on S^1. Therefore the Furstenberg transformation $h(\zeta_1, \zeta_2) = (\exp(2\pi i\theta)\zeta_1, \exp(2\pi i g(\zeta_1))\zeta_1\zeta_2)$ is not ergodic with respect to Lebesgue measure. (See the discussion before Proposition 1 in §4 of [**9**].) So h is not uniquely ergodic.

Define $g_0\colon S^1 \to \mathbb{R}$ by $g_0(\zeta) = g(e^{-2\pi i\theta}\zeta) - \theta$. One checks that the automorphism $f \mapsto f \circ h^{-1}$ of $A_0 = C(S^1 \times S^1)$ is equal to $\alpha_{0,-\theta,-1,-g_0}$. It follows from Lemma 1.3 that $C^*(\mathbb{Z}, S^1 \times S^1, h) \cong C^*(\mathbb{Z}, A_{-\theta}, \alpha_{-\theta,0,1,g_0})$. Accordingly, the C^*-algebra $C^*(\mathbb{Z}, A_{-\theta}, \alpha_{-\theta,0,1,g_0})$ has more than one tracial state. Since A_θ has a unique tracial state, it follows from Theorem 2.18 that the automorphism $\alpha_{-\theta,0,1,g_0}$ does not have the tracial Rokhlin property.

This version of the example does not have $\gamma \notin \mathbb{Q}$, but by applying Lemma 1.4 with $k = 0$ and $l = -1$, we easily find a continuous function $k\colon S^1 \to \mathbb{R}$ such that $C^*(\mathbb{Z}, A_{-\theta}, \alpha_{-\theta,0,1,g_0}) \cong C^*(\mathbb{Z}, A_{-\theta}, \alpha_{-\theta,\theta,1,k})$. So $C^*(\mathbb{Z}, A_{-\theta}, \alpha_{-\theta,\theta,1,k})$ does not have real rank zero, and $\alpha_{-\theta,\theta,1,k}$ does not have the tracial Rokhlin property.

4. *C^*-algebras of discrete subgroups of nilpotent Lie groups*

In this section, we consider the C^*-algebras $A_\theta^{5,3}$ and $A_\theta^{5,6}$ of [**26**], which are the 'largest' simple quotients of the C^*-algebras of certain discrete subgroups of five-dimensional nilpotent Lie groups. The corresponding results for the other four Lie groups considered in [**26**] follow from already known results. We give details at the end of this section. But those results do not apply to $A_\theta^{5,3}$ and $A_\theta^{5,6}$.

PROPOSITION 4.1. *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the C^*-algebra $A_\theta^{5,3}$ of §3 of [**26**] is isomorphic to a crossed product $C^*(\mathbb{Z}, B, \alpha)$ in which B is the C^*-algebra of a smooth minimal Furstenberg transformation on $S^1 \times S^1$ and has a unique tracial state τ, and α has the tracial Rokhlin property.*

https://doi.org/10.1017/S0143385706000277 Published online by Cambridge University Press

Proof. Let $\lambda = \exp(2\pi i\theta)$. Then $A_\theta^{5,3}$ is the universal C^*-algebra generated by unitaries u, v, w, x satisfying the relations

$$uv = xvu, \quad uw = wu, \quad ux = \lambda xu,$$
$$vw = \lambda wv, \quad vx = xv, \quad \text{and} \quad wx = xw.$$

(In [**26**], the element x is included in the relations but not among the generators. However, the first relation implies that x is in the C^*-algebra generated by the other unitaries.)

We take B to be the universal C^*-algebra generated by unitaries u, v, x satisfying the relations

$$uv = xvu, \quad ux = \lambda xu, \quad \text{and} \quad vx = xv.$$

We have $B \cong C^*(\mathbb{Z}, S^1 \times S^1, h)$, where $h\colon S^1 \times S^1 \to S^1 \times S^1$ is the smooth Furstenberg transformation given by $h(\zeta_1, \zeta_2) = (\lambda^{-1}\zeta_1, \lambda\zeta_1^{-1}\zeta_2)$, so that $h^{-1}(\zeta_1, \zeta_2) = (\lambda\zeta_1, \zeta_1\zeta_2)$. Theorem 2.1 of [**8**] and the remark after it imply that h is uniquely ergodic and minimal. Unique ergodicity implies that B has a unique tracial state τ.

One easily checks that there is a unique automorphism α of B satisfying

$$\alpha(u) = u, \quad \alpha(v) = \lambda v, \quad \text{and} \quad \alpha(x) = x,$$

and that $C^*(\mathbb{Z}, B, \alpha) \cong A_\theta^{5,3}$ with the canonical unitary of the crossed product being sent to w^*.

Let π_τ be the Gelfand–Naimark–Segal representation associated to τ. We prove that, for any $k \neq 0$, the automorphism $\overline{\alpha}^k$ of $\pi_\tau(B)''$ induced by α^k is outer. Suppose that $\overline{\alpha}^k$ is inner, so $\overline{\alpha}^k = \mathrm{Ad}(z)$ for some unitary $z \in \pi_\tau(B)''$. We apply Lemma 3.1 with $I = \mathbb{Z}^3$ and $y_{l,m,n} = u^l v^m x^n$ for $l, m, n \in \mathbb{Z}$, and write

$$z = \sum_{l,m,n\in\mathbb{Z}} \lambda_{l,m,n} u^l v^m x^n$$

with convergence in $\|\cdot\|_{2,\tau}$. Then, also with convergence in $\|\cdot\|_{2,\tau}$,

$$\sum_{l,m,n\in\mathbb{Z}} \lambda_{l,m,n} u^l v^m x^{n+1} = zx = xz = \sum_{l,m,n\in\mathbb{Z}} \lambda_{l,m,n} x u^l v^m x^n$$
$$= \sum_{l,m,n\in\mathbb{Z}} \lambda_{l,m,n} \lambda^{-l} u^l v^m x^{n+1}.$$

Uniqueness of the series representation and $\lambda^{-l} \neq 1$ for $l \neq 0$ imply $\lambda_{l,m,n} = 0$ for $l \neq 0$. Therefore

$$z = \sum_{m,n\in\mathbb{Z}} \lambda_{0,m,n} v^m x^n.$$

So z commutes with v, contradicting $\alpha^k(v) = \lambda^k v$. This shows that $\overline{\alpha}^k$ is outer.

It follows from [**24**] or [**22**] that B has tracial rank zero. It now follows from Theorem 2.17 that α has the tracial Rokhlin property. □

The crossed product decomposition of Proposition 4.1 is not the same as the one in Theorem 3.1(1) of [**26**], but can be obtained from it by forming the partial crossed product by one of the two homeomorphisms used there.

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COROLLARY 4.2. *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $A_\theta^{5,3}$ be as in* §3 *of* [**26**]. *Then:*

(1) *$A_\theta^{5,3}$ is simple;*
(2) *$A_\theta^{5,3}$ has a unique tracial state;*
(3) *$A_\theta^{5,3}$ has real rank zero;*
(4) *$A_\theta^{5,3}$ has stable rank one;*
(5) *the order on projections over $A_\theta^{5,3}$ is determined by traces;*
(6) *$A_\theta^{5,3}$ satisfies the local approximation property of Popa* [**40**] *(is a Popa algebra in the sense of Definition 1.2 of* [**3**]*).*

Proof. Since the algebra B in Proposition 4.1 has tracial rank zero (by [**24**], or by [**22**]), it follows from Theorem 3.4 of [**18**] that B has real rank zero and stable rank one, and from Theorems 5.8 and 6.8 of [**19**] that the order on projections over B is determined by traces. The proof is then the same as for Corollary 3.5. □

Parts (1) and (2) of Corollary 4.2 are proved in [**26**], but the other properties are new.

Remark 4.3. The same results hold for the slightly more complicated versions of these algebras which appear in [**28**]. For $r_1, r_2, r_3, r_4, r_5 \in \mathbb{Z}$ let $A_\theta^{5,3}(r_1, r_2, r_3, r_4, r_5)$ be the universal C^*-algebra generated by unitaries u, v, w, x satisfying the relations

$$uv = \lambda^{r_2} x^{r_1} vu, \quad uw = \lambda^{r_5} wu, \quad ux = \lambda^{r_3} xu,$$
$$vw = \lambda^{r_4} wv, \quad vx = xv, \quad \text{and} \quad wx = xw.$$

(In [**28**], the exponents r_1, r_2, r_3, r_4, r_5 are called $\alpha, \beta, \gamma, \delta, \varepsilon$.) Theorem 2 of [**28**] asserts that if

$$r_1, r_3, r_4 > 0, \quad 0 \leq r_5 \leq \tfrac{1}{2} \gcd(r_3, r_4), \quad \text{and} \quad 0 \leq r_2 \leq \tfrac{1}{2} \gcd(r_1, r_3, r_4, r_3),$$

then $A_\theta^{5,3}(r_1, r_2, r_3, r_4, r_5)$ is simple. (These conditions are stated in Theorem 1 of [**28**]. The last two conditions are really just normalization conditions, and can be omitted in what follows.)

We apply the same analysis to these algebras. The algebra B of Proposition 4.1 is now the universal C^*-algebra generated by unitaries u, v, x satisfying the relations

$$uv = \lambda^{r_2} x^{r_1} vu, \quad ux = \lambda^{r_3} xu, \quad \text{and} \quad vx = xv.$$

The homeomorphism h is now

$$h(\zeta_1, \zeta_2) = (\lambda^{-r_3} \zeta_1, \lambda^{r_1 r_3 - r_2} \zeta_1^{-r_1} \zeta_2).$$

Since r_1 and r_3 are non-zero, it is still a smooth uniquely ergodic minimal Furstenberg transformation. The automorphism α is now determined by

$$\alpha(u) = \lambda^{-r_5} u, \quad \alpha(v) = \lambda^{r_4} v, \quad \text{and} \quad \alpha(x) = x.$$

Since $r_4 \neq 0$, the proof that α has the tracial Rokhlin property is identical. Thus, the conclusions of Corollary 4.2 hold for these algebras as well. Parts (1) and (2) are proved in [**28**], but the others are new.

We now turn to $A_\theta^{5,6}$.

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PROPOSITION 4.4. *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the C^*-algebra $A_\theta^{5,6}$ of §6 of* [**26**] *is isomorphic to a crossed product $C^*(\mathbb{Z}, B, \alpha)$ in which B is the C^*-algebra of a smooth minimal Furstenberg transformation on $S^1 \times S^1$ and has a unique tracial state τ, and α has the tracial Rokhlin property.*

Proof. Let $\lambda = \exp(2\pi i\theta)$. Then $A_\theta^{5,6}$ is the universal C^*-algebra generated by unitaries u, v, w, x satisfying the relations

$$uv = wvu, \quad uw = xwu, \quad ux = \lambda xu,$$
$$vw = \lambda wv, \quad vx = xv, \quad \text{and} \quad wx = xw.$$

(In [**26**], the elements w and x are included in the relations but not among the generators. However, the first two relations imply that w and x are in the C^*-algebra generated by u and v.)

We take B to be the universal C^*-algebra generated by unitaries u, w, x satisfying the relations

$$uw = xwu, \quad ux = \lambda xu, \quad \text{and} \quad wx = xw.$$

As in the proof of Proposition 4.1, this C^*-algebra is the crossed product by a uniquely ergodic minimal smooth Furstenberg transformation on $S^1 \times S^1$. One easily checks (and this is the content of Theorem 6.1(1) of [**26**]) that there is a unique automorphism α of B satisfying

$$\alpha(u) = w^*u, \quad \alpha(w) = \lambda w, \quad \text{and} \quad \alpha(x) = x,$$

and that $C^*(\mathbb{Z}, B, \alpha) \cong A_\theta^{5,6}$ with the canonical unitary of the crossed product being sent to v.

Let τ be the unique tracial state on B, and let π_τ be the associated Gelfand–Naimark–Segal representation. We prove that, for $k \neq 0$, the automorphism $\overline{\alpha}^k$ of $\pi_\tau(B)''$ induced by α^k is outer. Suppose $\overline{\alpha}^k$ is inner, so $\overline{\alpha}^k = \mathrm{Ad}(z)$ for some unitary $z \in \pi_\tau(B)''$. We apply Lemma 3.1 with $I = \mathbb{Z}^3$ and $y_{l,m,n} = u^l w^m x^n$ for $l, m, n \in \mathbb{Z}$, and write

$$z = \sum_{l,m,n \in \mathbb{Z}} \lambda_{l,m,n} u^l w^m x^n$$

with convergence in $\|\cdot\|_{2,\tau}$. Then, also with convergence in $\|\cdot\|_{2,\tau}$,

$$\sum_{l,m,n \in \mathbb{Z}} \lambda_{l,m,n} u^l w^m x^{n+1} = zx = xz = \sum_{l,m,n \in \mathbb{Z}} \lambda_{l,m,n} x u^l w^m x^n$$
$$= \sum_{l,m,n \in \mathbb{Z}} \lambda_{l,m,n} \lambda^{-l} u^l w^m x^{n+1}.$$

Uniqueness of the series representation and $\lambda^{-l} \neq 1$ for $l \neq 0$ imply $\lambda_{l,m,n} = 0$ for $l \neq 0$. Therefore

$$z = \sum_{m,n \in \mathbb{Z}} \lambda_{0,m,n} w^m x^n.$$

So z commutes with w, contradicting $\alpha^k(w) = \lambda^k w$. This shows that $\overline{\alpha}^k$ is outer.

It follows from [**24**] or [**22**] that B has tracial rank zero. It now follows from Theorem 2.17 that α has the tracial Rokhlin property. □

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COROLLARY 4.5. *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let $A_\theta^{5,6}$ be as in* §6 *of* [**26**]. *Then:*

(1) *$A_\theta^{5,6}$ is simple;*
(2) *$A_\theta^{5,6}$ has a unique tracial state;*
(3) *$A_\theta^{5,6}$ has real rank zero;*
(4) *$A_\theta^{5,6}$ has stable rank one;*
(5) *the order on projections over $A_\theta^{5,6}$ is determined by traces;*
(6) *$A_\theta^{5,6}$ satisfies the local approximation property of Popa.*

Proof. The proof is the same as for Corollary 4.2. □

As for $A_\theta^{5,3}$, Parts (1) and (2) of Corollary 4.5 are proved in [**26**], but the other properties are new.

Since this is not done in [**26**] (which was written before some of the relevant results were known), we explain how to show that the other 'largest' simple quotients of the C^*-algebras of the discrete subgroups of five-dimensional nilpotent Lie groups satisfy the properties we have proved for $A_\theta^{5,3}$ and $A_\theta^{5,6}$. In fact, for three of the cases, the C^*-algebras are known to be classifiable simple C^*-algebras with tracial rank zero.

THEOREM 4.6. *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the C^*-algebras $A_\theta^{5,1}$ of* §1 *of* [**26**] *and the C^*-algebras $A_\theta^{5,5}$ of* §5 *of* [**26**] *are simple separable nuclear C^*-algebras with tracial rank zero which satisfy the Universal Coefficient Theorem. For $\theta_1, \theta_2 \in \mathbb{R}$ such that $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Q}$, the C^*-algebras $A_{\theta_1,\theta_2}^{5,2}$ of* §2 *of* [**26**] *are simple separable nuclear C^*-algebras with tracial rank zero which satisfy the Universal Coefficient Theorem.*

Proof. Theorem 1.1(1) of [**26**] easily gives $A_\theta^{5,1} \cong A_\theta \otimes A_\theta$, the tensor product of two copies of the same irrational rotation algebra. This algebra is a simple higher dimensional non-commutative torus, so the desired result follows from the main theorem of [**38**]. (Also see the long unpublished preprint [**37**]. There are earlier results which cover this particular case, such as Theorem 3.11 of [**7**].)

Define a diffeomorphism $h\colon (S^1)^3 \to (S^1)^3$ by

$$h(\zeta_1, \zeta_2, \zeta_3) = (\exp(2\pi i\theta)\zeta_1, \zeta_1\zeta_2, \zeta_2\zeta_3).$$

Theorem 5.1(1) of [**26**] gives an isomorphism $A_\theta^{5,5} \cong C^*(\mathbb{Z}, (S^1)^3, h)$. The diffeomorphism h is minimal and uniquely ergodic by Theorem 2.1 of [**8**] and the remark after it, so $A_\theta^{5,5}$ is simple and has a unique tracial state τ. Clearly $C^*(\mathbb{Z}, (S^1)^3, h)$ contains a copy of the irrational rotation algebra A_θ, so $\tau_*\colon K_0(C^*(\mathbb{Z}, (S^1)^3, h)) \to \mathbb{R}$ has dense range. The conclusion for $A_\theta^{5,5}$ therefore follows from [**22**] or from [**24**] (also see [**23**]).

Theorem 2.1(1) of [**26**] gives an isomorphism of $A_{\theta_1,\theta_2}^{5,2}$ with the crossed product of $S^1 \times S^1$ by the rotation by $(\exp(2\pi i\theta_1), \exp(2\pi i\theta_2))$. This rotation is uniquely ergodic and minimal because $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Q}$. The desired conclusion for $A_{\theta_1,\theta_2}^{5,2}$ now follows in the same way as for $A_\theta^{5,5}$. (This algebra is also a higher dimensional non-commutative torus, so one could also use [**38**] as was done for $A_\theta^{5,1}$.) □

THEOREM 4.7. *For $\theta_1, \theta_2 \in \mathbb{R}$ such that $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Q}$, let $A_{\theta_1,\theta_2}^{5,4}$ be as in* §4 *of* [**26**]. *Then:*

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(1) $A_{\theta_1,\theta_2}^{5,4}$ *is simple;*
(2) $A_{\theta_1,\theta_2}^{5,4}$ *has a unique tracial state;*
(3) $A_{\theta_1,\theta_2}^{5,4}$ *has real rank zero;*
(4) $A_{\theta_1,\theta_2}^{5,4}$ *has stable rank one;*
(5) *the order on projections over* $A_{\theta_1,\theta_2}^{5,4}$ *is determined by traces;*
(6) $A_{\theta_1,\theta_2}^{5,4}$ *satisfies the local approximation property of Popa.*

Proof. Theorem 4.1(1) of [**26**] identifies $A_{\theta_1,\theta_2}^{5,4}$ as the crossed product of the irrational rotation algebra $A_{-\theta_2}$ by the automorphism determined on the standard generators by $v \mapsto \exp(2\pi i\theta_1)v$ and $u \mapsto vu$. Since $1, \theta_1, \theta_2$ are linearly independent over $\mathbb{Q}$, this automorphism is conjugate to an automorphism covered by Corollary 3.5, and the result follows from that corollary. □

We end with a problem. The algebras in Propositions 4.1 and 4.4 and in Theorems 4.6 and 4.7 are all twisted group C^*-algebras using suitable cocycles. (See the note in §3 of [**26**].) In fact, the groups are nilpotent, and, under conditions which cover all these examples, the twisted group C^*-algebras have been shown in Theorem 1.7 of [**33**] to be simple and have unique tracial states. A unified treatment would be nice, and the following problem thus arises.

Problem 4.8. Let G be a countable nilpotent group, and let σ be a 2-cocycle on G. Assume the hypotheses of Theorem 1.7 of [**33**]. Does it follow that the twisted group C^*-algebra $C^*(G, \sigma)$ has real rank zero and stable rank one, and that the order on projections over $C^*(G, \sigma)$ is determined by traces? More generally, does it follow that $C^*(G, \sigma)$ has tracial rank zero, as one expects for crossed products of simple unital C^*-algebras with tracial rank zero by automorphisms with the tracial Rokhlin property?

One might hope for an approach using ideas related to the tracial Rokhlin property. This cannot be done directly, because we are looking at the C^*-algebra of a group acting trivially on the C^*-algebra $\mathbb{C}$. For example, the results of [**29**] may well hold for an action twisted by a cocycle, but proving this would not solve the problem. A proof that such C^*-algebras have tracial rank zero would give a new proof of the classification of simple higher dimensional non-commutative tori. The only known proof [**38**] (also see [**37**]) relies on the realization of some simple higher dimensional non-commutative tori as successive crossed product by $\mathbb{Z}$, in such a way that the intermediate crossed products are all simple and the actions of $\mathbb{Z}$ beyond the first all have the Rokhlin property. (See the proof of Corollary 6.6 of [**14**]. Later work [**21**] shows that the tracial Rokhlin property would suffice. Note that there is a gap in the proof of Theorem 2.9 of [**23**]. This gap is fixed in [**21**].) The general case is obtained from this case by consideration of fixed point algebras of actions of finite groups which have the tracial Rokhlin property.

Acknowledgements. Both authors would like to thank Masaki Izumi for valuable discussions, and the first author would also like to thank Kazunori Kodaka. The authors further thank the referee for useful suggestions. This research was partially supported by JSPS Grant for Scientific Research No. 14540217(c)(1), and by NSF grants DMS-0070776 and DMS-0302401.

https://doi.org/10.1017/S0143385706000277 Published online by Cambridge University Press

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