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Multicolour Discrepancies

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In this article we introduce combinatorial multicolour discrepancies and generalize several classical results from 2-colour discrepancy theory to c colours ($c \geqslant 2$). We give a recursive method that constructs c-colourings from approximations of 2-colour discrepancies. This method works for a large class of theorems, such as the 'six standard deviations' theorem of Spencer (1985), the Beck–Fiala (1981) theorem, the results of Matoušek, Wernisch and Welzl (1994) and Matoušek (1995) for bounded VC-dimension, and Matoušek and Spencer's (1996) upper bound for the arithmetic progressions. In particular, the c-colour discrepancy of an arbitrary hypergraph (n vertices, m hyperedges) is

$$O\left(\sqrt{\frac{n}{c}\log m}\right).$$

If $m = O(n)$, then this bound improves to

$$O\left(\sqrt{\frac{n}{c}\log c}\right).$$

On the other hand there are examples showing that discrepancy in c colours can not be bounded in terms of two-colour discrepancies in general, even if c is a power of 2. For the linear discrepancy version of the Beck–Fiala theorem, the recursive approach also fails.

Here we extend the method of floating colours via tensor products of matrices to multicolourings, and prove multicolour versions of the Beck–Fiala theorem and the Bárány–Grinberg theorem. Using properties of the tensor product we derive a lower bound for the c-colour discrepancy of general hypergraphs. For the hypergraph of arithmetic progressions in $\{1,\dots,n\}$ this yields a lower bound of $\frac{1}{25\sqrt{c}}\sqrt[4]{n}$ for the discrepancy in c colours. The recursive method shows an upper bound of $O(c^{-0.16}\sqrt[4]{n})$.

1. Introduction

Combinatorial discrepancy theory deals with the problem of partitioning the vertices of a hypergraph (set system) in such a way that all hyperedges are split into roughly

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equal-sized parts. Discrepancy measures the deviation of an optimal partition to an ideal one, that is, one where all edges contain the same number of vertices in each class of the partition.

Usually we represent the partition by a colouring, that is, a mapping from the vertices into some set such that the classes of equal images form the partition classes. In this setting, most results known so far only deal with two colours. Recent results from communication complexity (*e.g.*, [2]) further motivate the study of multicolour discrepancies.

In this article we give two methods for constructing multicolourings with low discrepancy. If the given hypergraph fulfils some hereditary property, we can construct multicolourings from 2-colourings having low discrepancy. This works for a large class of hypergraphs: for hypergraphs $\mathscr{H}$ with n vertices and n hyperedges we obtain an upper bound of $O(\sqrt{\frac{n}{c}})$ for the discrepancy in c colours. This extends Spencer's famous 'six standard deviations' result to arbitrary numbers of colours. Along the same lines we receive another extension of Spencer's result: we find that in this situation the discrepancy with respect to a given weight $(p, 1-p), p \in [0,1]$ is $O(\sqrt{pn})$.

We also apply the recursive method to prove multicolour versions of the Beck–Fiala theorem [7], the results of Matoušek, Wernisch and Welzl [26] and Matoušek [22] for bounded VC-dimension, and an upper bound of $O(c^{-0.16}n^{1/4})$ for the multicolour discrepancy of the hypergraph of arithmetic progressions, extending the bound of Matoušek and Spencer [25].

The recursive method fails for the linear discrepancy version of the Beck–Fiala theorem and the Bárány–Grinberg theorem. Here we extend the method of floating colours to multicolourings and use this to prove similar results for any number of colours.

Finally, introducing a tensor product representation for multicolour discrepancy, we extend a lower bound result for the discrepancy function due to Lovász and Sós to the multicolour case: we show that the c-colour discrepancy is at least

$$\sqrt{\frac{n(c-1)}{mc^2}\lambda_{\min}(A^\top A)},$$

where A is the incidence matrix of the hypergraph and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. Applying this to the hypergraph of arithmetic progressions $\mathscr{H}_n$, we get a lower bound for the c-colour discrepancy of $\Omega(c^{-0.5}n^{1/4})$. Spencer's examples of hypergraphs constructed from Hadamard matrices have n vertices and n hyperedges and show a c-colour discrepancy of $\Omega(\sqrt{\frac{n}{c}})$. Thus our bounds for the arithmetic progressions and the 'six standard deviations' situation are rather close. In particular, they are sharp for fixed numbers of colours (apart from constants).

2. Preliminaries

2.1. Two-colour discrepancy

Let $\mathscr{H} = (X, \mathscr{E})$ denote a finite *hypergraph*, *i.e.*, X is a finite set (called *vertices* or *nodes*) and $\mathscr{E}$ is a family of subsets of X (called *hyperedges*). A partition into two classes can be represented by a *colouring* $\chi : X \to \{-1, +1\}$. We call -1 and $+1$ *colours*. The colour-classes $\chi^{-1}(-1)$ and $\chi^{-1}(+1)$ form the partition. The imbalance of a hyperedge $E \in \mathscr{E}$ can

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be expressed as $\chi(E) := \sum_{x\in E} \chi(x)$. The *discrepancy* of $\mathscr{H}$ is defined by

$$\operatorname{disc}(\mathscr{H}) = \min_{\chi:X\to\{-1,+1\}} \max_{E\in\mathscr{E}} |\chi(E)| \tag{2.1}$$

This concept may be generalized to matrices in a natural way. Let $A = (a_{ij})$ be any $(m \times n)$-matrix and set

$$\operatorname{disc}(A) := \min_{\chi\in\{-1,1\}^n} \|A\chi\|_\infty. \tag{2.2}$$

Let $X = \{x_1, \ldots, x_n\}$, $\mathscr{E} = \{E_1, \ldots, E_m\}$, and define a matrix $A = (a_{ij})$ by $a_{ij} = 1$ if $x_j \in E_i$, and $a_{ij} = 0$ otherwise. A is called the incidence matrix of $\mathscr{H}$. We have $\operatorname{disc}(A) = \operatorname{disc}(\mathscr{H})$, so discrepancy of matrices is indeed a more general concept (if, of course, we restrict ourselves to zero–one-matrices, both concepts are equivalent). Sometimes even for hypergraphs, the matrix notion is more convenient.

There are two related notions. The *linear discrepancy* of an arbitrary matrix A is defined by

$$\operatorname{lindisc}(A) := \max_{p\in[-1,1]^n} \min_{\chi\in\{-1,1\}^n} \|A(p-\chi)\|_\infty. \tag{2.3}$$

Linear discrepancy can be regarded as a measure of how well a fractional solution can be rounded to an integer solution. Some authors define linear discrepancy by

$$\max_{p\in[0,1]^n} \min_{\chi\in\{0,1\}^n} \|A(p-\chi)\|_\infty. \tag{2.4}$$

Both versions differ only by the constant factor 2. A special case of the second version is the weighted discrepancy, which refers to the problem of splitting the edges in an arbitrary ratio $p : 1-p$ (instead of $0.5 : 0.5$ as in the discrepancy problem). We define the weighted discrepancy in 2 colours by

$$\operatorname{wd}(A, 2) := \max_{p\in[0,1]} \min_{\chi\in\{0,1\}^n} \|A(\mathbf{1}_n p - \chi)\|_\infty, \tag{2.5}$$

where $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbb{R}^n$.

Finally, the *hereditary discrepancy* is

$$\operatorname{herdisc}(A) := \max_{J\subseteq[n]} \operatorname{disc}((a_{ij})_{i\in[m], j\in J}), \tag{2.6}$$

where $[n] := \{1, \ldots, n\}$.

All notions can be translated to hypergraphs in a natural way, so $\operatorname{herdisc}(\mathscr{H})$ is the maximum discrepancy of all induced subgraphs.

To shorten notation we will write $A_0 \leqslant A$ to indicate that the matrix A_0 consists of some columns of the matrix A. Similarly, for hypergraphs we will write $\mathscr{H}_0 \leqslant \mathscr{H}$ if $\mathscr{H}_0$ is an induced subgraph of $\mathscr{H}$.

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Proposition 2.1. *The following relations between the different notions hold:*

(i) $\operatorname{disc}(A) \leqslant \operatorname{herdisc}(A)$,
(ii) $\operatorname{disc}(A) \leqslant \operatorname{lindisc}(A)$,
(iii) $\operatorname{wd}(A,2) \leqslant \frac{1}{2}\operatorname{lindisc}(A)$,
(iv) $\operatorname{lindisc}(A) \leqslant 2\operatorname{herdisc}(A)$.

Relations (i) to (iii) are trivial, while the relation between the linear and the hereditary discrepancy is not. Relation (iv) was discovered in [9] and [21]. It is not completely clear how sharp this inequality is. The first author recently gave a slight improvement in [12]. For our purposes (iv) is sufficient. Matoušek investigated the opposite problem of bounding the hereditary discrepancy in terms of the linear discrepancy in [24].

An excellent survey of classical and recent results in discrepancy theory is the article of Beck and Sós [8]. For a more thorough treatment we refer to the books by Beck and Chen [6] and Matoušek [23]. Discrepancy theory is put into a broader context in the new book by Chazelle [11].

2.2. Multicolour discrepancy

Until now little was known about the discrepancy problem for partitions into more than two classes. We have found the following two results in the literature.

Theorem 2.2. (Beck and Fiala [7]) *Given $n \geqslant 5$ finite sets, it is possible to partition their union into r parts $A_1, \ldots, A_r$ for any positive integer r in such a way that, for each i, j and k,*

$$\big||E_i \cap A_j| - |E_i \cap A_k|\big| < 24\sqrt{2n\ \log(2n)}.$$

Theorem 2.3. (Beck and Sós [8]) *Let $\mathscr{H}$ be any hypergraph such that the incidence matrix of $\mathscr{H}$ is unimodular. Then, for any number $c \in \mathbb{N}$, there is a c-partition $X = X_1 \dot\cup \cdots \dot\cup X_c$ of X such that, for any edge $E \in \mathscr{E}$ and $i \in [c]$,*

$$\left||E \cap X_i| - \frac{|E|}{c}\right| < 1.$$

Let us introduce some notation concerning c-colour discrepancies. A c-colouring of $\mathscr{H}$ is simply a mapping $\chi : X \to M$, where M is any set of cardinality c. For convenience, we normally have $M = [c] := \{1, \ldots, c\}$. Sometimes a different set M will be useful. Note that, in applications to communication complexity, M can be a finite abelian group [2]. The basic idea of measuring the deviation from the average motivates the definitions of the *discrepancy of an edge $E \in \mathscr{E}$ in colour $i \in M$ with respect to χ* by

$$\operatorname{disc}_{\chi,i}(E) := \left||\chi^{-1}(i) \cap E| - \frac{|E|}{c}\right|, \tag{2.7}$$

the *discrepancy of $\mathscr{H}$ with respect to χ* by

$$\operatorname{disc}(\mathscr{H}, \chi, c) := \max_{i \in M, E \in \mathscr{E}} \operatorname{disc}_{\chi,i}(E), \tag{2.8}$$

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and the *discrepancy of $\mathscr{H}$ in c colours* by

$$\operatorname{disc}(\mathscr{H}, c) := \min_{\chi: X \to [c]} \operatorname{disc}(\mathscr{H}, \chi, c). \tag{2.9}$$

Remark 1. We immediately deduce $\operatorname{disc}(\mathscr{H}, 2) = \frac{1}{2}\operatorname{disc}(\mathscr{H})$.

To add some further motivation to what follows, let us give an example which shows that a hypergraph may have very different discrepancies in different numbers of colours.

Example. Let $k \in \mathbb{N}$ and $n = 4k$. Set $\mathscr{H}_n = ([n], \{X \subseteq [n] \,|\, |X \cap [\frac{n}{2}]| = |X \setminus [\frac{n}{2}]|\})$. Obviously, $\mathscr{H}_n$ has 2-colour discrepancy zero, but $\operatorname{disc}(\mathscr{H}_n, 4) = \frac{1}{8}n$.

Proof. Let $\chi : [n] \to [4]$ be any 4-colouring. Let $i \in [4]$ be a colour such that $|\chi^{-1}(i)| \leqslant \frac{1}{4}n$. Then there are sets $E_1 \subseteq [\frac{n}{2}]$, $E_2 \subseteq [n] \setminus [\frac{n}{2}]$ such that $|E_j| = \frac{1}{4}n$ and $\chi^{-1}(i) \cap E_j = \emptyset$. Thus $E_1 \cup E_2$ is an edge in $\mathscr{H}$ and has discrepancy $\frac{1}{8}n$ in colour i. On the other hand, $\chi : x \mapsto \lceil \frac{4x}{n} \rceil$ is a colouring having discrepancy $\frac{1}{8}n$. □

In fact, such examples exist for nearly any two numbers of colours. Unless c_1 divides c_2, there are hypergraphs $\mathscr{H}_n$ on n vertices having discrepancy $\Theta(n)$ in c_1 colours and zero discrepancy in c_2 colours. This has been investigated in [16].

In the above notion we cannot express discrepancies simply by sums of colours. As this is sometimes very practical and a step towards the matrix concept, we describe the colour $i \in [c]$ by a vector $m^{(i)} \in \mathbb{R}^c$ defined by

$$m^{(i)}_j := \begin{cases} \frac{c-1}{c} & \text{if } i = j, \\ -\frac{1}{c} & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{disc}(\mathscr{H}, \chi, c) = \max_{E \in \mathscr{E}} \left\| \sum_{x \in E} m^{(\chi(x))} \right\|_\infty . \tag{2.10}$$

Set $M_c := \{m^{(i)} | i \in [c]\}$. Apparently, we have

$$\operatorname{disc}(\mathscr{H}, c) = \min_{\chi: X \to M_c} \max_{E \in \mathscr{E}} \left\| \sum_{x \in E} \chi(x) \right\|_\infty . \tag{2.11}$$

As for 2 colours, the notion of multicolour discrepancy has a natural extension to matrices. Let $A \in \mathbb{R}^{m \times n}$ be any matrix. Let $\overline{A}$ be the matrix which results from replacing every a_{ij} in A by $a_{ij} I_c$, where I_c denotes the identity matrix of dimension c. Identifying a $\chi : [n] \to M_c$ by a cn-dimensional vector in the natural way, we get

$$\operatorname{disc}(A, c) := \min_{\chi: [n] \to M_c} \left\| \overline{A}\chi \right\|_\infty . \tag{2.12}$$

The other notions of discrepancy transform to the multicolour case in a similar way. Set $\overline{M_c} = \{\sum_{i \in [c]} \lambda_i m^{(i)} | \lambda \in [0,1]^c, \sum_{i \in [c]} \lambda_i = 1\}$, the convex hull of M_c. For $p \in \overline{M_c}$ set $\overline{p} : [n] \to \overline{M_c}; i \mapsto p$. We define the (*weighted*) *discrepancy of A with respect to the weight p*

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and the colouring χ by

$$\mathrm{wd}(A,\chi,p) := \left\|\overline{A}(\overline{p}-\chi)\right\|_\infty, \tag{2.13}$$

the (*weighted*) *discrepancy of* A *with respect to the weight* p by

$$\mathrm{wd}(A,c,p) := \min_{\chi:[n]\to M_c} \mathrm{wd}(A,\chi,p) \tag{2.14}$$

and the *weighted discrepancy of* A by

$$\mathrm{wd}(A,c) := \max_{p\in\overline{M}_c} \mathrm{wd}(A,c,p). \tag{2.15}$$

There is an equivalent way to define weighted discrepancy which puts more emphasis on the aspect of weights. Let E_c denote the standard basis of $\mathbb{R}^c$ and let $\overline{E}_c$ denote its convex hull, that is, all $p \in [0,1]^c$ such that $\|p\|_1 = 1$. We have

$$\mathrm{wd}(A,c) = \max_{p\in\overline{E}_c} \min_{\chi:[n]\to E_c} \left\|\overline{A}(\overline{p}-\chi)\right\|_\infty. \tag{2.16}$$

Note that this is an extension of the definition of $\mathrm{wd}(\mathscr{H},2)$ in equation (2.5). For hypergraphs this translates to

$$\mathrm{wd}(\mathscr{H},c) = \min_{\chi:X\to M_c} \max_{j\in[c],E\in\mathscr{E}} \left||E\cap\chi^{-1}(j)| - p_j|E|\right|.$$

The linear discrepancy in c colours can be defined by

$$\mathrm{lindisc}(A,c) := \max_{p:[n]\to\overline{M}_c} \min_{\chi:[n]\to M_c} \left\|\overline{A}(p-\chi)\right\|_\infty. \tag{2.17}$$

Finally, the hereditary discrepancy in c colours is

$$\mathrm{herdisc}(A,c) := \max_{A_0\leqslant A} \mathrm{disc}(A_0,c). \tag{2.18}$$

All these notions will be defined for hypergraphs as well, by taking the incidence matrix of the hypergraph. For example, for a hypergraph $\mathscr{H}$ with incidence matrix A we have $\mathrm{lindisc}(\mathscr{H}) := \mathrm{lindisc}(A)$. As in Remark 1, these other discrepancy notions are identical to the usual notions up to the constant factor of 2. When citing 2-colour results we use the conventional notation, which has no parameter c in it, *e.g.*, $\mathrm{herdisc}(\mathscr{H})$, so there is no danger of confusion.

2.3. Tensor products

As it will be convenient to substitute matrices into each another in the way described above, let us analyse this briefly. For any two matrices $A_k \in \mathbb{C}^{m_k\times n_k}$, $k = 1,2$, the tensor (or Kronecker) product $A_1 \otimes A_2$ is the matrix $B = (b_{ij}) \in \mathbb{C}^{m_1m_2\times n_1n_2}$ such that

$$b_{(i_1-1)m_1+i_2,(j_1-1)n_1+j_2} = a_{i_1j_1}a_{i_2j_2}$$

for all $i_k \in [m_k]$, $j_k \in [n_k]$, $k = 1,2$. Hence B is produced by replacing every entry a_{ij} of A_1 by $a_{ij}A_2$.

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Lemma 2.4. *The following laws hold for the tensor product.*

(i) *Associativity: all matrices* A, B, C *satisfy* $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
(ii) *Distributivity with* $+$*: for all matrices* A, B, C *such that* $A + B$ *is defined, we have* $(A + B) \otimes C = A \otimes C + B \otimes C$ *and* $C \otimes (A + B) = C \otimes A + C \otimes B$.
(iii) *'Mixed Product Rule':* $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$ *for all matrices* A, B, C, D *such that* AB *and* CD *are defined.*
(iv) $\otimes$ *is compatible with inversion:* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ *for all nonsingular matrices* A *and* B.
(v) *The (complex) eigenvalues of* $A \otimes B$ *are exactly the products of an eigenvalue of* A *and one of* B.
(vi) $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A)\operatorname{rank}(B)$.
(vii) $\det(A \otimes B) = (\det A)^{n_B}(\det B)^{n_A}$ *for all matrices* $A \in \mathbb{C}^{n_A \times n_A}$ *and* $B \in \mathbb{C}^{n_B \times n_B}$.

Some patience and knowledge of linear algebra is enough to prove Lemma 2.4. Most books on multilinear algebra will contain these results in chapters concerning tensor products of linear mappings and matrices. An elementary approach can be found in [18].

In the tensor product notation, equation (2.12), for instance, transforms to

$$\operatorname{disc}(\mathscr{H}, c) := \min_{\chi: X \to M_c} \left\| (A \otimes I_c)\chi \right\|_\infty. \tag{2.19}$$

2.4. The basic probabilistic bound

An elementary probabilistic approach is to consider a random colouring. We colour each vertex independently with a random colour. Using the so-called Chernoff bound, we prove that with positive probability our random colouring is balanced to a certain extent. Let $\mathscr{H} = (V, \mathscr{E})$ be any hypergraph. Set $m := |\mathscr{E}|$ and $s = \max_{E \in \mathscr{E}} |E|$. For the 2-colour case it is well known that $\operatorname{disc}(\mathscr{H}) \leqslant \sqrt{2s \ln(2m)}$ holds ([1, Theorem 12.1.1]). For c colours we have the following result.

Proposition 2.5. $\operatorname{disc}(\mathscr{H}, c) \leqslant \sqrt{\frac{1}{2} s \ln(2mc)}$.

Proof. Define a random c-colouring χ by independently picking a random colour uniformly distributed from $[c]$ for every vertex $v \in V$. Define random variables $X_{i,v}$ by

$$X_{i,v} := \begin{cases} \frac{c-1}{c} & \text{if } \chi(v) = i, \\ -\frac{1}{c} & \text{otherwise,} \end{cases}$$

for all $v \in V$, $i \in [c]$. Set $X_{i,E} := \sum_{v \in E} X_{i,v}$ for all $E \in \mathscr{E}$, $i \in [c]$. From [1, Theorem A.1.4], we know

$$P(|X_{i,E}| > \alpha) < 2e^{-2\alpha^2/|E|}$$

for any real $\alpha > 0$.

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For $\alpha = \sqrt{\frac{1}{2}s \ln(2mc)}$ we have

$$\begin{aligned} P(\forall i \in [c], E \in \mathscr{E} : |X_{i,E}| \leqslant \alpha) &= 1 - P(\exists i \in [c], E \in \mathscr{E} : |X_{i,E}| > \alpha) \\ &\geqslant 1 - \sum_{i \in [c], E \in \mathscr{E}} P(|X_{i,E}| > \alpha) \\ &> 1 - \sum_{i \in [c], E \in \mathscr{E}} 2e^{-2\alpha^2/|E|} \\ &\geqslant 1 - cm\, 2e^{-2\alpha^2/s} = 0 \end{aligned}$$

by choice of α. Hence, with positive probability, our random χ has discrepancy not greater than $\sqrt{\frac{1}{2}s \ln(2mc)}$; thus such a colouring exists. □

3. Recursive colouring

For some 2-colour discrepancy results the proofs seem to rely heavily on the fact that only two colours are used. One example is Spencer's $O(\sqrt{n})$ bound for hypergraphs having n vertices and edges. A key step in the proof is to construct a low discrepancy partial colouring $\chi := \frac{1}{2}(\chi_1 - \chi_2)$ from two colourings χ_1, χ_2 with $\chi_1(E) \approx \chi_2(E)$ for all $E \in \mathscr{E}$. It is not clear to us how this idea can be generalized to c colours.

As the partial colouring method has been a major breakthrough in 2-colour discrepancy theory, it is desirable to have a similar method for c colours as well. What we do in this section is not partial colouring, *i.e.*, enlarging the partition classes by successively colouring points, but recursive 2-colouring, *i.e.*, successively enlarging the number of partition classes. The basic idea is to find a suitable 2-colouring of X with colour classes X_1, X_2 and then to iterate this process on the subhypergraphs induced by X_1 and X_2. If the weighted 2-colour discrepancies of suitable induced subhypergraphs are bounded, such a recursive method can be analysed, even if c is not a power of 2. This will lead to a generalization of the 'six standard deviations' theorem of Spencer [29], the discrepancy bound of Beck–Fiala [7] and the bounds using the primal and dual shatter function of Matoušek, Wernisch and Welzl [26] and Matoušek [22].

At the end of this long section we will show the limits of the recursive approach. For example, for the linear discrepancy in c colours, recursive methods are rather weak, and we need other methods, which will be introduced in the next section.

3.1. The recursive method

The following lemma analyses a single step in the recursion. It shows that an imbalance inflicted in the first step of the recursion is evenly split up in the remainder of the partitioning process.

We call a function $p : C \to [0,1]$ a *weight* of the set C of colours if $\sum_{i \in C} p_i = 1$. For $D \subseteq C$ set $p(D) := \|p_{|D}\|_1 = \sum_{i \in D} p_i$.

Lemma 3.1. *Let C be a set of colours with $c = |C|$ and let $\{C_1, C_2\}$ be a partition of C. Let p be a weight of C. Set $q_j = p(C_j)$, $j \in [2]$. Let $\chi_0 : X \to [2]$ be a 2-colouring of X. Set*

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$X_j := \chi_0^{-1}(j)$, $j \in [2]$. Let $\chi_j : X_j \to C_j$ be any colourings. Set $\chi := \chi_1 \cup \chi_2$. For all $E \in \mathscr{E}$, $j \in [2]$ and $i \in C_j$ the discrepancy of E with respect to the colour i, the colouring χ and the weight p is

$$\big||E \cap \chi^{-1}(i)| - p_i|E|\big| \leqslant \tfrac{p_i}{q_j}\big||E \cap X_j| - q_j|E|\big| + \big||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j|\big|.$$

In particular,

$$\mathrm{wd}(\mathscr{H}, c, p) \leqslant \max_{j\in[2], i\in C_j}\Big(\tfrac{p_i}{q_j}\mathrm{wd}(\mathscr{H}, 2, (q_1, q_2)) + \max_{\mathscr{H}_0 \leqslant \mathscr{H}} \mathrm{wd}\big(\mathscr{H}_0, |C_j|, \tfrac{1}{q_j}p_{|C_j}\big)\Big).$$

Proof. Let $j \in [2]$, $i \in C_j$, $E \in \mathscr{E}$. Then

$$\begin{aligned}
&\big||E \cap \chi^{-1}(i)| - p_i|E|\big| \\
&\quad = \big||E \cap X_j \cap \chi_j^{-1}(i)| - p_i|E|\big| \\
&\quad \leqslant \big||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j|\big| + \big|\tfrac{p_i}{q_j}|E \cap X_j| - p_i|E|\big|.
\end{aligned}$$

If the χ_j, $j = 0, 1, 2$ are chosen such that $\big||E \cap X_j| - q_j|E|\big| \leqslant \mathrm{wd}(\mathscr{H}, 2, (q_1, q_2))$ and $\big||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j|\big| \leqslant \mathrm{wd}(\mathscr{H}_{|X_j}, |C_j|, \tfrac{1}{q_j}p_{|C_j})$ for all $E \in \mathscr{E}$, $j \in [2]$, $i \in C_j$, then the second claim follows from the first. □

As this section is quite lengthy, here is a short overview of what is going to come. We first analyse recursive colouring assuming that we have a uniform bound on the weighted 2-colour discrepancies of the induced subhypergraphs. We derive a first result for the weighted c-colour discrepancy and then improve it in the case of equi-weighted discrepancy. Finally we replace the uniformity assumption with the assumption that subhypergraphs on n_0 vertices have weighted discrepancy $O(n_0^\alpha)$ for some $\alpha \in]0, 1[$. With this stronger precondition we get a number of beautiful results, among them a near-tight c-colour analogue of Spencer's 'six standard deviations' theorem.

3.2. Weighted discrepancy

In the following two subsections we analyse the case when all induced subgraphs have a common bound on all weighted discrepancies in two colours. This is an important case for two reasons. Firstly, the proof of some results on 2-colour discrepancy provides some information about the weighted discrepancy of the induced subgraphs (*e.g.*, in the Beck–Fiala setting). Secondly, the linear discrepancy and thus also the weighted discrepancies of all subgraphs are bounded by the hereditary discrepancy.

Remark 2. From Proposition 2.1 we deduce that, for all induced subhypergraphs $\mathscr{H}_0$ of $\mathscr{H}$, we have $\mathrm{wd}(\mathscr{H}_0, 2) \leqslant \mathrm{herdisc}(\mathscr{H})$.

Hence a bound on the hereditary discrepancy is also sufficient to apply the main theorems of these two subsections. Bounds on the hereditary discrepancy are often encountered in situations where the partial colouring method is used in the 2-colour case – for the simple reason that the uncoloured points induce a subhypergraph which has to be coloured in the next iteration.

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It will be convenient to represent the iterated partitioning of the set of colours C by a binary tree. We call a binary rooted tree $T = (V_T, E_T)$ a *partition tree* for C, if the following conditions are satisfied: the root of T is C, all nodes are subsets of C, all leaves are singletons of C and each pair of son nodes forms a partition of their common father node. For every colour $i \in C$ there is a unique path $C = C_0^{(i)} \supset C_1^{(i)} \supset \cdots \supset C_{k(i)}^{(i)} = \{i\}$ in the partition tree. We write $h(T)$ for the height of T, that is, the length of a longest path connecting a leaf and the root.

For a colour $i \in C$ set

$$v(T,p,i) := \sum_{l=1}^{k(i)} \frac{p_i}{p\left(C_l^{(i)}\right)} \quad \text{and} \quad v(T,p) = \max_{i \in C} v(T,p,i).$$

As the next theorem shows, these constants reflect the influence of the partition tree chosen for the recursive colouring process. In Lemmas 3.3 and 3.5 we will give partition trees for which these values (and hence the resulting discrepancy) is small.

Theorem 3.2. *Let* $\mathrm{wd}(\mathscr{H}_0, 2) \leqslant K$ *for all induced subgraphs* $\mathscr{H}_0$ *of* $\mathscr{H}$. *Let* C *be a set of colours with* $c = |C|$ *and let* p *be a weight of* C. *Let* $T = (V_T, E_T)$ *be a partition tree of* C. *Then there is a colouring* $\chi : X \to C$ *such that, for all colours* $i \in C$ *and all* $E \in \mathscr{E}$, *we have*

$$\left| |E \cap \chi^{-1}(i)| - p_i |E| \right| \leqslant K v(T,p,i).$$

In particular, $\mathrm{wd}(\mathscr{H}, p, c) \leqslant K v(T,p)$.

Proof. We use induction on the height $h(T)$ of T. For $h(T) = 0$ we have just one colour, and both sides of the inequality become zero. Let T be of height $h(T) > 0$ and assume that the theorem is true for all partition trees of height strictly less than $h(T)$. Let C_1 and C_2 be the sons of C in T. Set $q_j := p(C_j) = \sum_{k \in C_j} p_k$, $j = 1, 2$. By assumption there is a 2-colouring $\chi_0 : X \to [2]$ such that

$$\left| |E \cap \chi_0^{-1}(j)| - q_j |E| \right| \leqslant \mathrm{wd}(\mathscr{H}, 2, (q_1, q_2)) \leqslant K \tag{3.1}$$

holds for all $j \in [2]$ and $E \in \mathscr{E}$. Put $X_j := \chi_0^{-1}(j)$, $j = 1, 2$. Let T_j denote the subtree having C_j as its root. Then the hypergraph $\mathscr{H}_{|X_j}$, together with the set of colours C_j, the weight $\frac{1}{q_j} p_{|C_j}$ and the partition tree T_j, fulfils the assumption of this theorem. By induction there are colourings $\chi_j : X_j \to C_j$, $j \in 1, 2$ such that

$$\left| |E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{1}{q_j} p_i |E \cap X_j| \right| \leqslant K v\left(T_j, \tfrac{1}{q_j} p_{|C_j}, i\right) \leqslant K \sum_{l=2}^{k(i)} \frac{\frac{p_i}{q_j}}{\frac{1}{q_j} p\left(C_l^{(i)}\right)} \tag{3.2}$$

for all $i \in C_j$. Set

$$\chi = \chi_1 \cup \chi_2 : x \mapsto \begin{cases} \chi_1(x) & \text{if } x \in X_1, \\ \chi_2(x) & \text{otherwise.} \end{cases}$$

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Let $j \in [2]$ and $i \in C_j$. Then $C_1^{(i)} = C_j$ and $q_j = p(C_1^{(i)})$. Let $E \in \mathscr{E}$. From (3.1), (3.2) and Lemma 3.1, we get

$$\begin{aligned}\big||E \cap \chi^{-1}(i)| - p_i|E|\big| &\leqslant \big||E \cap X_j \cap \chi_j^{-1}(i)| - \tfrac{p_i}{q_j}|E \cap X_j|\big| + \tfrac{p_i}{q_j}\big||E \cap X_j| - q_j|E|\big| \\ &\overset{(3.1),\ (3.2)}{\leqslant} \sum_{l=2}^{k(i)} \frac{\frac{p_i}{q_j}}{\frac{1}{q_j}p\big(C_l^{(i)}\big)}K + \tfrac{p_i}{q_j}K \\ &= K\sum_{l=1}^{k(i)} \frac{p_i}{p\big(C_l^{(i)}\big)} = Kv(T,p,i).\end{aligned}$$

Hence χ satisfies the claim. □

In the following corollary we give a first upper bound on the constant $v(T,p,i)$. An improvement in the case of equi-weighted discrepancy will be discussed in more detail in Section 3.3.

Lemma 3.3. *In the situation of Theorem* 3.2*, there is a partition tree* T *such that*

$$v(T,p,i) < 4$$

for all $i \in [c]$. *Thus* $\mathrm{wd}(\mathscr{H},p,c) < 4K$.

Proof. Recursively, we construct a partition tree T for C with $v(T,p) \leqslant 4$. We start with the tree consisting of the unique node C. For a leaf C_0 of cardinality greater than 1 let us define sons by the following rule. If there is a colour $i \in C_0$ with weight $p_i \geqslant \frac{1}{2}p(C_0)$, then the sons of C_0 will be $\{i\}$ and $C_0 \setminus \{i\}$. Otherwise partition C_0 in any way (C_1, C_2) such that $p(C_j) \in [\frac{1}{3}p(C_0), \frac{2}{3}p(C_0)]$. Repeat this process until all leaves are singletons. The resulting tree T is a partition tree for C. All father–son pairs (C_0, C_1) in the resulting tree fulfil $\frac{2}{3}p(C_0) \geqslant p(C_1)$ or $|C_1| = 1$ and $p(C_0) > p(C_1)$. In the notation of Theorem 3.2 we have $p(C_{k(i)}^{(i)}) = p_i$, $p(C_{k(i)-1}^{(i)}) \geqslant p_i$ and $p(C_{k(i)-1-l}^{(i)}) \geqslant (\frac{3}{2})^l p_i$ for all $l \in [k(i)-1]$. Now

$$v(T,p) \leqslant \max_{i\in C} \sum_{l=1}^{k(i)} \frac{p_i}{p\big(C_l^{(i)}\big)} \leqslant \max_{i\in C} p_i \left(\frac{1}{p_i} + \sum_{l=0}^{k(i)-2} \frac{1}{\big(\frac{3}{2}\big)^l p_i}\right) \leqslant 1 + \sum_{l=0}^{h(T)-2} \big(\tfrac{2}{3}\big)^l < 4,$$

and Theorem 3.2 gives the bound $\mathrm{wd}(\mathscr{H},p,c) \leqslant 4K$. □

3.3. Equi-weighted discrepancy

In this subsection we consider the case of equi-weighted discrepancy in c colours. Hence our assumptions are identical to the ones from the preceding subsection, except that we always have $p = \frac{1}{c}\mathbf{1}_c$. In this case only the size of the colour sets is important, as all colours are equivalent. Therefore the following simpler structure can be investigated.

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A *partition tree* for a positive integer n is a binary tree $T = (V_T, E_T)$ together with a labelling $l : V_T \to [n]$ such that the following conditions are satisfied.

- The root r is labelled $l(r) = n$.
- For every non-leaf v with sons s_1 and s_2, we have $l(v) = l(s_1) + l(s_2)$.
- The leaves are labelled 1.

Note that we can no longer assume l to be injective. For a path $P : r = v_0^{(i)}, v_1^{(i)}, \ldots, v_{k(i)}^{(i)}$ connecting the root r and a leaf $v_{k(i)}^{(i)}$ labelled i, we call

$$v(T,P) = \sum_{l=1}^{k(i)} \frac{1}{l\left(v_l^{(i)}\right)}$$

the value of P and $v(T)$ the maximum $v(T,P)$ over all these paths P. Finally, $v(n)$ is the minimum $v(T)$ over all partition trees T of n.

There is a natural correspondence between partition trees for sets of colours and for positive integers. Let $T = (V_T, E_T)$ denote a partition tree for the set of colours C. Define a labelling $l_T : V_T \to [|C|]; v \mapsto |v|$. Then T together with l_T is a partition tree for $|C|$.

Now let T together with l denote a partition tree for a positive integer c. Let C be any set of colours such that $|C| = c$. We construct a partition tree T^* for C such that $l_{T^*} = l$. Define $f : V_T \to 2^{[c]}$ recursively: set $f(r) = C$ for the root r of T. For every node v with sons s_1 and s_2 such that $f(v)$ is already defined, choose $f(s_1)$ to be any subset of $f(v)$ of size $l(s_1)$ and $f(s_2) = f(v) \setminus f(s_1)$. Note that f is injective, and by replacing every $v \in V_T$ by $f(v)$ we get a partition tree T^* for C. Clearly, $l_{T^*} = l$.

Furthermore, we have

$$v\left(T^*, \tfrac{1}{c}\mathbf{1}_c\right) = \max_{i \in C} \tfrac{1}{c} \sum_{l=1}^{k(i)} \frac{1}{\frac{1}{c}\left|C_l^{(i)}\right|} = \max_{i \in C} \sum_{l=1}^{k(i)} \frac{1}{l\left(v_l^{(i)}\right)} = v(T).$$

Corollary 3.4. *Let* $\mathrm{wd}(\mathscr{H}_0, 2) \leqslant K$ *for all induced subgraphs* $\mathscr{H}_0$ *of* $\mathscr{H}$. *Then* $\mathrm{disc}(\mathscr{H}, c) \leqslant v(c)K$.

Proof. Let $T = (V_T, E_T)$ together with l be a partition tree for c such that $v(T) = v(c)$. We build T^* as above and apply Theorem 3.2 on T^* and $p = \frac{1}{c}\mathbf{1}_c$:

$$\mathrm{disc}(\mathscr{H}, c) = \mathrm{wd}(\mathscr{H}, p, c) \leqslant Kv(T^*, p) = Kv(T) = Kv(c). \qquad \square$$

The exact calculation of $v(c)$ seems to be a difficult task. In particular, the optimal partition trees are generally not of minimal height. Put $\lfloor c \rfloor_2 := 2^{\lfloor \log_2 c \rfloor}$ and $\lceil c \rceil_2 := 2^{\lceil \log_2 c \rceil}$. Let $n_1(c)$ denote the number of ones in the binary expansion of c (*e.g.*, $n_1(9) = 2$). We give a lower bound and an upper bound on $v(c)$. If c is a power of 2, both bounds coincide.

Lemma 3.5. *For all* $c \in \mathbb{N}$, $c \geqslant 2$, *we have*

$$2 - \tfrac{2}{\lceil c \rceil_2} \leqslant v(c) \leqslant 2 + (n_1(c) - 3)\tfrac{1}{\lceil c \rceil_2}.$$

In particular, $v(c) \leqslant 2.0005$.

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Proof. Let $T = (V_T, E_T)$ together with l be any partition tree for c. Then there is a path $v_0, \dots, v_k$ of length $k \geqslant \log_2 \lceil c \rceil_2$ such that v_k is a leaf and $l(v_{i-1}) \leqslant 2l(v_i)$ for all $i \in [k]$. Thus

$$\sum_{i=1}^{k} \frac{1}{l(v_i)} \geqslant \sum_{i=0}^{k-1} 2^{-i} = 2 - \frac{1}{2^{k-1}} \geqslant 2 - \frac{1}{\lceil c \rceil_2}.$$

For the upper bound we recursively construct a partition tree T for c. For a vertex v labelled $\sum_{i\in[k]} a_i 2^k \neq 1$, $a_i \in \{0,1\}$, we add sons $s_1(v)$ and $s_2(v)$ labelled $l(s_1(v)) = 2^{\min\{i\in[k]\mid a_i=1\}}$ and $l(s_2(v)) = l(v) - l(s_1(v))$, if $l(v)$ is not a power of two, and labelled $l(s_1(v)) = l(s_2(v)) = \frac{1}{2}l(v)$ otherwise. Immediately we see that we only need to investigate the path $P : r, s_2(r), s_2(s_2(r)), \dots$ (if r denotes the root of T), because the labels of all other paths occur also on this path. Thus $v(P)$ is maximal. The labels of the first $n_1(c)$ vertices of P are greater than or equal to $\lfloor c \rfloor_2$, so their contribution to $v(P)$ is not greater than $(n_1(c) - 1)\frac{1}{\lfloor c \rfloor_2}$. The rest of the vertices are labelled by $\frac{2}{\lfloor c \rfloor_2}, \frac{4}{\lfloor c \rfloor_2}, \dots$ up to 1. This sums up to $2 - \frac{2}{\lfloor c \rfloor_2}$ and the inequality is proved.

The last assertion is clear for $c \geqslant c_0 := 2^{15} - 1$, as

$$(n_1(c) - 3)\frac{1}{\lfloor c \rfloor_2} \leqslant \frac{\log_2(\lfloor c \rfloor_2) - 2}{\lfloor c \rfloor_2} \leqslant \frac{\log_2(\lfloor c_0 \rfloor_2) - 2}{\lfloor c_0 \rfloor_2}.$$

For the remaining small numbers, $v(c)$ can be computed in $O(c^2)$-time and attains its maximum value for $c = 909$, namely $v(909) \approx 2.000450$. □

Now Corollary 3.4 and Lemma 3.5 yield the following result.

Theorem 3.6. *Let* $\mathrm{wd}(\mathscr{H}_0, 2) \leqslant K$ *for all induced subgraphs* $\mathscr{H}_0$ *of* $\mathscr{H}$. *Then* $\mathrm{disc}(\mathscr{H}, c) \leqslant 2.0005K$ *holds for any number* c *of colours.* □

We apply Theorem 3.6 in the Beck–Fiala setting, and get the following.

Theorem 3.7. *For any hypergraph* $\mathscr{H}$ *we have*

$$\mathrm{disc}(\mathscr{H}, c) < v(c)\,\Delta(\mathscr{H}) \leqslant 2.0005\,\Delta(\mathscr{H}).$$

Proof. The Beck–Fiala theorem states that $\mathrm{lindisc}(\mathscr{H}) < 2\Delta(\mathscr{H})$ holds for any hypergraph $\mathscr{H}$. In particular, we have $\mathrm{wd}(\mathscr{H}_0, 2) \leqslant \frac{1}{2}\,\mathrm{lindisc}(\mathscr{H}_0) < \Delta(\mathscr{H}_0) \leqslant \Delta(\mathscr{H})$ for all induced subhypergraphs $\mathscr{H}_0$ of $\mathscr{H}$. From Corollary 3.4 and Lemma 3.5 we conclude $\mathrm{disc}(\mathscr{H}, c) \leqslant v(c)\mathrm{wd}(\mathscr{H}, 2) < 2.0005\Delta$. □

A similar result is proved in Section 4. Note that Theorem 3.6 also yields the following bounds.

- For any hypergraph $\mathscr{H} = (V, \mathscr{E})$ with $n := |V| = |\mathscr{E}|$ sufficiently large, we have

$$\mathrm{disc}(\mathscr{H}, c) < 12\sqrt{n}.$$

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- Let $\mathscr{H} = (V, \mathscr{E})$ be a hypergraph on n points. Let $d > 1$. If $\pi_{\mathscr{H}} = O(m^d)$, then $\operatorname{disc}(\mathscr{H}, c) = O(n^{\frac{1}{2} - \frac{1}{2d}})$. If $\pi^*_{\mathscr{H}} = O(m^d)$, then $\operatorname{disc}(\mathscr{H}, c) = O(n^{\frac{1}{2} - \frac{1}{2d}} \log n)$. In both cases the implicit constants are independent of c.
- The hypergraph $\mathscr{A}_n$ of arithmetic progressions in $[n]$ fulfils

$$\operatorname{disc}(\mathscr{A}_n, c) \leqslant v(c) C \sqrt[4]{n} \leqslant 2.0005 C \sqrt[4]{n},$$

where C is the constant of Matoušek and Spencer such that $\operatorname{disc}(\mathscr{A}_n) \leqslant C\sqrt[4]{n}$.

Using the fact that in these cases the discrepancies of smaller induced subhypergraphs are decreasing, we improve these bounds in the next two subsections.

The following example shows that the recursive approach is nearly optimal in the general case. Let $n = kc$ for some $k \in \mathbb{N}$. Set $\mathscr{H} = ([n], \binom{[n]}{k}) = ([n], \{E \subseteq [n] \mid |E| = k\})$. Any c-colouring for $\mathscr{H}$ produces a monochromatic hyperedge which has discrepancy $k(1 - \frac{1}{c})$. Hence $\operatorname{disc}(\mathscr{H}, c) \geqslant k(1 - \frac{1}{c})$. Now let $(p, 1-p)$ be any 2-colour weight. Assume without loss of generality that $p \leqslant \frac{1}{2}$. Put $\chi : [n] \to [2]; i \mapsto 2$. Now each hyperedge has weighted discrepancy $k(1-p)$ with respect to χ and $(p, 1-p)$. Thus $\operatorname{wd}(\mathscr{H}, c) \leqslant \frac{1}{2}k$, and of course this holds as well for any induced subhypergraph $\mathscr{H}_0$ of $\mathscr{H}$. This shows

$$\operatorname{disc}(\mathscr{H}, c) \geqslant 2\left(1 - \tfrac{1}{c}\right) \max_{\mathscr{H}_0 \leqslant \mathscr{H}} \operatorname{wd}(\mathscr{H}_0, 2).$$

In particular, the recursive method yields optimal colourings in this case if c is a power of 2, and it is asymptotically optimal for $c \to \infty$.

3.4. Refined recursive colouring

In this subsection we extend the recursive approach to make use of the additional assumption that subhypergraphs on fewer vertices have smaller discrepancy. This is a natural assumption as many results are of this type (see Section 3.5, where we prove their multicolour analogies).

Roughly speaking, we show that if the 2-colour discrepancy of the subhypergraphs on n_0 vertices is bounded by $O(n_0^\alpha)$, then the c-colour discrepancy is bounded by $O((\frac{n}{c})^\alpha)$. It seems a little surprising that this bound is achievable by a recursive approach, as the first step in the recursion will find a 2-colouring for the whole hypergraph with discrepancy guarantee $O(n^\alpha)$ only. We still get the $O((\frac{n}{c})^\alpha)$-discrepancy for the final colouring because imbalances inflicted in earlier rounds of the recursion are split up in a balanced manner by later steps (*cf.* Lemma 3.1). It turns out that this effect even exceeds the effect of decreasing discrepancy of smaller subhypergraphs. The last step of the recursion, where we search for colourings for hypergraphs on roughly $\frac{2n}{c}$ vertices, is therefore crucial.

There are two points, however, that need further attention. Firstly, as in the case where we only assumed a uniform bound on the discrepancies of the induced subhypergraphs, this simple approach only works if the number of colours is a power of 2. This is the reason why we have to use weighted discrepancies again.

A second point is that, to use the assumption of decreasing discrepancies, we need to make sure that the vertex sets considered actually become smaller. Unfortunately, in general we do not know the size of the colour classes generated by a low-discrepancy colouring. If the whole vertex set is a hyperedge, we know at least that the sizes of the

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colour classes deviate from the desired value by at most the discrepancy guarantee. This is not too bad if the discrepancy is relatively small, but even then, keeping track of these deviations during the recursion is tedious. Better bounds seem achievable by the cleaner approach of only investigating *fair* colourings, that is, those which have discrepancy less than one on the set of all vertices.

To ease notation let us define the following. Let $p \in [0,1]^c$ be a c-colour weight and $\mathscr{H} = (X, \mathscr{E})$ a hypergraph. We say that χ *is a fair* p*-colouring of* $\mathscr{H}$ *having discrepancy at most* d_i *in colour* $i \in [c]$ to denote that

- χ is a c-colouring of $\mathscr{H}$;
- χ is fair with respect to p, that is, for all $i \in [c]$ we have $||\chi^{-1}(i)| - p_i|X|| \leqslant 1$;
- the discrepancy of $\mathscr{H}$ with respect to χ and p in colour $i \in [c]$ is at most d_i.

The following remark eases work with the fractional parts. Let us call a weight $p \in [0,1]^c$ *integral* with respect to $\mathscr{H}$ (or $\mathscr{H}$-integral for short) if all $p_i, i \in [c]$ are multiples of $\frac{1}{|X|}$. From the definition it is clear that a fair colouring χ with respect to an integral weight p fulfils $|\chi^{-1}(i)| = p_i|X|$ for all colours $i \in [c]$. Suppose we know that, for a given hypergraph and for all integral weights p, there is a fair p-colouring that has discrepancy at most k. Then there are fair colourings having discrepancy at most $k+1$ for any weight: For an arbitrary weight p there is an integral weight p' such that $|p_i - p_i'| < \frac{1}{|X|}$ holds for all $i \in [c]$. Therefore, a fair colouring with respect to p' is also fair with respect to p, and its discrepancy with respect to p is larger (if at all) than the one with respect to p' by less than one. For these reasons we may restrict ourselves to the more convenient case that all weights are integral.

Using the following recolouring argument, we can transform arbitrary colourings into fair colourings.

Lemma 3.8. *Let* $\mathscr{H} = (X, \mathscr{E})$ *be a hypergraph such that* $X \in \mathscr{E}$. *Let* p *be a* 2*-colour weight. Then any* 2*-colouring* χ *of* $\mathscr{H}$ *can be modified in* $O(|X|)$ *time into a fair* p*-colouring* $\overline{\chi}$ *such that*

$$\operatorname{wd}(\mathscr{H}, \overline{\chi}, p) \leqslant 2 \operatorname{wd}(\mathscr{H}, \chi, p).$$

Proof. Let χ be a colouring such that $\operatorname{wd}(\mathscr{H}, \chi, p) = \operatorname{wd}(\mathscr{H}, c, p)$. Set $x := q|X| - |\chi^{-1}(1)|$. Since X is an edge in $\mathscr{H}$, $|x| \leqslant \operatorname{wd}(\mathscr{H}, c, p)$. Let $\overline{\chi}$ denote a colouring arising from χ by changing the colour of $\lfloor |x| \rfloor$ points in such a way that $|q|X| - |\overline{\chi}^{-1}(1)|| < 1$. Now $\overline{\chi}$ is a fair colouring with respect to the weight $(q, 1-q)$. For an edge $E \in \mathscr{E}$, we compute

$$\begin{aligned}
&|\, q|E| - |\overline{\chi}^{-1}(1) \cap E|| \\
&\quad \leqslant |\, q|E| - |\chi^{-1}(1) \cap E|| + ||\chi^{-1}(1) \cap E| - |\overline{\chi}^{-1}(1) \cap E|| \\
&\quad \leqslant |\, q|E| - |\chi^{-1}(1) \cap E|| + \lfloor |x| \rfloor \\
&\quad \leqslant 2 \operatorname{wd}(\mathscr{H}, c, p).
\end{aligned}$$

□

Lemma 3.8 requires the whole vertex set to be a hyperedge. Fortunately, most discrepancy results are relatively robust concerning the addition of a single hyperedge. In these cases we may just replace the hypergraph under consideration by the one obtained from adding X as additional edge.

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To analyse our recursive algorithm we need the following constants. Let $\alpha \in]0,1[$. For each $p \in]0,1[$ define $v_\alpha(p)$ by

$$v_\alpha(p) = \max\left\{ \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j \,\middle|\, k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \tfrac{2}{3}\right], q_k \in [0,1], \prod_{j=1}^{k} q_j = p \right\}.$$

Set

$$c_\alpha := \frac{2}{2^{1-\alpha}-1}\left(1 + \frac{1}{1-\left(\frac{2}{3}\right)^{(1-\alpha)}}\right).$$

Then we have the following.

Lemma 3.9. *Let* $\alpha \in]0,1[$.

(i) *Let* $0 < p < q \leqslant \frac{2}{3}$. *Then* $q^\alpha v_\alpha(\frac{p}{q}) + q^\alpha \frac{p}{q} \leqslant v_\alpha(p)$.
(ii) *For all* $p \in [0,1]$, $\frac{2}{2^{1-\alpha}-1} v_\alpha(p) \leqslant c_\alpha p^\alpha$.

Proof. Let $k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \frac{2}{3}\right], q_k \in [0,1]$ such that $\prod_{j=1}^{k} q_j = \frac{p}{q}$ and

$$v_\alpha\left(\tfrac{p}{q}\right) = \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j.$$

With $q_0 := q$ we have

$$\begin{aligned} q^\alpha v_\alpha\left(\tfrac{p}{q}\right) + q^\alpha \tfrac{p}{q} &= q_0^\alpha \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j + q_0^\alpha \prod_{j=1}^{k} q_j \\ &= \sum_{i=0}^{k} \prod_{j=0}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j \leqslant v_\alpha(p), \end{aligned}$$

since $\prod_{j=0}^{k} q_j = q \prod_{j=1}^{k} q_j = p$. This is (i).

Let $k \in \mathbb{N}, q_1, \ldots, q_{k-1} \in \left[0, \frac{2}{3}\right], q_k \in [0,1]$ such that $\prod_{j=1}^{k} q_j = p$ and

$$v_\alpha(p) = \sum_{i=1}^{k} \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j.$$

For $i \in [k]$ set $x_i := \prod_{j=1}^{i} q_j^\alpha \prod_{j=i+1}^{k} q_j$. Then $x_k = p^\alpha$ and $x_{k-1} \leqslant x_k$. For $i \in [k-2]$ we have

$$\frac{x_i}{x_{i+1}} = \frac{q_{i+1}}{q_{i+1}^\alpha} = q_{i+1}^{1-\alpha} \leqslant \left(\tfrac{2}{3}\right)^{1-\alpha},$$

and hence $x_{k-1-i} \leqslant \left(\frac{2}{3}\right)^{(1-\alpha)i} x_k$. Thus

$$\tfrac{2}{2^{1-\alpha}-1} v_\alpha(p) = \tfrac{2}{2^{1-\alpha}-1} \sum_{i=1}^{k} x_i \leqslant \tfrac{2}{2^{1-\alpha}-1} \left(1 + \sum_{i=0}^{k-2} \left(\tfrac{2}{3}\right)^{(1-\alpha)i}\right) x_k < c_\alpha p^\alpha. \qquad \square$$

Here is the precise setting we investigate in this section.

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Assumption (Decreasing Discrepancies Assumption). Let $\mathscr{H} = (X, \mathscr{E})$ be a hypergraph. Set $n := |X|$. Let $p_0, \alpha \in]0,1[$ and $D > 0$. For all $X_0 \subseteq X$ such that $|X_0| \geqslant p_0|X|$ and all $q \in [0,1]$ such that $(q, 1-q)$ is $\mathscr{H}_{|X_0}$-integral, there is a fair $(q, 1-q)$-colouring χ of $\mathscr{H}_{|X_0}$ having discrepancy at most $D|X_0|^\alpha$.

In addition to what we already explained, there is one further detail involved in our assumption. As we do recursive partitioning, we never need a discrepancy result concerning induced subhypergraphs on fewer than roughly $\frac{n}{c}$ vertices (in the equi-weighted case). This observation will be useful in some applications, *e.g.*, in the case $|\mathscr{E}| = |X|$.

Concerning computational complexity, there are two possible measures. We can count how many 2-colourings have to be computed, or how often a 2-colouring for a vertex has to be found. The latter is useful if the complexity of computing the 2-colourings is proportional to the number of vertices of the induced subhypergraph as in Theorem 3.13.

Theorem 3.10. *Suppose that the Decreasing Discrepancies Assumption holds. Then, for each $\mathscr{H}$-integral weight $p \in [0,1]^c$, there is a fair p-colouring χ of $\mathscr{H}$ such that the discrepancy is at most $\frac{2}{2^{1-\alpha}-1} D v_\alpha(p_i) n^\alpha \leqslant D c_\alpha(p_i n)^\alpha$ in all those colours $i \in [c]$ such that $p_i \geqslant p_0$. Such colourings can be obtained by computing at most $(c-1)\lceil \log_2(\frac{1}{p_0}) \rceil$ colourings, as in the Decreasing Discrepancies Assumption. At most $3n \log_{1.5}(\frac{1}{p_0})$ times a colour for a vertex has to be computed.*

For the proof we first show a stronger bound for the 2-colour discrepancy with respect to a weight $(q, 1-q)$, if q is small.

Lemma 3.11. *Suppose that the Decreasing Discrepancies Assumption holds. Then, for each $k \in \mathbb{N}$ such that $p = (2^{-k}, 1-2^{-k})$ is an $\mathscr{H}$-integral weight and $2^{-k} \geqslant p_0$, a fair p-colouring χ having discrepancy at most*

$$\mathrm{wd}(\mathscr{H}, \chi, p) \leqslant \sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} D n^\alpha$$

can be computed from k colourings, as in the Decreasing Discrepancies Assumption. This requires $\sum_{i=0}^{k-1} 2^{-i} n \leqslant 2n$ times computing a colour for a vertex.

Proof. We proceed by induction. For $k = 1$, there is nothing to show. Let $k > 1$. Let $\chi_0 : X \to [2]$ be a fair $(0.5, 0.5)$-colouring having discrepancy at most Dn^α. Set $X_1 := \chi_0^{-1}(1)$. Let $\chi_1 : X_1 \to [2]$ be a fair $(2^{-k+1}, 1-2^{-k+1})$-colouring. Note that $(2^{-k+1}, 1-2^{-k+1})$ is integral for $\mathscr{H}_{|X_1}$. By induction we may assume that χ_1 has discrepancy at most $\sum_{i=0}^{k-2} 2^{-k+2+i} 2^{-\alpha i} D(\frac{n}{2})^\alpha$. Define a colouring $\chi : X \to [2]$ by $\chi(x) = 1$ if and only if $\chi_0(x) = 1$ and $\chi_1(x) = 1$. Then χ is a fair $(2^{-k}, 1-2^{-k})$-colouring. Using a similar argument as in Lemma 3.1, we compute the discrepancy of an edge $E \in \mathscr{E}$ with respect to$(2^{-k}, 1-2^{-k})$

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in colour 1:

$$\begin{aligned}
&\big||E\cap\chi^{-1}(1)|-2^{-k}|E|\big|\\
&\quad=\big||E\cap\chi_0^{-1}(1)\cap\chi_1^{-1}(1)|-2^{-k}|E|\big|\\
&\quad\leqslant\big||E\cap\chi_0^{-1}(1)\cap\chi_1^{-1}(1)|-2^{-k+1}|E\cap\chi_0^{-1}(1)|\big|+\big|2^{-k+1}|E\cap\chi_0^{-1}(1)|-2^{-k}|E|\big|\\
&\quad\leqslant\big||(E\cap X_1)\cap\chi_1^{-1}(1)|-2^{-k+1}|E\cap X_1|\big|+2^{-k+1}\big||E\cap\chi_0^{-1}(1)|-0.5|E|\big|\\
&\quad\leqslant\sum_{i=0}^{k-2}2^{-k+2+i}2^{-\alpha i}D\left(\tfrac{n}{2}\right)^{\alpha}+2^{-k+1}Dn^{\alpha}\\
&\quad=\sum_{i=0}^{k-1}2^{-k+1+i}2^{-\alpha i}Dn^{\alpha}.
\end{aligned}$$

As 2-colourings have the same discrepancy in both colours, this proves Lemma 3.11. □

From our assumptions on $\mathscr{H}$ it is clear that the assertion of Lemma 3.11 also holds for any induced subhypergraph $\mathscr{H}_{|X_0}$ of $\mathscr{H}$ as long as $2^{-k}|X_0|\geqslant p_0|X|$. We use this fact to extend Lemma 3.11 to arbitrary weights.

Lemma 3.12. *Suppose that the Decreasing Discrepancies Assumption holds. For each $\mathscr{H}$-integral weight $(q,1-q)$, $p_0\leqslant q\leqslant\frac{1}{2}$, there is a fair $(q,1-q)$-colouring χ having discrepancy at most*

$$\operatorname{wd}(\mathscr{H},\chi,p)\leqslant\frac{2}{2^{1-\alpha}-1}D\,(qn)^{\alpha}.$$

A colouring of this kind can be computed by $\lceil\log_2(\frac{1}{q})\rceil$ times computing a colouring as in the Decreasing Discrepancies Assumption. This requires at most $3n$ times computing a colour for a vertex.

Proof. Let $k\in\mathbb{N}_0$ be maximal subject to the condition that $q'=2^kq\leqslant 1$. Since $(q,1-q)$ is $\mathscr{H}$-integral, so is $(q',1-q')$. According to the Decreasing Discrepancies Assumption there is a fair $(q',1-q')$-colouring $\chi_0:X\to[2]$ having discrepancy at most Dn^{α}. From $|\chi_0^{-1}(1)|=q'|X|$ we have $\frac{q}{q'}|\chi_0^{-1}(1)|=q|X|\in\mathbb{N}_0$. Hence $(\frac{q}{q'},1-\frac{q}{q'})$ is $(\mathscr{H}_{|\chi_0^{-1}(1)})$-integral. By Lemma 3.11 we may compute a fair $(\frac{q}{q'},1-\frac{q}{q'})$-colouring $\chi_1:\chi_0^{-1}(1)\to[2]$ that has discrepancy at most $\sum_{i=0}^{k-1}2^{-k+1+i}2^{-\alpha i}D(q'n)^{\alpha}$. Define a colouring $\chi:X\to[2]$ by $\chi(x)=1$ if and only if $\chi_0(x)=1$ and $\chi_1(x)=1$. Then χ is a fair $(q,1-q)$-colouring. For an edge $E\in\mathscr{E}$ we compute its discrepancy in colour 1:

$$\begin{aligned}
&\big||E\cap\chi^{-1}(1)|-q|E|\big|\\
&\quad=\big||E\cap\chi_0^{-1}(1)\cap\chi_1^{-1}(1)|-q|E|\big|\\
&\quad\leqslant\big||E\cap\chi_0^{-1}(1)\cap\chi_1^{-1}(1)|-\tfrac{q}{q'}|E\cap\chi_0^{-1}(1)|\big|+\big|\tfrac{q}{q'}|E\cap\chi_0^{-1}(1)|-q|E|\big|\\
&\quad=\big||E\cap\chi_0^{-1}(1)\cap\chi_1^{-1}(1)|-2^{-k}|E\cap\chi_0^{-1}(1)|\big|+2^{-k}\big||E\cap\chi_0^{-1}(1)|-q'|E|\big|\\
&\quad\leqslant\sum_{i=0}^{k-1}2^{-k+1+i}2^{-\alpha i}D(q'n)^{\alpha}+2^{-k}Dn^{\alpha}
\end{aligned}$$

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$$< \sum_{i=0}^{k-1} 2^{-k+1+i} 2^{-\alpha i} D(q'n)^\alpha + 2^{-k} 2^\alpha D(q'n)^\alpha$$
$$< 2q'^\alpha \frac{2^{-\alpha k}}{2^{1-\alpha}-1} Dn^\alpha = \frac{2}{2^{1-\alpha}-1} D(qn)^\alpha.$$

Note that if $q' = 1$, then we may compute χ directly using Lemma 3.11. Therefore the computation of χ requires $\lceil \log_2(\frac{1}{q}) \rceil$ times computing a colouring assured by the Decreasing Discrepancies Assumption. Computing χ_0 means computing a colour for n vertices. By Lemma 3.11, χ_1 can be computed by at most $2q'n$ times computing a colour for a vertex. To get χ we therefore computed at most $3n$ times a colour for a vertex. This proves Lemma 3.12. □

Proof of Theorem 3.10. To make the recursion work properly we need to fix a set C of colours at the beginning. A weight then is a vector $p = (p_i)_{i \in C}$ indexed by the colours, or, more formally, a function $p : C \to [0,1]$, such that $\|p\|_1 = \sum_{i \in C} p_i = 1$. To avoid trivial cases we shall always assume that no colour $i \in C$ has the weight $p_i = 0$.

We analyse the following recursive algorithm.

Input: A hypergraph $\mathscr{H} = (X, \mathscr{E})$ fulfilling the Decreasing Discrepancies Assumption, a set C of at least 2 colours and an $\mathscr{H}$-integral weight function $p : C \to [0,1]$.

Output: A colouring $\chi : X \to C$ as in Theorem 3.10.

(1) Choose a partition $\{C_1, C_2\}$ of the set of colours C such that $\|p_{|C_1}\|_1, \|p_{|C_2}\|_1 \leqslant \frac{2}{3}$ or C_1 contains a single colour with weight at least $\frac{1}{3}$. Set $(q_1, q_2) := (\|p_{|C_1}\|_1, \|p_{|C_2}\|_1)$.

(2) Following Lemma 3.12, compute a fair (q_1, q_2)-colouring $\chi_0 : X \to [2]$ that has discrepancy at most $\frac{2}{2^{1-\alpha}-1} D(q_i n)^\alpha$ in colour $i = 1, 2$ if $q_i \geqslant p_0$. Set $X_i := \chi^{-1}(i)$ for $i = 1, 2$.

(3) For $i = 1, 2$ do

if: $|C_i| > 1$,

then: by recursion compute a fair $\frac{1}{q_i} p_{|C_i}$-colouring $\chi_i : X_i \to C_i$ for $\mathscr{H}_{|X_i}$ having discrepancy at most $\frac{2}{2^{1-\alpha}-1} Dv_\alpha(\frac{p_j}{q_i})(q_i n)^\alpha$ in each colour $j \in C_i$ such that $p_j \geqslant p_0$

else: if $C_i = \{j\}$ for some $j \in C$, choose $\chi_i : X_i \to \{j\}$ as the constant mapping.

(4) Return $\chi : X \to C$ defined by $\chi(x) := \chi_1(x)$, if $x \in X_1$, and $\chi(x) := \chi_2(x)$, if $x \in X_2$, for all $x \in X$.

We prove that our algorithm produces a colouring as claimed in Theorem 3.10 and also fulfils the complexity statements. Suppose by induction that this holds for sets of less than c colours. We analyse the algorithm starting from an input as described above with $|C| = c$.

We first show correctness. For step (1) note that both C_1 and C_2 are non-empty and that $q_2 \leqslant \frac{2}{3}$ holds. Therefore, by Lemma 3.12 and induction the colourings χ_i, $i = 0, 1, 2$ can be computed as desired in steps (2) and (3). Let $E \in \mathscr{E}$, $i \in [2]$ and $j \in C_i$ such that $p_j \geqslant p_0$. If $|C_i| > 1$, then

$$\begin{aligned} &\big||E \cap \chi^{-1}(j)| - p_j|E|\big| \\ &= \big||E \cap \chi_0^{-1}(i) \cap \chi_i^{-1}(j)| - p_j|E|\big| \\ &\leqslant \big||E \cap \chi_0^{-1}(i) \cap \chi_i^{-1}(j)| - \tfrac{p_j}{q_i}|E \cap \chi_0^{-1}(i)|\big| + \big|\tfrac{p_j}{q_i}|E \cap \chi_0^{-1}(i)| - p_j|E|\big| \end{aligned}$$

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$$\leqslant \left||(E\cap X_i)\cap\chi_i^{-1}(j)| - \tfrac{p_j}{q_i}|E\cap X_i|\right| + \tfrac{p_j}{q_i}\left||E\cap\chi_0^{-1}(i)| - q_i|E|\right|$$
$$\leqslant \tfrac{2}{2^{1-\alpha}-1}Dv_\alpha\left(\tfrac{p_j}{q_i}\right)(q_in)^\alpha + \tfrac{p_j}{q_i}\tfrac{2}{2^{1-\alpha}-1}D(q_in)^\alpha$$
$$\leqslant \tfrac{2}{2^{1-\alpha}-1}Dv_\alpha(p_j)n^\alpha,$$

by Lemma 3.9(i). On the other hand, if C_i contains a single colour j, then $p_j = q_i$ and

$$\left||E\cap\chi^{-1}(j)| - p_j|E|\right| = \left||E\cap\chi_0^{-1}(i)| - q_i|E|\right|$$
$$\leqslant \tfrac{2}{2^{1-\alpha}-1}D(q_in)^\alpha$$
$$\leqslant \tfrac{2}{2^{1-\alpha}-1}Dv_\alpha(p_j)n^\alpha.$$

This is the correctness statement.

Concerning the complexity, note that the computation of χ_0 takes at most $\lceil\log_2(\frac{1}{p_0})\rceil$ and (by induction) that of the χ_i takes at most $(|C_i|-1)\lceil\log_2(\frac{q_i}{p_0})\rceil$ colourings, as in the Decreasing Discrepancies Assumption. These comprise no more than $(c-1)\lceil\log_2(\frac{1}{p_0})\rceil$ colourings altogether.

By Lemma 3.12 we compute a colour at most $3n$ times for a vertex in step (2). If $|C_i|>1$ for both $i=1,2$, then $q_i\leqslant\frac{2}{3}$, and computing χ_i involves at most $3q_in\log_{1.5}(\frac{q_i}{p_0})\leqslant 3q_in\log_{1.5}(\frac{2}{3p_0})$ times computing a colour for a vertex. Altogether this makes at most $3n+3q_1n\log_{1.5}(\frac{q_1}{p_0})+3q_2n\log_{1.5}(\frac{q_2}{p_0})\leqslant 3n\big(1+\log_{1.5}(\frac{2}{3p_0})\big)=3n\log_{1.5}(\frac{1}{p_0})$ times computing a colour for a vertex. If $|C_i|=1$ then there is nothing to do to get χ_i, and the respective term just vanishes in the calculation above. □

3.5. Applications of the refined recursive colouring approach

We are now ready to prove c-colour versions of a series of discrepancy results.

3.5.1. General hypergraphs. Let $\mathscr{H}=(X,\mathscr{E})$ denote an arbitrary hypergraph. Set $n:=|X|$ and $m:=|\mathscr{E}|$ for convenience. The approach of Proposition 2.5 shows that a random colouring generated by colouring each vertex independently with each colour with probability $\frac{1}{c}$ has discrepancy at most

$$\sqrt{\tfrac{1}{2}n\,\ln(4mc)},$$

with probability at least $\frac{1}{2}$. This yields a randomized algorithm computing such a colouring, by repeatedly generating and testing such a random colouring until its discrepancy is at most

$$\sqrt{\tfrac{1}{2}n\,\ln(4mc)}.$$

In this subsection we show that, via the recursive approach of Theorem 3.10, a better bound can be achieved. In particular, the discrepancy tends to decrease for larger numbers of colours.

Theorem 3.13. *Let p denote an $\mathscr{H}$-integral c-colour weight. Set $p_0:=\min\{p_i|i\in[c]\}$. Then a c-colouring χ having discrepancy at most $45\sqrt{p_in\,\ln(4m)}$ in colour $i\in[c]$ can be computed*

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in expected time $O(nm \log(\frac{1}{p_0}))$. *In particular, a c-colouring* χ *such that*

$$\operatorname{disc}(\mathscr{H}, \chi, c) \leqslant 45\sqrt{\tfrac{n}{c} \ln(4m)} + 1$$

can be computed in expected time $O(nm \log c)$.

Proof. There is little to do for $m = 1$, so let us assume that $m \geqslant 2$. We show that the colourings required by the Decreasing Discrepancies Assumption can be computed in expected time $O(|X_0|m)$. Let $\overline{\mathscr{H}}$ denote the hypergraph obtained from $\mathscr{H}$ by adding the whole vertex set as an additional hyperedge. Let $X_0 \subseteq X$ and $(q, 1-q)$ be a 2-colour weight. Let $\chi : X_0 \to [2]$ be a random colouring independently colouring the vertices with probabilities $P(\chi(x) = 1) = q$ and $P(\chi(x) = 2) = 1 - q$ for all $x \in X_0$. A standard application of the Chernoff inequality (*cf.* [1]) shows that

$$(*) \qquad \operatorname{wd}(\overline{\mathscr{H}}_{|X_0}, \chi, (q, 1-q)) \leqslant \sqrt{\tfrac{1}{2}|X_0| \ln(4m)}$$

holds with probability at least $\frac{m-1}{2m}$. Hence, by repeatedly generating and testing these random colourings until $(*)$ holds, we obtain a randomized algorithm computing such a colouring with expected running time $O(nm)$. By Lemma 3.8 we get a fair $(q, 1-q)$-colouring for $\mathscr{H}_{|X_0}$ having discrepancy at most $\sqrt{2|X_0| \ln(4m)}$. Hence, for $\alpha = \frac{1}{2}$, $D = \sqrt{2 \ln(4m)}$ and arbitrary p_0, the colourings required in the Decreasing Discrepancies Assumption can be computed in expected time $O(|X_0|m)$.

Therefore we may apply Theorem 3.10 with $p_0 = \min\{p_i | i \in [c]\}$. The discrepancy bounds follow from $c_\alpha \leqslant 31.15$. Computing such a colouring involves $O(\log(\frac{1}{p_0})n)$ times computing a colour for a vertex. As this can be done in expected time $O(m)$, we have the claimed bound of $O(nm \log(\frac{1}{p_0}))$. □

The following remarks concern the theorem and its proof above. For the complexity guarantee we assumed that the complexity contribution of computing the 2-colourings dominates the remaining operations of the recursive algorithm given in the proof of Theorem 3.10. This is justified by the fact that we may assume $c \leqslant n$, since integrality ensures $p_i \geqslant \frac{1}{|X|}$ for all colours $i \in C$.

A second point is that the constant of 45 could be improved by a more careful way of generating the random 2-colourings. In particular, by taking a random fair colouring we could avoid the extra factor of 2 inflicted by Lemma 3.8. This, though, requires an analysis of the hypergeometric distribution, which is considerably more difficult than ours.

Finally let us remark that the construction of the 2-colourings can be derandomized with standard derandomization techniques like an algorithmic version of the Chernoff–Hoeffding inequality (*cf.* [32] or [31]). Thus the colourings in Theorem 3.13 can be computed by a deterministic algorithm as well.

3.5.2. Six standard deviations. The celebrated 'six standard deviations' result due to Spencer [29] states that there is a constant K such that, for all hypergraphs $\mathscr{H} = (X, \mathscr{E})$ having n vertices and $m \geqslant n$ edges, we have

$$\operatorname{disc}(\mathscr{H}) \leqslant K\sqrt{n \ln\left(\tfrac{2m}{n}\right)}.$$

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The interesting case is of course the one where $m = O(n)$ and thus $\mathrm{disc}(\mathscr{H}) = O(\sqrt{n})$. For m significantly larger than n this result is outnumbered (due to the implicit constants) by a simple random colouring. The title 'Six standard deviations suffice' of this paper comes from the fact that, for sufficiently large $n = m$, $\mathrm{disc}(\mathscr{H}) \leqslant 6\sqrt{n}$ holds.

Using the relation between discrepancies respecting a particular weight and hereditary discrepancy (Remark 2) and the recolouring argument (Lemma 3.8), we derive the following lemma from Spencer's result.

Lemma 3.14. *For any $X_0 \subseteq X$ and $\mathscr{H}_{|X_0}$-integral weight $p = (q, 1-q)$, there is a fair p-colouring χ of $\mathscr{H}_{|X_0}$ that has*

$$\mathrm{wd}(\mathscr{H}_{|X_0}, \chi, p) \leqslant 2K\sqrt{|X_0|\,\ln\left(\tfrac{2m+2}{|X_0|}\right)}.$$

Proof. Let $X_0 \subseteq X$. Then any induced subgraph of $\mathscr{H}_{|X_0}$ has discrepancy at most

$$K\sqrt{|X_0|\,\ln\left(\tfrac{2m}{|X_0|}\right)},$$

simply because Spencer's bound is monotone in the number of vertices. From Remark 2, we have

$$\mathrm{wd}(\mathscr{H}_{|X_0}, 2, (q, 1-q)) \leqslant \mathrm{herdisc}(\mathscr{H}_{|X_0}) \leqslant K\sqrt{|X_0|\,\ln\left(\tfrac{2m}{|X_0|}\right)}.$$

It remains to show the existence of a fair colouring. Let $\overline{\mathscr{H}}$ denote the hypergraph arising from $\mathscr{H}$ by adding the set X as an additional edge (unless of course $X \in \mathscr{E}$ already holds). Then $\overline{\mathscr{H}}_{|X_0}$ has at most $m+1$ edges, and from the previous paragraph we know

$$\mathrm{wd}(\overline{\mathscr{H}}_{|X_0}, 2, (q, 1-q)) \leqslant K\sqrt{|X_0|\,\ln\left(\tfrac{2m+2}{|X_0|}\right)}.$$

Lemma 3.8 now proves the claim. □

Lemma 3.14 and Theorem 3.10 yield the following.

Theorem 3.15. *Let $\mathscr{H} = (X, \mathscr{E})$ denote a hypergraph having n vertices and $m \geqslant n$ edges and $p \in [0,1]^c$ an integral weight. Set $p_0 := \min_{i \in [c]} p_i$. Then there is a fair p-colouring having discrepancy at most*

$$63K\sqrt{p_i n\,\ln\left(\tfrac{2m+2}{p_0 n}\right)}$$

in colour i.

In particular, when $|X| = |\mathscr{E}| = n$ we have

$$\mathrm{disc}(\mathscr{H}, c) \leqslant O\left(\sqrt{\tfrac{n}{c}\,\ln c}\right).$$

Proof. By Lemma 3.14 we may apply Theorem 3.10 with $\alpha = \frac{1}{2}$,

$$D = 2K\sqrt{\ln\left(\tfrac{2m+2}{p_0 n}\right)}$$

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and p_0. This yields a fair p-colouring with discrepancy at most $Dc_\alpha\sqrt{p_i n}$ in colour $i \in [c]$. The claim follows from $c_\alpha \leqslant 31.15$. □

This is quite close to the optimum. Theorem 5.2 shows a class of hypergraphs such that $|X| = |\mathscr{E}| = n$ and $\operatorname{disc}(\mathscr{H}, c) = \Omega(\sqrt{\frac{n}{c}})$. Again we should remark that we did not try to optimize the constant.

From combining Lemma 3.12 and Lemma 3.14, we derive the following.

Corollary 3.16. *Let $\mathscr{H} = (X, \mathscr{E})$ denote a hypergraph such that $|X| = |\mathscr{E}| =: n$ and $(q, 1-q)$ is an integral 2-colour weight. Assume $q \leqslant \frac{1}{2}$. Then the weighted discrepancy* $\operatorname{wd}(\mathscr{H}, 2, (q, 1-q))$ *is at most*

$$10K\sqrt{qn\,\ln\left(\tfrac{3}{q}\right)}.$$

3.5.3. Arithmetic progressions. A third classical example is the hypergraph of arithmetic progressions on the first n numbers. This is probably the most famous of the few nontrivial examples where discrepancy is well understood. For $a, d, l \in \mathbb{N}$, let $A_{adl} := \{a + id \mid 0 \leqslant i \leqslant l-1\}$ denote the arithmetic progression with starting point a, difference d and length l. Let $\mathscr{E}_n$ denote the set of all arithmetic progressions in $[n]$, that is, $\mathscr{E}_n = \{A_{adl} \cap [n] | a, d, l \in [n]\}$. Set $\mathscr{A}_n = ([n], \mathscr{E}_n)$.

Roth [27] proved the celebrated lower bound $\operatorname{disc}(\mathscr{A}_n) = \Omega(n^{1/4})$. Roth himself believed that this bound was too small and that the discrepancy actually should be close to $n^{1/2}$. This was disproved by Sárközy [28], who showed an upper bound of $O(n^{1/3+\varepsilon})$. Inventing the partial colouring method, Beck [5] showed a nearly tight bound of $O(n^{1/4}(\log n)^{5/2})$. Finally Matoušek and Spencer [25] solved the discrepancy problem for $\mathscr{A}_n$ by proving the asymptotically tight upper bound $O(n^{1/4})$.

This bound holds for any fixed number of colours. Moreover, we prove that the discrepancy decreases for larger numbers of colours.

Theorem 3.17. *For an absolute constant C' the following holds. Let $p \in [0,1]^c$ be a weight. Then there is a fair colouring of $\mathscr{A}_n$ with respect to p, having discrepancy at most $C' p_i^{0.16} n^{0.25}$ in each colour i such that $p_i \geqslant n^{0.25}$. In particular,*

$$\operatorname{disc}(\mathscr{A}_n, c) = O(c^{-0.16} n^{0.25})$$

holds for $c \leqslant n^{0.25}$ colours.

Proof. From [25, Lemma 5.3] we have that an induced subgraph $\mathscr{H}_0 = (\mathscr{A}_n)_{|X_0}$ of $\mathscr{A}_n$ on $|X_0| = \rho n \geqslant n^{0.25}$ vertices has discrepancy at most $C_1 \rho^{0.16} n^{0.25}$ for some absolute constant C_1. We first show that $\operatorname{herdisc}(\mathscr{H}_0) \leqslant 2C_1 \rho^{0.16} n^{0.25}$.

Let $\mathscr{H}_1 = (X_1, \mathscr{E}_1)$ be an induced subhypergraph of $\mathscr{H}_0$. If $|X_1| \geqslant n^{0.25}$, we are done, by Matoušek and Spencer's lemma. Let us therefore assume $|X_1| < n^{0.25}$. We show that $(\mathscr{H}_1)_{|[\frac{n}{2}]}$ and $(\mathscr{H}_1)_{|[n]\setminus[\frac{n}{2}]}$ have discrepancy at most $C_1 \rho^{0.16} n^{0.25}$, and conclude $\operatorname{disc}(\mathscr{H}_1) \leqslant$

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$2C_1\rho^{0.16}n^{0.25}$. Consider the hypergraph

$$\mathscr{H}_2 := \mathscr{H}_{(X_1\cap[\frac{n}{2}])\cup\{n-n^{0.25}+|X_1\cap[\frac{n}{2}]|+1,\dots,n\}}.$$

This hypergraph has exactly $n^{0.25} \leqslant \rho n$ vertices and thus discrepancy at most $C_1\rho^{0.16}n^{0.25}$. As every edge of $(\mathscr{H}_1)_{|[\frac{n}{2}]}$ is also an edge of $\mathscr{H}_2$, we conclude $\operatorname{disc}((\mathscr{H}_1)_{|[\frac{n}{2}]}) \leqslant C_1\rho^{0.16}n^{0.25}$. A similar argument shows

$$\operatorname{disc}\big((\mathscr{H}_1)_{|[n]\setminus[\frac{n}{2}]}\big) \leqslant C_1\rho^{0.16}n^{0.25}.$$

Thus $\operatorname{herdisc}(\mathscr{H}_0) \leqslant 2C_1\rho^{0.16}n^{0.25}$. The relation between the linear and hereditary discrepancy yields that all weighted discrepancies of $\mathscr{H}_0$ are bounded by $2C_1\rho^{0.16}n^{0.25}$. As $[n]$ is an arithmetic progression, we may apply Lemma 3.8 and conclude that twice this discrepancy may be achieved by a fair colouring respecting the underlying weight.

Thus we may apply Theorem 3.10 with $D = 4C_1n^{0.09}$, $\alpha = 0.16$ and $p_0 = n^{0.25}$, which proves our claim. □

Theorem 3.17 has a nice corollary in two colours extending Matoušek and Spencer's bound to a more general counterpart of Roth's theorem. Though often only the (ordinary) discrepancy result is cited, Roth's famous paper actually shows a lower bound for the weighted discrepancy in two colours. For all $p \in [0,1]$,

$$\operatorname{disc}(\mathscr{A}_n,(p,1-p),2) = \Omega\big(p^{1/2}n^{1/4}\big).$$

Theorem 3.17 applied to $c = 2$ shows an upper bound of

$$\operatorname{disc}(\mathscr{A}_n,(p,1-p),2) = O\big(p^{0.16}n^{1/4}\big).$$

3.5.4. Bounded shatter functions. The recursive approach also generalizes results of Matoušek, Wernisch and Welzl [26] and Matoušek [22] connecting discrepancy with the primal shatter function $\pi_{\mathscr{H}}$ and dual shatter function $\pi^*_{\mathscr{H}}$ of a hypergraph. Note that this also yields a discrepancy bound in terms of the VC-dimension $\dim(\mathscr{H})$ of $\mathscr{H}$. Vapnik and Chervonenkis [33] showed $\pi_{\mathscr{H}} \in O(n^{\dim(\mathscr{H})})$.

Theorem 3.18. *Let $\mathscr{H} = (X,\mathscr{E})$ be a hypergraph on n points. Let $d > 1$. If $\pi_{\mathscr{H}} = O(m^d)$, then $\operatorname{disc}(\mathscr{H},c) = O((\frac{n}{c})^{\frac{1}{2}-\frac{1}{2d}})$. If $\pi^*_{\mathscr{H}} = O(m^d)$, then $\operatorname{disc}(\mathscr{H},c) = O((\frac{n}{c})^{\frac{1}{2}-\frac{1}{2d}}\log n)$. In both cases the implicit constants are independent of c.*

Proof. Clearly the assumptions on the shatter functions are hereditary, in the sense that a shatter function of an induced subhypergraph is less than or equal to that of the whole hypergraph. They are also very robust: adding the whole vertex set as an additional edge changes the primal shatter function by at most 1, and does not change the dual shatter function. Without loss of generality we may therefore assume $X \in \mathscr{E}$. The remainder of the proof is standard: bound the weighted discrepancies of the induced subhypergraphs using Remark 2, buy fairness at the price of a factor of 2 (Lemma 3.8) and apply Theorem 3.10. □

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3.6. Summary: recursive colouring

We have seen that a recursive approach is very effective in situations where we can bound the weighted discrepancies of induced subhypergraphs. We always get a uniform bound from the hereditary discrepancy of $\mathscr{H}$ (Remark 2) and often find a bound decreasing for smaller induced subhypergraphs.

There are many situations where the recursive approach is the only result we have. We do not have a direct proof for a result like Theorem 3.15 or Theorem 3.17. We feel that the original proof relies heavily on the fact that only two colours are considered.

Surprisingly, the recursive approach and direct methods are sometimes nearly equally effective. An example is the (equi-weighted) multicolour discrepancy in the case of bounded degree (as in Beck and Fiala's theorem). The direct approach of Section 4 yields $\operatorname{disc}(\mathscr{H}, c) \leqslant 2\Delta(\mathscr{H})$; the recursive one gives $\operatorname{disc}(\mathscr{H}, c) \leqslant v(c)\Delta(\mathscr{H})$ for constants $v(c) \in [2(1 - \frac{1}{c}), 2.0005]$. Both ways are constructive. For c tending to infinity both methods give the same bound.

On the other hand the recursive approach is limited. We can get results on weighted discrepancies, but we do not get a nice bound on the linear discrepancy, *e.g.*, in the Beck–Fiala setting.

A second point to keep in mind is that, to apply recursion, we need a 2-colour result on the hereditary discrepancy, even when c is a power of 2. See the example in Section 2.

Recently, the first author proved a converse result [15]: if the hereditary discrepancy in c colours is bounded, we may construct c_2 colourings (and in particular 2-colourings) with low discrepancy. Together with Remark 2 and Corollary 3.4, this shows that the hereditary discrepancy in two numbers of colours deviates by at most a constant factor (depending on the numbers of colours, but not on the hypergraph).

4. Vector colouring

In this section we extend the Beck–Fiala theorem and the Bárány–Grinberg theorem to any number of colours. In the 2-colour case both are proved using 'floating colours', *i.e.*, colours initially floating in $[-1, 1]$ are successively changed to colours in $\{-1, 1\}$. Linear algebra is the key tool there. For the c-colour case we need vector colours and tensor products as well. In the Beck–Fiala situation we will derive a bound independent of the number of colours (and twice the bound of the original result), whereas in the Bárány–Grinberg case our bound is $(c - 1)$ times the original bound (and thus coincides with the original result in the case $c = 2$).

4.1. Beck–Fiala theorem

Let $\Delta(\mathscr{H}) := \max_{x \in X} |\{E \in \mathscr{E} | x \in E\}|$ denote the maximum degree of the hypergraph $\mathscr{H}$. This is one of the few parameters of a hypergraph which give a good bound on the discrepancy. The Beck–Fiala theorem states that $\operatorname{disc}(\mathscr{H}) < 2\Delta(\mathscr{H})$ for any hypergraph $\mathscr{H}$ (*cf.* [7]).

It is quite easy to see from the proof that this bound can be improved to $2\Delta(\mathscr{H}) - 2$. With more effort Martin Helm [20] further improved the bound to $2\Delta(\mathscr{H}) - 3$. The much stronger conjecture of Beck and Fiala is as follows.

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Conjecture. $\operatorname{disc}(\mathscr{H}) \in O(\sqrt{\Delta(\mathscr{H})})$.

This conjecture remains far from being proved. Good results in this direction are Srinivasan [30] and Banaszczyk [3], though neither succeeds in avoiding a logarithmic dependence of the number of vertices.

Beck and Fiala actually proved a more general result. For any matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, let $\|A\|_1 := \max_{j\in[n]} \sum_{i\in[m]} |a_{ij}|$ denote the operator norm induced by the 1-norm.

Theorem 4.1. ([7]) *For any matrix $A \in \mathbb{R}^{m\times n}$,*

$$\operatorname{lindisc}(A) < 2\|A\|_1.$$

For c colours we prove the following.

Theorem 4.2. *For any matrix $A \in \mathbb{R}^{m\times n}$,*

$$\operatorname{lindisc}(A, c) < 2\|A\|_1.$$

Note that the linear discrepancy versions of the Beck–Fiala theorem do not allow the improvements cited above. They rely on the fact that the incidence matrix of a hypergraph is a 0–1-matrix. A multicolour version of the Beck–Fiala theorem for hypergraphs exploiting this fact was recently given by Biedl, Čenek, Chan, Demaine, Demaine, Fleischer and Wang [10].

The following very elementary remark plays a crucial role in the proofs of both the generalized Beck–Fiala theorem and the Bárány–Grinberg theorem.

Lemma 4.3. *Let $x \in \overline{M}_c$. Assume that there is a $j' \in [c]$ such that $x_{j'} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$. Then there is at least a second index j'' (different from j') such that $x_{j''} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$.*

Proof. By assumption we have $cx_{j'} \notin \mathbb{Z}$. As $c\sum_{j\in[c]} x_j = 0 \in \mathbb{Z}$ by definition of $\overline{M}_c$, there exists a $j'' \in [c]$, $j' \neq j''$, such that $cx_{j''} \notin \mathbb{Z}$. In particular, $x_{j''} \notin \{-\frac{1}{c}, \frac{c-1}{c}\}$. □

Proof of Theorem 4.2. Set $\Delta := \|A\|_1$ and $\overline{A} = (\overline{a}_{ij}) := A \otimes I_c$. Note that $\Delta = \|\overline{A}\|_1$. Let $p : [n] \to \overline{M}_c$. Set $\chi = p$. We will successively change χ to a mapping $[n] \to M_c$. Again we regard p and χ as cn-dimensional vectors.

Put $J := \{j \in [cn] | \chi_j \notin \{-\frac{1}{c}, \frac{c-1}{c}\}\}$, and call the columns from J floating (the others fixed). Set $I := \{i \in [cm] | \sum_{j\in J} |\overline{a}_{ij}| > 2\Delta\}$, and call the rows from I active (the others ignored). We will ensure that during the rounding process the following conditions are fulfilled (this is clear for the start, because $\chi = p$):

(i) $(\overline{A}(p - \chi))_{|I} = 0$, *i.e.*, all active rows have discrepancy zero, and
(ii) all colours are in $\overline{M}_c$ – in particular we have $\sum_{k=0}^{c-1} \chi_{cj-k} = 0$ for all $j \in [n]$.

Note that (ii) is the crucial difference to the 2-colour case, where we only need a condition of type (i). This increases the number of equations investigated below and is the reason why the multicolour bound is twice the 2-colour bound.

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Let us assume that the rounding process is at a stage where J and I are as above and (i) and (ii) hold. If there is no floating colour, *i.e.*, $J = \emptyset$, then all χ_j, $j \in [cn]$, are in $\{-\frac{1}{c}, \frac{c-1}{c}\}$ and χ has the desired form.

Hence assume that there are still floating colours. We consider the system of equations

$$\sum_{k=0}^{c-1} \chi_{cj-k} = 0,\ j \in [n] \text{ such that } c(j-1)+k \in J \text{ for some } k \in [c]. \tag{4.1}$$

By Lemma 4.3, in every equation of (4.1) there are at least two floating variables $\chi_{j'}, \chi_{j''}$, *i.e.*, $j', j'' \in J$. Thus (4.1) is a system of at most $\frac{1}{2}|J|$ equations.

We have

$$|J|\,\Delta \geqslant \sum_{j\in J}\sum_{i\in I} |\overline{a}_{ij}| = \sum_{i\in I}\sum_{j\in J} |\overline{a}_{ij}| > |I|\,2\Delta,$$

hence $|J| > 2|I|$. We conclude that the system

$$\begin{aligned} &\overline{A}_{|I\times J}\ \ \chi_{|J} = 0, \\ &\sum_{k=0}^{c-1} \chi_{cj-k} = 0,\ j \in [n] \text{ such that } c(j-1)+k \in J \text{ for some } k \in [c] \end{aligned} \tag{4.2}$$

consists of at most $|I| + \frac{1}{2}|J| < |J|$ equations and hence is under-determined (taking just the $\chi_j, j \in J$ as variables). Thus there is a nontrivial solution $x \in \mathbb{R}^{|J|}$ for (4.2). We extend x to $x_E \in \mathbb{R}^{cn}$ by

$$(x_E)_j := \begin{cases} x_j & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

By (ii) and the definition of J, all variables $\chi_j, j \in J$ are in $]-\frac{1}{c}, \frac{c-1}{c}[$. Thus there is a $\lambda > 0$ such that at least one component of $\chi + \lambda x_E$ becomes fixed and all colours are still in $\overline{M}_c$, *i.e.*, $\chi + \lambda x_E \in \overline{M}_c^n$. Note that $\chi + \lambda x_E$ also fulfils (i) since $(\overline{A}x_E)_{|I} = 0$. Set $\chi := \chi + \lambda x_E$. Since (i), (ii) are fulfilled for this new χ, we can continue this rounding process until all χ_j, $j \in [cn]$ are in $\{-\frac{1}{c}, \frac{c-1}{c}\}$.

We show $\|\overline{A}(p-\chi)\|_\infty < 2\Delta$. Let $i \in [cm]$. Let $\chi^{(0)}$ and $J^{(0)}$ denote the values of χ and J when the row i first became ignored. We have $\chi_j^{(0)} = \chi_j$ for all $j \notin J^{(0)}$ and $|\chi_j^{(0)} - \chi_j| < 1$ for all $j \in J^{(0)}$. Note that $\sum_{j\in J^{(0)}} |\overline{a}_{ij}| \leqslant 2\Delta$, since i is ignored. Thus

$$|\big(\overline{A}(p-\chi)\big)_i| = |\big(\overline{A}\big(p-\chi^{(0)}\big)\big)_i + \big(\overline{A}\big(\chi^{(0)}-\chi\big)\big)_i| = \left|0 + \sum_{j\in J^{(0)}} \overline{a}_{ij}\big(\chi_j^{(0)} - \chi_j\big)\right| < 2\Delta.$$

□

For the c-colour discrepancy we have the following.

Corollary 4.4. $\operatorname{disc}(\mathscr{H}, c) < 2\Delta(\mathscr{H})$.

Note that this result is very similar to Theorem 3.7.

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4.2. Bárány–Grinberg theorem

For the remainder of this section let $\|\cdot\|$ denote any norm on $\mathbb{R}^d$. The Bárány–Grinberg theorem states the following result.

Theorem 4.5. ([4]) *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$ and, for any k, let $v_1, v_2, \ldots, v_k$ be a finite sequence of vectors in $\mathbb{R}^n$ for which $\|v_i\| \leqslant 1$ for all $i = 1, \ldots, k$. Then there are signs $\varepsilon_i \in \{-1, +1\}, i = 1, \ldots, k$ such that, for all $l \in [k]$, we have*

$$\left\| \sum_{i=1}^{l} \varepsilon_i v_i \right\| < 2n.$$

This seems to be similar to the Beck–Fiala theorem, but has a slightly different flavour. Here partial sums are considered, and we may choose any norm for the input and the discrepancy. The Beck–Fiala theorem formulated in terms of a vector sequence states that, for any vectors $v_1, \ldots, v_k$ of $\|\cdot\|_1$-norm at most one, there are signs $\varepsilon_i \in \{-1, +1\}, i = 1, \ldots, k$ such that $\left\|\sum_{i=1}^{k} \varepsilon_i v_i\right\|_\infty < 2$. Thus neither theorem is a special case of the other.

The signs -1 and $+1$ are a convenient way to represent a partition. From this point of view the Bárány–Grinberg theorem states that, under the given assumptions, there is a 2-partition (I_1, I_2) of the set $X = \{v_1, \ldots, v_k\}$ such that, for any subset $X_0 = \{v_1, \ldots, v_l\}$,

$$\left\| \sum_{v \in I_j \cap X_0} v - \tfrac{1}{2} \sum_{v \in X_0} v \right\| < n$$

holds for both $j = 1, 2$. This motivates the following definition.

Definition. (Discrepancy of sets and vector sequences) Let X be a finite set of vectors in $\mathbb{R}^n$ and $\mathscr{P} = \{I_1, \ldots, I_c\}$ a c-partition of X. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. We define the *discrepancy of the set X* w.r.t. $\mathscr{P}$ and $\|\cdot\|$ by

$$\operatorname{disc}(\mathscr{P}, \|\cdot\|) := \max_{j \in [c]} \left\| \sum_{v \in I_j} v - \tfrac{1}{c} \sum_{v \in X} v \right\|.$$

Given a subset $X_0 \subseteq X$, set $\mathscr{P}_{|X_0} := \{I_1 \cap X_0, \ldots, I_c \cap X_0\}$. Let $v_1, v_2, \ldots, v_k$ be a finite sequence of vectors and $\mathscr{P} = \{I_1, \ldots, I_c\}$ be a c-partition of $\{v_1, v_2, \ldots, v_k\}$. We define the *discrepancy of the sequence $v_1, v_2, \ldots, v_k$* w.r.t. $\mathscr{P}$ and $\|\cdot\|$ by

$$\operatorname{disc}((v_l)_{l \in [k]}, \mathscr{P}, \|\cdot\|) := \max_{l \in [k]} \operatorname{disc}(\mathscr{P}_{|\{v_1, \ldots, v_l\}}, \|\cdot\|).$$

In this notation the Bárány–Grinberg theorem states that there is a 2-partition $\mathscr{P} = \{I_1, I_2\}$ such that $\operatorname{disc}((v_l)_{l \in [k]}, \mathscr{P}, \|\cdot\|) < n$. We define a norm $\|\cdot\|_c$ on $\mathbb{R}^{cn}$ by

$$\|w\|_c := \max_{j \in [c]} \|w_{|\{j, j+c, \ldots, j+(n-1)c\}}\|,$$

where we write $w_{|\{j, j+c, \ldots, j+(n-1)c\}}$ for the n-dimensional vector $(w_j, w_{j+c}, \ldots, w_{(n-1)c})^\top$.

Lemma 4.6. *Let $X \subseteq \mathbb{R}^n$ be a finite set of vectors and $\mathscr{P} = \{I_1, \ldots, I_c\}$ be any c-partition of X. Let $\chi : X \to [c]$ be the corresponding colouring (i.e., for all $v \in X, l \in [c]$ we have*

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$\chi(v) = l$ *if and only if* $v \in I_l$). *Then the discrepancy of* X *w.r.t.* $\mathscr{P}$ *and* $\|\cdot\|$ *is*

$$\operatorname{disc}(\mathscr{P}, \|\cdot\|) = \left\| \sum_{v \in X} v \otimes m^{(\chi(v))} \right\|_c .$$

Proof. Remember that by definition

$$m_j^{(\chi(v))} = \begin{cases} 1 - \frac{1}{c} & \text{if } \chi(v) = j, \\ -\frac{1}{c} & \text{otherwise.} \end{cases}$$

Thus

$$\left(v \otimes m^{(\chi(v))}\right)\Big|_{\{j, j+c, \ldots, j+(n-1)c\}} = m_j^{(\chi(v))} v \tag{4.3}$$

and

$$\sum_{v \in X} m_j^{\chi(v)} v = \sum_{\substack{v \in X \\ \chi(v)=j}} \left(1 - \tfrac{1}{c}\right) v - \sum_{\substack{v \in X \\ \chi(v) \neq j}} \tfrac{1}{c} v = \sum_{\substack{v \in X \\ \chi(v)=j}} v - \tfrac{1}{c} \sum_{v \in X} v. \tag{4.4}$$

So

$$\begin{aligned} \left\| \sum_{v \in X} v \otimes m^{(\chi(v))} \right\|_c &= \max_{j \in [c]} \left\| \left(\sum_{v \in X} v \otimes m^{(\chi(v))} \right)\Big|_{\{j, j+c, \ldots, j+(n-1)c\}} \right\| \\ &\overset{(4.3)}{=} \max_{j \in [c]} \left\| \sum_{v \in X} m_j^{(\chi(v))} v \right\| \\ &\overset{(4.4)}{=} \max_{j \in [c]} \left\| \sum_{v \in I_j} v - \frac{1}{c} \sum_{v \in X} v \right\| \\ &= \operatorname{disc}(\mathscr{P}, \|\cdot\|). \end{aligned}$$

□

We are now ready to prove the following multicolour version.

Theorem 4.7. *Let* $\|\cdot\|$ *be any norm on* $\mathbb{R}^n$ *and* $v_1, v_2, \ldots, v_k$ *be a finite sequence of vectors in* $\mathbb{R}^n$ *such that* $\|v_i\| \leqslant 1$ *for all* $i = 1, \ldots, k$. *Then there is a* c*-partition* $\mathscr{P} = \{I_1, \ldots, I_c\}$ *of* $\{v_1, v_2, \ldots, v_k\}$ *such that*

$$\operatorname{disc}((v_l)_{l \in [k]}, \mathscr{P}, \|\cdot\|) < (c-1)n.$$

Proof. We may assume $k > n$. By Lemma 4.6 it suffices to show the existence of a colouring $\chi : [k] \to M_c$ such that $\| \sum_{i \in [l]} v_i \otimes \chi^{(i)} \|_c < (c-1)n$ for all $l \in [k]$.

As in the proof of the Bárány–Grinberg theorem, we give an algorithmic construction of χ. At the beginning define $A := [n]$ and $\chi_j^{(i)} := 0$ for all $i \in [k]$, $j \in [c]$. Let us call those $\chi_j^{(i)}$ where $i \in A$ and $\chi_j^{(i)} \notin \{\frac{c-1}{c}, -\frac{1}{c}\}$ *variables* and the corresponding colour vector $\chi^{(i)}$ *active*. Hence at the beginning we have cn variables and n active colour vectors. Furthermore all colour vectors $\chi^{(i)}, i \in [k]$ are in $\overline{M_c}$ and we have $\sum_{i \in A} v_i \otimes \chi^{(i)} = 0$.

We repeat the following rounding process. Set $A_0 := \{i \in [k] | \exists j \in [c] : \chi_j^{(i)} \notin \{\frac{c-1}{c}, -\frac{1}{c}\}\}$, the set of indices of active colour vectors. We try to find a nontrivial solution to the

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system of equations

$$\sum_{i\in A} v_i \otimes \chi^{(i)} = 0, \qquad \sum_{j\in[c]} \chi_j^{(i)} = 0 \quad \text{for all } i \in A_0. \tag{4.5}$$

Let n' be the number of variables and m' the rank of the system (4.5). By Lemma 4.3, each active vector contains at least two variables, so $n' \geqslant 2|A_0|$. On the other hand, $m' \leqslant (c-1)n + |A_0|$, since $\sum_{j\in[c]} \chi_j^{(i)} = 0$ for all $i \in [k]$ holds at any stage of the rounding process.

If there is no nontrivial solution to (4.5), then there are at most m' variables. From $2|A_0| \leqslant n' \leqslant m' \leqslant (c-1)n + |A_0|$, we conclude $|A_0| \leqslant (c-1)n$.

If there are still vectors that have not been active, *i.e.*, $A \neq [k]$, we increase the number of active vectors by setting $A := A \cup \{\max(A)+1\}$ and continue the rounding process, considering the updated system (4.5). If $A = [k]$ we terminate the rounding process by changing the remaining variables to $\frac{c-1}{c}$ or $-\frac{1}{c}$ in any way such that all $\chi^{(i)}$ are in M_c.

If there is a nontrivial solution to (4.5), then we can change χ in the way that some variables become $\frac{c-1}{c}$ or $-\frac{1}{c}$ and all variables stay in $[-\frac{1}{c}, \frac{c-1}{c}]$ in the same fashion as in the proof of Beck and Fiala. Note that the conditions $\chi^{(i)} \in \overline{M_c}$ for all $i \in [k]$ and $\sum_{i\in A} v_i \otimes \chi^{(i)} = 0$ are still satisfied. Hence we can continue the rounding process.

For the analysis let $l \in [k]$. Let $\widetilde{\chi}^{(1)}, \ldots, \widetilde{\chi}^{(k)}$ denote the value of the colour vectors at that stage of the rounding process when $A = [l]$ and no nontrivial solution to (4.5) can be found. Let $\widetilde{A_0}$ denote the value of A_0 at this stage. Let $\chi_f^{(1)}, \ldots, \chi_f^{(k)}$ denote the final values of the colour vectors. From above we know $|\widetilde{A_0}| \leqslant (c-1)n$. Since $\chi^{(i)} \in \overline{M_c}$, we have $\|\widetilde{\chi}^{(i)} - \chi_f^{(i)}\|_\infty < 1$ for all $i \in [l]$. Furthermore $\widetilde{\chi}^{(i)} = \chi_f^{(i)}$ holds if $i \notin \widetilde{A}_0$, since an inactive vector never becomes active again. By (4.5) we also have the equation $\sum_{i\in[l]} v_i \otimes \widetilde{\chi}^{(i)} = 0$.

Now

$$\begin{aligned}
\left\| \sum_{i\in[l]} v_i \otimes \chi^{(i)} \right\|_c &\leqslant \underbrace{\left\| \sum_{i\in[l]} v_i \otimes \widetilde{\chi}^{(i)} \right\|_c}_{=0 \text{ by } (4.5)} + \left\| \sum_{i\in[l]} v_i \otimes \left(\chi^{(i)} - \widetilde{\chi}^{(i)}\right) \right\|_c \\
&= \left\| \sum_{i\in\widetilde{A}_0} v_i \otimes \left(\chi^{(i)} - \widetilde{\chi}^{(i)}\right) \right\|_c \\
&= \max_{j\in[c]} \left\| \sum_{i\in\widetilde{A}_0} \left(v_i \otimes \left(\chi^{(i)} - \widetilde{\chi}^{(i)}\right)\right)\big|_{\{j, j+c, \ldots, j+(n-1)c\}} \right\| \\
&= \max_{j\in[c]} \left\| \sum_{i\in\widetilde{A}_0} v_i \left(\chi^{(i)} - \widetilde{\chi}^{(i)}\right)_j \right\| \\
&< \sum_{i\in\widetilde{A}_0} \|v_i\| \\
&\leqslant (c-1)n.
\end{aligned}$$

□

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5. Lower bounds

In this section we give a general lower bound and analyse two prominent examples: Hypergraphs arising from Hadamard matrices and arithmetic progressions. We start with the c-colour version of a result attributed to Lovász and Sós in [8].

Theorem 5.1. *Let* $A \in \mathbb{R}^{m\times n}$. *Then* $\operatorname{disc}(A,c) \geqslant \sqrt{\frac{n(c-1)}{mc^2}\lambda_{\min}(A^\top A)}$.

Proof. Let $\chi : [n] \to M_c$ be an optimal colouring with respect to c-colour discrepancy. Then

$$\begin{aligned}
\operatorname{disc}(A,c) &= \|(A\otimes I_c)\chi\|_\infty \\
&\geqslant \frac{1}{\sqrt{cm}}\|(A\otimes I_c)\chi\|_2 \\
&\geqslant \frac{1}{\sqrt{cm}}\|\chi\|_2\sqrt{\lambda_{\min}((A\otimes I_c)^\top(A\otimes I_c))} \\
&\overset{\text{Lemma 2.4(iii)}}{=} \frac{1}{\sqrt{cm}}\sqrt{\frac{n(c-1)}{c}}\sqrt{\lambda_{\min}((A^\top A)\otimes I_c)} \\
&\overset{\text{Lemma 2.4(v)}}{=} \sqrt{\frac{n(c-1)}{mc^2}}\sqrt{\lambda_{\min}(A^\top A)}.
\end{aligned}$$

□

5.1. Hadamard matrices

Hypergraphs corresponding to Hadamard matrices show that Spencer's 'six standard deviations' result is best possible apart from constant factors. The following theorem extends this result to c colours.

Theorem 5.2. *There is a universal constant* $K > 0$ *such that, for an infinite sequence of* $n \in \mathbb{N}$, *there is a hypergraph with* n *vertices and* n *edges and discrepancy at least* $K\sqrt{\frac{n}{c}}$.

Proof. Let $n \in \mathbb{N}$ be such that there exists a Hadamard matrix H of dimension n, *i.e.*, $H \in \{+1,-1\}^{n\times n}$, and all rows of H are pairwise orthogonal. By multiplying some rows by -1 we may assume that all entries of the first column v_1 are 1. Let $v_2,\dots,v_n$ denote the remaining columns. Set $A = \frac{1}{2}(H+J)$, where J is the $n\times n$ matrix consisting of ones only. A is the incidence matrix of a hypergraph $\mathscr{H}$ of n edges on n vertices. We show that $\mathscr{H}$ has the desired discrepancy.

Let $\chi : [n] \to M_c$ be any colouring. Let $i \in [c]$ be such that

$$\left|\chi^{-1}\left(m^{(i)}\right) \setminus \{1\}\right| \geqslant \tfrac{n-1}{c}. \tag{5.1}$$

For all $j \in [c]$ set $\chi_j : [n] \to \{-\frac{1}{c}, \frac{c-1}{c}\}; k \mapsto \chi(k)_j$. Let $a_1,\dots,a_n$ be the row vectors of

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A, and for $x, y \in \mathbb{R}^n$ let $x \cdot y$ be the usual inner product in $\mathbb{R}^n$. Then

$$\begin{aligned}\operatorname{disc}_\chi(\mathscr{H}, c) &= \|(A \otimes I_c)\chi\|_\infty \\ &= \|(a_1 \cdot \chi_1, \ldots, a_1 \cdot \chi_c, \ldots, a_n \cdot \chi_1, \ldots, a_n \cdot \chi_c)^\top\|_\infty \\ &\geqslant \|(a_1 \cdot \chi_i, \ldots, a_n \cdot \chi_i)^\top\|_\infty \\ &= \|A\chi_i\|_\infty \\ &\geqslant \tfrac{1}{\sqrt{n}}\|A\chi_i\|_2.\end{aligned}$$

By (5.1) we have

$$|\{k \in [n] \setminus \{1\} | \chi_i(k) = \tfrac{c-1}{c}\}| \geqslant \tfrac{n-1}{c}. \tag{5.2}$$

By definition of A there is a $\lambda \in \mathbb{R}$ such that $A\chi_i = \sum_{k=2}^n \frac{1}{2}\chi_i(k)v_k + \lambda v_1$. Since the $v_1, \ldots, v_n$ are pairwise orthogonal, we have

$$\begin{aligned}\|A\chi_i\|_2 &= \sqrt{\sum_{k=2}^n \chi_i(k)^2 \left\|\tfrac{1}{2}v_k\right\|_2^2 + \lambda^2\|v_1\|_2^2} \\ &\geqslant \sqrt{\sum_{k=2}^n \chi_i(k)^2 \left\|\tfrac{1}{2}v_k\right\|_2^2} \\ &= \tfrac{1}{2}\sqrt{n}\sqrt{\sum_{k=2}^n \chi_i(k)^2} \\ &\geqslant \tfrac{1}{2}\sqrt{n}\sqrt{\tfrac{n-1}{c}\left(\tfrac{c-1}{c}\right)^2 + \tfrac{(n-1)(c-1)}{c}\left(-\tfrac{1}{c}\right)^2} \quad \text{(by (5.2))} \\ &= \tfrac{1}{2}\sqrt{n}\sqrt{\tfrac{(n-1)(c-1)}{c^2}}.\end{aligned}$$

Hence $\operatorname{disc}(\mathscr{H}, c) \geqslant \frac{1}{2}\sqrt{\frac{(n-1)(c-1)}{c^2}}$. □

5.2. Arithmetic progressions

In this section we give a lower bound for the c-colour discrepancy of the arithmetic progressions. We refer to Section 3.5.3 for an introduction to this problem.

Theorem 5.3. *The hypergraph of arithmetic progressions fulfils*

$$\operatorname{disc}(\mathscr{H}_n, c) \geqslant 0.04\frac{1}{\sqrt{c}}\sqrt[4]{n}.$$

Proof. For the lower bound we will follow the approach of [8]. Set $k = \lfloor\sqrt{\frac{1}{6}n}\rfloor$. Let $\mathscr{E}$ be the set of arithmetic progressions of length k and difference less than $6k$ computed modulo n (hence our arithmetic progressions may be over-wrapped from n to 1 at most once). Every arithmetic progression of $\mathscr{E}$ is a union of at most two arithmetic progressions from $\mathscr{E}_n$, so the discrepancy of $\mathscr{H}_n$ is at least half the discrepancy of $\mathscr{E}$.

Recall that a matrix is called a circulant if the ith row can be obtained from the first by shifting it $i-1$ times to the right. Let us enumerate the arithmetic progressions in $\mathscr{E}$ in

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a way that, if i is not divisible by n, then $E_{i+1} = E_i + 1$ (always computed modulo n), *i.e.*, E_{i+1} is E_i shifted right by one. Thus the incidence matrix $A = (a_{ij}) \in \{0,1\}^{6kn \times n}$ defined by $a_{ij} = 1$ if and only if $j \in E_i$ consists of $6k$ circulant sub-matrices. As the sum and product of two circulant matrices is also circulant, $A^\top A$ is circulant. The eigenvectors of circulant matrices are known to be of the form $(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1})^\top$, where ε is an nth root of unity. Thus we find that the minimum eigenvalue $\lambda_{\min}(A^\top A)$ of $A^\top A$ is greater than $\frac{1}{4}k^2$.

Using Theorem 5.1 we have $\operatorname{disc}(([n], \mathscr{E}_n), c)^2 \geqslant \frac{n(c-1)}{6knc^2}\frac{1}{4}k^2 = \frac{(c-1)k}{24c^2}$. Hence

$$\begin{aligned}\operatorname{disc}(\mathscr{H}_n, c) &\geqslant 0.5 \operatorname{disc}(([n], \mathscr{E}), c)\\ &\geqslant \sqrt{\frac{c-1}{96\sqrt{6}c^2}}\sqrt[4]{n}\\ &\geqslant 0.0652\sqrt{\frac{c-1}{c^2}}\sqrt[4]{n}\\ &\geqslant 0.04\sqrt{\frac{1}{c}}\sqrt[4]{n}.\end{aligned}$$

□

We may remark that the lower bound can also be proved using harmonic analysis, as in [34, 17].

6. Open problems

From this paper several open problems arise, some of which we would like to emphasize here. For the hypergraph of arithmetic progressions and the 'six standard deviations' situation, we gave upper and lower bounds for the c-colour discrepancy. For fixed number c of colours, these bounds are optimal apart from constant factors. Concerning the influence of the number of colours, we showed that these discrepancies decrease with larger numbers of colours. Still, our bounds display a multiplicative gap of $O(c^{0.34})$ and $O(\sqrt{\log c})$, respectively. Reducing these gaps seems to be a nice problem. For the arithmetic progressions this should also yield more information about the weighted discrepancy in two colours, reducing the gap between Roth's bound and ours.

Another interesting question is whether there is a direct proof for Spencer's 'six standard deviations' result for the multicolour discrepancy problem or, more generally, a suitable generalization of the partial colouring method. A negative answer would suggest that discrepancy in 2 colours is a rather special situation compared to discrepancy in arbitrary numbers of colours.

A problem field only touched upon in this paper is that of linear discrepancies. The following observation suggests that they might behave differently in more than 2 colours. Consider a totally unimodular $m \times n$ matrix A. Various proofs show that $\operatorname{lindisc}(A, 2) \leqslant 1$ holds. The sharp bound of $1 - \frac{1}{n+1}$ was recently proved in [14]. For more than 2 colours, it is not difficult to find a totally unimodular matrix such that even $\operatorname{lindisc}(A, c) > 1$ holds (*e.g.*, in [13] an example was given that fulfils $\operatorname{lindisc}(A, 3) \geqslant 1 + \frac{1}{9}$).

Recently, Hebbinghaus, Schoen and Srivastav [19] introduced the notion of positive c-colour discrepancy, and proved for two colours a tight discrepancy bound for hyperplanes

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in the r-dimensional vector space $\mathbb{F}_2^r$. Whether a tight bound also exists for the multicolour discrepancy, is an open problem.

References

[1] Alon, N. and Spencer, J. H. (1992) *The Probabilistic Method*, Wiley, New York.
[2] Babai, L., Hayes, T. P. and Kimmel, P. G. (1998) The cost of the missing bit: Communication complexity with help. In *Proc. 30th STOC*, pp. 673–682.
[3] Banaszczyk, W. (1998) Balancing vectors and Gaussian measures of n-dimensional convex bodies. *Random Struct. Alg.* **12** 351–360.
[4] Bárány, I. and Grinberg, V. S. (1981) On some combinatorial questions in finite-dimensional spaces. *Linear Alg. Appl.* **41** 1–9.
[5] Beck, J. (1981) Roth's estimate of the discrepancy of integer sequences is nearly sharp. *Combinatorica* **1** 319–325.
[6] Beck, J. and Chen, W. L. (1987) *Irregularities of Distribution*, *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge.
[7] Beck, J. and Fiala, T. (1981) 'Integer making' theorems. *Discrete Appl. Math.* **3** 1–8.
[8] Beck, J. and Sós, V. T. (1995) Discrepancy theory. In *Handbook of Combinatorics* (R. Graham, M. Grötschel and L. Lovász, eds), pp. 1405–1446.
[9] Beck, J. and Spencer, J. (1984) Integral approximation sequences. *Math. Programming* **30** 88–98.
[10] Biedl, T., Čenek, E., Chan, T., Demaine, E., Demaine, M., Fleischer, R. and Wang, M. (2002) Balanced k-colorings. *Discrete Math.* **254** 19–32.
[11] Chazelle, B. (2000) *The Discrepancy Method*, Princeton University.
[12] Doerr, B. (2000) Linear and hereditary discrepancy. *Combin. Probab. Comput.* **9** 349–354.
[13] Doerr, B. (2000) Multi-color discrepancies. Dissertation, Christian-Albrechts-Universität zu Kiel.
[14] Doerr, B. (2001) Linear discrepancy of totally unimodular matrices. In *Proc. 12th Annual ACM–SIAM Symposium on Discrete Algorithms*.
[15] Doerr, B. (2002) Balanced coloring: Equally easy for all numbers of colors? In *STACS 2002* (H. Alt and A. Ferreira, eds), Vol. 2285 of *Lecture Notes in Computer Science*, Springer, Berlin/Heidelberg, pp. 112–120.
[16] Doerr, B. (2002) Discrepancy in different numbers of colors. *Discrete Math.* **250** 63–70.
[17] Doerr, B., Srivastav, A. and Wehr, P. (1998) Discrepancies of Cartesian products of arithmetic progressions. Technical Report 98–42, Berichtsreihe des Mathematischen Seminars der Universität Kiel.
[18] Graham, A. (1981) *Kronecker Products and Matrix Calculus*, Ellis Horwood.
[19] Hebbinghaus, N., Schoen, T., Srivastav, A. (2002) Tight discrepancy of hyperplanes in finite vector spaces. Technical Report, Mathematisches Seminar, Universität zu Kiel.
[20] Helm, M. (1999) On the Beck–Fiala theorem. *Discrete Math.* **207** 73–87.
[21] Lovász, L., Spencer, J. and Vesztergombi, K. (1986) Discrepancies of set-systems and matrices. *Europ. J. Combin.* **7** 151–160.
[22] Matoušek, J. (1995) Tight upper bound for the discrepancy of halfspaces. *Disc. Comput. Geom.* **13** 593–601.
[23] Matoušek, J. (1999) *Geometric Discrepancy*, Springer, Berlin.
[24] Matoušek, J. (2000) On the linear and hereditary discrepancies. *Europ. J. Combin.* **21** 519–521.
[25] Matoušek, J. and Spencer, J. (1996) Discrepancy in arithmetic progressions. *J. Amer. Math. Soc.* **9** 195–204.
[26] Matoušek, J., Wernisch, L. and Welzl, E. (1993) Discrepancy and ε-approximations for bounded VC-dimension. *Combinatorica* **13** 455–466.

https://doi.org/10.1017/S0963548303005662 Published online by Cambridge University Press

[27] Roth, K. F. (1964) Remark concerning integer sequences. *Acta Arithmetica* **9** 257–260.
[28] Sárközy, A. (1974) In *Probabilistic Methods in Combinatorics* (P. Erdős and J. Spencer, eds), Akadémia Kiadó, Budapest.
[29] Spencer, J. (1985) Six standard deviations suffice. *Trans. Amer. Math. Soc.* **289** 679–706.
[30] Srinivasan, A. (1997) Improving the discrepancy bound for sparse matrices: Better approximations for sparse lattice approximation problems. In *Proc. Eighth Annual ACM–SIAM Symposium on Discrete Algorithms* (New Orleans, LA, 1997), ACM, New York, pp. 692–701.
[31] Srivastav, A. (2001) Derandomization in combinatorial optimization. In *Handbook of Randomized Computing* (S. Rajasekaran, P. M. Pardalos, J. H. Reif, and J. D. P. Rolim, eds) Kluwer Academic Publishers, Chapter 18, pp. 731–842.
[32] Srivastav, A. and Stangier, P. (1996) Algorithmic Chernoff–Hoeffding inequalities in integer programming. *Random Struct. Alg.* **8** 27–58.
[33] Vapnik, V. N. and Chervonenkis, A. Ya. (1971) On the uniform convergence of relative frequencies of events to their probabilities. *Theory Prob. Appl.* **16** 264–280.
[34] Wehr (Knieper), P. (1997) The discrepancy of arithmetic progressions. Dissertation, Institut für Informatik, Humboldt-Universität zu Berlin.

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