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Ray solution of a backward–forward parabolic problem for data-handling systems†

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A problem for a backward–forward parabolic partial differential equation with a small diffusion coefficient ε is analyzed. The problem arises in describing the stochastic behaviour of a data-handling system with a large number of sources which turn on or off at random. We solve the problem asymptotically for ε small by using the ray method and the boundary layer method.

1 Introduction

An equation like $xu_t = u_{xx}$ is parabolic, with the direction of increasing t being the time-like direction for $x > 0$ and the direction of decreasing t being time-like for $x < 0$. Such an equation is called a backward–forward parabolic equation. Forward parabolic equations are well-posed when initial data are given, and backward equations are well-posed when final data are given. Backward–forward equations require initial data where increasing t is time-like, and final data where decreasing t is timelike [1]. Since it is difficult to solve such problems, we shall present an asymptotic method for doing so based upon the smallness of some parameter ε in the problem. The method leads to the introduction of certain curves called rays, just as in geometrical optics, so it is called a *ray method*. Examples of the use of this method in other diffusion problems are given in Knessl & Keller [2, 3].

Backward–forward parabolic partial differential equations arise in the study of queues and data-handling systems with stochastic sources, in exit time problems, and in advection-diffusion processes. A particular data-handling system, studied by Knessl & Morrison [4], leads to the following problem:

$$\begin{aligned} (\xi - a)\overline{F}_y &= \overline{F}_{\xi\xi} + (\xi\overline{F})_\xi, \quad y > 0, \ -\infty < \xi < \infty \\ \overline{F}(\xi, 0) &= 0, \quad \xi > a \\ \int_{-\infty}^{\infty} \overline{F}(\xi, \infty) d\xi &= 1. \end{aligned} \tag{1.1}$$

Practically the same problem arose in the work of Hagan, Doering & Levermore [5] on a mean exit time problem. Similar problems are considered in earlier works [6]–[12], and

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their mathematical theory is dealt with by Weinberger & Keller [13], and the references therein. We shall solve (1.1) asymptotically for large values of a.

In Knessl & Morrison [4], $\overline{F}(\xi, y)d\xi$ is the steady-state probability that a normalized buffer backlog does not exceed y, and that the normalized number of 'on' sources lies in the interval ξ to $\xi + d\xi$. The constant $a > 0$ is the normalized excess of the buffer handling rate over the mean number of 'on' sources. It must be positive for a steady-state to exist.

The problem (1.1) was solved explicitly by Knessl & Morrison [4] in the form of an infinite series. That representation is evaluated asymptotically for large a by Knessl [14] in a following paper, where the formulation of the problem and a discussion of the results is also presented. Here we obtain the same asymptotic solution directly by using the ray method [15], the boundary layer method, and asymptotic matching [16]. These methods are applicable to other similar problems for which the explicit solutions cannot be found.

2 The ray expansion

To analyse (1.1) for large a, it is convenient to introduce the new variables $\eta = \xi/a$, $x = y/a$, the new function $F(\eta, x) = \overline{F}(\xi, y)$, and the small parameter $\varepsilon = a^{-2}$. Then (1.1) becomes the following problem for $F(\eta, x)$:

$$(\eta - 1)F_x = \varepsilon F_{\eta\eta} + (\eta F)_\eta, \quad x \geqslant 0, \quad -\infty < \eta < \infty \tag{2.1}$$

$$F(\eta, 0) = 0, \quad \eta > 1 \tag{2.2}$$

$$\int_{-\infty}^{\infty} F(\eta, \infty)\, d\eta = \varepsilon^{1/2}. \tag{2.3}$$

The initial condition at $x = 0$ is given only for $\eta > 1$, where the positive x direction is the time-like direction for (2.1). In writing the normalization condition (2.3) we have assumed that the limit $F(\eta, \infty)$ exists. This limit satisfies the ordinary differential equation obtained by setting $F_x = 0$ in (2.1), and the solution of that equation satisfying (2.3) is the Gaussian

$$F(\eta, \infty) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\eta^2}{2\varepsilon}\right). \tag{2.4}$$

To solve this problem for ε small, we shall use the ray method [15]. Thus we consider solutions of (2.1) which have the asymptotic form

$$F(\eta, x) = C(\varepsilon) \exp\left[\psi(\eta, x)/\varepsilon\right] \left\{K(\eta, x) + \varepsilon K^{(1)}(\eta, x) + \cdots\right\}. \tag{2.5}$$

We substitute (2.5) into (2.1) and equate the coefficients of the lowest power of ε to get the eikonal equation for ψ:

$$(\eta - 1)\psi_x = \psi_\eta^2 + \eta\psi_\eta. \tag{2.6}$$

Equating the coefficients of the next power of ε yields the transport equation for K:

$$(\eta - 1)K_x = (2\psi_\eta + \eta)K_\eta + \left(\psi_{\eta\eta} + 1\right)K. \tag{2.7}$$

We solve (2.6) by introducing the characteristic curves or rays $[x(t), \eta(t)]$, written in terms of a parameter t. The characteristic ordinary differential equations for a ray are:

$$\dot{x} = \eta - 1, \quad \dot{\eta} = -\eta - 2\psi_\eta \tag{2.8}$$

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$$\dot{\psi} = \psi_x \dot{x} + \psi_\eta \dot{\eta} = (\eta - 1)\psi_x - (\eta + 2\psi_\eta)\psi_\eta = -\psi_\eta^2 \tag{2.9}$$

$$\dot{\psi}_x = 0, \quad \dot{\psi}_\eta = \psi_\eta - \psi_x. \tag{2.10}$$

Here a dot denotes d/dt, and we have used (2.8) and (2.6) to simplify the right side of (2.9). Integrating (2.10) yields

$$\psi_x = A, \quad \psi_\eta = A + (B - A)\, e^t, \tag{2.11}$$

where A and B are constant on a ray.

First we consider the family of rays emanating from the point $\eta = 1$, $x = 0$, which is the endpoint of the interval in which the initial condition is imposed. We use (2.11) for ψ_η in (2.8) and solve for $\eta(t)$ requiring $\eta(0) = 1$, to get

$$\eta = -2A + (A - B)\, e^t + e^{-t}(A + B + 1). \tag{2.12}$$

Then we use (2.12) in (2.8) and solve for $x(t)$, subject to $x(0) = 0$, to obtain

$$x = -(2A + 1)t + 2B + 1 + (A - B)\, e^t - (A + B + 1)\, e^{-t}. \tag{2.13}$$

Next we use (2.11) in (2.9) and solve for $\psi(t)$ with the initial value $\psi(0) = -1/2$:

$$\psi = -A^2 t - 2A(B - A)(e^t - 1) - \tfrac{1}{2}(B - A)^2(e^{2t} - 1) - \tfrac{1}{2}. \tag{2.14}$$

The choice of $\psi(0)$ is not restrictive because the factor $C(\varepsilon)$ in (2.5) is still undetermined.

The constant B in (2.11) can be determined by substituting (2.11) into the eikonal equation (2.6) at $\eta = 1$, $x = 0$, corresponding to $t = 0$. This yields $B^2 + B = 0$, which has the two solutions $B = 0$ and $B = -1$. The solution $B = 0$ must be rejected because when it is used in (2.13) it gives $x(t) \approx -t^2/2$ for small t, which is outside the domain $x \geqslant 0$. Therefore, $B = -1$ so that $\psi_\eta = A - (A + 1)\, e^t$. Then (2.12)–(2.14) become

$$\begin{aligned} x &= -1 - (1 + 2A)t + (A + 1)\, e^t - Ae^{-t}, \\ \eta &= (A + 1)\, e^t + Ae^{-t} - 2A. \end{aligned} \tag{2.15}$$

$$\psi = -A^2 t + 2A(A + 1)(e^t - 1) - \tfrac{1}{2}(A + 1)^2(e^{2t} - 1) - \tfrac{1}{2}, \tag{2.16}$$

By using (2.15) we can rewrite (2.16) in the following form, which is convenient for comparison with certain other results:

$$\psi + \tfrac{1}{2}\eta^2 - Ax = A\left(\tfrac{3}{2}A + 1\right) + tA(A + 1) + \tfrac{1}{2}A^2 e^{-2t} - A(A + 1)e^t - A^2 e^{-t}. \tag{2.17}$$

For $t \geqslant 0$ and each value of A, (2.15) determines a ray in the (x, η) plane, which starts from (0,1) at $t = 0$. For small positive t and any value of A, (2.15) becomes $\eta - 1 \approx t$, $x \approx t^2/2$. Thus, all the rays of the family labelled by A leave the point (0,1) along half of the parabola $2x = (\eta - 1)^2$, $\eta > 1$. As $A \to \infty$ and $t \to 0$ with $At \to \infty$, (2.15) yields $x \approx At^3/3$, $\eta - 1 \approx At^2$ so that $x \approx (\eta - 1)^{3/2}/(3A^{1/2})$, which is also useful for comparison with other results.

In Figure 1, the rays (2.15) are shown for several values of A. For $A > -1$, both x and η increase to $+\infty$ as $t \to +\infty$ with $\eta/x \to 1$. For $A = 0$, $A = -1/2$, and $A = -1$ we can

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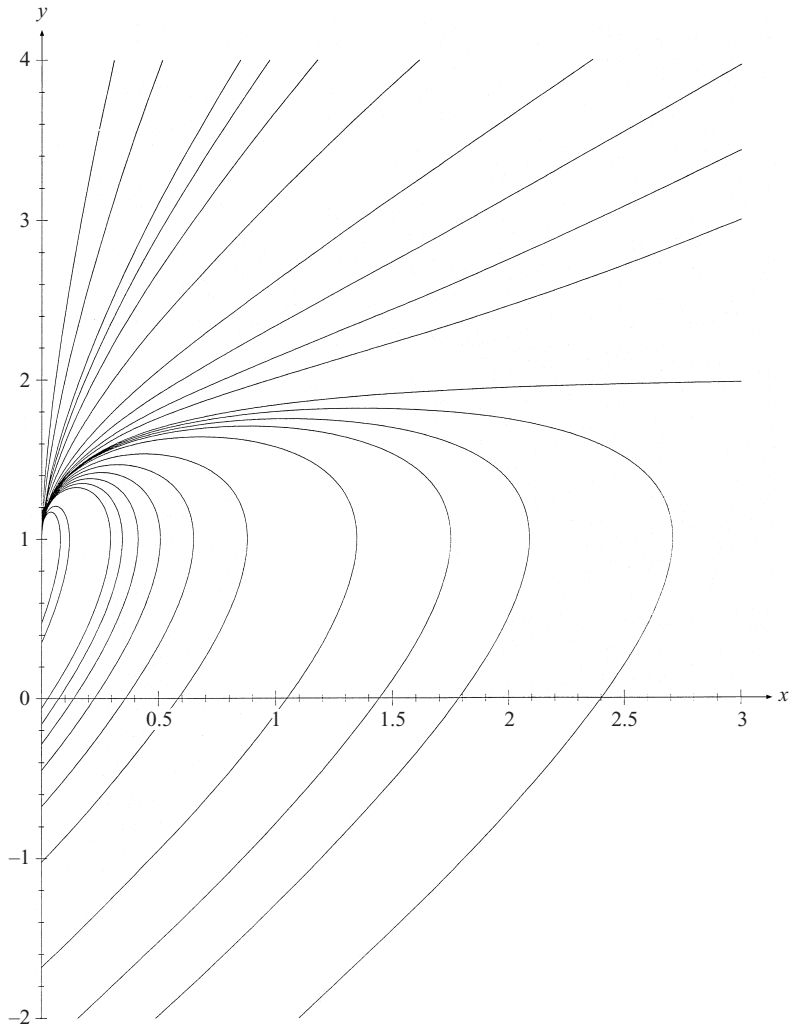


FIGURE 1. A sketch of the rays from the corner.

eliminate t from (2.15) and obtain the explicit equations

$$\begin{aligned} &A = 0 : x = \eta - 1 - \log\eta \equiv X_0(\eta), \quad \eta \geqslant 1 \\ &A = -\tfrac{1}{2} : x^2 - \eta^2 + 2(x+\eta) = 1, \quad \eta \geqslant 1 \\ &A = -1 : x = 1 - \eta - \log(2-\eta), \quad \eta \geqslant 1. \end{aligned} \tag{2.18}$$

The ray $A = -1$ has the asymptote $\eta = 2$ as $x \to +\infty$. For any $A > -1$ and $t > 0$, we have both $\dot{x} > 0$ and $\dot{\eta} > 0$. For $A = -\frac{1}{2}$ the ray is part of a hyperbola.

For $A < -1$, $\eta(t)$ increases at t increases from $t = 0$ to $t = t_1(A) = \frac{1}{2}\log\left(\frac{-A}{-A-1}\right)$ where $\dot{\eta} = 0$, while $x(t)$ increases for $0 < t < t_2(A) = 2t_1(A)$. Then $x(t)$ decreases until $t = t_\star(A) > t_2(A)$ where $x[t_\star(A)] = 0$. Thus $t_\star(A)$ is the unique positive solution of $1 + (1+2A)t_\star = (A+1)e^{t_\star} - Ae^{-t_\star}$, which exists only if $A < -1$. For $t > t_\star(A)$ the ray lies in the half-plane $x < 0$. Some rays with $A < -1$ are shown in Figure 1 up to the point

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where they leave the domain $x \geqslant 0$. At $t = t_2(A)$ the value of $\eta[t_2(A)] = 1$, so all rays are parallel to the η-axis at $\eta = 1$. At $t = t_1(A)$ where $\dot{\eta}[t_1(A)] = 0$, we have

$$x + 1 = -(1 + 2A)t = -\left[1 - \frac{\eta^2}{2(\eta - 1)}\right] \log\left(\frac{\eta}{2 - \eta}\right), \quad \eta > 1. \tag{2.19}$$

This is the locus of points where the rays are parallel to the x-axis.

Along the special rays $A = 0, -1/2, -1$ we can give ψ in the following explicit forms:

$$\begin{aligned} A = 0 &: \ \psi = -\tfrac{1}{2}\eta^2 \\ A = -\tfrac{1}{2} &: \ \psi = \tfrac{1}{8} - \tfrac{1}{8}(\eta^2 + x^2 + 2x\eta) - \tfrac{1}{2}(\eta + x) - \tfrac{1}{4}\sinh^{-1}(\eta - 1) \\ A = -1 &: \ \psi = -\tfrac{1}{2} + \log(2 - \eta) = \tfrac{1}{2} - x - \eta. \end{aligned} \tag{2.20}$$

For any $A \neq -1$ we can solve (2.15) for t in terms of A and η:

$$t = \log\left[\frac{A + \eta/2 \pm \sqrt{A(\eta - 1) + \eta^2/4}}{A + 1}\right]. \tag{2.21}$$

The upper sign applies for $A > -1$ and $A < -1$, $t < t_1(A)$, while the lower sign applies for $A < -1$, $t > t_1(A)$. We can use (2.21) to eliminate t from (2.15) and (2.17) with the results:

$$x + 1 \mp 2\sqrt{(\eta - 1)A + \eta^2/4} + (2A + 1)\log\left[\frac{A + \eta/2 \pm \sqrt{(\eta - 1)A + \eta^2/4}}{A + 1}\right] = 0 \tag{2.22}$$

$$\begin{aligned} \psi + \tfrac{1}{2}\eta^2 - Ax = \tfrac{1}{4}\eta^2 + \tfrac{1}{2}A \mp (A + \eta/2)\sqrt{(\eta - 1)A + \eta^2/4} \\ + A(A + 1)\log\left[\frac{A + \eta/2 \pm \sqrt{(\eta - 1)A + \eta^2/4}}{A + 1}\right]. \end{aligned} \tag{2.23}$$

Now we shall solve the transport equation (2.7) by using (2.8) and (2.9) to write it as an ordinary differential equation along a ray:

$$\dot{K} = (\psi_{\eta\eta} + 1)K. \tag{2.24}$$

From (2.11) with $B = -1$ we get $\psi_{\eta\eta} = (1 - e^t)A_\eta - (A + 1)e^t t_\eta$. To find A_η and t_η we differentiate the two equations (2.15) with respect to η

$$\begin{aligned} (e^t + e^{-t} - 2)A_\eta + \left([A + 1]\, e^t - Ae^{-t}\right) t_\eta = 1 \\ (e^t - e^{-t} - 2t)A_\eta + \left([A + 1]\, e^t + Ae^{-t} - 2A - 1\right) t_\eta = 0. \end{aligned} \tag{2.25}$$

We solve (2.25) for A_η and t_η, use them to evaluate $\psi_{\eta\eta}$, and then we can write (2.24) as

$$\frac{\dot{K}}{K} = \frac{4A + 2 - 2tAe^{-t} - (A + 1)e^t - (3A + 1)e^{-t}}{8A + 4 + 2t\,[(A + 1)\, e^t - Ae^{-t}] - (4A + 3)e^t - (4A + 1)e^{-t}}. \tag{2.26}$$

After division of the numerator and denominator of the right side of (2.6) by $2e^t$, the numerator becomes equal to $-\frac{1}{2}$ times the derivative of the denominator. Thus, the general solution of (2.26) is

$$K = K_0(A)\left|(A + 1)\, t - \left(2A + \tfrac{3}{2}\right) + (4A + 2)e^{-t} - \left[A\,(t + 2) + \tfrac{1}{2}\right] e^{-2t}\right|^{-1/2}. \tag{2.27}$$

The function $K_0(A)$ is arbitrary.

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So far we have determined the exponent ψ and the leading amplitude K of the solution (2.5), except for the function $K_0(A)$ in (2.27). In the next section we will determine it and the factor $C(\varepsilon)$ in (2.5) by matching (2.5) to a boundary layer solution valid in a neighbourhood of the end point (0,1). Before that, we will find another solution of (2.1) which satisfies the boundary condition (2.4) at $x = \infty$.

To this end we assume that for ε small, the solution has the asymptotic form

$$F(\eta, x) = \exp\left[\phi(\eta, x)/\varepsilon\right]\left\{L(\eta, x) + \varepsilon L^{(1)}(\eta, x) + \cdots\right\}. \tag{2.28}$$

It follows that ϕ satisfies (2.6) and L satisfies (2.7). We impose the boundary condition (2.4) at a finite value $x = x_1 > 0$, and then let $x_1 \to +\infty$. From (2.28) and (2.24) we get

$$\phi(\eta, x_1) = -\frac{\eta^2}{2}, \quad L(\eta, x_1) = \frac{1}{\sqrt{2\pi}}, \quad \eta < 1. \tag{2.29}$$

We shall solve for ϕ by using rays which start at $x = x_1$. Let $s < 1$ denote the initial value of η at $x = x_1$. The solution of the characteristic equations (2.8)–(2.10) with $(x, \eta) = (x_1, s)$ at $t = 0$, is

$$\begin{aligned} \phi_x &= \overline{A}, \quad \phi_\eta = \overline{A} + (\overline{B} - \overline{A})e^t, \\ \eta &= -2\overline{A} + (\overline{A} - \overline{B})e^t + (\overline{A} + \overline{B} + s)e^{-t}, \\ x &= -(2\overline{A} + 1)t + (\overline{A} - \overline{B})e^t - (\overline{A} + \overline{B})e^{-t} + 2\overline{B} + x_1 + s(1 - e^{-t}). \end{aligned} \tag{2.30}$$

From (2.29) we get $\phi_\eta = -\eta = -s$ at $x = x_1$, and from (2.30) at $t = 0$ we get $\phi_\eta = \overline{B}$, so $\overline{B} = -s$. Upon using the expressions above for ϕ_x and ϕ_η in the eikonal equation (2.6) at $x = x_1$, we get the relation $(s-1)\overline{A} = \overline{B}^2 + s\overline{B}$, and since $\overline{B} = -s$ we see that $\overline{A} = 0$. Thus (2.30) simplifies to $\phi_x = 0$, $\phi_\eta = -se^t$ and

$$\eta = se^t, \quad x = x_1 - t + s(e^t - 1). \tag{2.31}$$

Finally, using (2.9) we have $\dot{\phi} = -\phi_\eta^2 = -s^2e^{2t}$. We integrate this equation from $t = 0$, where (2.29) holds, and use (2.31) to get

$$\phi(\eta, x) = -\frac{\eta^2}{2}. \tag{2.32}$$

Since $\phi_{\eta\eta} = -1$, we find that the solution of the transport equation (2.7) satisfying (2.29) is $L(\eta, x) = (2\pi)^{-1/2}$. By using this and (2.32) in (2.28) we obtain

$$F(\eta, x) = (2\pi)^{-1/2}e^{-\eta^2/2\varepsilon}. \tag{2.33}$$

This is an exact solution of (2.1), and it remains unchanged as $x_1 \to \infty$.

It remains to determine the domain $R(x_1)$ covered by the rays from $x = x_1$, where (2.33) is valid, and the limit R of $R(x_1)$ as $x_1 \to +\infty$. By eliminating t from (2.31) we can write the rays as

$$\begin{aligned} &x = x_1 + \log s - s + \eta - \log \eta, \quad 0 < s < 1 \\ &x = x_1 - t, \quad s = 0 \\ &x = x_1 + \log(-s) - s + \eta - \log(-\eta), \quad s < 0. \end{aligned} \tag{2.34}$$

On each ray with $s < 0$, we have $\eta < 0$, $\dot{x} < 0$, and $\dot{\eta} < 0$. These rays fill the strip $\{0 \leqslant x \leqslant x_1,\ \eta < 0\}$. The ray with $s = 0$ lies on the segment $\{0 \leqslant x \leqslant x_1,\ \eta = 0\}$. The

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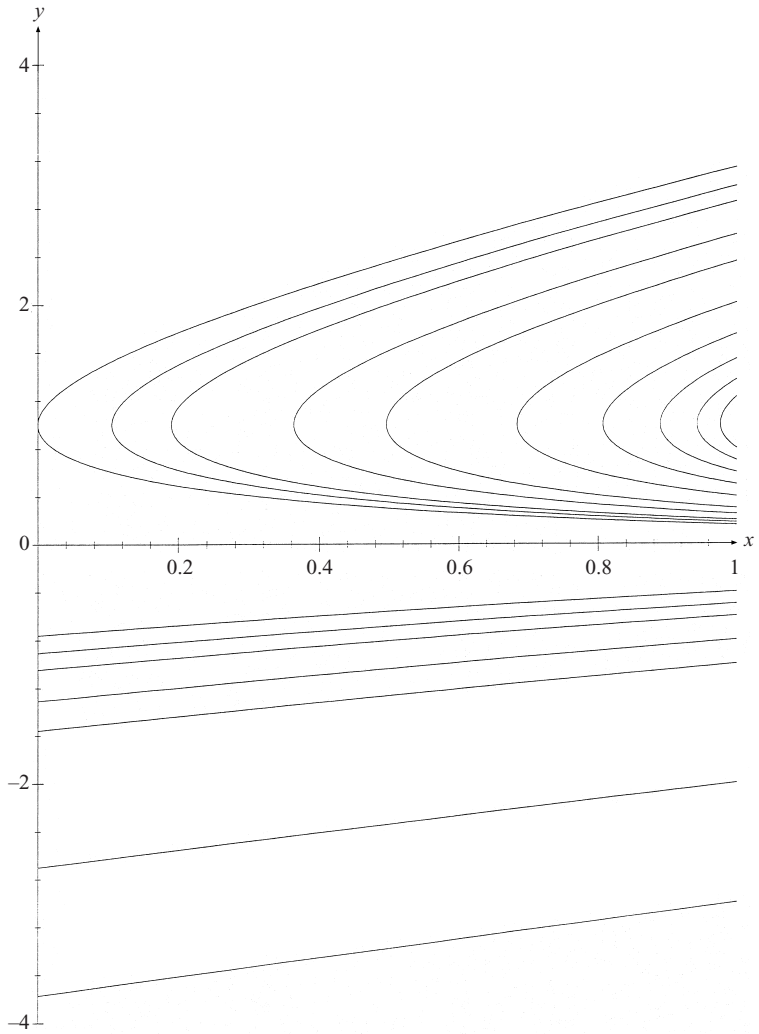


FIGURE 2. A sketch of the rays from $x = 1$.

rays with s small and positive have $\dot{x} < 0$, $\dot{\eta} > 0$ and they hit the η-axis between $\eta = 0$ and $\eta = 1$. At a critical value $\tilde{s}$ the ray is tangent to the η-axis at $\eta = 1$, so $\tilde{s}$ satisfies the equation

$$0 = x_1 + \log \tilde{s} - \tilde{s} + 1. \tag{2.35}$$

For $\tilde{s} < s < 1$ the rays do not reach the η-axis. Instead, they start with $\dot{x} < 0$ and $\dot{\eta} > 0$ until $\dot{x} = 0$ where $\eta = 1$. Thereafter, they continue into the region $\eta > 1$ with $\dot{x} > 0$ and eventually return to $x = x_1$ at some point where $\eta > 1$. The limiting ray with $s = 1$ is vertical at $x = x_1$. Some rays with $s < 0$ and $0 < s < 1$ are shown in Figure 2. We shall designate the critical ray with $s = \tilde{s}$ as $x = X_0(\eta)$. It is obtained by using (2.35) in (2.34), and is given by

$$x = X_0(\eta) \equiv \eta - \log \eta - 1. \tag{2.36}$$

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Summarizing, the rays from $x = x_1$, $\eta \leqslant 1$ fill the domain

$$R(x_1) = \{x, \eta |\ 0 \leqslant x \leqslant x_1,\ \eta \leqslant 1\} \bigcup \{x, \eta | \eta - \log \eta - 1 \leqslant x \leqslant x_1,\ \eta \geqslant 1\}. \tag{2.37}$$

As $x_1 \to \infty$, $R(x_1) \to R$ given by

$$R = \{x, \eta |\ 0 \leqslant x,\ \eta \leqslant 1\} \bigcup \{x, \eta | \eta - \log \eta - 1 \leqslant x,\ \eta \geqslant 1\}. \tag{2.38}$$

We have found the ray expansion (2.5) defined in $x \geqslant 0$, $-\infty < \eta < \infty$ and the expansion (2.33) valid in R defined by (2.38). We add them together to represent F in the asymptotic forms

$$F(\eta, x) \sim C(\varepsilon) K(\eta, x) \exp\left[\psi(\eta, x)/\varepsilon\right] + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\eta^2}{2\varepsilon}\right), \quad (x, \eta) \in R, \tag{2.39 a}$$

$$F(\eta, x) \sim C(\varepsilon) K(\eta, x) \exp\left[\psi(\eta, x)/\varepsilon\right], \quad (x, \eta) \in R^C. \tag{2.39 b}$$

Here ψ is given by (2.17) or (2.23), K by (2.27) and R^C is the complement of R in the half plane $x \geqslant 0$. It is defined by

$$R^C = \{x, \eta | 0 \leqslant x \leqslant \eta - \log \eta - 1,\ \eta \geqslant 1\}. \tag{2.40}$$

In the next section we shall determine $C(\varepsilon)$ and $K_0(A)$.

The region R^C is a shadow for the rays coming from $x = \infty$, $\eta \leqslant 1$, while the rays from $x = 0$, $\eta = 1$ have no shadow. There is a boundary layer expansion which describes the transition across the shadow boundary $x = X_0(\eta)$. We shall determine it in the next section.

3 Boundary layer expansion along the shadow boundary

First we shall find the boundary layer expansion in the neighbourhood of $x = 0$, $\eta = 1$ where the shadow boundary is tangent to the η-axis. We introduce the stretched variables u, ζ and the new function $\mathscr{F}(\zeta, u)$ defined by

$$u = x/\varepsilon^2, \quad \zeta = (\eta - 1)/\varepsilon, \quad F(\eta, x) = e^{-1/2\varepsilon} \mathscr{F}(\zeta, u)\left[1 + o(1)\right]. \tag{3.1}$$

The scalings have been chosen to make all three terms in (2.1) balance. The factor $e^{-1/2\varepsilon}$ is introduced because $\psi(1, 0) = -1/2$, so (2.39 *a*) and (2.39 *b*) contain this factor at $(1, 0)$. Use of (3.1) in (2.1) yields the leading order equation and boundary condition

$$\zeta \mathscr{F}_u = \mathscr{F}_{\zeta\zeta} + \mathscr{F}_\zeta, \quad \mathscr{F}(\zeta, 0) = 0 \text{ for } \zeta > 0. \tag{3.2}$$

To simplify (3.2) we set $\mathscr{F}(\zeta, u) = e^{-\zeta/2} H(\zeta, u)$ in it and obtain

$$\zeta H_u = H_{\zeta\zeta} - H/4. \tag{3.3}$$

We seek a solution of (3.3) in the form

$$H(\zeta, u) = \frac{1}{2\pi i} \int_C e^{u\omega} k(\zeta, \omega)\, d\omega. \tag{3.4}$$

By using (3.4) in (3.3), we see that (3.3) is satisfied if

$$k_{\zeta\zeta} - \tfrac{1}{4} k = \zeta \omega k. \tag{3.5}$$

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This equation can be transformed into the Airy equation, and the solution of it which is bounded as $\zeta \to +\infty$ is

$$k(\zeta, \omega) = a(\omega) Ai\left(\frac{\zeta\omega + 1/4}{\omega^{2/3}}\right). \tag{3.6}$$

To match the expressions for F in (3.1) and (2.39) we set $\zeta = 0$ for convenience. In (2.39 *a*) the Gaussian term has the value $(2\pi)^{-1/2}\ e^{-1/2\varepsilon}$ at $\eta = 1$. From (3.1) and the definition of H, we have $F \sim \mathscr{F} = e^{-1/2\varepsilon}\ H(0, u)$ at $\zeta = 0$, and H is given by (3.4) with k in (3.6). Matching $e^{-1/2\varepsilon} H(0, u)$ to $(2\pi)^{-1/2} e^{-1/2\varepsilon}$ yields

$$\frac{1}{2\pi i}\int_C e^{u\omega} k(0, \omega) d\omega \to \frac{1}{\sqrt{2\pi}} \text{ as } u \to +\infty. \tag{3.7}$$

It follows from (3.7) that $k(0, \omega) = a(\omega) Ai\left[1/(4\omega^{2/3})\right]$ must have a simple pole at $\omega = 0$ with residue $(2\pi)^{-1/2}$. For ω near zero the asymptotic form of Ai is [17]

$$Ai\left(\frac{1}{4\omega^{2/3}}\right) \sim \frac{1}{\sqrt{2\pi}}\ \omega^{1/6} \exp\left(-\frac{1}{12\omega}\right), \quad \omega \to 0. \tag{3.8}$$

Therefore, we write $a(\omega)$ in the form

$$a(\omega) = \exp\left(\frac{1}{12\omega}\right)\ \omega^{-7/6}\ \tilde{a}(\omega). \tag{3.9}$$

Then $\tilde{a}(0) = 1$ to give the correct residue.

By combining the preceding results we have

$$\mathscr{F}(\zeta, u) = e^{-1/2\varepsilon}\ e^{-\zeta/2} \frac{1}{2\pi i}\int_C \frac{e^{u\omega}}{\omega^{7/6}}\ \tilde{a}(\omega) \exp\left(\frac{1}{12\omega}\right) Ai\left(\frac{\zeta\omega + 1/4}{\omega^{2/3}}\right)\ d\omega. \tag{3.10}$$

We choose C to be a vertical contour with $Re(\omega) > 0$. To apply the boundary condition $\mathscr{F}(\zeta, 0) = 0$ for $\zeta > 0$ we set $u = 0$ in (3.10) and let $\zeta \to 0^+$. Then the contour integral vanishes provided that $\tilde{a}(\omega)\omega^{-1/6} \to 0$ as $\omega \to \infty$. We assume that $\tilde{a}(\omega)$ is an entire function, and then this requirement forces $\tilde{a}(\omega)$ to be constant. Matching showed that $\tilde{a}(0) = 1$, so $\tilde{a}(\omega) \equiv 1$. Thus, (3.10) becomes

$$\mathscr{F}(\zeta, u) = e^{-1/2\varepsilon}\ e^{-\zeta/2} \frac{1}{2\pi i}\int_C \frac{e^{u\omega}}{\omega^{7/6}} \exp\left(\frac{1}{12\omega}\right) Ai\left(\frac{\zeta\omega + 1/4}{\omega^{2/3}}\right)\ d\omega. \tag{3.11}$$

Next we match $\mathscr{F}$ given by (3.11) with (2.39 *b*). We expand the latter for $(x, \eta) \to (0, 1)$ in R^C and the former for $u \to \infty$, $\zeta \to \infty$ with $2u < \zeta^2$, which is the condition that (u, ζ) lie in R^C. First we find from (2.15) and (2.17) that $\varepsilon^{-1}\psi(\eta, x) = \varepsilon^{-1}\psi(1 + \varepsilon\zeta,\ \varepsilon^2 u) \sim -(2\varepsilon)^{-1} - \zeta$. Near $(x, \eta) = (0, 1)$, t is small and $K(\eta, x)$ given by (2.27) can be expanded, with the result $K(\eta, x) \sim (3/t^3)^{1/2} K_0(A)$. From (2.15) we find that near the curve $2u = \zeta^2$, A and t are given by

$$A \sim -\tfrac{3}{2}\left[x - \tfrac{1}{2}(\eta - 1)^2\right]/(\eta - 1)^3, \quad t \sim \eta - 1. \tag{3.12}$$

Thus (2.39 *b*) yields

$$e^{\psi/\varepsilon} C(\varepsilon) K(\eta, x) \sim e^{-\frac{1}{2\varepsilon} - \zeta}\ \frac{3^{1/2} C(\varepsilon)}{(\eta - 1)^{3/2}}\ K_0\left[\frac{3}{2}\ \cdot\ \frac{(\eta - 1)^2/2 - x}{(\eta - 1)^3}\right]. \tag{3.13}$$

To evaluate the integral in (3.11) we let u and $\zeta \to \infty$ with $u\zeta^{-2}$ fixed. We replace Ai by

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its asymptotic form (3.8) and set $\omega = (X^2-1)/4\zeta$ to obtain

$$\mathscr{F} \sim \exp\left(-\frac{1}{2\varepsilon}\right) e^{-\zeta/2} \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_C \frac{2\sqrt{X}}{X^2-1} \exp\left(\zeta\left[\frac{u}{4\zeta^2}(X^2-1) + \frac{1-X^3}{3(X^2-1)}\right]\right) dX. \tag{3.14}$$

The contour C goes from $\infty e^{-i\pi/4}$ to $\infty e^{i\pi/4}$ with $Re(X) > 1$. For $\zeta \to \infty$ with $\zeta^2/u = w$ fixed we can evaluate the integral by the saddle point method. The saddle point is at $X_\star(w)$ defined by

$$X_\star(w) = \frac{w}{3} - 1 + \tfrac{1}{3}(w^2+6w)^{1/2}, \quad w = \zeta^2/u. \tag{3.15}$$

$Re(X_\star(w))$ is greater than one only for $w > 2$, and then the saddle point method yields

$$\mathscr{F} \sim \exp\left(-\frac{1}{2\varepsilon}\right) e^{-\zeta/2} \frac{1}{2\pi} \frac{2\sqrt{3}}{X_\star - 1} \sqrt{\frac{X_\star+1}{X_\star+3}} \frac{1}{\sqrt{\zeta}} \exp\left\{-\zeta\left[\frac{X_\star}{6} + \frac{2}{3(X_\star+1)}\right]\right\}. \tag{3.16}$$

The curve $w = 2$, or $\zeta^2 = 2u$, is just the local form near $\eta = 1$ of the curve $x = X_0(\eta)$, which is the boundary between R and R^C. For $w < 2$ the pole at $X = 1$ in (3.14) contributes to the asymptoic expansion of the integral.

From (3.15) for $w-2$ small, $X_\star(w) \sim 1 + 3(w-2)/4$. Then the exponents in (3.16) add up to $-1/2\varepsilon - \zeta$. The other factors in (3.16) become

$$\begin{aligned} \frac{3^{1/2}}{\pi\zeta^{1/2}(X_\star-1)}\left(\frac{X_\star+1}{X_\star+3}\right)^{1/2} &\sim \frac{\sqrt{3}}{\pi\sqrt{2\zeta}} \frac{4}{3(w-2)} \\ &= \frac{\sqrt{2}}{\pi\sqrt{3}} \frac{2}{(\zeta^2-2u)} \frac{u}{\sqrt{\zeta}} \sim \frac{1}{2\pi}\sqrt{\frac{2}{3}} \frac{\zeta^{3/2}}{\zeta^2/2-u}. \end{aligned} \tag{3.17}$$

Now we set $\zeta = (\eta-1)/\varepsilon$ and $u = x/\varepsilon^2$ in (3.17), and include the exponential to get

$$\mathscr{F} \sim e^{-\frac{1}{2\varepsilon}-\zeta} \frac{(2/3)^{1/2}\,\varepsilon^{1/2}(\eta-1)^{3/2}}{2\pi\left[(\eta-1)^2/2 - x\right]}. \tag{3.18}$$

This agrees with (3.13) provided that $C(\varepsilon)$ and $K_0(A)$ are given by

$$C(\varepsilon) = \varepsilon^{1/2}, \quad K_0(A) = \frac{1}{2^{3/2}\pi A}. \tag{3.19}$$

This completes the determination of the asymptotic solution (2.5) corresponding to rays from the end point $x = 0$, $\eta = 1$.

Finally, we shall find the boundary layer solution in the region $\eta > 1$ near the curve $x = X_0(\eta)$ defined by (2.36). On this curve $A = 0$, so $K_0(A)$ given by (3.19) is singular there. Hence (2.5) is not valid on this curve because $K(\eta, x)$ is infinite there. We introduce the stretched variable Y and the new function $G(\eta, Y)$ defined by

$$Y = \frac{x - X_0(\eta)}{\sqrt{\varepsilon}}, \quad F(\eta, x) = G(\eta, Y) \exp\left(-\frac{\eta^2}{2\varepsilon}\right). \tag{3.20}$$

Then (2.1) yields for G the equation

$$\left(1 - \frac{1}{\eta}\right)^2 G_{YY} - \eta G_\eta + \sqrt{\varepsilon}\left[-\frac{1}{\eta^2} G_Y - 2\left(1 - \frac{1}{\eta}\right) G_{Y\eta}\right] + \varepsilon G_{\eta\eta} = 0. \tag{3.21}$$

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Writing $G = G^0 + o(1)$ we find that G^0 satisfies the simpler equation

$$\left(1 - \frac{1}{\eta}\right)^2 G^0_{YY} = \eta G^0_\eta. \tag{3.22}$$

By matching F given by (3.20) with (2.39) we find that G^0 must have the limits

$$G^0 \to \frac{1}{\sqrt{2\pi}}, \quad Y \to +\infty; \quad G^0 \to 0, \quad Y \to -\infty. \tag{3.23}$$

To solve (3.22) subject to (3.23) we assume that $G^0 = G^0(v)$ is a function of the similarity variable $v = Y/v(\eta)$ where $v(\eta)$ is not yet determined. From (3.22) we get

$$(1 - \eta^{-1})^2 v^{-2} G^0_{vv} = -\left(\frac{\eta v'}{v}\right) v\, G^0_v. \tag{3.24}$$

We can eliminate η by choosing v to satisfy the equation

$$(1 - \eta^{-1})^2 v^{-2} = \eta v'/v. \tag{3.25}$$

We choose $v(1) = 0$, which is necessary for F to match the solution (3.11), and the solution of (3.25) is

$$v(\eta) = \left[2\log\eta + \frac{4}{\eta} - \frac{1}{\eta^2} - 3\right]^{1/2}. \tag{3.26}$$

Now (3.24) is just $G^0_{vv} = -vG^0_v$, and its solution satisfying the conditions (3.23) is

$$G^0(\eta, Y) = \frac{1}{2\pi}\int_{-\infty}^{Y/v(\eta)} e^{-z^2/2}\, dz. \tag{3.27}$$

Thus from (3.20) the boundary layer solution for $x - X_0(\eta) = O(\varepsilon^{1/2})$ is

$$F(\eta, x) \sim e^{-\eta^2/2\varepsilon} \cdot \frac{1}{2\pi}\int_{-\infty}^{\frac{x - X_0(\eta)}{\varepsilon^{1/2} v(\eta)}} e^{-z^2/2}\, dz, \quad \eta > 1. \tag{3.28}$$

For $x > X_0(\eta)$ this result matches with the Gaussian term in (2.39 *a*), and near $x = 0$, $\eta = 1$ it matches with the local solution (3.11). For $x < X_0(\eta)$, the result matches with (2.39 *b*).

4 Conclusion

We have obtained the asymptotic form of the solution $F(\eta, x)$ of the boundary value problem (2.1)–(2.3) for ε small. The asymptotic solution has four different representations in four regions of the half-plane $x \geqslant 0$. In the region R^C above the shadow boundary $x = X_0(\eta)$, $\eta \geqslant 1$, F is given by (2.39 *b*) which is constructed by the ray method. Only one ray family is present there, and for it the exponent ψ becomes infinite like a negative multiple of $1/x$ as $x \to 0$. Therefore F and all its x derivatives vanish as $x \to 0$ for $\eta > 1$. In the region R below the shadow boundary, F is represented as the sum of two terms in (2.39 *a*), both of which are found by the ray method. These two ray expansions determine the solution at $x = 0$ for $\eta < 1$. Near the shadow boundary but away from $x = 0$, F is given by the boundary layer expression (3.28). Near the point (0,1) where the shadow boundary is tangent to the line $x = 0$, F is given by the local expression (3.11).

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The same methods of analysis can be used to solve other problems involving backward–forward parabolic equations.

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