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Dissipation in contact dynamics†

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Abstract. Moser's celebrated theorem guarantees that every diffeomorphism of a closed manifold can be isotoped to a volume-preserving one. We show that this statement cannot be extended into the contact category: some connected components of contactomorphism groups contain no volume-preserving maps. Thus, the dissipation of volume appears for purely topological reasons. This phenomenon can be considered from different viewpoints: geometric (isometric action of the contact mapping class group on the moduli space of contact forms), topological (action in symplectic homology) and dynamical (propagation of trajectories for symplectic maps). We define a numerical invariant—a contact Lyapunov exponent—which leads to a quantitive version of the above-mentioned result.

1. *Introduction and results*

A diffeomorphism of a smooth orientable closed manifold is called *conservative* if it has an invariant measure given by a strictly positive continuous density with respect to a smooth volume form. For diffeomorphisms, there are no global topological obstructions to conservativity. Moser's celebrated theorem [**Mo**] implies that every diffeomorphism can be isotoped to a volume-preserving one. Our starting observation is that, in contrast to this, in the contact category such obstructions do exist.

For a closed manifold X consider the space of co-oriented contact elements $\mathbb{P}_+T^*X$ endowed with the standard contact structure (see, for example, [**MS**]). Given a contact diffeomorphism f of $\mathbb{P}_+T^*X$, write I_f for the inverse of the induced automorphism of the first cohomology group $H^1(\mathbb{P}_+T^*X, \mathbb{R})$ (so the mapping $f \to I_f$ is a homomorphism of the corresponding groups). We start with the following result, which deals with the case when X is the n-dimensional torus $\mathbb{T}^n$.

THEOREM 1.1. *Let f be a conservative contactomorphism of $\mathbb{P}_+T^*\mathbb{T}^n$, $n \geq 3$. Then I_f is a periodic automorphism: $I_f^m = \mathrm{id}$ for some $m \in \mathbb{N}$.*

One can easily show that this result is sharp in the following sense: every periodic automorphism of $H^1(\mathbb{P}_+T^*\mathbb{T}^n, \mathbb{R})$ preserving integer cohomology can be represented as I_f for some conservative contactomorphism f (see §2). The case of the 2-torus $\mathbb{T}^2$ is

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slightly more involved and will be considered separately in Theorem 2.2. Below we give a quantitive version of Theorem 1.1 (see Theorem 1.3) as well as an extension to general manifolds X (see Theorem 1.4).

The lack of conservation (which we call *dissipation*) can be measured in the following way (cf. [**KH**, §5.1]). Let f be a contactomorphism of a closed contact manifold (M, ξ), where the contact structure ξ is assumed to be co-oriented. Pick a contact form λ on M, which means that $\xi = \text{Kernel}\,\lambda$ and λ agrees with the co-orientation. Note that $(f^*)^{-1}\lambda/\lambda$ is a non-vanishing function on M. Put $F = \log|(f^*)^{-1}\lambda/\lambda|$. For every $x \in M$ and $k \in \mathbb{N}$ define

$$R_k(f, \lambda, x) = \log\left|\frac{(f^*)^{-k}\lambda}{\lambda}(x)\right| = \sum_{i=0}^{k-1} F(f^{-i}x)$$

and consider a sequence of real numbers

$$r_k(f, \lambda) = \max_{x \in M} |R_k(f, \lambda, x)|.$$

This sequence is clearly subadditive, hence the limit

$$\chi(f) = \lim_{k \to +\infty} \frac{1}{k} r_k(f, \lambda)$$

exists. A contactomorphism f of (M, ξ) is said to be *elliptic* if the sequence $r_k(f, \lambda)$ is bounded and *hyperbolic* if $\chi(f) > 0$. There are contactomorphisms such that the sequence r_k has intermediate growth (for instance, $r_k \sim \log k$), but we will not focus on them here.

Since any other contact form λ' on (M, ξ) satisfies $\lambda' = u\lambda$ for some positive function u on M, we have

$$|r_k(f, \lambda) - r_k(f, \lambda')| \leq \text{constant}$$

for all k. Therefore the type of f and the value of $\chi(f)$ for hyperbolic f does not depend on the specific choice of a contact form λ.

For a hyperbolic contactomorphism f, the quantity $\chi(f)$ is a natural measure for the rate of the dissipation of volume. Indeed, assume that $\dim M = 2n + 1$. Then every volume form σ on M can be written (up to a sign) as $\sigma = \lambda \wedge (d\lambda)^n$ for some contact form λ. Hence

$$(f^*)^{-1}\sigma/\sigma = ((f^*)^{-1}\lambda/\lambda)^{n+1},$$

and therefore

$$\chi(f) = \frac{1}{n+1} \lim_{k \to \infty} \max_{x \in M} \left|\frac{1}{k} \log\left|\frac{(f^*)^{-k}\sigma}{\sigma}(x)\right|\right|.$$

From a different viewpoint, $\chi(f)$ can be interpreted as the leading Lyapunov exponent in the direction transversal to the contact structure. In fact, there always exists a point $x \in M$ such that the ratio $R_k(f, \lambda, f^k x)/k$ converges as $k \to +\infty$ and the absolute value of the limit equals $\chi(f)$ (see Appendix A for the proof and §5 below for further discussions).

PROPOSITION 1.2. *A conservative contactomorphism is elliptic.*

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Indeed, take a contact form λ on M. Assume that f is a co-orientation-preserving conservative contactomorphism (the co-orientation-reversing case can be treated similarly), so f preserves a volume form $u\lambda \wedge (d\lambda)^n$ for some continuous positive function u. A simple calculation shows that then f preserves the continuous contact form $u^{1/(n+1)}\lambda$, and it is easy to conclude that f is elliptic.

Our next goal is to find a lower bound for $\chi(f)$ in terms of topological data. For a linear automorphism I of a finite-dimensional real vector space put

$$s(I) = \max |\log |c||,$$

where c runs over all (complex) eigenvalues of I. If $s(I) > 0$, the automorphism I is said to be *hyperbolic*.

THEOREM 1.3. *Let f be a contactomorphism of $\mathbb{P}_+T^*\mathbb{T}^n$, $n \geq 2$. Assume that I_f is hyperbolic. Then f is hyperbolic and $\chi(f) \geq s(I_f)$.*

Let us turn to the case when X is a general closed connected manifold. For a diffeomorphism e of X write Γ_e for the connected component of the group of all contactomorphisms of $\mathbb{P}_+T^*X$ containing the canonical lift of e to $\mathbb{P}_+T^*X$. We are able to prove a version of Theorems 1.1 and 1.3 under an additional assumption that the contactomorphism f lies in Γ_e. As compensation, we estimate $\chi(f)$ in terms of its action on the fundamental group rather than homology. Denote by $\mathcal{C}$ the set of non-trivial conjugacy classes of $\pi_1(X)$. Fix a system of generators of $\pi_1(X)$. Given $\gamma \in \mathcal{C}$, write $\ell(\gamma)$ for the minimal word length of an element representing γ. Let $J : \mathcal{C} \to \mathcal{C}$ be the map induced by e. Put

$$\bar{s}(J) = \sup_{\gamma \in \mathcal{C}} \limsup_{n \to \infty} \frac{1}{n} \log \ell(J^n\gamma).$$

It is easy to see that this quantity does not depend on the choice of a system of generators. J is said to be *hyperbolic* if $\bar{s}(J) > 0$.

THEOREM 1.4. *Assume that J is hyperbolic. Then every contactomorphism $f \in \Gamma_e$ is hyperbolic and $\chi(f) \geq \bar{s}(J)$. In particular, f is non-conservative.*

Denote by I the inverse of the automorphism of $H^1(X, \mathbb{R})$ induced by e. It is easy to see that $\bar{s}(J) \geq s(I)$.

COROLLARY 1.5. *Assume that I is hyperbolic. Then every contactomorphism $f \in \Gamma_e$ is hyperbolic and $\chi(f) \geq s(I)$.*

Here is an example where Theorem 1.4 gives a better estimate than Corollary 1.5. Let X be a closed oriented surface of genus ≥ 2. There exists a pseudo-Anosov diffeomorphism of X acting *trivially* on $H^1(X, \mathbb{R})$ (see, for example, [**LP**]). On the other hand, every pseudo-Anosov diffeomorphism induces a hyperbolic map $J : \mathcal{C} \to \mathcal{C}$ (see [**FLP**]), so $\bar{s}(J) > s(I) = 0$.

Note also that the assumption that $f \in \Gamma_e$ can be omitted when X is a closed surface of genus ≥ 2. A recent result by Giroux [**Gi**] states that in this case *every* co-orientation-preserving contactomorphism of $\mathbb{P}_+T^*X$ is contact isotopic to the canonical lift of some diffeomorphism e of X and hence is contained in Γ_e.

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There are several possible viewpoints on the dissipation phenomenon described above. One of them is closely related to *propagation* of trajectories for symplectomorphisms of cotangent bundles. We use it in the next section in order to prove Theorems 1.1 (and its version for the 2-torus) and 1.3. Another approach based on the isometric action of the contact mapping class group on the moduli space of contact forms is presented in §3 where we prove Corollary 1.5. This action is studied with the use of symplectic shapes introduced by Sikorav and Eliashberg. In §4 we prove Theorem 1.4 using Floer–Hofer Symplectic Homology Theory. Section 5 contains some speculations and open problems, and in particular we discuss some duality between symplectic homology and shapes. Let us emphasize that our decision to present a separate proof of Corollary 1.5 is due to the fact that it is much more transparent than the more general argument given in §4. Finally, in Appendix A we explain why the quantity $\chi(f)$ should be interpreted as the leading Lyapunov exponent.

2. *Symplectic propagation*

Our proof of the theorems stated above starts with the following *symplectization* procedure (see, for example, [**MS**]). For a closed manifold X consider its cotangent bundle $\pi : T^*X \mapsto X$ endowed with the standard symplectic form $\Omega = dp \wedge dq$. Write Z for the zero section $\{p = 0\}$, and put $T_0^*X = T^*X \setminus Z$. The group $\mathbb{R}_+$ acts naturally on T_0^*X by $(p, q) \mapsto (cp, q)$, $c \in \mathbb{R}_+$, and the corresponding space of orbits is naturally identified with $\mathbb{P}_+T^*X$. Write $\tau : T_0^*X \mapsto \mathbb{P}_+T^*X$ for the natural projection. Let us look more carefully at this fibration. A point $(p, q) \in T_0^*X$ represents a contact element (that is a tangent hyperplane to X) $l = \text{Kernel}(p)$ at a point $q \in X$ co-oriented by p, so $\tau(p, q) = (l, q)$. Furthermore, p is considered as a covector, this time, on $\mathbb{P}_+T^*X$ at a point $\tau(p, q)$ which sends any tangent vector $(\dot{l}, \dot{q})$ to $p(\dot{q})$. Its co-oriented kernel is precisely the co-oriented contact hyperplane on $\mathbb{P}_+T^*X$. In view of this discussion, every co-orientation-preserving contactomorphism f of $\mathbb{P}_+T^*X$ lifts to a unique $\mathbb{R}_+$-equivariant diffeomorphism ϕ of T_0^*X given by

$$\phi(p, q) = ((f^*)^{-1}p, f(\tau(p, q))),$$

which in addition turns out to be symplectic with respect to Ω. Conversely, every $\mathbb{R}_+$-equivariant symplectomorphism of T_0^*X is the lift of a unique contact transformation.

Furthermore, contact forms λ are in a one-to-one correspondence with smooth sections of the bundle $T_0^*X \to \mathbb{P}_+T^*X$. Fix such a λ, and for a point $z = (p, q) \in T_0^*X$ set $|z| = p/\lambda$, where λ is taken at the point $\tau(z)$.

The construction above allows us to translate contact dissipation described in §1 into symplectic language. The next proposition is an immediate consequence of the definitions, and its proof is left to the reader.

PROPOSITION 2.1. *Let f be a co-orientation-preserving contactomorphism of $\mathbb{P}_+T^*X$, and let ϕ be its lift to an $\mathbb{R}_+$-equivariant symplectomorphism of T_0^*X.*

- *f is elliptic if and only there exist constants $C > c > 0$ such that $c|z| < |\phi^k(z)| < C|z|$ for every $z \in T_0^*X$ and $k \in \mathbb{N}$.*

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- *f is hyperbolic if and only if there exists $c > 0$ such that*

$$\sup_{z \in T_0^* X} \left| \log \frac{|\phi^k(z)|}{|z|} \right| > ck$$

for all sufficiently large $k \in \mathbb{N}$. In this case $\chi(f) \geq c$.

Thus the fact that a contactomorphism f is non-elliptic can be interpreted as the existence of unbounded trajectories for its symplectic lift ϕ, while hyperbolicity of f corresponds to the existence of trajectories which propagate exponentially fast (cf. Appendix A).

Our approach to the study of symplectic propagation described in §§2 and 3 is based on symplectic topology of Lagrangian submanifolds in cotangent bundles. Denote by $\mathcal{L}$ the space of all embedded Lagrangian submanifolds $L \subset T^*X$ diffeomorphic to X such that the projection $\pi\big|_L : L \to X$ induces an isomorphism $i_L : H^1(X, \mathbb{R}) \to H^1(L, \mathbb{R})$. For a Lagrangian submanifold $L \in \mathcal{L}$ denote by $a(L) \in H^1(X, \mathbb{R})$ the pre-image of its Liouville class† under i_L. The role played by Lagrangian submanifolds becomes especially transparent in the case when X is the torus $\mathbb{T}^n$.

Proof of Theorems 1.1 and 1.3 for $n \geq 3$. Assume without loss of generality that our contactomorphism f preserves the co-orientation of the standard contact structure on $\mathbb{P}_+ T^*X$ (otherwise consider f^2 instead of f). Since $n \geq 3$, there is a natural identification

$$H^1(T_0^*\mathbb{T}^n, \mathbb{R}) = H^1(\mathbb{P}_+ T^*\mathbb{T}^n, \mathbb{R}) = H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n,$$

where $\mathbb{R}^n$ stands for a fiber of $T^*\mathbb{T}^n$. With this identification I_f is an automorphism of $\mathbb{R}^n$ which preserves the lattice $H^1(\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}^n$. Denote by $\|w\|$ the Euclidean norm of a vector in $\mathbb{R}^n$. Choose a contact form λ on $\mathbb{P}_+ T^*\mathbb{T}^n$ such that $|(p, q)| = \|p\|$ for all $(p, q) \in T_0^*\mathbb{T}^n$.

Let ϕ be the symplectic lift of f to a symplectomorphism of $T_0^*\mathbb{T}^n$. For a vector $v \in \mathbb{R}^n \setminus \{0\}$ consider the 'flat' Lagrangian torus $L_v = \{p = v\}$. Observe that $a(\phi^k(L_v)) = I_f^k(v)$ for all $k \in \mathbb{N}$.

Now we use a fundamental result due to Gromov [**Gr1**]: given $w \in \mathbb{R}^n$, every Lagrangian torus $L \in \mathcal{L}$ with $a(L) = w$ intersects L_w. Pick a sequence of vectors v_k with $\|v_k\| = 1$ and set $w_k = I_f^k(v_k)$. We conclude that $\phi^k(L_{v_k})$ intersects L_{w_k} for all $k \in \mathbb{N}$.

Suppose now that I_f is not periodic. Since the operator norm of I_f^k goes to infinity‡ with k, there exists a sequence v_k as above such that $w_k \to \infty$ as $k \to \infty$. Therefore there exists a sequence z_k with $|z_k| = 1$ such that $|\phi^k(z_k)| \to \infty$. Hence f is non-elliptic in view of the first part of Proposition 2.1, and therefore f is non-conservative in view of Proposition 1.2. This completes the proof of Theorem 1.1.

† The Liouville class of a Lagrangian submanifold L is represented by the restriction of the canonical 1-form $p\,dq$ to L, see [**MS**].

‡ Indeed, the automorphism I_f preserves the lattice $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$. Given $C > 0$, there is only a finite number of such automorphisms with operator norm $\leq C$ (since there is only a finite number of frames in $H^1(X, \mathbb{Z})$ consisting of vectors whose lengths do not exceed a given constant). Thus if the sequence $\|I_f^k\|$ contains an infinite bounded subsequence then $I_f^{k_1} = I_f^{k_2}$ for some $k_1 \neq k_2$, so I_f is periodic.

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Assume now that I_f is hyperbolic. Then the sequence v_k can be chosen in such a way that $|\log \|w_k\|| \geq s(I_f)k$ for all sufficiently large $k \in \mathbb{N}$. Hence we get a sequence of points z_k with $|z_k| = 1$ such that

$$|\log |\phi^k(z_k)|| \geq s(I_f)k$$

for all sufficiently large $k \in \mathbb{N}$. Now the second part of Proposition 2.1 yields the hyperbolicity of f, as well as the inequality $\chi(f) \geq s(I_f)$. This completes the proof of Theorem 1.3 for $n \geq 3$. □

It remains to handle the case $n = 2$. Identify $\mathbb{P}_+T^*\mathbb{T}^2$ with the 3-torus

$$\mathbb{T}^3 = \{p_1^2 + p_2^2 = 1\} \subset T^*\mathbb{T}^2.$$

Endow $\mathbb{T}^3$ with coordinates (θ, q_1, q_2), where q_1, q_2 are coordinates in the base $\mathbb{T}^2$, while θ is the angular coordinate in the fiber. The standard contact structure on $\mathbb{T}^3$ is given by Kernel λ, where

$$\lambda = \cos 2\pi\theta \, dq_1 + \sin 2\pi\theta \, dq_2.$$

Identify $H^1(\mathbb{T}^3, \mathbb{R})$ with the space $\mathbb{R}^3$ generated by the classes $[d\theta], [dq_1], [dq_2]$, and consider the line V generated by $[d\theta]$. It was shown in [**EP1**] that an automorphism $I \in SL(3, \mathbb{Z})$ of $\mathbb{R}^3$ can be represented by a contactomorphism (that is, $I = I_f$ for some contactomorphism f) if and only if $I(V) = V$. Such an automorphism acts in the following way: $I([dx]) = \alpha[dx] + \beta[dy] + l[d\theta]$ and $I([dy]) = \gamma[dx] + \delta[dy] + m[d\theta]$ where the matrix

$$A_I = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

belongs to $\mathrm{GL}(2, \mathbb{Z})$ and l, m are arbitrary integers. Here is an analog of Theorem 1.1 for the 2-torus.

THEOREM 2.2. *If an automorphism $I \in SL(3, \mathbb{Z})$ represents a conservative contactomorphism of $\mathbb{P}_+T^*\mathbb{T}^2$, the matrix A_I is periodic.*

Proof of Theorems 2.2 and 1.3 for $n = 2$. One should proceed exactly as in the case $n \geq 3$ presented above, replacing I_f by A_I and taking into account that $s(I) = s(A_I)$ and that the Liouville class of a flat torus L_v transforms as follows: $a(\phi(L_v)) = A_I(v)$. □

We conclude this section with a remark about the sharpness of our results for tori. Consider first the case $n \geq 3$. Every periodic automorphism $I \in \mathrm{GL}(n, \mathbb{Z})$ of $H^1(\mathbb{P}_+T^*\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$ is represented by a periodic contactomorphism f which preserves the co-orientation. Indeed, take the matrix $(I^{\mathrm{T}})^{-1}$ (where I^{T} stands for the transposed matrix) and consider it as an algebraic automorphism of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Now define f as the canonical lift of $(I^{\mathrm{T}})^{-1}$ to $\mathbb{P}_+T^*\mathbb{T}^n$. Every periodic co-orientation-preserving contactomorphism has an invariant contact form (use the obvious averaging procedure), and therefore has a smooth invariant measure.

In the case $n = 2$ periodicity of the matrix A_I does not imply periodicity of the automorphism I. It turns out that in this case there exist volume-preserving contactomorphisms f with non-periodic I_f.

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PROPOSITION 2.3. *Every automorphism* $I \in \mathrm{SL}(3,\mathbb{Z})$ *of* $H^1(\mathbb{T}^3,\mathbb{R})$ *with periodic* A_I *can be represented by a volume-preserving contactomorphism.*

Proof. Indeed, each such I represents the action in cohomology of a diffeomorphism of the form $A^m B^k C$, where C is the canonical lift to $\mathbb{T}^3$ of a periodic automorphism of $\mathbb{T}^2$,

$$A(\theta, q_1, q_2) = (\theta, q_1 + \theta, q_2) \quad \text{and} \quad B(\theta, q_1, q_2) = (\theta, q_1, q_2 + \theta).$$

Obviously, C is a contactomorphism of $\mathbb{T}^3$. Note that C preserves a volume form $\sigma = u(\theta)\, d\theta\, dq_1\, dq_2$ for some function u. It suffices to show that A and B are homotopic to contactomorphisms which preserve the volume form $d\theta\, dq_1\, dq_2$ as well as the family of tori $\{\theta = \text{const}\}$, and therefore preserve σ as well. We explain this for A (the argument for B is analogous). In fact, the required volume-preserving contactomorphism f homotopic to A is given by the following explicit formula:

$$f(\theta, q_1, q_2) = \left(\theta, q_1 + \theta - \frac{1}{4\pi}\sin 4\pi\theta, q_2 + \frac{1}{4\pi}\cos 4\pi\theta\right).$$

This completes the proof. □

We refer the reader to a forthcoming article [**BPS**] for a study of symplectic propagation in the context of Hamiltonian dynamics.

3. *Contact mapping class group and shapes*

The argument presented in §2 does not work for general manifolds X where, for instance, it can happen that every Lagrangian submanifold $L \in \mathcal{L}$ intersects the zero section $Z \subset T^*X$, so the action of the symplectic lift of a contactomorphism on Lagrangians is not defined! We get around this difficulty by using the *symplectic shapes* introduced in [**S1**], [**S2**] and [**E1**]. Denote by $\mathcal{L}_0$ the subset of $\mathcal{L}$ consisting of all those L which are Lagrangian isotopic to the zero section. Given a domain $U \subset T^*X$ put

$$\mathrm{Shape}(U) = \{a(L) \mid L \subset U,\ L \in \mathcal{L}_0\} \subset H^1(X,\mathbb{R}).$$

The following elementary properties of shapes are important for our purposes. First of all, shapes are monotone in the following sense: if $U \subset U'$ then $\mathrm{Shape}(U) \subset \mathrm{Shape}(U')$. Furthermore, for $U \subset T^*X$ and $c > 0$ one has $\mathrm{Shape}(cU) = c\,\mathrm{Shape}(U)$. Finally, shapes behave nicely under symplectomorphisms ψ of T^*X preserving the zero section. Identify $H^1(T^*X,\mathbb{R})$ with $H^1(X,\mathbb{R})$ and write I_ψ for the inverse of the automorphism of $H^1(X,\mathbb{R})$ induced by ψ. Then $\mathrm{Shape}(\psi U) = I_\psi\,\mathrm{Shape}(U)$. This follows from two obvious facts: $a(\psi(L)) = I_\psi a(L)$ for every $L \in \mathcal{L}_0$, and $\psi(\mathcal{L}_0) = \mathcal{L}_0$. Here one should use the fact that ψ preserves the zero section.

Let us formulate the last, crucial, property of shapes established by Sikorav with the use of Lagrangian intersection theory (see [**S2**, Proposition 2.6]). Fix a contact form λ on $\mathbb{P}_+T^*X$. Consider the subset

$$U_\lambda = \{(p,q) \in T_0^*X \mid p/\lambda < 1\} \cup Z \subset T^*X. \tag{1}$$

Obviously, $\mathrm{Shape}(U_\lambda)$ is an open starshaped subset of $H^1(X,\mathbb{R})$. Sikorav proved that in addition this subset is bounded.

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Let (M, ξ) be a closed contact manifold with a co-oriented contact structure. Denote by Γ the group of all contactomorphisms of (M, ξ) which preserve the co-orientation, and by Γ_0 the connected component of the identity. Consider the *contact mapping class group*

$$G = \Gamma / \Gamma_0.$$

Let $\mathcal{F}$ be the space of all contact forms on M. Put $W = \mathcal{F}/\Gamma_0$, and consider the action of the contact mapping class group G on W given by the standard action of diffeomorphisms on forms (cf. [**S2**, §3.4]). It turns out that G acts by isometries of a natural pseudo-distance on W, and moreover this action carries non-trivial information on contact dissipation.

The pseudo-distance, say d, is defined as follows. Take two elements $a, b \in W$. Define

$$d(a, b) = \inf_{\alpha, \beta} \max_{x \in M} \left| \log \frac{\alpha}{\beta}(x) \right|,$$

where the infimum is taken over all pairs of contact forms $\alpha, \beta \in \mathcal{F}$ representing the classes a and b respectively†.

The group G acts on W by isometries of d. For an element $g \in G$ consider its *displacement*

$$\mathrm{disp}(g) = \lim_{k \to +\infty} k^{-1} d(a, g^k a), \quad a \in W.$$

Obviously, the limit exists and does not depend on the choice of the point $a \in W$.

PROPOSITION 3.1. *Let $f \in \Gamma$ be a contactomorphism representing a class $g \in G$.*

- *If f is elliptic, then for every $a \in W$ the trajectory $\{g^k a\}$, $k \in \mathbb{N}$, is bounded.*
- *If $\mathrm{disp}(g) > 0$, then f is hyperbolic and $\chi(f) \geq \mathrm{disp}(g)$.*

This is an immediate consequence of the definitions.

Our next goal is to prove Corollary 1.5 with the use of this language. Let $M = \mathbb{P}_+ T^* X$. For a contact form λ consider the set U_λ defined in (1). $\mathrm{Shape}(U_\lambda)$ does not alter when one considers $(f^*)^{-1}\lambda$ instead of λ for $f \in \Gamma_0$ (see the proof of Proposition 3.2 below), so for $a \in W$ we have a well-defined subset of $H^1(X, \mathbb{R})$ denoted by $\mathrm{Shape}(a)$. This subset is open, starshaped and bounded. Denote by V the space of all open, bounded, starshaped domains of $H^1(X, \mathbb{R})$. It carries a natural metric $\delta(A, B) = \inf \log c$, where the infimum is taken over all numbers $c > 1$ such that $B \subset cA$ and $A \subset cB$. Clearly, the map $\mathrm{Shape} : (W, d) \to (V, \delta)$ does not increase the (pseudo-)distance.

Denote by D the group of all linear automorphisms of $H^1(X, \mathbb{R})$ preserving the lattice $H^1(X, \mathbb{Z})$. Consider the usual mapping class group

$$K = \mathrm{Diff}(X) / \mathrm{Diff}_0(X)$$

of the manifold X. We have two natural homomorphisms:

† It would be interesting to decide whether d is a non-degenerate distance on W or not. For instance, as Kai Cieliebak pointed out, d is non-degenerate when M is the circle S^1 equipped with the zero-dimensional co-oriented contact structure. In this case contact forms are simply the volume forms on the circle while the group Γ_0 coincides with $\mathrm{Diff}_0(S^1)$. Since any two volume forms are diffeomorphic if and only if their volumes coincide, we conclude that $W = \mathbb{R}_+$. Furthermore, (W, d) is isometric to the Euclidean line $\mathbb{R}$, where the isometry is given by $x \to \log x$.

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- $j : K \to G$, which takes a diffeomorphism of X to its canonical lift to a contactomorphism of $\mathbb{P}_+ T^*X$;
- $I : K \to D$, which takes a diffeomorphism to the inverse of the induced automorphism in cohomology.

PROPOSITION 3.2. *For every $u \in K$ and $a \in W$ we have* $\mathrm{Shape}(j(u)a) = I(u)\,\mathrm{Shape}(a)$.

For the proof, we need the following auxiliary fact which will be useful in the next section as well. Let ϕ be an $\mathbb{R}_+$-equivariant symplectomorphism of T_0^*X, and let ψ be a symplectomorphism of T^*X preserving the zero section whose restriction to T_0^*X is $\mathbb{R}_+$-equivariant. Assume that ϕ and $\psi\big|_{T_0^*X}$ are isotopic through $\mathbb{R}_+$-equivariant symplectomorphisms of T_0^*X. Take any contact form λ on $\mathbb{P}_+ T^*X$ and put $U = U_\lambda$.

PROPOSITION 3.3. *There exists $\epsilon \in (0, 0.5)$ and a symplectomorphism $h : T^*X \to T^*X$ such that $h \equiv \phi$ outside $0.5U$ and $h \equiv \psi$ in ϵU.*

Proof. Write $\phi = \psi\theta$, where θ is a $\mathbb{R}_+$-equivariant symplectomorphism of T_0^*X. Then there exists an isotopy θ_t, $t \in [0, 1]$, with $\theta_0 = \mathrm{id}$ and $\theta_1 = \theta$ consisting of $\mathbb{R}_+$-equivariant symplectomorphisms of T_0^*X. It suffices to show that there exists a symplectomorphism σ of T^*X which coincides with θ outside $0.5U$ and equals the identity in ϵU. Indeed, then $h = \psi\sigma$ is as required. To construct σ, choose $0.01 > \epsilon > 0$ such that

$$\cup_t \theta_t(T^*X \setminus 0.4U) \subset T^*X \setminus 4\epsilon U.$$

Let $B : T^*X \to \mathbb{R}$ be a cut-off function which vanishes on $2\epsilon U$ and equals 1 outside $3\epsilon U$. The path θ_t is generated by a degree-one homogeneous Hamiltonian function $A(z, t)$ on T_0^*X. Consider the Hamiltonian isotopy σ_t, $t \in [0; 1]$, generated by the Hamiltonian $B(z)A(z, t)$. Clearly, $\sigma = \sigma_1$ is as required. □

Proof of Proposition 3.2. Take a contact form λ representing some $a \in W$. Let e be a diffeomorphism of X representing u, and let f be a contactomorphism representing $j(u)$. Denote by ψ the canonical lift of e to a symplectomorphism of T^*X, and by ϕ the canonical lift of f to an $\mathbb{R}_+$-equivariant symplectomorphism of T_0^*X. Proposition 3.3 yields the existence of a symplectomorphism h of T^*X which coincides with ϕ outside $0.5U_\lambda$ and preserves the zero section. Looking at the action of h on the boundary of U we conclude that $h(U) = U_\beta$, where $\beta = (f^*)^{-1}\lambda$ is the contact form representing $j(u)a \in W$. Since for every Lagrangian submanifold $L \in \mathcal{L}_0$ one has $a(h(L)) = I(u)a(L)$ we conclude that $\mathrm{Shape}(U_\beta) = I(u)\,\mathrm{Shape}(U)$. □

Proof of Corollary 1.5. For every $a \in W$

$$d(a, j(u)^k a) \geq \delta(\mathrm{Shape}(a), I(u)^k\,\mathrm{Shape}(a)).$$

Since I is hyperbolic, the expression on the right-hand side grows asymptotically as $s(I)k$. Therefore, $\mathrm{disp}(j(u)) \geq s(I) > 0$, so Proposition 3.1 implies that every contactomorphism f representing $j(u)$ is hyperbolic with $\chi(f) \geq s(I)$. □

It would be interesting to explore further the isometric action of the mapping class group G on the space W. Let us mention also that one can define a canonical isometric action of G on the metric space Z introduced in [**EP2**] and prove a direct analog of Proposition 3.1 in this case.

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4. *Using symplectic homology*

Let us recall very briefly the symplectic homology theory, see [**CFH, V, C**]. For a conjugacy class $\gamma \in \mathcal{C}$ denote by Λ_γ the space of smooth loops $S^1 \to T^*X$ whose projections to X represent γ. Let Φ be the set of all smooth Hamiltonians $F : T^*X \times S^1 \to \mathbb{R}$ such that there exist a compact subset $Q \subset T^*X$, a strictly convex function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ and a contact form λ such that $F(p,q,t) = \kappa(p/\lambda)$ for all $(p,q) \in T^*X \setminus Q$. For $F \in \Phi$ consider the action functional

$$A : \Lambda_\gamma \to \mathbb{R}, \quad A(z) = \int_z p\,dq - F\,dt.$$

Its critical points are precisely 1-periodic orbits of the Hamiltonian vector field ξ given by $i_\xi \Omega = -dF$. Fix $a \in \mathbb{R}$, and set $E^a_\gamma(F)$ to be the homology of the Floer–Morse complex of the set $\{z \in \Lambda_\gamma,\ A(z) < a\}$ associated to the negative gradient flow of A. We work with $\mathbb{Z}_2$-coefficients, so $E^a_\gamma(F)$ is a vector space over $\mathbb{Z}_2$. There are the following natural maps between these spaces:

- $E^a_\gamma(F) \to E^b_\gamma(F)$ for all $F \in \Phi$ and all pairs $a, b \in \mathbb{R}$ with $a < b$;
- $E^a_\gamma(F_1) \to E^a_\gamma(F_2)$ for all $a \in \mathbb{R}$ and all pairs of functions $F_1, F_2 \in \Phi$ with $F_1 \le F_2$.

Using the second natural map, one defines the symplectic homology $E^a_\gamma(U)$ of a bounded domain $U \subset T^*X$ as the (unique!) direct limit of the spaces $E^a_\gamma(F)$, where F runs over the set $\{F \in \Phi,\ F|_{U \times S^1} \le 0\}$ directed by the natural partial order $\le$ on functions. The natural maps above respect the direct limit and give rise to the following natural maps:

- $E^a_\gamma(U) \to E^b_\gamma(U)$ for $a < b$;
- $E^a_\gamma(V) \to E^a_\gamma(U)$ for $U \subset V$.

Define $E^\infty_\gamma(U)$ as the direct limit of $E^a_\gamma(U)$ when $a \to +\infty$. Let us list some properties of symplectic homology. (All of them except (3) are simple consequences of definitions.)

(1) $E^a_\gamma(cU) = E^{ca}_\gamma(U)$ for all $c \in \mathbb{R}_+$ and $a \in \mathbb{R} \cup \{+\infty\}$.

(2) Let $h : T^*X \to T^*X$ be a symplectomorphism of T^*X which is $\mathbb{R}_+$-equivariant outside a compact set. Denote by $J : \mathcal{C} \to \mathcal{C}$ the map induced by h. Then h induces an isomorphism

$$E^a_\gamma(U) \to E^a_{J\gamma}(h(U)).$$

(3) (Viterbo [**V**], see also [**W**] for some related results.) Let $U = U_\lambda$ be the domain associated to a contact form λ on $\mathbb{P}_+T^*X$ defined by formula (1) of §3. Then $E^\infty_\gamma(U) = H_*(\Lambda_\gamma, \mathbb{Z}_2)$.

(4) Let λ be a contact form on $\mathbb{P}_+T^*X$. Put

$$l_\gamma(\lambda) = \inf_z \int_z p\,dq,$$

where the infimum is taken over all closed characteristics of the hypersurface $\{p = \lambda\} \subset T^*X$. Here each closed characteristic is equipped with the natural orientation. Then $E^a_\gamma(U_\lambda) = 0$ for every $a < l_\gamma(\lambda)$.

For a contact form λ on $\mathbb{P}_+T^*X$ define $\nu_\gamma(\lambda)$ as the infimum of those $a \in \mathbb{R}$ for which the natural map $E^a_\gamma(U_\lambda) \to E^\infty_\gamma(U_\lambda)$ does not vanish. The following properties of ν are important for our purposes.

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(5) The quantity $\nu_\gamma(\lambda)$ is finite (see (3) above) and satisfies $\nu_\gamma(\lambda) \geq l_\gamma(\lambda)$ (see (4) above).

(6) If $\lambda_1 \leq \lambda_2$, then $\nu_\gamma(\lambda_1) \leq \nu_\gamma(\lambda_2)$. Indeed, $U_{\lambda_1} \subset U_{\lambda_2}$, and the desired statement follows from the sequence of maps

$$E^a_\gamma(U_{\lambda_2}) \to E^a_\gamma(U_{\lambda_1}) \to E^\infty_\gamma(U_{\lambda_1}) = E^\infty_\gamma(U_{\lambda_2}).$$

Here all the arrows as well as their composition are natural maps considered above.

(7) $\nu_\gamma(c\lambda) = c\nu_\gamma(\lambda)$ for all $c \in \mathbb{R}_+$.

(8) Let $h : T^*X \to T^*X$ be a symplectomorphism of T^*X which is $\mathbb{R}_+$-equivariant outside a compact set. Denote by $J : \mathcal{C} \to \mathcal{C}$ the map induced by h. Let λ_1 and λ_2 be contact forms on $\mathbb{P}_+T^*X$ such that $h(U_{\lambda_2}) = U_{\lambda_1}$. Then $\nu_{J\gamma}(U_{\lambda_1}) = \nu_\gamma(U_{\lambda_2})$. This is an immediate consequence of (2).

Proof of Theorem 1.4. Take a contactomorphism $f \in \Gamma_e$ and consider its lift ϕ to T^*_0X. Take any contact form λ on $\mathbb{P}_+T^*X$. Modifying ϕ in a neighbourhood of the zero section and extending it to the zero section as in Proposition 3.3, we get a symplectomorphism h which preserves the zero section such that $h(U_\lambda) = U_{(f^*)^{-1}\lambda}$. Using properties (6)–(8) we get that for every $\gamma \in \mathcal{C}$

$$d([\lambda], [(f^*)^{-1}\lambda]) \geq \left|\log \frac{\nu_{J\gamma}(\lambda)}{\nu_\gamma(\lambda)}\right|,$$

where $[\lambda]$ and $[(f^*)^{-1}\lambda]$ are the classes of the corresponding contact forms in the moduli space W and d is the pseudo-metric on W defined in §3.

Thus Proposition 3.1 combined with (5) yields the inequality

$$\chi(f) \geq \sup_\gamma \limsup_{n\to\infty} \frac{1}{n} \log \nu_{J^n\gamma}(\lambda) \geq \sup_\gamma \limsup_{n\to\infty} \frac{1}{n} \log l_{J^n\gamma}(\lambda)$$

which is valid for every contact form λ on $\mathbb{P}_+T^*X$. Denote the expression on the right-hand side by K_λ. Take λ to be the contact form associated to a Riemannian metric on X. Then $l_\gamma(\lambda)$ is the length of the shortest loop in the free homotopy class γ. It follows from [**Gr2**, Proposition 3.22] that, for every system of generators in $\pi_1(X)$, there exists $c > 1$ such that

$$c^{-1}\ell(\gamma) \leq l_\gamma(\lambda) \leq c\ell(\gamma),$$

where $\ell(\gamma)$ is the minimal word length of an element of $\pi_1(X)$ representing γ (see §1). Thus $K_\lambda = \bar{s}(J)$. This completes the proof. □

5. *Discussion and open problems*

We start our discussion with the following well-known fact. For a diffeomorphism f of a manifold M, define its leading Lyapunov exponent by

$$\mathrm{Lyap}(f) = \lim_{k\to\infty} \frac{1}{k} \max_{x\in M} |\log |D_x(f^k)||,$$

where $|D_x f|$ denotes the operator norm of the differential of f at a point x with respect to some fixed Riemannian metric on M (see Remark A.2 below).

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FACT 5.1. *A diffeomorphism inducing a hyperbolic automorphism in the mth (co)homology of a manifold has a strictly positive leading Lyapunov exponent.*

Our results described in §1 can be considered from two slightly different viewpoints. First, they refine the statement of 5.1 for contactomorphisms of $M = \mathbb{P}_+T^*X$ in the case $m = 1$. Indeed, the invariant $\chi(f)$ can be considered as the Lyapunov exponent in the direction transversal to the contact structure, so *a priori* it can be smaller than the leading Lyapunov exponent $\mathrm{Lyap}(f)$, and hence might vanish. We show that this is not the case.

From another viewpoint, Corollary 1.5 is a *contactization* of 5.1 in the case $m = 1$. Indeed, let e be a diffeomorphism of the base X acting hyperbolically in (co)homology. Let $\tilde{e}$ be its canonical lift to the contactomorphism of $\mathbb{P}_+T^*X$. Then obviously $\chi(\tilde{e}) = \mathrm{Lyap}(e) > 0$. Corollary 1.5 extends this inequality to the whole connected component of $\tilde{e}$ in the group of contactomorphisms.

The proof of Fact 5.1 is based on a transparent (classical) geometric idea. Since f acts hyperbolically in homology, the volume of the images of certain m-cycles in M under the iterations of f grows exponentially fast, thus the norm of $D(f^k)$ must grow exponentially fast, so $\mathrm{Lyap}(f) > 0$. The approach we used in §4 can be considered as a combination of this simple geometric idea (in the case $m = 1$) with Floer–Hofer symplectic homology theory which enabled us to use symplectic action on T^*X as a substitute for the length of loops on X.

It seems likely that Theorem 1.4 remains true without the assumption on the isotopy class of a contactomorphism. There is hope that this can be done along the lines of §4 but with the use of contact homology theory (see [**E2, EGH**]) instead of symplectic homology. Moreover, it would be interesting to apply contact homology to the study of dissipation on contact manifolds which are more sophisticated than spaces of co-oriented contact elements. For instance, one can expect that the hyperbolicity of a contactomorphism can be detected in terms of its action on contact homology.

The link between shapes and symplectic homology is not totally clear at the moment, however there is an indication that these notions can be considered as dual ones. From a naive viewpoint the duality looks as follows: shapes are responsible for Lagrangian submanifolds of T^*X, and thus are related to closed 1-forms on X, while symplectic homology deals with closed orbits on $\mathbb{P}_+T^*X$ whose projections define 1-cycles on X. An elaboration of this idea leads to the following problem. Given a contact form λ on $\mathbb{P}_+T^*X$, symplectic homology gives rise to a function $N : H_1(X, \mathbb{Z}) \to [0; +\infty)$ defined by $N(\beta) = \inf_\gamma \nu_\gamma(\lambda)$, where γ runs over all free homotopy classes of loops representing β. It seems likely that one can stabilize this function by considering

$$N_\infty(\beta) = \lim_{n\to\infty} \frac{1}{n} N(n\beta).$$

Extend N_∞ to $H_1(X, \mathbb{Q})$ by linearity and define a subset $B_\lambda \subset H_1(X, \mathbb{R})$ as the closure of $\{N_\infty < 1\}$. Is there any relation between B_λ and $\mathrm{Shape}(U_\lambda)$? For instance, when λ is a contact form associated to a Riemannian metric on X, there is strong evidence that both sets are convex and dual to one another in the usual sense (see [**PS**]). Let us sketch one result in this direction which holds true for every contact form λ on $\mathbb{P}_+T^*X$. Namely,

$$\nu_\gamma(\lambda) \geq (b, \gamma) \tag{2}$$

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for all $\gamma \in \mathcal{C}$ and $b \in \text{Shape}(U_\lambda)$. This translates into the language developed above as $(b, \beta) \leq 1$ for every $b \in \text{Shape}(U_\lambda)$ and $\beta \in B_\lambda$. For the proof of inequality (2) assume that $L \subset U = U_\lambda$ is a Lagrangian submanifold which is Lagrangian isotopic to the zero section and whose Liouville class equals b. There exist $\epsilon > 0$ and a symplectomorphism h of T^*X such that $h(\epsilon U) \subset U$ and $[h^*(pdq) - pdq] = b$. Moreover, one can choose such an h so that outside a compact subset of T^*X one has $h(p, q) = (p + \theta(q), q)$, where θ is a closed 1-form on X representing b. One can show that h induces a shift of real grading on symplectic homology: $E^a_\gamma(h(V)) = E^{a-(b,\gamma)}_\gamma(V)$ for all a and V. Since h takes ϵU inside U we have

$$\nu_\gamma(\epsilon\lambda) + (b, \gamma) \leq \nu_\gamma(\lambda).$$

Using the fact that $\nu_\gamma(\epsilon\lambda) = \epsilon\nu_\gamma(\lambda)$ and taking $\epsilon > 0$ to be arbitrarily small we get inequality (2).

We conclude with the following problem which is absolutely open. Is it possible to contactize 5.1 for $m \geq 2$? More precisely, we ask the following.

Question 5.2. Let e be a diffeomorphism of a closed *simply connected* manifold X which acts hyperbolically on $H^m(X, \mathbb{R})$ for some $m \geq 2$ (for instance, $X = S^3 \times S^3$ and $m = 3$). Let f be a contactomorphism of $\mathbb{P}_+T^*X$ which is isotopic to the canonical lift of e through contactomorphisms. Is it true that f is non-conservative/hyperbolic?

A. *Appendix. $\chi(f)$ as the leading contact Lyapunov exponent*

Let f be a hyperbolic contactomorphism of a contact manifold M, and λ be a contact form. Recall that we write

$$R_k(f, \lambda, x) = \log \left| \frac{(f^*)^{-k}\lambda}{\lambda}(x) \right|, \quad k \in \mathbb{N}.$$

Our goal is to prove the following (probably, obvious to experts) statement.

PROPOSITION A.1. *There exists a point $x \in M$ such that the limit*

$$R(f, \lambda, x) = \lim_{k \to +\infty} \frac{1}{k} R_k(f, \lambda, f^k x)$$

exists and moreover $|R(f, \lambda, x)| = \chi(f)$.

This fact was pointed out to us by D. Burago who suggested a nice argument of a combinatorial nature. In the proof below combinatorics is replaced by ergodic theory (see also [**BPS**] where a similar method is applied in a different context).

Note that if the limit $R(f, \lambda, x)$ exists then its absolute value does not exceed $\chi(f)$. This justifies the title of the Appendix.

Proof. The proof is divided into several steps.

(1) Introduce a natural numerical invariant associated to an invariant measure of a contactomorphism. Let $\mathcal{P}$ be the space of Borel probability measures on M endowed with the weak-$*$ topology. For a contactomorphism f write $\mathcal{P}(f)$ for the set of f-invariant measures. Given $\mu \in \mathcal{P}(f)$ and a contact form λ put

$$r(\mu) = \int \log |(f^*)^{-1}\lambda/\lambda| \, d\mu.$$

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Since any other contact form on M is given by $u\lambda$ where u is a positive function, we see that $r(\mu)$ does not depend on the particular choice of λ.

(2) We claim that for every $\epsilon > 0$, there exists an f-invariant measure μ with $|r(\mu)| \geq \chi(f) - \epsilon$. Indeed, denote $F = \log |(f^*)^{-1}\lambda/\lambda|$, and put

$$F_k = \frac{1}{k} \log |(f^*)^{-k}\lambda/\lambda|.$$

Then F_k is simply the Birkhoff sum:

$$F_k = \frac{1}{k} \sum_{i=0}^{k-1} F \circ f^{-i}.$$

By definition of $\chi(f)$, there exists a sequence of points $x_k \in M$ so that $|F_k(x_k)| \geq \chi(f) - \epsilon$. Without loss of generality assume that there exists an infinite subset $K \subset \mathbb{N}$ such that $F_k(x_k) \geq \chi(f) - \epsilon$ for all $k \in K$. Denote by δ_{ik} the Dirac measure concentrated at $f^{-i}(x_k)$. Note that $F_k(x_k) = \int F \, d\mu_k$, where

$$\mu_k = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{ik}.$$

The sequence μ_k, $k \in K$, contains a subsequence μ_j which converges to an f-invariant measure, say, μ. Clearly

$$r(\mu) = \lim \int F \, d\mu_j \geq \chi(f) - \epsilon.$$

The claim follows.

(3) The set $\mathcal{P}(f)$ is convex and compact, and the map $r : \mathcal{P}(f) \to \mathbb{R}$ is affine and continuous. Hence its image is a closed interval, say $[r_-(f); r_+(f)]$. It follows from step 2 that it contains either $\chi(f)$ or $-\chi(f)$ (or both). Assume without loss of generality that the first possibility holds so that $r_+(f) \geq \chi(f)$.

(4) Let us show that there exists an *ergodic* f-invariant measure, say, $\mu \in \mathcal{P}(f)$ with $r(\mu) = r_+(f)$. Indeed, the pre-image of $r_+(f)$ under the map r is a convex compact subset of $\mathcal{P}(f)$. Let μ be its extremal point. One can easily see that then μ is an extremal point of $\mathcal{P}(f)$ as well, and hence μ is ergodic. Applying the Birkhoff Ergodic Theorem we get that

$$R(f, \lambda, x) = \lim_{k \to +\infty} F_k(f^k x) = r(\mu) = r_+(f)$$

for almost all (in the sense of μ) points $x \in M$. Combining this with the definition of $\chi(f)$ one gets $\chi(f) \geq r_+(f)$. The converse inequality is true as well (see step (3)). Therefore $r_+(f) = \chi(f)$, and thus we have found a point $x \in M$ with $R(f, \lambda, x) = \chi(f)$. □

Remark A.2. Let f be a diffeomorphism of a closed manifold M. We claim that the quantity $\mathrm{Lyap}(f)$ introduced in §5 can be interpreted as the leading Lyapunov exponent. Indeed, take any Riemannian metric on M and let λ be the corresponding contact form on $\mathbb{P}_+T^*M$. Write $\|p\|$ for the Riemannian length of a covector p on M. Let $\tilde{f}$ be the canonical lift of f to $\mathbb{P}_+T^*M$. Take a co-oriented contact element (l, q) and denote by

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$p(l,q) \in T_q^*M$ the unit covector whose kernel equals l and which co-orients l in the positive direction. One can readily check that

$$\frac{(\tilde{f}^*)^{-k}\lambda}{\lambda}(\tilde{f}^k(l,q)) = \|(f^*)^{-k}p(l,q)\|. \tag{3}$$

It follows immediately that $\mathrm{Lyap}(f) = \chi(\tilde{f})$. Suppose now that $\mathrm{Lyap}(f) > 0$. Combining (3) with Proposition A.1 we conclude that there exists $(l,q) \in \mathbb{P}_+T^*M$ so that

$$\lim_{k\to+\infty}\frac{1}{k}|\log\|(f^*)^{-k}p(l,q)\|| = \mathrm{Lyap}(f).$$

The claim follows.

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