GENERIC DERIVATIONS ON ALGEBRAICALLY BOUNDED STRUCTURES II: MODEL THEORETICAL PROPERTIES

ANTONGIULIO FORNASIERO AND GIUSEPPINA TERZO

Abstract. Let T be an algebraically bounded theory. We consider the $L(\bar{\delta})$-expansions of T by a tuple $\bar{\delta}$ of derivations (which may be commuting or not). We investigate the model completion of either of the above theories, whose existence has been established in [21], with particular attention to its model-theoretic properties, including ω-stability, simplicity, open core, and elimination of imaginaries.

§1. Introduction. We study the model-theoretic properties of the model completion of the theory of algebraically bounded fields expanded with derivations, continuing the foundational work in [21]. Algebraically bounded fields, encompassing structures like algebraically closed, real closed, and p-adically closed fields, provide a rich framework for exploring the interplay between field-theoretic and model-theoretic algebraic notions. Throughout this article, we denote by $\mathbb{K}$ a structure that expands a field of characteristic 0. Recall that $\mathbb{K}$ is *algebraically bounded* if the model-theoretic algebraic closure and the field-theoretic algebraic closure coincide in every structure elementarily equivalent to $\mathbb{K}$; for more details, examples, and main properties (see [13]).

Let $\mathbb{K}$ be algebraically bounded, L be the language of $\mathbb{K}$, and T be its theory. In order to study derivations on $\mathbb{K}$, we denote by $\bar{\delta} = \{\delta_1, \dots, \delta_k\}$ new unary functions symbols.

Assume that T is model complete. In [21] we proved that, for every $k \in \mathbb{N}$, the $L^{\bar{\delta}}$-expansions of T, $T^{\bar{\delta},nc}$ (saying that $\bar{\delta}$ is a k-tuple of derivations) and $T^{\bar{\delta}}$ (saying that $\bar{\delta}$ is a k-tuple of *commuting* derivations) have a model completion. We denote by $T_g^{\bar{\delta},?}$ the model completion of either of the above theories. In the preceding work [21], we began investigating the model-theoretic properties of $T_g^{\bar{\delta},?}$: we showed that if T is stable (resp., dependent), then $T_g^{\bar{\delta},?}$ is also stable (resp., dependent).

The above results include and extend a long series of results in the literature (see [21] for a short survey: here we mention [49] that shows that large model complete fields admit generic derivations). There are also examples of theories which do not admit generic derivations (see [22]).

In this article, we extend the investigation of $T_g^{\bar{\delta},?}$, with a specific focus on model-theoretic properties such as stability, ω-stability, simplicity, and elimination of

Received July 30, 2025 UTC.

2020 *Mathematics Subject Classification.* Primary 03C60, 12H05, 12L12, Secondary 03C10.

Keywords and phrases. derivation, algebraically bounded, model completion, imaginaries, simplicity, open core.

© The Author(s), 2026. Published by Cambridge University Press on behalf of The Association for Symbolic Logic.
0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10229

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

imaginaries (EI); we also address other questions raised in [21]. Multiple authors have examined these properties in several particular cases; here we consider general setting of algebraically bounded fields with several generic derivations. For instance, in [38], the authors demonstrated that $\mathrm{DCF}_{0,m,nc}$, the model completion of the theory of fields with non-commuting derivations, is stable but not ω-stable, eliminates imaginaries, and is uniformly finite. Building upon these results, [33, 37] further generalized some of the properties established in [38].

In this article, we pay particular attention to the following questions:

1) When is $T_g^{\bar{\delta},?}$ simple or totally transcendental?
2) Which are the imaginaries of $T_g^{\bar{\delta},?}$?
3) If T is equipped with a definable topology, is T the open core of $T_g^{\bar{\delta},?}$?
4) Is $T_g^{\bar{\delta},?}$ uniformly finite?
5) What is a "generic" object?
6) Does $T_g^{\bar{\delta},?}$ have a dimension function in the sense of [13]?
7) What is the theory of the restriction of $T_g^{\bar{\delta},?}$ where we forget the derivation but we keep the field of constants?

We will give a full answer to 1) (Sections 7 and 9 and [20]). In Section 7 we prove that if T is simple, then $T_g^{\bar{\delta},?}$ is simple. [38] showed that the model completion of the theory of fields with non-commuting operators is simple, eliminates imaginaries, is uniformly finite, and under some additional assumptions it is stable.

In [20] the first author together with E. Kaplan and A. Matthews further extend the results exposed here, proving the following.

THEOREM A.

- T *is rosy iff it is super-rosy of thorn rank* 1; *in this case*, T *has Geometric EI*;
- T *is simple iff it is supersimple or SU-rank* 1;
- T *is stable iff it is equal to the theory of pure algebraically closed fields, expanded by some constants* (*we will simply write "* $T = \mathrm{ACF}_0$*"*).

Combining the above with the results in this article, we get that $T_g^{\bar{\delta},?}$ is stable iff T is stable iff $T = \mathrm{ACF}_0$. In the case the derivations commute, one can improve stability with ω-stability. Furthermore, in the commuting case when T is stable the theory $T_g^{\bar{\delta},?}$ coincides with the theory of differentially closed fields with several commuting derivations $\mathrm{DCF}_{0,m}$; while in the noncommuting case it yields Moosa–Scanlon's theory $\mathrm{DCF}_{0,m,nc}$.

We have some partial answers to 2). We show that if T has (Geometric) EI and it is either stable or supersimple, then $T_g^{\bar{\delta},?}$ also has (Geometric) EI: Theorems 7.5 and 7.9 and Corollary 7.10: together with Theorem A, we get that if T is simple then $T_g^{\bar{\delta},?}$ has GEI. We also show that if T has a definable topology satisfying some general conditions, then $T_g^{\bar{\delta},?}$ eliminates imaginaries relative to T: see Corollary 11.11; we generalize results in [32]. We conjecture that a stronger result holds than what we manage to prove (see Conjectures 3.7 and 6.7).

We give a complete answer to 3) (Section 11 in particular Corollary 11.11).

For 4) we will give a positive answer in a separate paper.

For 5) we give some possible answers (Section 13), where we study the Polish space of all derivations on a countable model $M \models T$ (under some "bigness" condition

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

on M) and show that generic derivations form a dense $\mathcal{G}_\delta$-set among all possible derivations.

For 6) we give a complete answer in Section 12: $T_g^{\bar{\delta},?}$ has a dimension function iff the derivations commute. We also generalize the result in [14] on "coincidence of dimensions" (see also [6, 24] for previous results in particular cases).

For 7), in Section 10 we study the field of constant $\mathcal{C}_{\bar{\delta}}$ of a model $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$. We show that $\langle \mathbb{K}, \mathcal{C}_{\bar{\delta}} \rangle$ is a lovely pair of geometric structures (in the sense of [4]), and we study the definable subsets of $\mathcal{C}_{\bar{\delta}}^n$.

In Sections 5 and 6 we study independence relations on model of T and of $T_g^{\bar{\delta},?}$. We introduce the independence relation $\mathop{\smile\!\!\!\!\!\!|}^{M,\delta}$ and prove the fundamental Extension Theorem 5.3 and Independence Theorem 6.1: they will make much easier to prove the results in Sections 7 and 11. We also characterize the algebraic closure inside models of $T_g^{\bar{\delta},?}$.

Let us mention here the recent work [42] where the authors study groups definable in models of $T_g^{\bar{\delta}}$: they show that any such group can be embedded in a T-definable group.

We conclude the article with several open questions, conjectures, and announcements of further work.

§2. Preliminaries and conventions. L is a language and T is a L-theory (later we will impose additional conditions on them).

When we say that $\mathfrak{C}$ is a "monster model", we mean that $\mathfrak{C}$ is a λ-saturated and λ-homogeneous model of T for a "sufficently big" cardinal $\lambda > \aleph_0 + |L|$ (other authors mean that $\mathfrak{C}$ is a proper class, and it is λ-saturated and λ-homogeneous for every cardinal λ).

By "small" subset/tuple of $\mathfrak{C}$ we mean of cardinality less than λ.

$\mathbb{K}$ will be an L-structure, and K will be either its domain, or, when $\mathbb{K}$ expands a field, its underlying field.

We say that $\mathbb{K}$ is **algebraically bounded** if it expands a field of characteristic 0 and, for every subset $A \subseteq K$ (not necessarily a substructure), the model theoretic closure of A (computed in a sufficiently saturated elementary extension of $\mathbb{K}$) coincides with the field theoretic algebraic closure of the field generated by A and $\mathrm{dcl}(\emptyset)$.

2.1. Generic derivations. This work explores the model theory of differential fields, specifically focusing on expansions of a base theory T by derivations.

Our assumptions for the whole article are the following:

- $\mathbb{K}$ is a structure expanding a field of characteristic 0.
- L is the language of $\mathbb{K}$ and $T = Th(\mathbb{K})$ is its model complete L-theory.
- $F := \mathrm{dcl}(\emptyset) \subseteq \mathbb{K}$.
- $\mathbb{K}$ is algebraically bounded over F.

By algebraic closure we always mean the T-algebraic closure; in particular, acl denotes the T-algebraic closure, and algebraic independence is understood with respect to T (equivalently, over F in the field-theoretic sense).

Under our assumptions, $\mathbb{K}$ is geometric: in the monster model $\mathfrak{C} \succ \mathbb{K}$, the operator acl satisfies the exchange property, and thus induces a matroid structure.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Moreover, $\mathbb{K}$ is equipped with a dimension function dim, assigning to each definable set (with parameters) a natural number and satisfying the axioms of [13].

We also consider the rank function rk associated with the matroid acl: for sets V and B, $\mathrm{rk}(V/B)$ is the cardinality of a basis of V over B. In particular, if $X \subseteq \mathfrak{C}^n$ is definable with parameters $\bar{b}$, then

$$\dim(X) = \max\{\mathrm{rk}(\bar{a}/\bar{b}) : \bar{a} \in X\}.$$

Let $\bar{\delta} := \langle \delta_1, \dots, \delta_k \rangle$. Let $\eta_1, \dots, \eta_k$ be derivations on F. We obtain two theories:

$T^{\bar{\delta}}$: the expansion of T saying that the δ_i are derivations which commute with each other, i.e., for every $i, j \leq k$, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, and that δ_i extends η_i for $i \leq k$;

$T^{\bar{\delta},nc}$: the expansion of T saying that the δ_i are derivations without any further conditions and that δ_i extends η_i for $i \leq k$.

Both theories have a model completion (remember that T is model complete) (see [21]). For convenience, we use $T_g^{\bar{\delta},?}$ to denote either of the model completions, both for commuting and non-commuting tuples of derivations.

$\langle \mathfrak{C}, \bar{\delta} \rangle$ is a monster model of $T_g^{\bar{\delta},?}$.

$\langle \mathbb{K}, \bar{\delta} \rangle$ is some model of $T_g^{\bar{\delta},?}$.

2.2. Notations and know results. We introduce some notation that we use in the sequel. We denote by Γ the free commutative (or non-commutative) monoid generated by $\bar{\delta}$, we can define the canonical partial order $\preceq$ given by $\beta \preceq \alpha\beta$, for all $\alpha, \beta \in \Gamma$.

REMARK 2.1. If Γ is the free non-commutative monoid generated by $\bar{\delta}$ then $\preceq$ is a well-founded partial order on Γ, but it is not a well-partial-order (i.e., there exist infinite anti-chains):

1) $\emptyset$ (i.e., the empty word, corresponding to the identity function on $\mathbb{K}$) is the minimum of Γ;
2) if $\alpha \preceq \beta$, then $\gamma\alpha \leq \gamma\beta$ and $\alpha\gamma \leq \beta\gamma$.

REMARK 2.2. If Γ is the free commutative monoid generated by $\bar{\delta}$, with the canonical partial order $\preceq$, it is isomorphic to $\mathbb{N}^k$.

Fix $n \in \mathbb{N}$, $\bar{x} = \langle x_1, \dots, x_n \rangle$, and denote by $\Gamma_n := \{\gamma x_i : \gamma \in \Gamma, i \leq n\}$: we consider Γ_n as a set of some $L^{\bar{\delta}}$-terms, as a set of functions from K^n to K, and as a set of indices. For example, $\delta_1\delta_2 x_3$ is in Γ_3 but $\delta_1\delta_2 x_3 + \delta_1 x_1$ is not. Given $\bar{a} \in \mathbb{K}^n$, we denote by $\mathrm{Jet}(\bar{a}) := \langle \gamma\bar{a} : \gamma \in \Gamma_n \rangle$.

Moreover, we denote by $\bar{x}_\Gamma$ a set of formal variables indexed by Γ_n: that is, for every $f \in \Gamma_n$ we have a variable $\bar{x}_f$. Given an $L(A)$-formula $\alpha(\bar{x}_\Gamma)$ we denote by $\alpha(\Gamma)$ the $L^{\bar{\delta}}(A)$-formula where we replaces each occurrence of a variable $\bar{x}_f$ with the corresponding term $f(\bar{x}) \in \Gamma_n$.

Let $J \subseteq \Gamma_n$ and $A \subset K$ be a "small" subset of K with $\bar{\delta}A \subseteq A$. Let $p(\bar{x})$ be a complete $L^{\bar{\delta}}$-type over A (in n variables). Define

$$p_J := \{\alpha : \alpha(\bar{x}_\Gamma)\ L(A)\text{-formula s.t. } \alpha(\Gamma) \in p\} \in S_L^J(A).$$

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

We say that some type $q \in S_L^J(A)$ is $\bar{\delta}$-compatible if there exists $s \in S^n_{L^{\bar{\delta}}}(A)$ such that $q = s_J$.

Many of the model-theoretic properties of T are inherited by $T_g^{\bar{\delta},?}$. The following results from [21], which plays a crucial role in our subsequent analysis, is particularly useful.

THEOREM 2.3. *For every $\bar{a}$ tuple in $\mathfrak{C}$ and* B *subset of $\mathfrak{C}$, the $L^{\bar{\delta}}$-type of $\bar{a}$ over* B *is uniquely determined by the* L*-tuple of* $\mathrm{Jet}(\bar{a})$ *over* $\mathrm{Jet}(B)$.

THEOREM 2.4.

1) *If* T *eliminates quantifiers, then $T_g^{\bar{\delta},?}$ eliminates quantifiers.*
2) *For every $L^{\bar{\delta}}$-formula $\alpha(\bar{x})$ there exists an* L*-formula $\beta(\bar{x}_\Gamma)$ such that*

$$T_g^{\bar{\delta},?} \models \forall \bar{x}\ \big(\alpha(\bar{x}) \leftrightarrow \beta(\Gamma)\big).$$

§3. Elimination of imaginaries. EI is a powerful concept in model theory that simplifies arguments by allowing us to work directly with elements of the model, rather than with equivalence classes of definable sets.

We remind what EI and some of its variants are as follows.

DEFINITION 3.1. T eliminates imaginaries (T has EI) if every imaginary is interdefinable with a real tuple: i.e., for every imaginary e there is a real tuple a such that $\mathrm{dcl}^{\mathrm{eq}}(e) = \mathrm{dcl}^{\mathrm{eq}}(a)$.

DEFINITION 3.2. T weakly eliminates imaginaries (T has WEI), if for every imaginary e there is a real tuple a such that e is definable over a and a is algebraic over e: that is, $e \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(e)$.

REMARK 3.3. Equivalently, a theory T admits weak EI (WEI) in the sense of B. Poizat (see [10, 53]) if, for any $\phi(\overline{x}, \overline{a}) \in L(\overline{a})$, we have the smallest algebraically closed set B such that $\phi(\overline{x}, \overline{a})$ is definable over B.

DEFINITION 3.4. T has geometric eliminates imaginaries (T has GEI), if for every imaginary e there is a real tuple a such that e is algebraic over a and a is algebraic over e: that is, $e \in \mathrm{acl}^{\mathrm{eq}}(a)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(e)$.

REMARK 3.5. By definition, it is clear that EI implies WEI, and that WEI implies GEI. The converse implication, namely, that WEI implies EI, holds when working over a field as is the case here (see [35]).

REMARK 3.6. In any structure containing at least two constants, as in our case, the definition of EI is equivalent to the Uniform EI (see [10] for definition and proof).

Regarding generic derivations, we formulated the following.

CONJECTURE 3.7 [21]. Let T be algebraically bounded. Then, $T_g^{\bar{\delta},?}$ has EI modulo T^{eq}.

In particular: if T has GEI, then $T_g^{\bar{\delta},?}$ also has GEI, and if T has EI, then $T_g^{\bar{\delta},?}$ also has EI.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

A few particular cases were already known, when T is one of the following:

- ACF_0: see [36, 38];
- RCF: see [19, 32] for a proof based on M. Tressl's idea, see also [7, 43] for different proofs.

We will prove two cases of the above conjecture:

- When T is simple: by Theorem A T has GEI; we then employ a result in [52] to show that $T_g^{\bar{\delta},?}$ also has GEI (see Section 7.1); if T has EI, we adapt a technique in [28] to show that $T_g^{\bar{\delta},?}$ also has EI (see Section 7.2).
- When T has a suitable definable topology: we employ a technique by M. Tressl to show that $T_g^{\bar{\delta},?}$ has EI modulo T^{eq} (see Section 11).

§4. Independence relations. To establish the primary results of this article, we remind the concept of independence relations and revisit key theorems related to this notion. We fix a monster model $\mathfrak{C} \models T$.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be a ternary relation on small subsets of $\mathfrak{C}$. We mostly use the definitions and nomenclature from [1]: in particular, we are interested in the cases when $\mathop{\smile\!\!\!\!\!\!|}$ is either a strict independence relation, or the relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ defined in [1, Definition 1.26], or Shelah-forking $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.

We remind for completeness the definition $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ introduced in [1].

Definition 4.1. The relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ (M-dividing independence) is defined as:
$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_C B$ iff for any C' s.t $C \subseteq C' \subseteq \mathrm{acl}(BC)$ then $\mathrm{acl}(AC') \cap \mathrm{acl}(BC') = \mathrm{acl}\, C'$.

For the remainder of this section, $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation on $\mathfrak{C}$.

Remark 4.2. If $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$.

Proof. By Existence, there exists $D \equiv_{BC} \mathrm{acl}(BC)$ s.t. $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$: notice that $D = \mathrm{acl}(BC)$, and therefore $A \mathop{\smile\!\!\!\!\!\!|}_{BC} \mathrm{acl}(BC)$. By transitivity, $A \mathop{\smile\!\!\!\!\!\!|}_C \mathrm{acl}(BC)$. ⊣

Notice that we don't assume that T has some form of EI, nor that $\mathop{\smile\!\!\!\!\!\!|}$ extends to $\mathfrak{C}^{eq}$. However, the setting in [1] often assumes implicitly to work in T^{eq}: to avoid confusion, we will introduce some additional nomenclature, following in part [52].

Definition 4.3. We say that an independence relation $\mathop{\smile\!\!\!\!\!\!|}$ is:

real-strict: if, for every $a \in \mathfrak{C}$ and $B \subset \mathfrak{C}$, $a \mathop{\smile\!\!\!\!\!\!|}_B a$ iff $a \in \mathrm{acl}(B)$;

strict: if $\mathop{\smile\!\!\!\!\!\!|}$ extends to an independence relation on $\mathfrak{C}^{eq}$ (which we also denote by $\mathop{\smile\!\!\!\!\!\!|}$), and for every $a \in \mathfrak{C}^{eq}$ and $B \subset \mathfrak{C}^{eq}$, $a \mathop{\smile\!\!\!\!\!\!|}_B a$ iff $a \in \mathrm{acl}^{\mathrm{eq}}(B)$;

real-canonical: if it is real-strict and, for every $A, B, C, D \subset \mathfrak{C}$ such that $B \supseteq C, D$

$$A \underset{C}{\smile\!\!\!\!\!\!|}\, B \wedge A \underset{D}{\smile\!\!\!\!\!\!|}\, B \implies A \underset{\mathrm{acl}(C)\cap\mathrm{acl}(D)}{\smile\!\!\!\!\!\!|}\, B;$$

canonical: if it is strict and, for every $A, B, C, D \subset \mathfrak{C}^{eq}$, such that $B \supseteq C, D$

$$A \underset{C}{\smile\!\!\!\!\!\!|}\, B \wedge A \underset{D}{\smile\!\!\!\!\!\!|}\, B \implies A \underset{\mathrm{acl}^{\mathrm{eq}}(C)\cap\mathrm{acl}^{\mathrm{eq}}(D)}{\smile\!\!\!\!\!\!|}\, B.$$

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Notice that if $\mathop{\smile\!\!\!\!\!\!|}$ is strict, then it is real-strict. If either $\mathfrak{C} = \mathfrak{C}^{eq}$, or $\mathfrak{C}$ has geometric EI (GEI), then $\mathop{\smile\!\!\!\!\!\!|}$ is real-strict iff it is strict; similarly, under the same assumption, $\mathop{\smile\!\!\!\!\!\!|}$ is real-canonical iff it is canonical.

FACT 4.4.

1) *There exist rosy theories with EI* (*even o-minimal*) *such that* $\mathop{\smile\!\!\!\!\!\!|}^{b}$ (*thorn-forking independence*) *is not* (*real*)*-canonical*: [1, *Appendix A*.3] *explains an example from* [34]; *see also* [52, *Remark* 3.5].
2) *There exists a theory with WEI* (*the "integral Urysohn space"*) *with more than one strict independence relation* (*see* [11]).

FACT 4.5 [52, Lemma 3.1]. *If* $\mathop{\smile\!\!\!\!\!\!|}$ *is strict and real-canonical, then* $\mathfrak{C}$ *has GEI.*

COROLLARY 4.6. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be real-strict and real-canonical. T.f.a.e.*:

1) $\mathfrak{C}$ *has GEI*;
2) $\mathop{\smile\!\!\!\!\!\!|}$ *is strict*;
3) $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{b}$.

LEMMA 4.7. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be a real-strict independence relation on* $\mathfrak{C}$. *If* $A \mathop{\smile\!\!\!\!\!\!|}_C B$, *then* $A \mathop{\smile\!\!\!\!\!\!|}^{M}_C B$.
In particular, if T *is simple and* $A \mathop{\smile\!\!\!\!\!\!|}^{f}_C B$, *then* $A \mathop{\smile\!\!\!\!\!\!|}^{M}_C B$.

PROOF. Let A, B, C, C' be as in Definition 4.1. Assume that $A \mathop{\smile\!\!\!\!\!\!|}_C B$. Let $d \in \mathrm{acl}(AC') \cap \mathrm{acl}(BC')$. We need to show that $d \in \mathrm{acl}(C')$:

$$A \underset{C}{\smile\!\!\!\!\!\!|} B \implies A \underset{C}{\smile\!\!\!\!\!\!|} \mathrm{acl}(BC) \implies A \underset{C}{\smile\!\!\!\!\!\!|} BC' \implies$$
$$\implies A \underset{C'}{\smile\!\!\!\!\!\!|} B \implies \mathrm{acl}(AC') \underset{C'}{\smile\!\!\!\!\!\!|} \mathrm{acl}(BC') \implies d \underset{C'}{\smile\!\!\!\!\!\!|} d.$$

Since $\mathop{\smile\!\!\!\!\!\!|}$ is real-strict, the latter implies that $d \in \mathrm{acl}(C')$. ⊣

FACT 4.8 [1, Section 3.1],[8].

1) *If* $\mathfrak{C}$ *is either stable or super-simple, then it has elimination of hyperimaginaries* (*EHI*);
2) *if* $\mathfrak{C}$ *is simple with EHI, then* $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *is canonical*;
3) *if* $\mathop{\smile\!\!\!\!\!\!|}$ *is canonical, then* $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{b}$.

For more on EHI in (simple) theories, see, e.g., [8, 9, 39]; there are no known examples of simple theories that do *not* eliminate hyperimaginaries; for us the relevant implication is the following fact.

FACT 4.9. *If* $\mathfrak{C}$ *is either stable or super-simple, then* $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *is canonical and* $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{b}$.

DEFINITION 4.10. $\mathop{\smile\!\!\!\!\!\!|}$ satisfies "Independence over Models" if:
For every $\bar{a}, \bar{b} \in \mathfrak{C}^n$, $\bar{a}', \bar{b}' \in \mathfrak{C}^m$, for every $M \prec \mathfrak{C}$ with M small, if

$$\bar{a} \underset{M}{\smile\!\!\!\!\!\!|} \bar{b}, \qquad \bar{a}' \underset{M}{\smile\!\!\!\!\!\!|} \bar{a}, \qquad \bar{b}' \underset{M}{\smile\!\!\!\!\!\!|} \bar{b}, \qquad \bar{a}' \equiv^{L}_{M} \bar{b}'$$

then there exists $\bar{c} \in \mathfrak{C}^m$ such that

$$\bar{c} \underset{M}{\smile\!\!\!\!\!\!|} \bar{a}\bar{b}, \qquad \bar{c} \equiv^{L}_{M\bar{a}} \bar{a}', \qquad \bar{c} \equiv^{L}_{M\bar{b}} \bar{b}'.$$

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

$\downarrow$ satisfies "Independence over Algebraically Closed (real) Substructures" (IACS) if the above is true when we let M vary among algebraically closed substructure of $\mathfrak{C}$. We say that T satisfies IACS if $\downarrow^{\mathrm{f}}$ satisfies it.

FACT 4.11 (Kim–Pillay: see [30], [48, Theorems 29.11 and 29.13]). *Let $\downarrow$ be an independence relation on $\mathfrak{C}$ that satisfies Independence over Models. Then, $\mathfrak{C}$ is simple and $\downarrow$ is Shelah-forking (on $\mathfrak{C}$).*

Conversely, if $\mathfrak{C}$ is simple, then $\downarrow^{f}$ is a strict independence relation satisfying Independence over Models.

FACT 4.12 [8]. *If $\mathfrak{C}$ is simple with EHI and EI, then $\downarrow^{f}$ satisfies IACS.*

$\downarrow$ can be extended to L-types: given $A \subseteq B$ small subsets of $\mathfrak{C}$ and an $L(B)$-type (in a small number of variables) $q(\bar{y})$, we write

$$q \underset{A}{\downarrow} B$$

if for *some* $\bar{c} \in \mathfrak{C}^{|\bar{y}|}$ realizing $q(\bar{y})$ we have that

$$\bar{c} \underset{A}{\downarrow} B.$$

Notice that the above is equivalent to:

"For *every* $\bar{c} \in \mathfrak{C}^{|\bar{y}|}$ realizing $q(\bar{y})$ we have that $\bar{c} \downarrow_A B$".

4.1. Independence relations on algebraically bounded structures. We analyze now the case when L extends the language of rings and T is algebraically bounded.

LEMMA 4.13. *Let $A, B, C \subseteq \mathfrak{C}$. T.f.a.e.:*

1) $A \downarrow^{M}_{C} B$;
2) *for every $A' \subseteq A$, if A' is algebraically independent over CF (where $F = dcl(\emptyset)$), then it is still algebraically independent over BC;*
3) *for every $A' \subseteq A$, if A' generates* A *(in the sense of the matroid* acl*) over BC, then it generates* A *over* B;
4) *for every $A' \subseteq A$ and $B' \subseteq B$, $\mathrm{tr.deg.}(A'B'/C) = \mathrm{tr.deg.}(A'/C) + \mathrm{tr.deg.}(B'/C)$.*

Moreover, $\downarrow^{M}$ is a real-strict independence relation on $\mathfrak{C}$ satisfying the strong finite character condition.

PROOF. For the "moreover" part, notice that 4 implies that $\downarrow^{\mathrm{M}}$ is symmetric, and the conclusion follows by [1, Theorem 2.39 and Exercise 2.6]. ⊣

FACT 4.14. *If $\mathfrak{C}$ is algebraically bounded and has GEI, then $\downarrow^{M} = \downarrow^{\mathfrak{p}}$ and $\mathfrak{C}$ is rosy, i.e., $\downarrow^{\mathfrak{p}}$ is an independence relation on $\mathfrak{C}^{eq}$; moreover, it is super-rosy of* þ*-rank* 1.[1]

PROOF. Since $\mathfrak{C}$ has GEI, $\downarrow^{\mathrm{M}}$ is a strict independence relation on $\mathfrak{C}^{eq}$, and therefore $\mathfrak{C}$ is rosy. For more details see [1]: in particular, [1, Definition 1.28].

$\mathfrak{C}$ is then super-rosy of rank 1 because $U^{\mathfrak{p}}(\bar{a}/B) = \mathrm{tr.deg.}(\bar{a}/FB)$, where $U^{\mathfrak{p}}$ is the þ-rank, tr. deg. is the transcendence degree, and F is the model-theoretic algebraic closure of the empty set. ⊣

[1] Later, we will prove a kind of converse to this fact.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

REMARK 4.15. Assume that $\mathfrak{C}$ is algebraically bounded, with GEI, simple, and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is canonical. Then, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.

PROOF. Since $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ is canonical, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$, and by the Fact 4.14 we have the conclusion. ⊣

§5. The extension theorem. The results presented in this section are of independent interest and will be utilized in the sequel. For now, K is some field of characteristic 0.

DEFINITION 5.1. We denote by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ the ternary relation on subsets of K induced by acl over F:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}_{C} B$$

iff, for every $\bar{a}$ finite subset of A, if $\bar{a}$ is algebraically independent over CF, then $\bar{a}$ remains algebraically independent over CFB.[2]

The following lemma is well-known; we provide a proof for completeness.

LEMMA 5.2 (Amalgamation). *Let B_0, B_1, and B_2 be subrings of* K *containing* F. *Let $\bar{\delta}_i$ be* k*-tuples of derivations on B_i, $i = 0, 1, 2$. Assume*:

1) *$\langle B_0, \bar{\delta}_0 \rangle$ is a common substructure of $\langle B_1, \bar{\delta}_1 \rangle$ and $\langle B_2, \bar{\delta}_2 \rangle$*;
2)

$$B_1 \mathop{\smile\!\!\!\!\!\!|}^{M}_{B_0} B_2.$$

Then, there exists a k*-tuple of derivations $\bar{\delta}$ on* K *which extends all the $\bar{\delta}_i$. Moreover, if each $\bar{\delta}_i$ commutes, then $\bar{\delta}$ commutes.*

PROOF. Let $\bar{b}_i$ be a transcendence basis of B_i over B_0, $i = 1, 2$. By 2), $\bar{b}_1$ and $\bar{b}_2$ are disjoint, and $\bar{b}_1 \cup \bar{b}_2$ is algebraically independent over B_0. Thus, there exists a k-tuple of derivations $\bar{\delta}$ on K extending $\bar{\delta}_0$ and such that

$$\bar{\delta}(\bar{b}_i) = \bar{\delta}_i(\bar{b}_i), \qquad i = 1, 2.$$

But then $\bar{\delta}$ extends also $\bar{\delta}_1$ and $\bar{\delta}_2$. If moreover each $\bar{\delta}_i$ commutes, let $\bar{c}$ be a transcendence basis of K over $B_0 \cup B_1 \cup B_2$. Let $\bar{\delta}$ be the unique k-tuple of derivations on K such that:

1) $\bar{\delta}$ extends $\bar{\delta}_0$;
2) $\bar{\delta}(\bar{b}_i) = \bar{\delta}_i(\bar{b}_i)$, $i = 1, 2$;
3) $\bar{\delta}(\bar{c}) = 0$.

Then, $\bar{\delta}$ commutes. ⊣

Now $\langle \mathbb{K}, \bar{\delta} \rangle$ is a monster model of $T_g^{\bar{\delta},?}$.

The following technical result has several applications: we will see some later; in [40, Lemma 5.8] the authors prove a version of it in order to study definable groups.

[2] When $\mathbb{K}$ is algebraically bounded, the above definition coincides with the one in Section 4.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

THEOREM 5.3 (Extension). *Let $A \subseteq B$ be "small" subsets of* K *with $\bar{\delta}A \subseteq A$ and $\bar{\delta}B \subseteq B$. Let $p(\bar{x})$ be a complete $L^{\bar{\delta}}$-type over* A (*in* n *variables*). *Let $q \in S_L^J(B)$ be some extension of p_J. If*

$$q \underset{A}{\overset{M}{\downarrow}} B,$$

then q *is $\bar{\delta}$-compatible* (*see Section 2.2*).

PROOF. W.l.o.g., A and B are also subrings of K. Let $\bar{a} \in K^n$ be a realization of p and $\bar{b} \in K^J$ be a realization of q. Since $\mathrm{Jet}(\bar{a})$ and $\bar{b}$ have the same L-type q over A, there exists ϕ an L-automorphism of $\mathbb{K}$ fixing A point-wise and mapping $\bar{b}$ to $\mathrm{Jet}(\bar{a})$.

Define $\bar{\varepsilon} := \phi^{-1} \circ \bar{\delta} \circ \phi$. Let

- $\bar{\delta}_0$ be the restriction of $\bar{\delta}$ to A;
- $\bar{\delta}_1$ be the restriction of $\bar{\delta}$ to B;
- $\bar{\delta}_2$ be the restriction of $\bar{\varepsilon}$ to $A[\bar{b}]$.

Notice that $\bar{\varepsilon}$ is a tuple of derivations on K which extends $\bar{\delta}_0$ and such that $\bar{\varepsilon}$ commutes if $\bar{\delta}$ commutes. Thus, by the amalgamation lemma, there exists a k-tuple $\bar{\lambda}$ of derivations on K which extends $\bar{\delta}_0$, $\bar{\delta}_1$, and $\bar{\delta}_2$, and such that $\bar{\lambda}$ commutes if $\bar{\delta}$ commutes. Thus, $\langle \mathbb{K}, \bar{\lambda} \rangle \models T^{\bar{\delta},?}$ ($\bar{\lambda}$ is not generic, in general). Let $\bar{b}_0 \in K^n$, interpreted in $\langle \mathbb{K}, \bar{\lambda} \rangle$, we have that $\mathrm{Jet}(\bar{b}_0) = \bar{b}$, and therefore $\mathrm{Jet}(\bar{b}_0)$ satisfies q. Thus, the partial $L^{\bar{\delta}}(B)$-type

$$u(\bar{x}) := T_g^{\bar{\delta},?} \cup \mathrm{Diag}_{L^{\bar{\delta}}}(B) \cup q(\mathrm{Jet}(\bar{x}))$$

is consistent. Since $\langle \mathbb{K}, \bar{\delta} \rangle$ is existentially closed and $|B|^+$-saturated, there exists $\bar{c} \in K^n$ satisfying u. ⊣

5.1. Relation with the framework of [33]. It is worth noting that, in view of Lemma 5.2, the theory $T^{\bar{\delta}}$ with respect to $(T, \overset{\mathrm{M}}{\downarrow})$ fits into the framework of derivation-like expansions introduced in [33].

Indeed, Lemma 5.2 yields the amalgamation property required in part (1) of the definition of derivation-like expansions. Moreover, since $\mathbb{K}$ is algebraically bounded, algebraic closure defines an independence relation, and, in characteristic 0, derivations extend uniquely to algebraic extensions. This immediately implies part (2) of the definition.

Furthermore, when $T = \mathrm{Th}(\mathbb{K})$ is simple, forking independence implies algebraic independence, and hence $T^{\bar{\delta}}$ is derivation-like also with respect to $(T, \overset{\mathrm{f}}{\downarrow})$.

As a consequence, several of our results can alternatively be obtained from the general theory developed in [33], and our Theorems 5.3, 6.1, 7.1, and 7.4 (as well as Corollary 6.2) can be recovered from their corresponding general results in that setting.

Nevertheless, we have chosen to include self-contained proofs of these results. This is partly for the convenience of the reader, and partly to maintain a uniform approach and notation throughout the article, avoiding the need to repeatedly translate between our setting and the abstract framework of [33].

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

§6. Algebraic closure and independence relations. In this section we examine the relations between algebraic closure, independence relations, and (geometric) EI. The results in this section will be used in Sections 7 and 12.

We fix $\langle \mathfrak{C}; \bar{\delta} \rangle$ monster model of $T_g^{\bar{\delta},?}$.

Let $\mathop{\smile\!\!\!\!\!\!|}$ be some ternary relation on subsets of $\mathfrak{C}$. We define the following ternary relation on subsets of $\mathfrak{C}$:

$$A \underset{C}{\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}} B \iff \mathrm{Jet}(A) \underset{\mathrm{Jet}(C)}{\mathop{\smile\!\!\!\!\!\!|}} \mathrm{Jet}(B).$$

Notice that we are not assuming that T eliminates imaginaries, or that $\mathop{\smile\!\!\!\!\!\!|}$ extends to $\mathfrak{C}^{eq}$).

THEOREM 6.1. *Assume that $\mathop{\smile\!\!\!\!\!\!|}$ is a real-strict independence relation on $\mathfrak{C}$ (see Def. 4.3). Then, $\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}$ is a real-strict independence relation on $\langle \mathfrak{C}, \bar{\delta} \rangle$.*

If moreover $\mathop{\smile\!\!\!\!\!\!|}$ is real-canonical, then $\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}$ is real-canonical.

PROOF. In [1], a list of axioms for independence relations is presented. The author subsequently demonstrates that the Extension Axiom is equivalent to the conjunction of the Existence Axiom and the Symmetry Axiom. Notably, among the axioms for independence ($\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}$) outlined in [1], only the Existence Axiom requires non-trivial verification. Thus, let $\bar{a}, \bar{b}, \bar{c}$ be small tuples in $\mathfrak{C}$. Since $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation, there exists a small tuple $\bar{d} \subset \mathfrak{C}$ s.t. $\bar{d} \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Jet}(\bar{c})} \mathrm{Jet}(\bar{b})$. Let $p := \mathrm{tp}^{L^{\bar{\delta}}}(\mathrm{Jet}(\bar{a}) / \mathrm{Jet}(\bar{c}))$ and $q := \mathrm{tp}^{L}(\bar{d} / \mathrm{Jet}(\bar{c})\, \mathrm{Jet}(\bar{b}))$. By assumption, q is a complete L-type extending p, and

$$q \underset{\mathrm{Jet}(\bar{c})}{\mathop{\smile\!\!\!\!\!\!|}} \mathrm{Jet}(\bar{b}).$$

Since $\mathop{\smile\!\!\!\!\!\!|}$ is real-strict, Lemma 4.7 implies that

$$q \underset{\mathrm{Jet}(\bar{c})}{\overset{\mathrm{M}}{\smile\!\!\!\!\!\!|}} \mathrm{Jet}(\bar{b})$$

and therefore, by the extension theorem, there exists $\bar{a}' \subset \mathfrak{C}$ s.t. $\mathrm{Jet}(\bar{a}')$ realizes q: hence,

$$\bar{a}' \equiv^{L^{\bar{\delta}}}_{\bar{c}} \bar{a} \ \wedge\ \bar{a}' \underset{\bar{c}}{\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}} \bar{b},$$

proving that $\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}$ satisfies Existence.

The fact that $\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}$ is real-strict is clear.

Finally, assume that $\mathop{\smile\!\!\!\!\!\!|}$ is real-canonical. Assume that

$$\bar{c} \underset{E\bar{b}}{\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}} \bar{a} \ \wedge\ \bar{c} \underset{E\bar{a}}{\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}} \bar{b}.$$

We have to prove that

$$\bar{c} \underset{\mathrm{acl}^{\bar{\delta}}(E\bar{a}) \cap \mathrm{acl}^{\bar{\delta}}(E\bar{b})}{\overset{\bar{\delta}}{\smile\!\!\!\!\!\!|}} \bar{a}\bar{b}.$$

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

W.l.o.g., we may assume that $E = \mathrm{acl}^{\bar{\delta}} E$. Thus, we have

$$\mathrm{Jet}(\bar{c}) \underset{E\ \mathrm{Jet}(\bar{b})}{\downarrow} \mathrm{Jet}(\bar{a}) \wedge \mathrm{Jet}(\bar{c}) \underset{E\ \mathrm{Jet}(\bar{a})}{\downarrow} \mathrm{Jet}(\bar{b}).$$

Since $\downarrow$ is real-canonical, we have

$$\mathrm{Jet}(\bar{c}) \underset{\mathrm{acl}(E\ \mathrm{Jet}(\bar{b}))\cap\mathrm{acl}(E\ \mathrm{Jet}(\bar{b}))}{\downarrow} \mathrm{Jet}(\bar{b}).$$

$\mathrm{acl}(E\ \mathrm{Jet}(\bar{b})) = \mathrm{acl}^{\bar{\delta}}(E\bar{b})$, and similarly for $\bar{a}$, we are done. ⊣

In particular, we can choose $\downarrow := \mathop{\downarrow}^{\mathrm{M}}$, and obtain that the induced relation $\mathop{\downarrow}^{M,\bar{\delta}}$ is a real-strict independence relation on $\langle \mathfrak{C}; \bar{\delta} \rangle$.

Corollary 6.2 (cf. [33, Corollary 3.16], [42, Corollary 1.4]).

1) *The algebraic closure on* $\langle \mathfrak{C}; \bar{\delta} \rangle$ *is given by*

$$\mathrm{acl}(\mathrm{Jet}(A)),$$

for every $A \subseteq \mathfrak{C}$.[3]

2) *If* $\langle \mathfrak{C}; \bar{\delta} \rangle$ *has GEI, then it is rosy, and* $\mathop{\downarrow}^{M,\bar{\delta}}$ *coincides with* þ*-forking* $\mathop{\downarrow}^{þ}$ *(for* $\langle \mathfrak{C}; \bar{\delta} \rangle$*).*

Proof. Let $\mathrm{acl}^{\bar{\delta}}$ be the algebraic closure according to $T_g^{\bar{\delta},?}$.

1) If $a \in \mathrm{acl}^{\bar{\delta}} B$, then, since $\mathop{\downarrow}^{M,\bar{\delta}}$ is an independence relation,

$$a \mathop{\downarrow}^{M,\bar{\delta}}_{B} a,$$

and in particular,

$$a \underset{\mathrm{Jet}(B)}{\mathop{\downarrow}^{\mathrm{M}}} a,$$

that is, $a \in \mathrm{acl}(\mathrm{Jet}(B))$.

Conversely, it is clear that $\mathrm{acl}(\mathrm{Jet}(B))$ must be contained in $\mathrm{acl}^{\bar{\delta}}(B)$.

2) Since $\mathop{\downarrow}^{M,\bar{\delta}}$ is a real-strict independence relation on $\langle \mathfrak{C}, \bar{\delta} \rangle$, we have that the latter is real-rosy; if moreover $\langle \mathfrak{C}, \bar{\delta} \rangle$ has GEI, it is rosy (see [1] for definitions and proofs: in particular, [1, Definition 1.26]). ⊣

6.1. GEI in algebraically bounded structures.

Proposition 6.3. *Let* K *be any field of characteristic* 0. *Define* $A \underset{C}{\downarrow} B$ *if, for every* $X \subseteq A$ *which is algebraically independent over* C *(in the field-theoretic sense),* X *remains algebraically independent over* BC. *Then,* $\downarrow$ *is real-canonical.*

Proof. Let $\tilde{K}$ be the field-theoretic algebraic closure of K. Given $C \subseteq K$, let $\tilde{C}$ be the field-theoretic algebraic closure of C inside $\tilde{K}$, and $\mathrm{acl}(C)$ be the field-theoretic algebraic closure of C inside K.

[3] $T_g^{\bar{\delta},?}$ is differentially algebraically bounded (see [50, Definition 4.1, Remark 4.21]; cf. [13, Section 2.25]).

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Let $D := C_1 \cap C_2$.

CLAIM 1.

$$\tilde{C}_1 \cap \tilde{C}_2 = \widetilde{\mathrm{acl}(C_1) \cap \mathrm{acl}(C_2)}.$$

First, w.l.o.g. we may assume that $C_i = \mathrm{acl}(C_i)$, for $i = 1, 2$. It is clear that

$$\tilde{D} \subseteq \tilde{C}_1 \cap \tilde{C}_2;$$

we have to prove the opposite inclusion. Let C be either C_1 or C_2. Let $a \in \tilde{C}_1 \cap \tilde{C}_2$ and $a_1, \ldots, a_\ell$ be the conjugates of a over K. Then, $a_j \in \tilde{C}$, $j = 1, \ldots, \ell$. Let $\sigma_t(x_1, \ldots, x_\ell)$ be one of the symmetric polynomials: we have that $\sigma_t(a_1, \ldots, a_\ell) \in \tilde{C} \cap K = C$. Thus, if $p(x) \in K[x]$ is the minimal polynomial of a over K, we have $p \in C[x]$, and therefore $p \in D[x]$.

Thus, $a \in \tilde{D}$, proving the claim.

Let $\tilde{\mathop{\smile}\limits^{\mid}}$ be the analogue of $\mathop{\smile}\limits^{\mid}$ inside $\tilde{K}$.

CLAIM 2. $\tilde{\mathop{\smile}\limits^{\mid}}$ *is real-canonical.*

In fact, $\tilde{\mathop{\smile}\limits^{\mid}}$ coincides with $\mathop{\smile}\limits^{\mid^{\mathrm{f}}}$ on $\tilde{K}$ seen as a pure field, and, since $\tilde{K}$ (using the fact that it is an algebraically closed field) is stable with EI, $\mathop{\smile}\limits^{\mid^{\mathrm{f}}}$ is real-canonical (see [1, Section 3.1]).

Assume now that $C_i \subseteq B$ for $i = 1, 2$, and $A \mathop{\smile}\limits^{\mid}_{C_i} B$ and $C_i \subseteq B$, for $i = 1, 2$.

We have $A \tilde{\mathop{\smile}\limits^{\mid}}_{C_i} B$ for $i = 1, 2$. Since $\tilde{\mathop{\smile}\limits^{\mid}}$ is real-canonical, we have that

$$A \tilde{\mathop{\smile}\limits^{\mid}}_{\tilde{C}_1 \cap \tilde{C}_2} B,$$

and therefore, by Claim 1,

$$A \tilde{\mathop{\smile}\limits^{\mid}}_{\tilde{D}} B,$$

which is equivalent to $A \mathop{\smile}\limits^{\mid}_{D} B$.

COROLLARY 6.4. $\mathop{\smile}\limits^{\mid^{M}}$ *is real-strict and real-canonical.* $\mathop{\smile}\limits^{\mid^{M,\bar{\delta}}}$ *is also real-strict and real-canonical.*

PROOF. This is an immediate consequence of Theorem 6.1 and Proposition 6.3. ⊣

Thus, by Corollary 4.6, we have the following theorem.

THEOREM 6.5. *T.f.a.e.*:

1) $\mathfrak{C}$ *has GEI*;
2) $\mathop{\smile}\limits^{\mid^{M}}$ *is strict*;
3) $\mathop{\smile}\limits^{\mid^{M}} = \mathop{\smile}\limits^{\mid^{\mathfrak{p}}}$;
4) $\mathfrak{C}$ *is super-rosy of* $\mathfrak{p}$*-rank* 1.

THEOREM 6.6. *T.f.a.e.*:

1) $\langle \mathfrak{C}; \bar{\delta} \rangle$ *has GEI*;
2) $\mathop{\smile}\limits^{\mid^{M,\bar{\delta}}}$ *is strict*;
3) $\mathop{\smile}\limits^{\mid^{M,\bar{\delta}}}$ *is equal to* $\mathfrak{p}$*-forking on* $\langle \mathfrak{C}; \bar{\delta} \rangle$.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

CONJECTURE 6.7. Assume that $\mathfrak{C}$ has GEI. Then, $\langle \mathfrak{C}; \bar{\delta} \rangle$ also has GEI.

PROPOSITION 6.8. *Let $\mathfrak{C}$ be a monster model of some theory. Assume that $\downarrow$ is an independence relation on $\mathfrak{C}$ satisfying the following conditions*:

I) $\downarrow$ *is real-canonical*;

II) $\downarrow$ *is "controlled by a rank"*: *there exist a linearly ordered set $\langle I; \leq \rangle$ and a function ρ associating an element of* I *to each type* (*over some small subset of the monster model $\mathfrak{C}$ in finitely many real variables*), *such that*:
   a) *ρ is invariant under automorphisms*;
   b) *ρ for every $\bar{a}$, B, C, $\rho(\bar{a}/B) \leq \rho(a/BC)$, with equality iff $\bar{a} \overset{f}{\downarrow}_B C$*;
   c) *for every non-empty set* X *which is type-definable over* A, *$\rho(\bar{x}/A)$ attains a maximum on* X (*e.g., ρ is upper semi-continuous*).

Let $e \in \mathfrak{C}^{eq}$, where $e = f(\bar{a})$ for some (*real finite tuple*) *$\bar{a} \in \mathfrak{C}^{\ell}$ and some function* f *which is definable without parameters. Let $E := \mathrm{acl}^{\mathrm{eq}}(e) \cap \mathfrak{C}$ and* P *be the set of realization of* $\mathrm{tp}(a/E)$ (*inside $\mathfrak{C}$*). *Then there exists $\bar{c} \in P$ such that $f(\bar{c}) = e$ and $\bar{a} \downarrow_E \bar{c}$.*

PROOF. Proposition 6.8 follows the proof in [28, (1.10)] and [38, Theorem 5.12]. As noticed in [28], Neumann's Lemma implies that there exists $\bar{b}$ conjugate of $\bar{a}$ over $E \cup \{e\}$ such that

$$\mathrm{acl}(E\bar{a}) \cap \mathrm{acl}(E\bar{b}) = \mathrm{acl}^{\mathrm{eq}}(Ee) \cap \mathfrak{C} = E \tag{1}$$

(see [15]). Let Q be the set of $\bar{b} \in P$ satisfying (1).

CLAIM 3. Q *is type-definable over $E\bar{a}$.*[4]

Since P is type-definable, it suffices to show that the set

$$X := \{\bar{x} \in \mathfrak{C}^{\ell} : \mathrm{acl}(E\bar{a}) \cap \mathrm{acl}(E\bar{x}) \supset E\}$$

is $\vee$-definable. For every $d \in \mathrm{acl}(E\bar{a}) \setminus E$, let $X_d := \{\bar{x} \in \mathfrak{C}^{\ell} : d \in \mathrm{acl}(E\bar{x})\}$. X_d is $\vee$-definable (over Ed) because it is a union of sets defined by formulae of the form $\alpha(\bar{c}, d, \bar{x}) \wedge |\alpha(\bar{c}, \mathfrak{C}, x)| \leq n$, as α varies among all L-formulae and n varies in $\mathbb{N}$. Therefore, $X = \bigcup_{d \in \mathrm{acl}(E\bar{a}) \setminus E} X_d$ is also $\vee$-definable (over $\mathrm{acl}(E\bar{a})$).

By assumption, there exists $\bar{b} \in Q$ such that $\rho(\bar{b}/E\bar{a})$ is maximum (inside Q).

Let $\bar{c} \models \mathrm{tp}(\bar{b}/E\bar{a})$ such that $\bar{a} \downarrow_{E\bar{a}} \bar{b}$. As in [28, (1.10)] (cf. [38, Theorem 5.12]), we see that

$$\bar{c} \underset{E\bar{b}}{\downarrow} \bar{a} \wedge \bar{c} \underset{E\bar{a}}{\downarrow} \bar{b}.$$

Since $\downarrow$ is real-canonical, and $\mathrm{acl}(E\bar{a}) \cap \mathrm{acl}(E\bar{b}) = E$, we have that

$$\bar{c} \underset{E}{\downarrow} \bar{a}\bar{b}$$

and therefore $\bar{c} \downarrow_E \bar{a}$, proving Proposition 6.8.

[4]This step was not clear to us: thanks to Itay Kaplan for explaining it.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

§7. Simplicity. We will now prove that if T is simple, then $T_g^{\bar\delta,?}$ is also simple. Our proof relies on the crucial notion of "Independence over Algebraically Closed Substructures" (IACS), which we define and explore in this section.

THEOREM 7.1 (Simplicity of $T_g^{\bar\delta,?}$). *If* T *is simple, then* $T_g^{\bar\delta,?}$ *is also simple.*

The above is non-trivial, but thanks to the preliminary work we can give an easy proof.

We fix $\langle \mathfrak{C}, \bar\delta\rangle$ monster model of $T_g^{\bar\delta,?}$. Assume that $\mathfrak{C}$ is simple. Let $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ be Shelah-forking relation on $\mathfrak{C}$, and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f},\delta}$ be the induced independence relation on $\langle \mathfrak{C}, \bar\delta\rangle$ as in Section 6.

Since we want to prove a stronger version of Theorem 7.1, we need the following definition.

DEFINITION 7.2. An independence relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfies "Independence over Algebraically Closed Substructures" (IACS) if:

(IACS) For every $\bar a, \bar b \in \mathfrak{C}^n$, $\bar a', \bar b' \in \mathfrak{C}^m$, for every $M \subset \mathfrak{C}$ with M small and algebraically closed, if

$$\bar a \mathop{\smile\!\!\!\!\!\!|}_M \bar b, \qquad \bar a' \mathop{\smile\!\!\!\!\!\!|}_M \bar a, \qquad \bar b' \mathop{\smile\!\!\!\!\!\!|}_M \bar b, \qquad \bar a' \equiv^L_M \bar b'$$

then there exists $\bar c \in \mathfrak{C}^m$ such that

$$\bar c \mathop{\smile\!\!\!\!\!\!|}_M \bar a\bar b, \qquad \bar c \equiv^L_{M\bar a} \bar a', \qquad \bar c \equiv^L_{M\bar b} \bar b'.$$

We say that T satisfies IACS if $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ satisfies it.

As we will see later, IACS is closely related to (weak) EI (cf. [28]).

REMARK 7.3. Obviously, IACS implies Independence over Models; the converse is not true: for instance, let $L = \{E\}$, where E is a binary relation, and T be the theory saying that E has two equivalence classes, both infinite; then, T is ω-stable and hence simple, therefore $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ satisfies Independence over Models, but it is easy to see that it does not satisfy IACS (take $M = \emptyset$). Indeed it is enough to consider four distinct elements a, a', b, b' with a', b' in two different equivalence classes.

THEOREM 7.4. *Assume that* T *is simple. Then,* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f},\delta}$ *is a real-strict independence relation on* $\langle \mathfrak{C}, \bar\delta\rangle$ *and it satisfies independence over models.*

Thus, $T_g^{\bar\delta,?}$ *is simple, and* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f},\delta}$ *is Shelah-forking on* $\langle \mathfrak{C}, \bar\delta\rangle$.

If moreover T *satisfies IACS, then also* $T_g^{\bar\delta,?}$ *satisfies IACS.*

PROOF. From Theorem 6.1 we have that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f},\delta}$ is a strict independence relation.

We have to show that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f},\delta}$ satisfies independence over models (resp., over algebraically closed substructures) if $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ satisfies it. Let $\bar a$, $\bar b$, $\bar a'$, $\bar b'$ be as above

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

and M be a small elementary $L^{\bar\delta}$-substructure (resp., small $L^{\bar\delta}$-algebraically closed substructure) of $\langle \mathfrak{C}, \bar\delta\rangle$. The assumptions become

$$\mathrm{Jet}(\bar a) \underset{M}{\overset{\mathrm{f}}{\downarrow}} \mathrm{Jet}(\bar b), \qquad \mathrm{Jet}(\bar a') \underset{M}{\downarrow} \mathrm{Jet}(\bar a),$$

$$\mathrm{Jet}(\bar b') \underset{M}{\overset{\mathrm{f}}{\downarrow}} \mathrm{Jet}(\bar b), \qquad \mathrm{Jet}(\bar a') \underset{M}{\equiv^{L^{\bar\delta}}} \mathrm{Jet}(\bar b').$$

By Fact 4.11 (applied to $\mathfrak{C}$), $\overset{\mathrm{f}}{\downarrow}$ satisfies independence over models (resp., by assumption, $\overset{\mathrm{f}}{\downarrow}$ satisfies IACS), and therefore there exists $\bar d \in M^{\Gamma_n}$ such that

$$\bar d \underset{\mathfrak{C}}{\overset{\mathrm{f}}{\downarrow}} \mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b), \qquad \bar d \underset{\mathfrak{C}\,\mathrm{Jet}(\bar a)}{\equiv^{L}} \mathrm{Jet}(\bar a'), \qquad \bar d \underset{\mathfrak{C}\,\mathrm{Jet}(\bar b)}{\equiv^{L}} \mathrm{Jet}(\bar b')$$

Let $q := \mathrm{tp}^L(\bar d/M\,\mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b))$, and p be restriction of q to $\mathfrak{C}$. Notice that p is realized by $\delta\bar a$, and it is therefore $\bar\delta$-compatible. Moreover, $\bar d \overset{\mathrm{f}}{\downarrow}_{\mathfrak{C}} \mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b)$ and therefore $q \overset{\mathrm{f}}{\downarrow}_{\mathfrak{C}} \mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b)$, and therefore

$$q \underset{\mathfrak{C}}{\overset{\mathrm{M}}{\downarrow}} \mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b).$$

Thus, by the extension theorem, q is $\bar\delta$-compatible, and it is therefore realized by $\mathrm{Jet}(\bar c)$ for some $\bar c \in \mathfrak{C}^m$. Thus, we have

$$\mathrm{Jet}(\bar c) \underset{\mathfrak{C}}{\overset{\mathrm{f}}{\downarrow}} \mathrm{Jet}(\bar a)\,\mathrm{Jet}(\bar b), \quad \mathrm{Jet}(\bar c) \underset{\mathfrak{C}\,\mathrm{Jet}(\bar a)}{\equiv^{L}} \mathrm{Jet}(\bar a'), \quad \mathrm{Jet}(\bar c) \underset{\mathfrak{C}\,\mathrm{Jet}(\bar b)}{\equiv^{L}} \mathrm{Jet}(\bar b')$$

which implies the conclusion. ⊣

7.1. GEI and simplicity.

THEOREM 7.5. *Assume that* T *is algebraically bounded, simple, with GEI and* $\overset{\mathrm{f}}{\downarrow}$ *is real-canonical* (*e.g.*, T *is either stable or supersimple*). *Then*:

1) $\overset{M}{\downarrow} = \overset{\mathrm{b}}{\downarrow} = \overset{\mathrm{f}}{\downarrow}$;
2) T *is super-simple of SU-rank* 1;
3) $\overset{M}{\downarrow}$ *is strict and real-canonical*;
4) *Shelah forking on* $T_g^{\bar\delta,?}$ *is equal to* $\overset{\mathrm{f},\delta}{\downarrow}$ *and is also strict and real-canonical*;
5) $T_g^{\bar\delta,?}$ *has GEI.*

PROOF. $\overset{\mathrm{M}}{\downarrow} = \overset{\mathrm{b}}{\downarrow}$ by Fact 4.14 and it is therefore strict. $\overset{\mathrm{M}}{\downarrow} = \overset{\mathrm{f}}{\downarrow}$ by Remark 4.15 $\overset{\mathrm{M}}{\downarrow}$ is real-canonical by Corollary 6.4.

$\overset{\mathrm{f},\delta}{\downarrow}$ is Shelah's forking by Theorem 7.4; it is therefore strict; it is real-canonical by Theorem 6.1.

$T_g^{\bar\delta,?}$ has GEI by 3) and Fact 4.5. ⊣

By Theorem A, we can deduce that the assumptions of Theorem 7.5 can be weakened to "T is algebraically bounded and simple."

7.2. Elimination of imaginaries and simplicity. In this section we give some sufficient conditions for $T_g^{\bar\delta,?}$ to have EI when T is simple (see Theorem 7.9 and Corollary 7.10); for previous results see [37, 38].

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Given a monster model $\mathfrak{C}$, we use the following convention: $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are finite tuples of real elements, A, B, C, D, E are small sets of real elements, while e will be an imaginary element; acl will always denote the algebraic closure inside $\mathfrak{C}$ (and not $\mathfrak{C}^{eq}$); when we consider two theories, T and $T_g^{\bar{\delta},?}$, we will use acl for the algebraic closure according to T, and $\text{acl}^{\bar{\delta}}$ for the algebraic closure according to $T_g^{\bar{\delta},?}$.

DEFINITION 7.6. We say that T is "good" if:

1) T is algebraically bounded;
2) T has EI;
3) T is supersimple.

REMARK 7.7. If T is good then T has EHI and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$ (Fact 4.8 and Theorem 6.5).

Moreover conditions 1) and 2) imply that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$ (Theorem 6.5).

Conditions 2) and 3) imply that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ satisfies IACS (Fact 4.12).

Finally, if T is good, then T is supersimple of SU rank 1.

REMARK 7.8. If we consider the conditions 1) and 2) of Definition 7.6 together with the assumption that the theory T is stable, then it is easy to see that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ which implies that T is superstable of SU rank 1 (a particular case of supersimplicity), and therefore it is good. By Theorem A, we can further weaken the assumption to conditions 1) and 2) plus T is simple.

THEOREM 7.9. *If* T *is good, then* $T_g^{\bar{\delta},?}$ *is simple,* $\mathop{\smile\!\!\!\!\!\!|}^{f,\delta}$ *is real-canonical and satisfies IACS, and* $T_g^{\bar{\delta},?}$ *has EI.*

We will prove the above theorem at the end of this section, after some consequences and preliminary results.

COROLLARY 7.10. *If* T *is algebraically bounded and stable with EI, then* $T_g^{\bar{\delta},?}$ *has EI.*

PROOF. If T is stable with EI, then it is superstable (of U-rank 1). (Theorem A gives that $T = \text{ACF}_0$). ⊣

We prove a more general result that we will use for the proof of Theorem 7.9.

PROPOSITION 7.11. *Let* $\mathfrak{C}$ *be a monster model satisfying the following conditions*:

1) $\mathfrak{C}$ *is simple*;
2) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *satisfies IACS*;
3) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *is real-canonical*;
4) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *is controlled by a rank* ρ, *as defined in II of Proposition* 6.8.

Then, $\mathfrak{C}$ *has WEI. If moreover* $\mathfrak{C}$ *expands a field, then* $\mathfrak{C}$ *has EI.*

PROOF. The proof keeps following the one in [28, (1.10)] and [38, Theorem 5.12].[5] Here are some more details.

[5]Notice that in their proof [38] do not mention that $\mathfrak{C}$ should have IACS: but this holds in their setting. [12, Fact 4.14] gives a similar result that might be easier to prove in this context.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Let e be an imaginary element. Write $e = f(\bar{a})$ for some (real finite tuple) $\bar{a} \in \mathfrak{C}^\ell$ and some function f which is definable without parameters. Let $E := \mathrm{acl}^{\mathrm{eq}}(e) \cap \mathfrak{C}$. Our thesis that U has WEI is equivalent to $e \in \mathrm{dcl}^{\mathrm{eq}}(E)$ (when $\mathfrak{C}$ expands a field, WEI implies EI see Remark 3.6).

Let P be the set of realization of $\mathrm{tp}(\bar{a}/E)$ (inside $\mathfrak{C}$). By Proposition 6.8, there exists $\bar{c} \in P$ such that $f(\bar{c}) = e$ and $\bar{a}$ and $\bar{c}$ are (forking)-independent over E.

Claim 4. f *is constant on* P.

Otherwise, there exists $\bar{d}' \in P$ such that $f(\bar{a}) \neq f(\bar{d}')$. Let $\bar{d}'', \bar{a}''$ be such that $\bar{d}''\bar{a}'' \mathop{\downarrow^{\mathrm{f}}}_E \bar{d}'\bar{a}$ and $\bar{d}''\bar{a}'' \equiv_E \bar{d}'\bar{a}$. Thus, $f(\bar{d}'') \neq f(\bar{a}'')$, $\bar{d}'' \equiv_E \bar{a}$, and $\bar{a}'' \equiv_E \bar{a}$. Thus, choosing either $\bar{d} := \bar{d}''$ or $\bar{d} := \bar{a}''$ we get $f(\bar{d}) \neq f(\bar{a})$ and and $\bar{d} \mathop{\downarrow^{\mathrm{f}}}_E \bar{a}$.

By IACS (since E is algebraically closed), (by taking $\bar{a} := \bar{a}$, $\bar{b} := \bar{d}$, $\bar{a}' := \bar{c}$, and $\bar{b}' := \bar{a}$ in Definition. 7.2) there exists $\bar{c}' \in \mathfrak{C}^\ell$ such that $\bar{c}' \equiv_{E\bar{d}} \bar{a}$ and $\bar{c}' \equiv_{E\bar{a}} \bar{c}$. Since $e = f(\bar{a}) \neq f(\bar{d})$, we get $f(\bar{c}') \neq e$; but since $e = f(\bar{a}) = f(\bar{c})$, we get $f(\bar{c}') = e$, absurd.

Claim 5. $e \in \mathrm{dcl}^{\mathrm{eq}}(E)$.

By Claim 4 and compactness, there exists an $L(E)$-definable set X such that $P \subseteq X$ and f is constant on X. Thus, e is defined by the $L(E)$-formula $\exists \bar{x} \in X\ f(\bar{x}) = y$. ⊣

Proof of Theorem 7.9. We have to show that T and $T_g^{\bar{\delta},?}$ satisfy the assumptions of Proposition 7.11.

For T, we define $\rho(\bar{a}/B)$ to be the rank given by the algebraic closure (notice that under our assumptions T is super-simple of rank 1, and ρ is equal to the SU-rank). Since $\rho(a_1, \dots, a_m)/B \leq m$, condition IIc of Proposition 6.8 is automatically true, and therefore T satisfies the assumptions of Proposition 7.11.

We consider now $T_g^{\bar{\delta},?}$: we have to show that the various properties of T that we need are inherited by $T_g^{\bar{\delta},?}$.

We have seen before that $T_g^{\bar{\delta},?}$ is simple with IACS and real-canonical.

We need to introduce a suitable function ρ and show that it satisfies the conditions in Proposition 7.11. We can use the analogue of the dimension function $\dim_{\mathcal{D}}$ in [38]. For every $r \in \mathbb{N}$, define $\rho(\bar{a}/B)(n)$ as the transcendence degree of $\mathrm{Jet}_n(\bar{a})$ over $\mathrm{acl}(\mathrm{Jet}(B))$, where

$$\mathrm{Jet}_n(\bar{a}) = \{\mu\bar{a} : \mu \in \Gamma \text{ and } |\mu| \leq n\}.$$

Thus, $\rho(\bar{a}/B) \in \omega^\omega$. If we endow ω^ω with the lexicographic ordering, then ρ is upper semi-continuous, and satisfies the conditions in Proposition 7.11. ⊣

§8. Distality. This section is based on a suggestion by Elliot Kaplan. Distality is a stronger condition than NIP, it characterizes NIP theories that are as far from stability as possible. For a formal definition and key properties of distal theories, see [2, 47]. In [21] we showed that if T is NIP, then $T_g^{\bar{\delta},?}$ is NIP. A similar results hold when T is distal.

Theorem 8.1. *If* T *is distal, then* $T_g^{\bar{\delta},?}$ *is distal.*

Proof. Immediate from Theorem 2.4 and [2, Proposition 7.1]. ⊣

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

§9. ω-stability. In [21], we showed that if the theory T is stable, then $T_g^{\bar{\delta},?}$ retains stability. The analogous result for ω-stability is false: here we give necessary and sufficient conditions for $T_g^{\bar{\delta},?}$ to be ω-stable. Recall that countable theory is totally transcendental iff the theory is ω-stable (see [46, p. 125]).

THEOREM 9.1. *$T_g^{\bar{\delta},?}$ is totally transcendental iff $T = \mathrm{ACF}_0$ (i.e., T is the theory of pure algebraically closed fields, possibly with some constants) and $\bar{\delta}$ commute.*

We first need the following lemma (cf. Theorem A).

LEMMA 9.2. *T is algebraically bounded and strongly minimal iff $T = \mathrm{ACF}_0$.*

PROOF. It is clear that, if $T = \mathrm{ACF}_0$, then T is algebraically bounded and strongly minimal.

Conversely, assume that T is algebraically bounded and strongly minimal. Let $\mathbb{K} \models T$. Since $\mathbb{K}$ is strongly minimal, as a field K is algebraically closed (see Macintyre's Theorem [44, Theorem 3.1]). By [27, Theorem 1], every $\mathbb{K}$-definable set is definable in the field language. ⊣

PROOF OF THEOREM 9.1 If $T = \mathrm{ACF}_0$, then $T_g^{\bar{\delta}}$ is ω-stable [36].

Let us now consider the converse. Let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$.

1) Assume that $\bar{\delta}$ do not commute. Let C_1 be the fixed field of δ_1. Notice that $\delta_2(C_1) = K$, since $\mathbb{K}$ is existentially closed. Thus, $\mathrm{MR}(C_1) = \mathrm{MR}(K)$. However, C_1 is a subgroup of K of infinite index, and therefore $\mathrm{MR}(K) = \infty$, proving that $\langle \mathbb{K}, \bar{\delta} \rangle$ is not ω-stable.
2) We have to prove that $T = \mathrm{ACF}_0$, so by Lemma 9.1 we assume that K is not strongly minimal. We want to show that $\langle \mathbb{K}, \bar{\delta} \rangle$ is not ω-stable. It suffices to show that, assuming that $L(K)$ is countable, there exist 2^ω $L^{\bar{\delta}}$-1-types over K. Then, let $X \subset K$ which is $\mathbb{K}$-definable and such that both X and $\mathbb{K} \setminus X$ are infinite. Fix $J \subseteq \mathbb{N}$ and consider the following partial $L^{\bar{\delta}}(K)$-type:

$$p_J(x) = \left(\delta_1^i x \in X : i \in J\right) \cup \left(\delta_1^i x \notin X : i \notin J\right).$$

Notice p_J is indeed a partial type and, for every $J \neq J'$, p_J and $p_{J'}$ are incompatible. Thus, $|S^1_{L^{\bar{\delta}}}(K)| \geq 2^\omega$. ⊣

§10. The field of constants. We give some interesting results on the field of constants. Before we introduce some notations and definitions.

DEFINITION 10.1. The *field of constants* is the set

$$\mathcal{C}_{\bar{\delta}} := \{a \in K : \delta_1(a) = \cdots = \delta_k(a) = 0\}.$$

We consider the reduct $\langle \mathbb{K}, \mathcal{C}_{\bar{\delta}} \rangle$ (i.e., the expansion of $\mathbb{K}$ with a unary predicate for $\mathcal{C}_{\bar{\delta}}$).

Observe that $\mathcal{C}_{\bar{\delta}}$ is **dense** in $\mathbb{K}$ w.r.t. the matroid acl: that is, for every $Z \subseteq K$ which is L-definable (remember that L is the signature of $\mathbb{K}$) with parameters in $\mathbb{K}$ and large, Z intersects $\mathcal{C}_{\bar{\delta}}$; moreover, $\mathcal{C}_{\bar{\delta}}$ is also L-algebraically closed in $\mathbb{K}$. Thus, $\langle \mathbb{K}, \mathcal{C}_{\bar{\delta}} \rangle$ is a **lovely pair of geometric structures** (in the sense of [4, 5]: see also [16]). Thus, we can apply the known results (see [4, 5, 16]).

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

LEMMA 10.2. *$\bar b \in K^\ell$ and $B := \mathrm{dcl}_L(\mathrm{Jet}(\bar b)) = \mathrm{dcl}_{L^{\bar\delta}}(\mathrm{Jet}(\bar b))$. Then, B is P-independent: that is, $B \mathop{\smile\!\!\!\!\!\!|}^{M}_{A} \mathcal{C}_{\bar\delta}$, where $A := B \cap P$.*

PROOF. Assume not: then, there exists $\bar d \in \mathrm{Jet}(\bar b)^n$ s.t. $\bar d$ is L-algebraically independent over A but the following system has a solution $\bar c \in (K^n)^m$:

$$\begin{cases} \delta \bar x = 0 \\ \bar x \neq 0 \\ \sum_{i,j} x_{ij} d^i_j = 0. \end{cases} \tag{1}$$

W.l.o.g., we may assume that $\langle \mathbb{K}, \bar\delta\rangle$ is a monster model. Let $\bar b' \equiv^{L^{\bar\delta}}_{A} \bar b$ s.t. $\bar b' \mathop{\smile\!\!\!\!\!\!|}^{M,\bar\delta}_{A} \bar c$. Let σ be an $L^{\bar\delta}$-automorphism of $\mathbb{K}$ over A mapping $\bar b$ to $\bar b'$ and $\bar d' := \sigma(\bar d)$. Then, $\bar d'$ is L-algebraically independent over A and therefore (1) (with $\bar d'$ in place of $\bar d$) cannot have a solution in K. But $\bar b$ and $\bar b'$ have the same $L^{\bar\delta}$-type, absurd. ⊣

THEOREM 10.3. *$\mathcal{C}_{\bar\delta}$ is an elementary* L*-substructure of $\mathbb{K}$.*

Let $\bar b \in K^\ell$ and $X \subseteq \mathcal{C}^n_{\bar\delta}$ be $L^{\bar\delta}$-definable with parameters $\bar b$. Then, there exists $Y \subseteq K^n$ which is L*-definable (in $\mathbb{K}$) with parameters $\mathrm{Jet}(\bar b)$ such that $X = Y \cap \mathcal{C}^n_{\bar\delta}$. If moreover $\bar b \in \mathcal{C}_{\bar\delta}$, then* X *is* L*-definable in $\mathcal{C}_{\bar\delta}$ with parameters $\bar b$: equivalently, there exists $Y \subseteq K^n$ which is* L*-definable in $\mathbb{K}$ with parameters $\bar b$ such that $X = Y \cap K^n$.*

PROOF. The theorem is a reformulation of [16, Lemma 8.22] (see also [4, Proposition 3.4]), using Lemma 10.2. ⊣

Notice that, by Corollary 6.2, $\mathcal{C}_{\bar\delta}$ is algebraically closed in $\mathbb{K}$ w.r.t. the $L^{\bar\delta}$-structure.

DEFINITION 10.4. A basic formula is a formula of the form

$$\exists \bar y \left(\bar y \in K^\ell \wedge \psi(\bar x, \bar y)\right),$$

where ψ is an L-formula. A basic set is a set definable by a basic formula (with parameters from $\mathbb{K}$).

THEOREM 10.5. *Let $Z \subseteq K^n$ be definable in $\langle \mathbb{K}, \mathcal{C}_{\bar\delta}\rangle$ with parameters from $\mathbb{K}$. Then, Z is a finite Boolean combination of basic sets, with the same parameters as Z.*

The Theorem is a reformulation of [4, Corollary 3.2] and [16, Theorem 8.5].

REMARK 10.6. Let $\langle A, B\rangle$ be a lovely pair of geometric structures, with $A \models T$. Then, there exists $B^* \succeq B$ and a derivation δ^* on B^* such that $\langle B^*, \delta^*\rangle \models T^\delta_g$ and $\langle B^*, A^*\rangle \succeq \langle B, A\rangle$, where $A^* := \mathfrak{C}_{\delta^*}$

PROOF. The theory T^{lovely} of lovely pairs of models of T is complete (see [4]). Let $\langle B^*, A^*\rangle \succeq \langle B, A\rangle$ be a 0-big (a.k.a. "splendid": see [26]). By bigness, there exists a derivation δ^* on B^* satisfying the conclusion. ⊣

[32, Section 5] use a particular case of the above remark to (re-)prove some results about lovely pairs.

For more results (in particular on imaginaries in $\langle \mathbb{K}, \mathcal{C}_{\bar\delta}\rangle$) see [4, 5, 16].

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

§11. Open core. The open core of a topological structure is the structure generated by the sets which are both open and definable. We investigate expansions of algebraically bounded topological structures with generic derivations: more precisely we prove that, under very weak assumptions, T is the open core of $T_g^{\bar{\delta},?}$ (Corollary 11.11).

By a topology τ on T we mean an assignment, to every $\mathbb{K} \models T$ and every $n \in \mathbb{N}$, of a topology $\tau_{n,\mathbb{K}}$ on K^n.

DEFINITION 11.1 [41]. τ is definable if, for every $n \in \mathbb{N}$, there exists an L-formula (without parameters) $\alpha_n(\bar{x}, \bar{y})$, where $\bar{x}$ is an n-tuple and $\bar{y}$ is a finite tuple, such that, for every $\mathbb{K} \models T$, we have that

$$\{\alpha(\mathbb{K}, \bar{b}) : \bar{b} \in K^{|\bar{y}|}\}$$

is a basis of open sets for $\tau_{n,\mathbb{K}}$. The family $\{\alpha(\mathbb{K}, \bar{b}) : \bar{b} \in K^{|\bar{y}|}\}$ is a definable basis for the topology.

If we give only a topology on K, we assume that the topology on K^n is the product topology.

DEFINITION 11.2. T is an open core of $T_g^{\bar{\delta},?}$ if, for every $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$ and every $n \in \mathbb{N}$, every τ-open $L^{\bar{\delta}}(K)$-definable subset of K^n is already $L(K)$-definable.

We give: sufficient conditions for T being an open core of $T_g^{\bar{\delta},?}$ (see Assumption 11.7 and Theorem 11.9) and for $T_g^{\bar{\delta},?}$ to have EI (see Theorem 11.10).

In an article in preparation we will present another approach, with slightly different assumptions and a very different proof.

Let $\mathbb{K} \models T$ and $n \in \mathbb{N}$. Let $X \subseteq K^n$ be an $L(K)$-definable set.

DEFINITION 11.3. Given $\bar{a} \in K^n$, the **Local Dimension** of X at $\bar{a}$ is

$$\text{loc-dim}_{\bar{a}}(X) := \min\{\dim(X \cap U) : U \text{ open definable set containing } \bar{a}\}.$$

The Local Dimension of X is

$$\text{loc-dim}(X) := \max\{\text{loc-dim}_{\bar{a}}(X) : \bar{a} \in X\}.$$

We say that X has **constant local dimension** d if, for every $\bar{a} \in X$, $\text{loc-dim}_{\bar{a}}(X) = d$.

REMARK 11.4. $\text{loc-dim}(X) \leq \dim(X)$.

DEFINITION 11.5. dim is local (w.r.t. to τ) if, for every $\mathbb{K} \models T$, for every $n \in \mathbb{N}$, and every $X \subseteq K^n L(K)$-definable, $\dim(X) = \text{loc-dim}(X)$.

[18] gives sufficient conditions for dim to be local. It gives also an example where it is not local (the Sorgenfrey plane).

FACT 11.6 [18]. *Assume that τ is definable, the topology on K^n is the product topology (for every $n \in \mathbb{N}$), $\langle K, + \rangle$ is a Hausdorff topological group (for every $\mathbb{K} \models T$), and every open definable set is large, i.e., it has the same dimension of ambient space.*

Then, dim *is local.*

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

ASSUMPTIONS 11.7. We assume that, for every $n, m \in \mathbb{N}$ and $\mathbb{K} \models T$,

1) τ is definable;
2) the projection map $K^n \times K^m \to K^n$ is continuous;
3) $\tau_{n,\mathbb{K}}$ is invariant under permutation of coordinates;
4) dim is local.

EXAMPLES 11.8. The following topologies satisfy the assumptions:

1) $T = \mathrm{RCF}$ with the usual Euclidean topology.
2) T is a theory of Henselian valued fields with the valuation topology.

THEOREM 11.9 (Open Core). T *is the open core of* $T_g^{\bar{\delta},?}$.

More precisely: let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$ *and* $A \subseteq K$ *s.t.* $\bar{\delta}(A) \subseteq A$. *Let* $X \subseteq K^n$ *be* $L^{\bar{\delta}}(A)$*-definable and closed. Then,* X *is* $L(A)$*-definable.*

We give a proof later in Section 11.1. Particular cases of Theorem 11.9 were already known (see [32, Section 6]).

THEOREM 11.10. *Assume moreover that* τ *satisfies the following condition*:

5) *If* X *is* $L(\mathbb{K})$*-definable and nonempty, then* $\dim(\overline{X} \setminus X) < \dim(X)$, *where* $\overline{X}$ *is the topological closure of* X.

Then, $T_g^{\bar{\delta},?}$ *has EI modulo* T^{eq}.

PROOF. With trivial modifications, the proof suggested by M. Tressl in [19] works. ⊣

For the following corollary, we spell all conditions explicitly.

COROLLARY 11.11. *Let* T *be algebraically bounded, with a definable topology* τ. *Assume that* τ *is a non-trivial and non-discrete ring topology. Then,* T *is the open core of* $T_g^{\bar{\delta},?}$.

If moreover τ *and satisfies Assumption* 5) *in Theorem* 11.10, *then* $T_g^{\bar{\delta},?}$ *has EI modulo* T^{eq}.

PROOF. Since τ is a non-trivial ring topology, it is Hausdorff (see, e.g., [51, Theorem 11.10]).

We only need to show that dim is τ-local.

By Fact 11.6, it suffices to show that every nonempty open subset of K^n is large.

For $n = 1$, since τ is Hausdorff and non-discrete, every nonempty open subset of K is infinite and hence large.

For $n > 1$, let $U \subseteq K^n$ be open and nonempty. Let $B_1, \dots, B_n \subseteq K$ be open and nonempty s.t. $B := B_1 \times \cdots \times B_n \subseteq K^n$. Since, by the case $n = 1$, each B_i is large, then B is large. ⊣

11.1. Proof of Theorem 11.9. We present some preliminary results before giving the proof.

Fix $n \in \mathbb{N}$ and $\bar{x} = \langle x_1, \dots, x_n \rangle$. We use the same notation as in Section 2.

We need to endow K^{Γ_n} with a topology. For every J finite subset of Γ_n of cardinality $m \in \mathbb{N}$, $\tau_{m,\mathbb{K}}$ is a topology on K^m and hence on K^J (it does not depend on how we enumerate the elements of J, since we assumed that $\tau_{m,\mathbb{K}}$ is invariant under

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

permutation of coordinates). We have a natural map $\Pi_J : K^{\Gamma_n} \to K^J$. We define the topology $\tau_{\mathbb{K},\Gamma_n}$ as the coarsest topology making all the maps Π_J continuous: equivalently, a basis of $\tau_{\mathbb{K},\Gamma_n}$ is given by

$$\{\Pi_J^{-1}(U) : U \subseteq K^J \text{ open}, J \subset_{fin} \Gamma_n\}.$$

Let $\mathbb{K} \models T$ be a monster model (i.e., λ-saturated and λ-homogeneous for some sufficiently large cardinal $\lambda > \aleph_0 + |L|$) and $A \subset K$ be a "small" subset (i.e., $|A| < \lambda$).

Let $J \subseteq \Gamma_n$ (J could be finite or infinite); again, we consider J as a set of indices and x_J as a tuple of variables (see Section 2.2)

A partial type $p(x_J)$ over A determines a (type-definable) subset $Z \subseteq K^J$. If J is finite, we can define the dimension of Z as

$$\dim(Z) := \min\{\dim(\alpha(\mathbb{K}) : \alpha \in p\} \in \mathbb{N}.$$

If however J is infinite, the dimension of Z (defined in the "obvious" way) may be infinite (e.g., $\dim(K^J)$ is infinite). We need to define when Z' is a large subset of Z even when $\dim(Z)$ is infinite.

Definition 11.12. Let $Y \subseteq Z$ be subsets of K^J which are type-definable (over some small $A \subset K$). We say that Y is *large in* Z if, for every $J' \subseteq J$ with J' finite, $\dim(\Pi_{J'}(Y)) = \dim(\Pi_{J'}(Z))$.

Given a partial type $p(x_J)$, we denote by $p(\mathbb{K})$ the set of realizations of p in $\mathbb{K}^J$, and $\dim(p(x_J)) := \dim(p(\mathbb{K}))$. Given $J' \subseteq J$, we denote by $p(J')$ the partial type corresponding to the set $\Pi_{J'}(p(\mathbb{K}))$.

Let $q \in S^J(A)$ be a complete type, of dimension d.

Lemma 11.13. *Assume that* J *is finite.*
Then, there is a family $\{X_i : i \in I\}$ *of* $L(A)$*-definable sets such that*:

1) $q(\mathbb{K}) = \bigcap_{i \in I} X_i$;
2) *each* X_i *has constant local dimension* d.

Proof. Let $\{Y_i : i \in I\}$ be a family of $L(A)$-definable sets such that: $q(\mathbb{K}) = \cap\{Y_i : i \in I\}$. Let Z be some $L(A)$-definable set of dimension d containing $q(\mathbb{K})$: by replacing Y_i with $Y_i \cap Z$, w.l.o.g. we may assume that all Y_i have dimension d. Given an $L(A)$-definable set $Y \subseteq K^J$ of dimension d, let

$$\text{Sing}(Y) := \{\bar{a} \in Y : \text{loc-dim}_{\bar{a}}(Y) < d\}$$
$$\text{and } \text{Reg}(Y) := Y \setminus \text{Sing}(Y).$$

Since dim is definable, $\text{Sing}(Y)$ and $\text{Reg}(Y)$ are $L(A)$-definable. Since dim is local, $\dim(\text{Sing}(Y)) < d$, and therefore $\dim(\text{Reg}(Y)) = d$ and $\text{Reg}(Y)$ has constant local dimension d.

Claim 6. *For every* $i \in I$, $q(\mathbb{K}) \subseteq \text{Reg}(Y_i)$.

Assume not: then, since q is a complete type over A, $q(\mathbb{K}) \subseteq \text{Sing}(Y_i)$, but then $\dim(q) \leq \dim(\text{Sing}(Y_i)) < d$, absurd.

Thus, we can define $X_i := \text{Reg}(Y_i)$.

Notice that in the above lemma we used that dim is definable.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

LEMMA 11.14. *Let $U \subseteq K^J$ be open and definable (with parameters in some small set* B *with $A \subseteq B \subset K$). If $U \cap q(\mathbb{K}^J)$ is nonempty, then it is large inside $q(\mathbb{K}^J)$.*

PROOF. Since U is definable, there is some $J_0 \subseteq J$ finite and $V \subseteq K^{J_0} L(A)$-definable and open, such that $U = \pi_{J_0}^{-1}(V)$. Let $J_0 \subseteq J' \subseteq J$, with J' finite. We need to prove that $\Pi_{J'}(U \cap q(K^{J'}))$ is large inside $\Pi_{J'}(q(\mathbb{K}))$.

The former is equal to $U' \cap q(\mathbb{K}(J'))$, where $U' := \Pi_{J'}(U')$: thus, by replacing U with U', J with J', and $q(x_J)$ with $q(x_{J'})$, we may assume that J' is finite. Thus, by Lemma 11.13, we may assume that $q(\mathbb{K}) = \bigcap_i X_i$, where each X_i is an $L(A)$-definable set of constant local dimension d. Thus, since $X_i \cap U$ is nonempty, $X_i \cap U$ has (local) dimension d, and therefore $\dim(\bigcap_i X_i \cap U) = d$. ⊣

PROOF OF THEOREM 11.9. W.l.o.g., we may assume that $\langle \mathbb{K}, \bar{\delta} \rangle$ is a monster model and A is "small". By Beth definability, it suffices to show that X is $L(A)$-invariant (i.e., that X is set-wise invariant under automorphisms of $\mathbb{K}$ as L-structure fixing A point-wise).

Let $Z := \mathrm{Jet}(X) \subseteq K^{\Gamma_n}$ and Y be the closure of Z (according to the topology $\Pi_{\mathbb{K},\Gamma_n}$). Since X is closed, we have that $X = \Pi_n(Y)$ (where Π_n is the projection onto the first n coordinates). Thus, it suffices to show that Y is $L(A)$-invariant.

Let $\bar{a}, \bar{a}' \in K^{\Gamma_n}$, with the same $L(A)$-type, and such that $\bar{a} \in Y$. We need to show that $\bar{a}' \in Y$. Since $\tau_{\mathbb{K},n}$ is definable and Y is closed, it suffices to prove the following.

CLAIM 7. *Let $U' \subseteq K^{\Gamma_n}$ be an $L(\mathbb{K})$-definable open set containing $\bar{a}'$. Then, U' intersects* Y.

Let ϕ be an $L(A)$-automorphism of $\mathbb{K}$ such that $\phi(\bar{a}') = \bar{a}$. Let $U := \phi(U')$: notice that U is an open $L(A)$-definable set containing $\bar{a}$; therefore, there exists $\bar{b} \in X$ such that $\mathrm{Jet}(\bar{b}) \in U$. Let $\bar{c}' := \phi^{-1}(\mathrm{Jet}(\bar{b}))$.

Let $q(\bar{x}_\Gamma)$ be the $L(A)$-type of $\mathrm{Jet}(\bar{b})$: by definition, it is also the $L(A)$-type of $\bar{c}'$. By Lemma 11.14, $q(\mathbb{K}) \cap U'$ is large inside $q(\mathbb{K})$. Thus, by the Extension Theorem, there exists $\bar{b}' \in K^n$ such that

$$\mathrm{Jet}(\bar{b}') \in q(\mathbb{K}) \cap U'.$$

Therefore, $\bar{b}' \in X$ and therefore $\mathrm{Jet}(\bar{b}') \in Z \cap U' \subseteq Y \cap U'$.

§12. Differential dimension.

12.1. The commutative case. Let $\langle A, \bar{\delta} \rangle$ be a field (of characteristic 0) with k commuting derivations. The derivations induce a matroid on A. Given $a \in A$, $Y \subseteq A$, and $X \subseteq A$, we define $a \in \bar{\delta}\text{-acl}_Y(X)$ if $\mathrm{Jet}(a)$ is not algebraically independent over $\mathrm{Jet}(X \cup Y)$.

As shown in [19], $\bar{\delta}\text{-acl}_Y$ is a matroid on A.[6]

Fix $\langle \mathfrak{C}, \bar{\delta} \rangle$ monster model of $T_g^{\bar{\delta}}$. We have the corresponding matroid $\bar{\delta}\text{-acl}(X) := \bar{\delta}\text{-acl}_F(X)$.

[6]A more general result is true. Let $\mathbb{A} := \langle A, \mathrm{cl} \rangle$ be a finitary matroid. Let $\bar{\delta}$ be a tuple of commuting quasi-endomorphisms of $\mathbb{A}$, in the sense of [19]. Given $X, Y \subseteq A$, define $\bar{\delta}\text{-}\, cl_Y(X)$ as the set of $a \in A$ such that a^{Jet} is not cl-independent over $\mathrm{Jet}(X)\,\mathrm{Jet}(Y)$. Then, $\bar{\delta}$- cl is a finitary matroid on A.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

THEOREM 12.1. $\bar{\delta}$-acl *is an existential matroid* (*in the sense of* [16]).

PROOF. We have to prove that $\bar{\delta}$-acl is definable and it satisfies existence (see [16, Section 3]).

The fact that $\bar{\delta}$-acl is definable means that, for every $A \subseteq \mathfrak{C}$ and $b \in \bar{\delta}\text{-acl}(A)$ there exists an $L^{\bar{\delta}}$-formula $\phi(\bar{x}, y)$ and $\bar{a} \in A^n$ such that $\langle \mathfrak{C}, \bar{\delta} \rangle \models \phi(\bar{a}, b)$ and, for every $\bar{a}', b'$ in $\mathfrak{C}$, if $\langle \mathfrak{C}, \bar{\delta} \rangle \models \phi(\bar{a}', b')$, then $b' \in \bar{\delta}\text{-acl}(\bar{a}')$. We can take as ϕ any formula witnessing that b^{Jet} is not algebraically independent over A.

For existence, let $A \subseteq B \subset \mathfrak{C}$ be subsets of small cardinality. Let $c \in \mathfrak{C}$ such that $c \notin \bar{\delta}\text{-acl}(A)$. We have to show that there exists $d \in \mathfrak{C}$ such that c and d have the same L^{δ}-type over A and $d \notin \bar{\delta}\text{-acl}(B)$.

Since $\mathop{\smile\!\!\!\!\!\!|}^{M,\bar{\delta}}$ satisfies existence, there exists $d \in \mathfrak{C}$ such that c and d have the same L^{δ}-type over A and $d \mathop{\smile\!\!\!\!\!\!|}^{M,\bar{\delta}}_{A} B$. Then, $\mathrm{Jet}(d)$ is algebraically independent over $\mathrm{Jet}(B)$: therefore, $d \notin \bar{\delta}\text{-acl}(B)$, proving that $\bar{\delta}$-acl is an existential matroid. ⊣

Thus, $\bar{\delta}$-acl induces a dimension function $\bar{\delta}$-dim on models of $T_g^{\bar{\delta}}$ (see [16]; see also [24]).

REMARK 12.2.

1) $\bar{\delta}$-acl is not the $T_g^{\bar{\delta}}$-algebraic closure: the former only contains the latter. For instance, the whole field of constants $\mathcal{C}_{\bar{\delta}}$ is in $\bar{\delta}\text{-acl}(\emptyset)$.
2) $\mathop{\smile\!\!\!\!\!\!|}^{M,\bar{\delta}}$ is not the independence relation induced by $\bar{\delta}$-acl, because $\mathop{\smile\!\!\!\!\!\!|}^{M,\bar{\delta}}$ is strict. For instance, if $a \in \mathcal{C}_{\bar{\delta}} \setminus \mathrm{acl}(F)$, then $a \not\mathop{\smile\!\!\!\!\!\!|}^{M,\bar{\delta}}_{\emptyset} a$.

LEMMA 12.3 (See [14]). *Let* $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta}}$. *Let* $Y \subseteq K^n$ *be* L*-definable* (*with parameters*). *Then*, $\dim(Y) = \bar{\delta}\text{-dim}(Y)$.

PROOF. By the properties of dimension functions (see [13]) it suffices to treat the case when $n = 1$ (the general case follows by induction on n). If $\dim(X) = 0$, then X is finite, and therefore $\bar{\delta}\text{-dim}(X) = 0$. If $\dim(X) = 1$, then $(X - X)/(X - X) = \mathbb{K}$, and therefore $\bar{\delta}\text{-dim}(X) = 1$. ⊣

The same proof gives a more general result.

PROPOSITION 12.4 (Invariance of dimension for fields). *Let* L *be a language expanding the language of rings, and* L^* *be an expansion of* L. *Let* A^* *be an* L^**-structure expanding a field, and* A *be its restriction to the language* L. *Assume that* $\dim^*$ *and* dim *be dimension functions on* A^* *and* A, *respectively. Then, for every* $X \subseteq A^n$ *which is* L*-definable* (*with parameters*), $\dim^*(X) = \dim(X)$.

Unlike in the case of lovely pairs, we cannot approximate L^{δ}-definable sets with L-definable sets.

REMARK 12.5. Let $\langle \mathbb{K}, \delta \rangle \models T_g^{\delta}$. Let $X \subseteq K$ be L^{δ}-definable (with parameters). If X is definable in the lovely pair $\langle \mathbb{K}, \mathcal{C}_{\bar{\delta}} \rangle$ (see Section 10), then there exists $Y \subseteq K$ which is L-definable and such that $\bar{\delta}\text{-dim}(X \Delta Y) < 1$ ([16, Proposition 8.36]). If not, such Y might not exist: for instance, let $\mathbb{K}$ be a real closed field, and $X := \{x \in K : \delta x > 0\}$.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

12.2. The non-commutative case. The assumption that the derivations commute cannot be dropped.

LEMMA 12.6. *If the derivations do not commute, then* $\bar{\delta}$-acl_Y *is not a matroid, because it is not transitive.*[7] *In fact, let* $k = 2$ *and* $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},nc}$. *Then, there exist* $a, b, c \in K$ *such that*:

1) a^Γ *is algebraically independent over* F;
2) $\delta_2 b = 0$ *and* $\delta_1 b = \delta_1 a$;
3) $c = a - b$.

Notice that $\delta_1 c = 0$. *Then,* $a \notin \bar{\delta}\text{-acl}(F)$, $b, c \in \bar{\delta}\text{-acl}(F)$, *but* $a \in \bar{\delta}\text{-acl}_F(b, c)$: *thus, transitivity fails.*

LEMMA 12.7. *For* $k \geq 2$, *models of* $T_g^{\bar{\delta},nc}$ *do not have a dimension function.*

PROOF. For simplicity, we do the case when $k = 2$. Let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},nc}$. Assume, by contradiction, that $\dim'$ is a dimension function on $\langle \mathbb{K}, \bar{\delta} \rangle$. Let $X := \{b \in K : \delta_1 b = 0\}$. Let $Y := \{c \in K : \delta_2 c = 0\}$. Notice that X and Y are L^δ-definable subfields of K of infinite index inside $\mathbb{K}$: thus, $\dim'(X) = \dim'(Y) = 0$.

CLAIM 8. $X + Y = K$.

Let $a \in K$. Let $b \in K$ such that $\delta_1 b = 0$ and $\delta_2 b = \delta_2 a$, and let $c = a - b$. Notice that $b \in X$ and $c \in Y$. Thus $X + Y = K$.

But $\dim'(X) = \dim'(Y) = 0$, and therefore $\dim'(K) = 0$, while the axioms of dimension require that $\dim'(K) = 1$.

§13. Genericity. We denote by K^K the set of all functions from K to K, and by $\mathrm{Der_K} \subset K^K$ the set of derivations on K extending η (remember that η is a fixed derivation on the field $F = dcl(\emptyset)$: see Section 2.1). The main references for this section is [25], from which our presentation is heavily inspired; for the background notions of descriptive set theory see [29].

For every $\bar{a}, \bar{b} \in K^n$, we define

$$B_{\bar{a},\bar{b}} := \{\delta \in K^K : \delta(\bar{a}) = \bar{b}\}.$$

For every L^δ-sentence ϕ with parameters in $\mathbb{K}$, we define

$$U_\phi := \{\delta \in K^K : \langle \mathbb{K}, \delta \rangle \models \phi\}.$$

The set K^K has two "canonical" topologies:

- The pro-discrete topology, whose basis of open sets is given by

  $$\{B_{\bar{a},\bar{b}} : \bar{a}, \bar{b} \in K^n, n \in \mathbb{N}\},$$

  and which we denote by τ_d, which is the topology induced by the product topology on K^K.

[7] Naturally, we use the free monoid Γ instead of the free commutative monoid Jet to define $\bar{\delta}$-acl in this situation.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

- The "first-order" topology, whose basis of open sets is given by

$$\{U_\phi : \phi \ L^\delta\text{-sentence with parameters in } \mathbb{K}\},$$

and which we denote by τ_{FO}.

REMARK 13.1. Another basis for τ_d is

$$\{U_\phi : \phi \text{ quantifier-free } L^\delta\text{-sentence with parameters in } \mathbb{K}\}.$$

In fact,

$$B_{\bar{a},\bar{b}} = U_{(\delta a_1 = b_1 \wedge \cdots \wedge \delta a_n = b_n)}.$$

For the remainder of this section, when we don't specify the topology, we mean τ_d. The following notions concerning specific formulas are useful for obtaining information about sets with respect to the topology.

DEFINITION 13.2. We say that an L^δ-sentence ϕ with parmeters $\bar{a}$ is "relatively quantifier free" if $\phi = \alpha(\mathrm{Jet}_\delta(\bar{a}))$ for some L-formula without parameters α. We say ϕ is "relatively existential" if

$$\phi = \exists \bar{x} \ \alpha(\mathrm{Jet}(\bar{a}), \mathrm{Jet}(\bar{x})),$$

for some L-formula without parameters α. We can define "relatively universal" and "relatively $\forall\exists$" L^δ-sentences with parameters.

LEMMA 13.3. *Let ϕ be an L^δ sentence with parameters.*

- *If ϕ is relatively quantifier free, then U_ϕ is clopen.*
- *If ϕ is relatively existential, then U_ϕ is open.*
- *If ϕ is relatively universal, then U_ϕ is closed.*
- *If ϕ is relatively $\forall\exists$, then U_ϕ is $\mathcal{G}_\delta$.*

PROOF. We do only the case when ϕ is relatively existential: the others are similar. Write $\phi = \exists \bar{y} \ \alpha(\bar{a}, \delta\bar{a}, \ldots, \delta^n\bar{a}, \bar{b}, \delta\bar{b}, \ldots, \delta^m\bar{b})$, for some L-formula ϕ. Then,

$$U_\phi = \bigcup \big(B_{\bar{a},\bar{a}_1} \cap B_{\bar{a}_1,\bar{a}_2} \cap \cdots \cap B_{\bar{a}_{n-1},\bar{a}_n} \cap B_{\bar{b},\bar{b}_1} \cap B_{\bar{b}_1,\bar{b}_2} \cap \cdots \cap B_{\bar{b}_{m-1},\bar{b}_m} : \bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_m \in K^{<\omega} \wedge \langle \mathbb{K}, \delta \rangle \models \alpha(\bar{a}, \bar{a}_1, \ldots, \bar{a}_n, \bar{b}, \bar{b}_1, \ldots, \bar{b}_m)\big). \qquad \dashv$$

For the remainder of this section, we assume that $\mathbb{K}$ and L are *countable*.

Thus, Der_K is τ_{FO}-closed and it is a τ_d- $\mathcal{G}_\delta$ inside K^K; we use the same names for the induced topologies on Der_K. Notice that K^K is a Polish space: therefore, Der_K is also a Polish space (see [29]). Thus, any two dense $\mathcal{G}_\delta$ subsets of Der_K always intersect.

Given $Z \subseteq K^n \times K^n$, we define

$$I_Z := \{\delta \in \mathrm{Der}_K : \exists \bar{b} \in K^n : \langle \bar{b}, \delta\bar{b} \rangle \in Z\}.$$

LEMMA 13.4. *For every $Z \subseteq K^n \times K^n$, I_Z is an open subset of* Der_K.

PROOF.

$$I_Z = \bigcup \big(B_{\bar{a},\bar{b}} : \bar{a} \in K^n, \bar{b} \in K^n, \langle \bar{a}, \bar{b} \rangle \in Z\big). \qquad \dashv$$

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

Let $\mathbb{G}$ be the family of derivations $\delta \in \mathrm{Der}_{\mathrm{K}}$ such that $\langle \mathbb{K}, \delta \rangle \models T_g^\delta$. Let $\mathcal{L}$ be the family of the sets $Z \subseteq K^{n+n}$ definable with parameters, such that $\Pi_n(Z)$ is large (for some $n \in \mathbb{N}$).

LEMMA 13.5. *$\mathbb{G}(M) = \bigcap_{Z \in \mathcal{L}} I_Z$. Moreover, $\mathbb{G}$ is a $\mathcal{G}_\delta$-subset of* $\mathrm{Der}_{\mathrm{K}}$.

PROOF. By the axiomatization T^δ_{wide} (where we introduced in [21, Definition 3.3]), $\mathbb{G}(M) = \bigcap_{Z \in \mathcal{L}} I_Z$. Each I_Z is open. By our assumptions, $\mathcal{L}$ is countable. ⊣

LEMMA 13.6. *On $\mathbb{G}$, τ_{FO} and τ_d coincide.*

PROOF. By elimination of quantifiers, every L^δ-sentence is equivalent, modulo T_g^δ, to a relatively quantifier-free sentence. The conclusion follows from Lemma 13.3. ⊣

LEMMA 13.7. *Assume that $\mathrm{rk}(K/F)$ is infinite. Then, for every $\bar{a}$ finite tuple in* K *and every* W *large subset of K^n which is* L*-definable with parameters, there exists $\bar{b} \in W$ which is algebraically independent over $F\bar{a}$.*

PROOF. By induction on n, it suffices to treat the case when $n = 1$. Let $b \in K \setminus \mathrm{acl}(F\bar{a})$. Since $W \subseteq K$ is large, then there exists $b_1, b_2, b_3, b_4 \in W$ such that $(b_1 - b_2)/(b_3 - b_4) = b$ and $b_3 \neq b_4$. Therefore, at least one of the b_i is not in $\mathrm{acl}(F\bar{a})$. ⊣

THEOREM 13.8. *There exists $\mathbb{K} \models T$ which is countable and of infinite rank over* F. *For any such $\mathbb{K}$, the set $\mathbb{G}$ is a dense subset of* $\mathrm{Der}_{\mathrm{K}}$.

Thus, in a precise topological sense, $\mathbb{G}$ is a *generic* set (notice that $\mathbb{G}$ is τ_{FO}-closed in $\mathrm{Der}_{\mathrm{K}}$).

PROOF. We have seen that each I_Z is open. It suffices to prove the following claim.

CLAIM 9. *For every $Z \in \mathcal{L}$, I_Z is dense.*

Let $Z \subseteq K^n \times K^n$. Let $B_{\bar{a},\bar{b}}$ be a nonempty basic open set. We have to verify that $I_Z \cap B_{\bar{a},\bar{b}}$ is nonempty. Let $\varepsilon \in B_{\bar{a},\bar{b}}$: that is, $\varepsilon \in \mathrm{Der}_{\mathrm{K}}$ and $\varepsilon\bar{a} = \bar{b}$. Let ε_0 be the restriction of ε to $\mathrm{acl}(F\bar{a})$. Let $\bar{c} \in \Pi_n(Z)$ be algebraically independent over $F\bar{a}$. We can extend ε_0 arbitrarily to $\bar{c}$; in particular, there exists $\delta \in \mathrm{Der}_{\mathrm{K}}$ such that δ extends ε_0 and $\delta\bar{c} \in Z$. Thus, $\delta \in I_Z \cap B_{\bar{a},\bar{b}}$.

The following theorem gives a "topological" criterion for when a differential system has a solution in models of T_g^δ.

THEOREM 13.9. *Let $\langle \mathbb{K}, \varepsilon \rangle \models T^\delta$. Assume that $\mathbb{K}$ countable and of infinite rank over* F. *Let $\bar{a} \in K^\ell$. Let $\mathrm{Der}_{\mathrm{K}}(\bar{a}, \varepsilon)$ be the set of derivations δ on* K *extending η and such that ε and δ coincide on $\mathrm{Jet}_\varepsilon(\bar{a})$, the Jet relative to the derivation ε.*

Let $Z \subseteq K^n \times K^n$ be L*-definable with parameters $\bar{a}$. Let δ be a derivation in* K *such that $\langle \mathbb{K}, \delta \rangle \models T_g^\delta$. T.f.a.e.*:

1) *I_Z is dense in $\mathrm{Der}_{\mathrm{K}}(\bar{a}, \varepsilon)$;*
2) *I_Z is nonempty;*
3) *$I_Z \cap \mathbb{G}$ is nonempty;*
4) *$\delta \in I_Z$.*

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

PROOF. First of all it is easy to see that (1) $\implies$ (2), and (4) $\implies$ (3) are obvious. Moreover (2) is equivalent to (3), since I_Z by Lemma 13.4 is open and $\mathbb{G}$ by Theorem 13.8 is dense in Der_K.

We prove first the case when $\bar{a}$ is empty (i.e., Z is L-definable without parameters and $\mathrm{Der}_K(\bar{a}, \varepsilon) = \mathrm{Der}_K$).

In this case, we can add another equivalent formulation to 4:

5) $\mathbb{G} \subseteq I_Z$.

Since T_g^δ is complete, and "$\delta \in I_Z$" can be expressed as a first-order sentence (without parameters), we have that (4) $\implies$ (5), the converse is trivial so (4) and (5) are equivalent. Therefore, (1) is equivalent to (2).

Let us consider now the case when $\bar{a}$ is non-empty. Let $F' := F[\mathrm{Jet}_\varepsilon(\bar{a})]$ and let η' be the restriction of ε to F'. We denote by $L' := L(F)$, and $T' := T \cup \mathrm{Diag}(F')$.

We can consider the theory T'^δ_g of generic derivations on K extending η': notice that $\langle \mathbb{K}, \delta \rangle \models T'^\delta_g$. We can apply the previous proof to T', since Z is now L'-definable in K without parameters (notice that we need to modify the definition of $\mathbb{G}$, since we are restricting the space of derivations to those extending η': however, we already proved the equivalence between (2) and (3)). We conclude in this way the proof. ⊣

Barbina and Zambella [3] deal with a similar situation: however, we cannot use their results, since to apply them to our setting we would need that $\mathbb{K}$ is countable and saturated. Maybe there could be a common refinement if one could weaken their assumption to $\mathbb{K}$ resplendent (since every countable consistent theory has a countable resplendent model: see [26]).

§14. Conjectures and open problems. In the article [22], we posed several conjectures. We now conclude the article with a list of additional open problems, remarks, and some further ideas.

14.1. Definable types. Let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$. Given a type $p \in S^n_{L^{\bar{\delta}}}(\mathbb{K})$, let $\bar{a}$ be a realization of p; we define $\tilde{p} \in S^{\Gamma_n}_L(\mathbb{K})$ as the L-type of $\mathrm{Jet}(\bar{a})$ over $\mathbb{K}$.

OPEN PROBLEM 14.1. *Is it true that* p *is definable iff* $\tilde{p}$ *is definable*? *We conjecture that it is true when* $T_g^{\bar{\delta},?} = T_g^{\bar{\delta}}$ *and* T *has NIP.*

14.2. Zariski closure. Given $X \subseteq K^n$, denote by X^{Zar} be the Zariski closure of X.

QUESTIONS 14.2 (See [23]).

1) *Let* $(X_i : i \in I)$ *be an* L*-definable family of subsets of* K^n. *Is* $(X_i^{Zar} : i \in I)$ *also* L*-definable*?
2) *Assume that* 1) *holds for* $\mathbb{K}$. *Let* $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$. *Let* $(X_i : i \in I)$ *be an* L^δ*-definable family of subsets of* K^n. *Is* $(X_i^{Zar} : i \in I)$ *also* L^δ*-definable*?

We have affirmative answers to both questions, which will be presented in a paper in preparation.

14.3. Kolchin polynomial. Let $\langle \mathfrak{C}, \bar{\delta} \rangle$ be a monster model of $T_g^{\bar{\delta}}$. Let $\bar{a} \in \mathfrak{C}^n$, $B \subseteq \mathfrak{C}$ such that $\bar{\delta} B \subseteq B$. There exists a polynomial $\omega_{\bar{a}|B}(t)$ such that, for n large

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

enough, $\mathrm{rk}(\mathrm{Jet}_n(\bar{a}) \mid B) = \omega_{\bar{a}|B}(n)$, where $\mathrm{Jet}_n(\bar{a}) = \{\mu\bar{a} : \mu \in \Gamma \text{ and } |\mu| \leq n\}$ (see [31]). The degree of the polynomial is at most k; denote by $\mu(\bar{a})$ the leading monomial of $\omega_{\bar{a}|K}$ (including its coefficient). Let $X \subseteq K^n$ be $L^{\bar{\delta}}$-definable with parameters $\bar{b}$: define

$$\mu(X) := \sup\big(\mu(\bar{a} \mid \mathrm{Jet}(\bar{b})) : \bar{a} \in X\big)$$
$$\omega_X := \sup\big(\omega_{\bar{a}|\mathrm{Jet}(\bar{b})} : \bar{a} \in X\big),$$

where the supremum for ω is taken inside $\mathbb{R}[t]$ w.r.t. the order $p > q$ if $\lim_{t\to+\infty} p(t) > q$ (cf. [23]), while the supremum of μ is taken w.r.t. the order $rt^d \leq r't^{d'}$ if $d < d'$ or $d = d'$ and $r \leq r'$ (cf. [17]). Notice that, by Section 6, $\mu(X)$ and ω_X are well-defined (i.e., they not depend on the choice of the parameters $\bar{b}$).

Conjecture 14.3 (See [23, 45]). The suprema in the definitions of ω_X and $\mu(X)$ are maxima. Moreover, ω and μ are definable in families: that is, for every $L^{\bar{\delta}}$-definable family $\big(X_i : i \in I\big)$ there exists a partition of I into finitely many definable set $I = I_1 \sqcup \cdots \sqcup I_m$ such that $\mu(X_i)$ and ω_{X_i} are constant on each I_j.

Open problem 14.4. *What is the "geometric" meaning of $\mu(X)$? Notice that, up to a multiplicative constant, the k^{th} coefficent of ω_X is equal to $\bar{\delta}$-$\dim(X)$.*

If Conjecture 14.3 is true, then the function $X \mapsto \mu(X)$ behaves like a dimension on L^{δ}-definable sets (with the difference that the values of μ are not natural numbers, but monomials: cf. [13, Section 2.5]).

Conjecture 14.5. Assume that $\mathfrak{C}$ is endowed with a topology τ satisfying some suitable conditions. Let $\tau_{\bar{\delta}}$ be the topology on $\mathfrak{C}$ induced by the embedding $\mathfrak{C} \to \mathfrak{C}^\Gamma$, $x \mapsto \mathrm{Jet}(x)$ (where $\mathfrak{C}^\Gamma$ is endowed with the product topology induced by τ). Denote by $\overline{X}^{\tau_{\bar{\delta}}}$ the $\tau_{\bar{\delta}}$-closure of X. Then, for every $X \subseteq \mathfrak{C}^n$ which is $L^{\bar{\delta}}$-definable and nonempty, $\overline{X}^{\tau_{\bar{\delta}}}$ is also $L^{\bar{\delta}}$-definable, and $\mu(\overline{X}^{\tau_{\bar{\delta}}} \setminus X) < \mu(X)$.

Acknowledgments. The authors thank Elliot Kaplan, Itay Kaplan, Noa Lavi, Silvain Rideau-Kikuchi, and Marcus Tressl. The authors thank the anonymous referee for the careful reading of the manuscript and for the constructive suggestions.

Funding. Both authors are members of the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA-INDAM). The authors acknowledge financial support from Istituto Nazionale di Alta Matematica (INdAM), particularly through the Intensive Period in Model Theory, "Model Theory and tame expansions of topological Fields," Naples, May19– July 18, 2025.

REFERENCES

[1] H. Adler, ***Explanation of independence***. Ph.D. thesis, Fakultät für Mathematik und Physik der Albert-Ludwigs-Universität Freiburg im Breisgau, 2005.

[2] M. Aschenbrenner, A. Chernikov, A. Gehret, and M. Ziegler, *Distality in valued fields and related structures*. ***Transactions of the American Mathematical Society***, vol. 375 (2022), no. 7, pp. 4641–4710.

[3] S. Barbina and D. Zambella, *Generic expansions of countable models*. ***Notre Dame Journal of Formal Logic***, vol. 53 (2012), no. 4, pp. 511–523.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

[4] A. Berenstein and E. Vassiliev, *On lovely pairs of geometric structures*. ***Annals of Pure and Applied Logic***, vol. 161 (2010), no. 7, pp. 866–878.

[5] G. J. Boxall, ***Lovely pairs and dense pairs of real closed fields***, Ph.D. thesis, University of Leeds, 2009.

[6] T. Brihaye, C. Michaux, and C. Rivière, *Cell decomposition and dimension function in the theory of closed ordered differential fields*. ***Annals of Pure and Applied Logic***, vol. 159 (2009), nos. 1–2, pp. 111–128.

[7] Q. Brouette, P. C. Kovacsics, and F. Point, *Strong density of definable types and closed ordered differential fields*. ***Journal of Symbolic Logic***, vol. 84 (2019), no. 3, pp. 1099–1117.

[8] S. Buechler, A. Pillay, and F. Wagner, *Supersimple theories*. ***Journal of the American Mathematical Society***, vol. 14 (2000), no. 1, pp. 109–124.

[9] E. Casanovas, ***Simple Theories and Hyperimaginaries***, first ed., Cambridge University Press, 2011. https://doi.org/10.1017/CBO9781139003728

[10] E. Casanovas and R. Farré, *Weak forms of elimination of imaginaries*. ***Mathematical Logic Quarterly***, vol. 50 (2004), no. 2, pp. 126–140.

[11] G. Conant, *A remark on strict Independence relations*. ***Archive for Mathematical Logic***, vol. 55 (2016), no. 3, pp. 535–544.

[12] C. d'Elbée, *Generic multiplicative endomorphism of a field*. ***Annals of Pure and Applied Logic***, vol. 176 (2025), no. 4, p. 103554.

[13] L. van den Dries, *Dimension of definable sets, algerbraic boundedness and Henselian fields*. ***Annals of Pure and Applied Logic***, vol. 45 (1989), pp. 189–209.

[14] P. E. Eleftheriou, O. L. Sánchez, and N. Regnault, *On coincidence of dimensions in closed ordered differential fields*. ***Notre Dame Journal of Formal Logic***, vol. 62 (2021), no. 2, pp. 257–268.

[15] D. M. Evans and E. Hrushovski, *On the automorphism groups of finite covers*. ***Annals of Pure and Applied Logic***, vol. 62 (1993), no. 2, pp. 83–112.

[16] A. Fornasiero, *Dimensions, matroids, and dense pairs of first-order structures*. ***Annals of Pure and Applied Logic***, vol. 162 (2011), no. 7, pp. 514–543.

[17] ———, *Hilbert polynomial of length functions*. ***Annali di Matematica Pura ed Applicata (1923-)***, vol. 204 (2025), no. 1, pp. 73–116.

[18] A. Fornasiero and I. Halupczok, *Dimension in topological structures: Topological closure and local property*. ***Contemporary Mathematics***, vol. 576 (2012), pp. 89–94.

[19] A. Fornasiero and E. Kaplan, *Generic derivations on o-minimal structures*. ***Journal of Mathematical Logic***, (2020). https://doi.org/10.1142/S0219061321500070.

[20] A. Fornasiero, E. Kaplan, and A. Matthews, *Geometric fields, ranks, and generic derivations*, preprint, 2026, arXiv:2605.22725.

[21] A. Fornasiero and G. Terzo, *Generic derivations on algebraically bounded structures*. ***The Journal of Symbolic Logic***, (2024), pp. 1–27. https://doi.org/10.1017/jsl.2024.57.

[22] ———, *Exponential fields: Lack of generic derivations*. ***Notre Dame Journal of Formal Logic***, vol. 66 (2025), no. 4, pp. 513–519.

[23] J. Freitag, O. L. Sánchez, and W. Li, *Effective definability of Kolchin polynomials*. ***Proceedings of the American Mathematical Society***, vol. 148 (2020), no. 4, pp. 1455–1466.

[24] N. Guzy and F. Point, *Topological differential fields and dimension functions*. ***The Journal of Symbolic Logic***, vol. 77 (2012), no. 4, pp. 1147–1164.

[25] G. Hjorth, *Group actions and countable models*, ***Logic Colloquium '99***, Lecture Notes in Logic, 17, Association for Symbolic Logic, Urbana, 2004, pp. 3–29. https://doi.org/10.1017/9781316755921.002

[26] W. Hodges, ***Model Theory***, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1993. (Visited 21 September, 2023).

[27] E. Hrushovski, *Strongly minimal expansions of algebraically closed fields*. ***Israel Journal of Mathematics***, vol. 79 (1992), nos. 2–3, pp. 129–151.

[28] E. Hrushovski and Z. Chatzidakis, *Model theory of difference fields*. ***Transactions of the American***, vol. 351 (1999), no. 8, pp. 2997–3071.

[29] A. S. Kechris, ***Classical Descriptive Set Theory***, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[30] B. Kim and A. Pillay, *From stability to simplicity*. ***The Bulletin of Symbolic Logic***, vol. 4 (1998), no. 1, pp. 17–36.

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press

[31] E. R. Kolchin, ***Differential Algebra and Algebraic Groups***, Pure and Applied Mathematics, 54, Academic Press, New York, 1973, pp. xviii+446.

[32] P. C. Kovacsics and F. Point, *Topological fields with a generic derivation*. ***Annals of Pure and Applied Logic***, vol. 174 (2023), no. 3, Article no. 103211, 38 pp.

[33] O. León Sánchez and S. Mohamed, *Neostability transfers in derivation-like theories*. ***Model Theory***, vol. 4 (2025), no. 2, pp. 177–201.

[34] J. Loveys and Y. Peterzil, *Linear O-minimal structures*. ***Israel Journal of Mathematics***, vol. 81 (1993), pp. 1–30. http://link.springer.com/article/10.1007/BF02761295

[35] D. Marker, M. Messmer, and A. Pillay, ***Model Theory of Fields***, Lecture Notes in Logic, Cambridge University Press, 2017. https://doi.org/10.1017/9781316716991

[36] T. McGrail, *The model theory of differential fields with finitely many commuting derivations*. ***The Journal of Symbolic Logic***, vol. 65 (2000), no. 2, pp. 885–913.

[37] S. Mohamed, *The uniform companion for fields with free operators in characteristic zero*. ***Journal of the London Mathematical Society***, vol. 113 (2026), no. 2, p. e70455.

[38] R. Moosa and T. Scanlon, *Model theory of fields with free operators in characteristic zero*. ***Journal of Mathematical Logic***, vol. 14 (2014), no. 2, p. 1450009, 43 pp.

[39] D. Palacín and F. O. Wagner, *Elimination of hyperimaginaries and stable Independence in simple CM-trivial theories*. ***Notre Dame Journal of Formal Logic***, vol. 54 (2013), nos. 3–4, pp. 541–551.

[40] Y. Peterzil, A. Pillay, and F. Point, *On definable groups in real closed fields with a generic derivation, and related structures*, preprint, 2023, arXiv:2208.08293.

[41] A. Pillay, *First order topological structures and theories*. ***Journal of Symbolic Logic***, vol. 52 (1987), no. 3, pp. 763–778.

[42] A. Pillay, F. Point, and S. Rideau-Kikuchi, *On groups definable in geometric fields with generic derivations*, preprint, 2025, arXiv:2503.13242 [math]. Pre-published.

[43] F. Point, *Ensembles définissables dans les corps ordonnés différentiellement clos*. ***Comptes Rendus. Mathématique***, vol. 349 (2011), nos. 17–18, pp. 929–933.

[44] B. Poizat, ***Stable Groups***, Mathematical Surveys and Monographs, 87, American Mathematical Society, Providence, 1987, 145 pp.

[45] C. Rivière, *Further notes on cell decomposition in closed ordered differential fields*. ***Annals of Pure and Applied Logic***, vol. 159 (2009), nos. 1–2, pp. 100–110.

[46] G. E. Sacks, ***Saturated Model Theory***, second ed., World Scientific, 2010, pp. xii+207. https://doi.org/10.1142/6974

[47] P. Simon, ***A Guide to NIP Theories***, Lecture Notes in Logic, 44, Association for Symbolic Logic, Cambridge Scientific Publishers, Chicago; Cambridge, 2015, pp. vii+156. https://www.normalesup.org/∼simon/NIP˙guide.pdf

[48] K. Tent and M. Ziegler, ***A Course in Model Theory***, Lecture Notes in Logic, 40, Association for Symbolic Logic, Cambridge University Press, La Jolla; Cambridge, 2012, pp. x+248.

[49] M. Tressl, *The uniform companion for large differential fields of characteristic 0*. ***Transactions of the American Mathematical Society***, vol. 357 (2005), no. 10, pp. 3933–3951.

[50] P. Z. Wang, *On groups and fields interpretable in NTP2 fields*, preprint, 2025. arXiv:2402.09143 [math]. Pre-published.

[51] S. Warner, ***Topological Fields***, New York: North-Holland, 1989.

[52] I. Yoneda, *Some remarks on CM-triviality*. ***Journal of the Mathematical Society of Japan***, vol. 61 (2009), no. 2. https://doi.org/10.2969/jmsj/06120379

[53] ———, *Pillay's alternative definition for weak elimination of imaginaries coincides with Poizat's original one*. ***RIMS Kokyuroku***, vol. 2218 (2022), pp. 59–63.

DIPARTIMENTO DI MATEMATICA E INFORMATICA "U. DINI"
UNIVERSITÀ DEGLI STUDI DI FIRENZE
ITALY
E-mail: antongiulio.fornasiero@gmail.com

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
UNIVERSITÀ DEGLI STUDI DI NAPOLI "FEDERICO II"
ITALY
E-mail: giuseppina.terzo@unina.it

https://doi.org/10.1017/jsl.2026.10229 Published online by Cambridge University Press